• Inverse Fourier Transform

Example: Suppose that

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$$

Evaluate $x[n]$

Ans:

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$$

$$= \frac{A}{1-ae^{-j\omega}} + \frac{B}{1-be^{-j\omega}}$$

$$1 = A(1-be^{-j\omega}) + B(1-ae^{-j\omega})$$

when $e^{-j\omega} = \frac{1}{b}$

$\Rightarrow$

$$1 = B\left(1-\frac{a}{b}\right)$$

$$1 = B\left(\frac{b-a}{b}\right) \Rightarrow B = \frac{a}{b-a}$$

When $e^{-j\omega} = \frac{1}{a}$

$$1 = A\left(1-\frac{b}{a}\right) \Rightarrow 1 = A\left(\frac{a-b}{a}\right) \rightarrow A = \frac{a}{a-b}$$

$$X(e^{j\omega}) = \frac{a}{a-b}\,\frac{1}{1-ae^{-j\omega}} + \frac{a}{b-a}\,\frac{1}{1-be^{-j\omega}}$$

$$= \frac{a}{a-b}(a)^n u[n] - \frac{a}{a-b}(b)^n u[n]$$

• Z- transform

- The Fourier Transform of a sequence $x[n]$ was defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

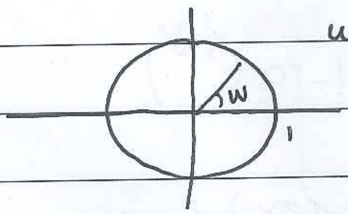
- The z-transform of sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

where z can be expressed in polar form as

$$z = r e^{j\omega}$$

$|z| = 1$



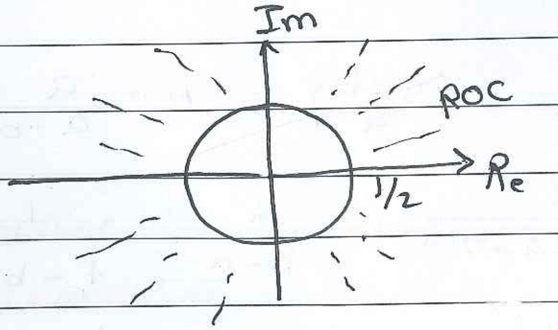
Example: Consider the following signal

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n]$$ [called right-sided exponential seq]

① Evaluate $X(z)$. ② Evaluate and Plot ROC

Ans: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}}$$



ROC $|z| > 1/2$

FT exist

Note that: FT converges (exist) only if z-transform ($X(z)$) converges at $|z|=1 \Rightarrow$ i.e, ROC contains unit circle ($|z|=1$).

Example: Consider the signal $x[n] = a^n u[n]$

ⓐ Evaluate $X(z)$
ⓑ Plot the ROC
ⓒ Does FT exists

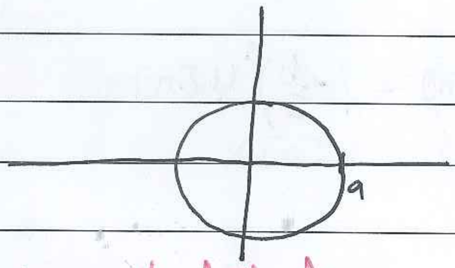
Ans:

$x(n) = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \frac{1}{1 - a z^{-1}}$$

$$= \frac{z}{z - a}$$



It can be noted that:

a. The Fourier Transform of $x[n]$ exists if $|a|<1$

b. For $|a|=1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

c. For $|a|>1$, the ROC does not include the unit circle, consistent with the fact that, for these values of a, the Fourier transform of the exponentially growing sequence $a^n u[n]$ does not converge.

Example: Consider the following signal $x[n] = -a^n u[-n-1]$ (called left-Sided Exponential sequence).

(a) Evaluate $X(z)$

(b) Plot ROC

(c) Does FT exists

Ans: $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

$$= 1 - \frac{1}{1 - a^{-1} z} \qquad |z| < |a|$$

the FT does not exists in case $|a| < 1$.

Example: Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

(a) Evaluate $X(z)$

(b) plot ROC

(c) Specify zeros and poles.

Ans: $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{3}\right)^n u[n]$

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}}$$

$$= \frac{z}{z - \frac{1}{2}} + \frac{z}{z + \frac{1}{3}}$$

$$= \frac{(z + \frac{1}{3}) z + z (z - 1/2)}{(z - 1/2)(z + 1/3)}$$

$$X(z) = \frac{z^2 + \frac{1}{3}z + z^2 - \frac{1}{2}z}{(z-1/2)(z+1/3)}$$

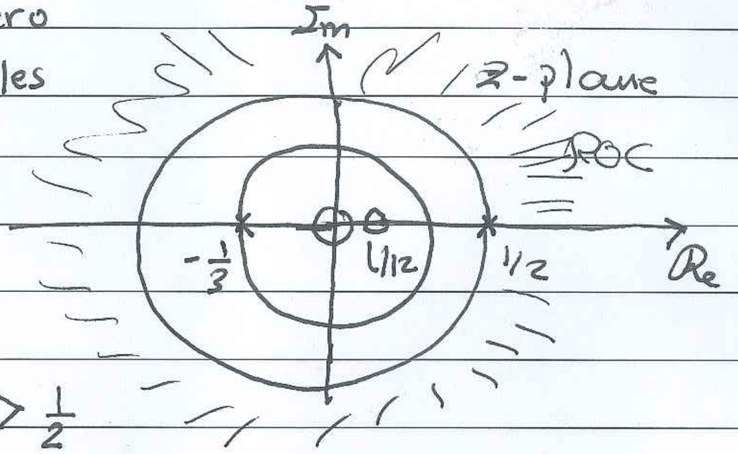
$$= \frac{2z^2 - \frac{1}{6}z}{(z-1/2)(z+1/3)}$$

$$= \frac{2z(z-1/12)}{(z-1/2)(z+1/3)} \begin{array}{l} \leftarrow \text{zeros} \\ \leftarrow \text{poles} \end{array}$$

let us defined :-

o zero

x poles



ROC $|z| > \frac{1}{2}$

Example: Consider the sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

ⓐ Evaluate $X(z)$

ⓑ plot ROC

Ans: