• Z-transform

• The Fourier Transform of a sequence $x[n]$ was defined as

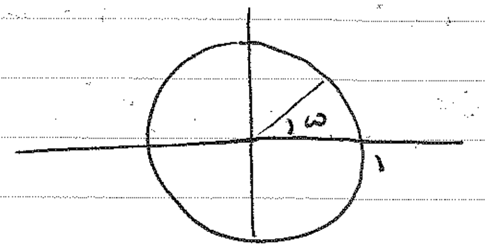
$$X(e^{j\omega}) = \sum_{n=\infty}^{\infty} x[n]\, e^{-j\omega n}$$

• The Z-transform of sequence $x[n]$ is defined as

$$X(z) = \sum_{n=\infty}^{\infty} x[n]\, z^{-n}$$

where z can be expressed in polar form as

$$z = r\, e^{j\omega} \quad ; \quad r = |z| = 1$$



unit circle

Example: Consider the following signal

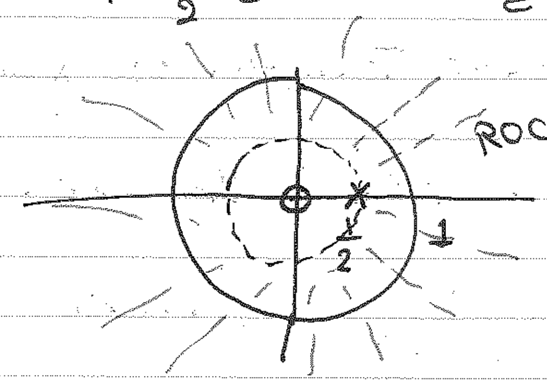
$$x[n] = \left(\frac{1}{2}\right)^n u[n] \quad \text{[called right-sided exponential sequence]}$$

1. Evaluate $X(z)$
2. Evaluate and Plot Region of convergence (ROC)

Ans:

$$X(z) = \sum_{n=\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$\Rightarrow X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-1/2}$$



$|z| > 1/2$

where $\circ$ denotes zero
$\times$ denotes pole

On the other hand, it can be noted that Fourier Transform (FT) converges (exist) only if z-transform ($X(z)$) converges at $|z|=1 \Rightarrow$ i.e, ROC contains unit circle ($|z|=1$)

Therefore, in our example, F.T exists since ROC contains unit circle ($|z|=1$).

Example: Consider the signal $x[n] = a^n u[n]$

1. Evaluate $X(z)$
2. Plot ROC
3. Does FT exists

Ans: $x[n] = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1-az^{-1}}$$

$$= \frac{z}{z-a}$$

It can be noted:

a. The Fourier Transform of $x[n]$ exist if $|a|<1$

b. For $|a|=1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1-z^{-1}}, \quad |z|>1$$

c. For $|a|>1$, the ROC does not include the unit circle, consistent with the fact that, for these values of a, the Fourier Transform of the exponentially growing sequence $a^n u[n]$ does not converge.

Example: Consider the following signal $x[n] = -a^n u[-n-1]$ (called left-sided Exponential Sequence).

1. Evaluate $X(z)$
2. Plot ROC
3. Does FT exists

Ans:
$$x[n] = -a^n u[-n-1]$$
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n}$$
$$= -\sum_{n=\infty}^{-\infty} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^{n} = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$
$$= 1 - \frac{1}{1-a^{-1}z}$$

$\Rightarrow$ ROC at $|z|<|a|$, where the FT does not exists in case $|a|<1$

Example: Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{3}\right)^n u[n]$$

1. Evaluate $X(z)$
2. Plot ROC
3. Specify zeros and poles

Ans:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{3}\right)^n u[n]$$

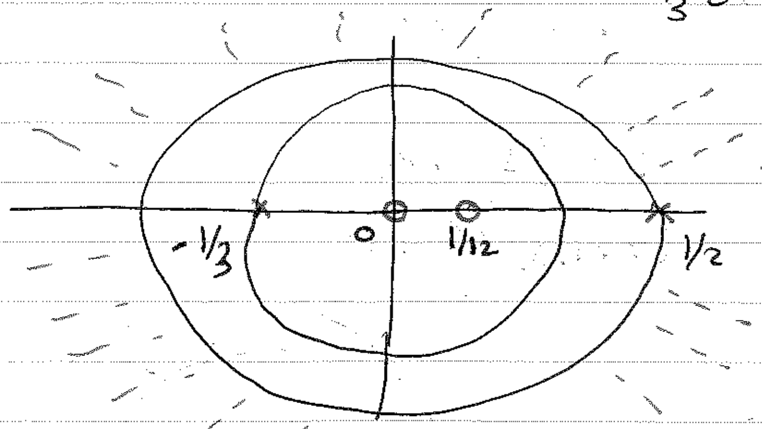
$$x[n] = x_1[n] + x_2[n]$$

$$X(z) = X_1(z) + X_2(z)$$

where

$$X_1(z) = Z\left[\left(\frac{1}{2}\right)^n u[n]\right] = \frac{1}{1 - \frac{1}{2}z^{-1}} \quad ; \quad |z| > 1/2$$

$$X_2(z) = Z\left[\left(\frac{-1}{3}\right)^n u[n]\right] = \frac{1}{1 + \frac{1}{3}z^{-1}} \quad ; \quad |z| > 1/3$$



ROC $|z| > 1/2$

o : zero
x : pole

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$= \frac{2z\,(z - 1/12)}{(z - 1/2)\,(z + 1/3)}$$

Example: Consider the sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

1. Evaluate $X(z)$
2. plot ROC

Ans:

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$= x_1[n] + x_2[n]$$

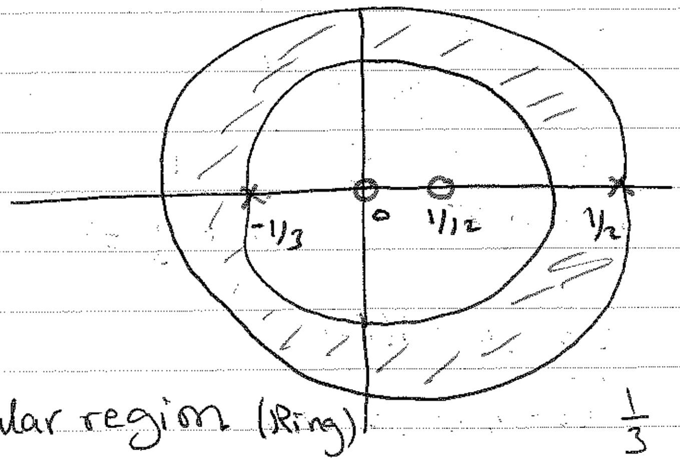
$$X(z) = X_1(z) + X_2(z)$$

where

$$X_1(z) = \frac{1}{1 + \frac{1}{3} z^{-1}} \quad ; \quad |z| > 1/3$$

$$X_2(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \quad ; \quad |z| < 1/2$$

$$X(z) = \frac{2z\,(z - 1/12)}{(z + 1/3)(z - 1/2)}$$



ROC of $X(z)$ is annular region (Ring) $\frac{1}{3} < |z| < \frac{1}{2}$

In addition, it can be noted that FT does not exist since ROC does not include unit circle ($|z|=1$).

Example (Finite Sequence): Consider the sequence

$$x[n] = \delta[n] + \delta[n-5]$$

Evaluate $X(z)$.

Ans: $x[n] = \delta[n] + \delta[n-5]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = 1 + z^{-5}$$

Example: Consider the following signal

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{o.w} \end{cases}$$

Evaluate $X(z)$

Ans:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}}$$

$$= \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

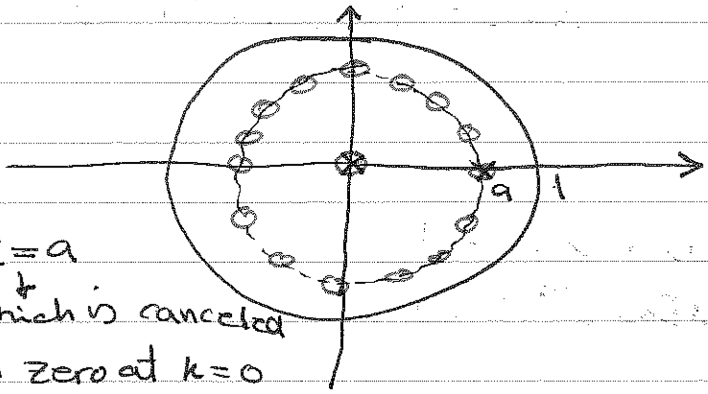
It can be noted:

- Since there are only finite number of non-zero terms, the sum will be finite as long as (az^{-1}) is finite

• For example, if we assume $N=16$ and a is real where, $0<|a|<1 \Rightarrow X(z)$ has N zeros at:

$$Z_k = a\, e^{j2\pi k/N}, \quad k=0,1,2,\ldots,N-1$$

which satisfy $z^N = a^N$



poles $z=0$, $z=a$
which is canceled with zero at $k=0$

In addition, it can be noted that there is one pole at $z=0$, and the remaining zeros at $Z_k = a\, e^{j2\pi k/N}$; $k=1,2,3,\ldots,N-1$

• Causality & stability

The system will be :

1. causal : • If $h(n)=0$ for $n<0$
 • has right-sided sequence
 • ROC is outside outermost pole.

2. stable : if the ROC includes the unit circle.

Example: Consider the following difference equation of the system

$$y[n] - \frac{1}{2}\, y[n-1] = x[n]$$

check the stability, and causality of the system.

- Inverse Fourier Transform

Example: Suppose that

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$$

Evaluate $x[n]$

Ans:

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$$

$$= \frac{A}{1-ae^{-j\omega}} + \frac{B}{1-be^{-j\omega}}$$

$\Rightarrow$

$$1 = A(1-be^{-j\omega}) + B(1-ae^{-j\omega})$$

when $e^{-j\omega} = \frac{1}{b}$

$\Rightarrow$

$$1 = B\left(1-\frac{a}{b}\right) \Rightarrow B = \frac{a}{b-a}$$

when $e^{-j\omega} = \frac{1}{a}$

$\Rightarrow$

$$1 = A\left(1-\frac{b}{a}\right) \Rightarrow A = \frac{a}{a-b}$$

$$X(e^{j\omega}) = \frac{a}{a-b}\,\frac{1}{1-ae^{-j\omega}} + \frac{a}{b-a}\,\frac{1}{1-be^{-j\omega}}$$

$$= \frac{a}{a-b}\,a^n u[n] - \frac{a}{a-b}\,b^n u[n]$$

If so, give the appropriate region of convergence.

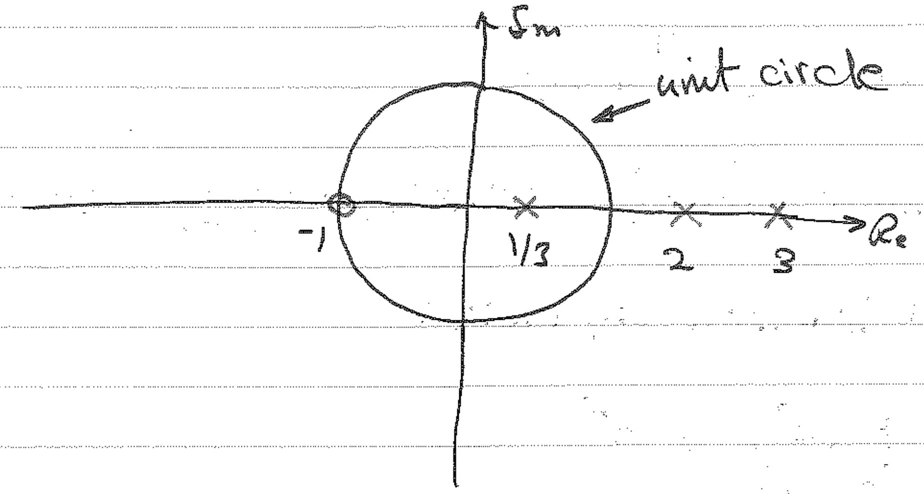


Fig: Problem # 2

- **The inverse z-transform**

In this section, we will consider some procedures, specifically the inspection method, partial fraction expansion, and power series expansions.

1. **Inspection Method**

In general,

$$a^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \;;\; |z|>|a|$$

$$-a^n u[-n-1] \stackrel{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \;;\; |z|<|a|$$

Example: Evaluate the inverse z-transform of the

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \; |z|>1/2$$

Ans: By inspection Method (From Table 3.1 Text box).

$$x(n) = \left(\frac{1}{2}\right)^n u[n]$$

and if we assume $|z|<1/2 \Rightarrow x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$

2. Partial Fraction Expansion

$$X(z) = \frac{P(z^{-1})}{Q(z^{-1})} = z^N$$

$$= \frac{\sum_{k=1}^{M} b_k z^{-k}}{\sum_{k=1}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=1}^{M} b_k z^{M-k}}{z^M \sum_{k=1}^{N} a_k z^{N-k}}$$

where we have M zeros and N poles at non-zero location in Z-plane.

In addition, it can be noted that, there will be either M−N poles at z=0 if M>N or N−M zeros at z=0 if N>M

In other words, z-transform of the form equation above always have the same number of poles and zeros in the finite z-plane, and there are no poles or zeros at $z=\infty$.

To obtain the partial fraction expansion of X(z), it is most convenient to note that X(z) could be expressed in the form

Ans: $y[n] - \frac{1}{2} y[n-1] = x[n]$

$$Y(z) - \frac{1}{2} Y(z) z^{-1} = X[z]$$

$$\left[1 - \frac{1}{2} z^{-1}\right] Y(z) = X[z]$$

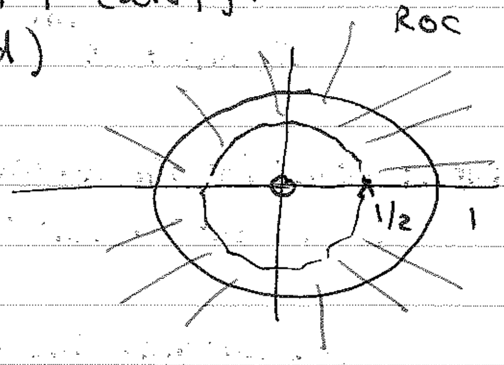
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

There are two choices for ROC

1. $|z| > \frac{1}{2}$

In this case, the system will be:

- Stable system (FT exist).
- Causal (Right-Sided)
- $h[n] = \left(\frac{1}{2}\right)^n u[n]$



2. $|z| < 1/2$

In this case, the system will be:

- Not stable (F.T does not exist)
- Non-causal
- $h[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$

Exercises:

Problem#1: Determine the z-transform, including the region of convergence, for each of the following sequences;

1. $x_1[n] = \left(\frac{1}{2}\right)^n u[-n]$

2. $x_2[n] = \delta[n]$

3. $x_3[n] = \delta[n+1]$

4. $x_4[n] = \left(\frac{1}{2}\right)^n (u[n] - u[n-10])$

5. $x_5[n] = \begin{cases} n & 0 \le n \le N-1 \\ N & N \le n \end{cases}$

6. $x_6[n] = \alpha^{|n|} \quad ; \; 0 < |\alpha| < 1$

Problem#2: Consider the z-transform $X(z)$ whose pole-zero plot is as shown below.

1. Determine the region of convergence of $X(z)$ if it is known that the Fourier Transform exists. For this case, determine whether the corresponding sequence $x[n]$ is right sided, left sided, or two sided.

2. How many possible two-sided sequences have the pole-zero plot shown in figure.

3. Is it possible for the pole-zero plot shown in figure to be associated with a sequence that is both stable and causal?

$$X(z) = \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Example: Consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})} \;, \quad |z| > 1/2$$

Evaluate $x(n)$.

Ans: By using Partial Fraction Expansion Method

$$X(z) = \frac{1}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})}$$

$$= \frac{A}{1 - \frac{1}{4} z^{-1}} + \frac{B}{1 - \frac{1}{2} z^{-1}}$$

$$1 = A(1 - \tfrac{1}{2} z^{-1}) + B(1 - \tfrac{1}{4} z^{-1})$$

at $z^{-1} = 2$

$\Rightarrow$

$$1 = B(1 - \tfrac{1}{2}) \Rightarrow B = 2$$

and when $z^{-1} = 4$

$$\Rightarrow 1 = A(1 - 2) \Rightarrow A = -1$$

$$\Rightarrow \quad X(z) = \frac{-1}{1 - \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

Example: Consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} \; ; \; |z|>1$$

Evaluate $x[n]$

ANS: Since $M=N=2$ and the poles are all first order, $X(z)$ can be represented as

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

where the constant B_0 can be found by long division

$$\begin{array}{r|l} & \quad 2 \\ \hline 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} & 1+2z^{-1}+z^{-2} \\ & \mp 2 \pm 3z^{-1} \mp z^{-2} \\ \hline & -1+5z^{-1} \end{array}$$

$$X(z) = 2 + \frac{5z^{-1}-1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$= 2 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

By using Partial fraction Expansion $\Rightarrow$

$A_1 = -9$ and $A_2 = 8$

$\Rightarrow$

$$x[n] = \mathcal{Z}^{-1}[X(z)] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n + 8\,(1)^n u[n]$$

- Power Series Expansion

If the z-transform is given as a power series in the form

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-2] z^{2} + x[-1] z + x[0] + x[1] z^{-1} + \cdots$$

we can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}.

Example:

Suppose $X(z)$ is given in the form

$$X(z) = z^2 \left(1 - \tfrac{1}{2} z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

Evaluate $x[n]$.

Ans: By multiplying the factors in

$$X(z) = z^2 \left(1 - \tfrac{1}{2} z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

we can express $X(z)$ as

$$X(z) = z^2 - \tfrac{1}{2} z - 1 + \tfrac{1}{2} z^{-1}$$

Therefore, by inspection, $x[n]$ is seen to be

$$x[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & \text{o.w.} \end{cases}$$

Equivalently

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1]$$

Example: Consider the z-transform

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$

Evaluate $x[n]$

Ans:

$$X(z) = \log(1 + az^{-1}) \quad ; \quad |z| > |a|$$

By using the power series expansion for $\log(1+x)$, with $|x| < 1$, we obtain

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

Therefore,

$$x[n] = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

Example: Consider the z-transform

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

Evaluate $x[n]$ by using long division Method

Ans:

$$\begin{array}{r|l} & 1 + az^{-1} + a^2 z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & 1 - az^{-1} \\ \hline & az^{-1} \\ & az^{-1} - a^2 z^{-2} \\ \hline & a^2 z^{-2} \cdots \end{array}$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

$$\Rightarrow \quad x[n] = a^n u[n]$$

- Differentiation of $X(z)$

$$n\,x[n] \xleftrightarrow{z} -z\frac{\partial X(z)}{\partial z}, \quad ROC = R_x$$

Proof:

$$X(z) = \sum_{n=\infty}^{\infty} x[n]\,z^{-n}$$

$$\frac{\partial X(z)}{\partial z} = \sum_{n=\infty}^{\infty} -n\,x[n]\,z^{-n-1}$$

$$-z\frac{\partial X(z)}{\partial z} = \sum_{n=\infty}^{\infty} n\,x[n]\,z^{-n}$$

$$= \mathcal{Z}\left[n\,x[n]\right]$$

Example: Consider the following sequence

$$X(z) = \log(1+az^{-1}), \quad |z| > |a|$$

Using differentiation Method to evaluate the inverse - z - transform of $X[z]$

Ans:

$$X(z) = \log(1+az^{-1})$$

$$\frac{\partial X(z)}{\partial z} = -\frac{az^{-2}}{1+az^{-1}}$$

From differentiation property

$$n\,x[n] \xleftrightarrow{z} -z\frac{\partial X(z)}{\partial z} = \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a|$$

Example: Consider the z-transform

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

Using long division method to evaluate $x[n]$

Ans:

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^{-2} - - - \\ \hline -a+z & z \\ & z - a^{-1}z^{2} \\ \cline{2-2} & a^{-1}z^{2} \end{array}$$

Therefore, $x[n] = -a^{n}u[-n-1]$

• z-transform Properties

In this section, we consider some of the most frequently used properties. In the following discussion, $X(z)$ denotes the z-transform of $x[n]$, and the ROC of $X(z)$ is indicated by R_x; i.e.,

$$x[n] \overset{z}{\longleftrightarrow} X(z) \qquad ROC = R_x$$

• Linearity

$$\mathcal{Z}\left[a x_1[n] + b x_2[n]\right] = \sum_{n=\infty}^{\infty} \left(a x_1[n] + b x_2[n]\right) z^{-n}$$

$$= \sum_{n=\infty}^{\infty} a x_1[n] z^{-n} + \sum_{n=\infty}^{\infty} b x_2[n] z^{-n}$$

$$= a X_1(z) + b X_2(z)$$

ROC contains $R_{x_1} \cap R_{x_2}$

Example: Consider

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{3}\right)^n u[n]$$

Evaluate $X(z)$

Ans:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{3}\right)^n u[n]$$
$$= x_1[n] + x_2[n]$$
$$X(z) = X_1(z) + X_2(z)$$

- Time shifting

$$x[n-n_0] \xleftrightarrow{z} z^{-n_0} X(z)$$

ROC $= R_x$ (except for the possible addition or deletion of $z=0$ or $z=\infty$).

$$\mathcal{Z}[x[n-n_0]] = \sum_{n=-\infty}^{\infty} x[n-n_0]\, z^{-n}$$

let $m = n-n_0 \Rightarrow n = m+n_0$

when $n=-\infty \Rightarrow m=-\infty$ and when $n=\infty \Rightarrow m=\infty$

$$\Rightarrow \mathcal{Z}[x[n-n_0]] = \sum_{m=-\infty}^{\infty} x[m]\, z^{-(m+n_0)} = z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]\, z^{-m}$$
$$= z^{-n_0} X(z).$$

Example: Consider the z-transform

$$X(z) = \frac{1}{z-\frac{1}{4}}, \quad |z| > 1/4$$

Evaluate $x[n]$

Ans: From the ROC, we identify this as corresponding to a right-sided sequence.

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}}, \quad |z| > 1/4$$

$$x(n) = \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

- Multiplication by an Exponential Sequence

$$z_0^n x[n] \xleftrightarrow{z} X(z/z_0) \qquad ROC = |z_0| R_x$$

Proof:

$$Z[z_0^n x[n]] = \sum_{n=\infty}^{\infty} z_0^n x[n] z^{-n}$$

$$= \sum_{n=\infty}^{\infty} x[n] (z/z_0)^{-n}$$

$$= X(z/z_0)$$

Example: Consider the following sequence

$$x[n] = r^n \cos(\omega_0 n) u[n]$$

Evaluate $X(z)$

Ans:

$$x[n] = r^n \cos(\omega_0 n) u[n]$$

$$= r^n \left(\frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}\right) u[n]$$

$$= \frac{1}{2} r^n e^{j\omega_0 n} u[n] + \frac{1}{2} r^n e^{-j\omega_0 n} u[n]$$

$$= \frac{1}{2} \cdot \frac{1}{1 - re^{j\omega_0} z^{-1}} + \frac{1}{1 - re^{-j\omega_0} z^{-1}}, \quad |z| > r$$

• Differentiation of $X(z)$

$$n\,x[n] \xleftrightarrow{z} -z\,\frac{\partial X(z)}{\partial z}, \qquad ROC = R_x$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$\frac{\partial X(z)}{\partial z} = \sum_{n=-\infty}^{\infty} -n\, x[n]\, z^{-n-1}$$

$$-z\,\frac{\partial X(z)}{\partial z} = \sum_{n=-\infty}^{\infty} n\, x[n]\, z^{-n}$$

$$= \mathcal{Z}\left[n\, x[n]\right]$$

Example: Consider the following sequence

$$X(z) = \log(1 + a z^{-1}), \qquad |z| > |a|$$

Using differentiation Method to evaluate the inverse - z - transform of $X[z]$

Ans:

$$X(z) = \log(1 + a z^{-1})$$

$$\frac{\partial X(z)}{\partial z} = -\frac{a z^{-2}}{1 + a z^{-1}}$$

From differentiation property

$$n\,x[n] \xleftrightarrow{z} -z\,\frac{\partial X(z)}{\partial z} = \frac{a z^{-1}}{1 + a z^{-1}}, \quad |z| > |a|$$

Example: Using differentiation property to determine the z-transform of the sequence

$$x[n] = n a^n u[n]$$

Ans:

$$X(z) = -z \frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right), \quad |z| > |a|$$

$$= \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

Therefore,

$$n a^n u[n] \xleftrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

- Conjugation of a Complex Sequence

The conjugation property is expressed as

$$x^*[n] \xleftrightarrow{z} X^*(z^*) \qquad ROC = R_x$$

- Time Reversal

By the time-reversal property,

$$x^*[-n] \xleftrightarrow{z} X^*(1/z^*), \qquad ROC = \frac{1}{R_x}$$

Example: Using the time reversal property

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \quad |z| < |a^{-1}|$$

- Convolution of Sequence

According to the convolution property

$$x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z)\, X_2(z) \quad \text{ROC} \quad R_{x_1} \cap R_{x_2}$$

To derive this property formally, we consider

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \right\} z^{-n}$$

If we interchange the order of summation,

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m] z^{-m} \right\} z^{-k}$$

$$Y(z) = X_1(z) X_2(z)$$

Example: Let $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$

Evaluate the z-transform of $y[n] = x_1[n] * x_2[n]$

Ans: $y[n] = x_1[n] * x_2[n]$

$$Y(z) = X_1(z)\, X_2(z)$$

where

$$X_1(z) = Z[a^n u[n]]$$, assume $|z|>|a|$ and $|a|<1$

$$X_1(z) = \frac{1}{1-az^{-1}}$$

$$X_2(z) = Z[u[n]]$$; assume $|z|>1$

$$X_2(z) = \frac{1}{1-z^{-1}}$$

$\Rightarrow$

$$Y(z) = \frac{1}{(1-az^{-1})(1-z^{-1})} = \frac{z^2}{(z-a)(z-1)}, \quad |z|>1$$

- Initial-Value Theorem

If $x[n]$ is zero for $n<0$ (i.e, if $x[n]$ is causal), then

$$x[0] = \lim_{z\to\infty} X(z)$$

• Impulse Response for Rational System Functions:

Any rational function of z^{-1} with any only first-order poles can be expressed in the form:

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

- Two classes of LTI system:

1. IIR: at least one non-zero pole at $H(z)$ is not cancelled by a zero. $\Rightarrow$ at least one term of $A_k (d_k)^n u[n]$, and so $h[n]$ will not be of finite length (i.e, will not be zero outside of finite interval), such systems are called Infinite Impulse Response (IIR) systems.

Example: show that the following system whose DFE is given by

$$y[n] - a y[n-1] = x[n] \,; \quad |a| < 1 \quad \text{and} \quad |z| > |a|$$

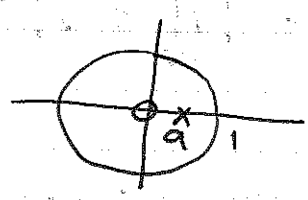
is IIR system

Ans: The Z-transform of transfer function

$$H(z) = \frac{1}{1 - a z^{-1}}$$

Since we have one pole $\Rightarrow$ The system is IIR.

$$h[n] = a^n u[n]$$



- **FIR:**
  - $H(z)$ has no poles except at $z=0$, i.e $N=0$
  - A partial fraction expansion is not possible.
  - $H(z)$ will be a polynomial in z^{-1}, i.e,

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} \quad (\text{assume } a_0 = 1)$$

  - $\cancel{h[n] = \sum_{k=0}^{M} b_k z^{-k}}$
  - $h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{o.w} \end{cases}$

- ALL-Pass Systems: $\left[\, |H_{ap}(e^{j\omega})| = 1 \,\right]$

  - All-Pass Systems mean that poles and zeros are in conjugate reciprocal pairs.

$$H_{ap}(\cancel{e^{j\omega}} z) = \prod_{i=1}^{P} \frac{z^{-1} - c_i^*}{1 - c_i z^{-1}} \quad ; \quad \text{poles: } c_i = r e^{j\Theta}, \quad \frac{1}{c_i^*} = \frac{1}{r} e^{j\Theta}$$

Example: Show that $|H_{ap}(e^{j\omega})| = 1$, assume $P = 1$

Proof:- $$|H_{ap}(e^{j\omega})| = \left| \frac{e^{-j\omega} - c^*}{1 - c e^{-j\omega}} \right| = \left| \frac{e^{-j\omega}(1 - c^* e^{j\omega})}{1 - c e^{-j\omega}} \right|$$

Assume $b = 1 - c e^{-j\omega} \Rightarrow$ $$= \frac{|b^*|}{|b|} = 1$$

It can be noted that: All pass systems form as product of

$$\frac{z^{-1} - c^*}{1 - c z^{-1}}$$

Minimum-Phase and All-Pass decomposition

To factor $H(z) = H_{min} \cdot H_{ap}(z)$

1. Take zeros that lie outside $|z| = 1$ (unit-circle) and move to $H_{ap}(z)$

2. Add poles to $H_{ap}(z)$ in conjugate reciprocal locations of zeros.

3. Put zeros $H_{min}(z)$ to cancel poles added to $H_{ap}(z)$

Example: Suppose $H(z) = H_1(z)\,(1 - \beta z^{-1})$; $|\beta| > 1$, decompose $H(z)$ into min-phase and all-phase.

Ans:

$$H(z) = H_1(z)\,(1 - \beta z^{-1})$$

$$= H_1(z)\,(-\beta)\,(z^{-1} - \tfrac{1}{\beta}) \cdot \frac{(1 - \frac{1}{\beta^*} z^{-1})}{(1 - \frac{1}{\beta^*} z^{-1})}$$

$$= \underbrace{H_1(z)\,(-\beta)\,(1 - \tfrac{1}{\beta^*} z^{-1})}_{H_{min}(z)} \; \underbrace{\frac{(z^{-1} - 1/\beta)}{(1 - 1/\beta^* z^{-1})}}_{H_{ap}(z)}$$

Note that: The min-phase portion of any system has a stable, causal inverse system.

Example: Decompose $H(z)$ into min-phase and all-pass.

$$H(z) = \left(1 - \frac{1}{0.9} z^{-1}\right)\left(1 + \frac{1}{0.9} z^{-1}\right)\left(1 - j0.7 z^{-1}\right)\left(1 + j0.7 z^{-1}\right)$$

$$= \frac{-1}{0.81}\left(z^{-1} - 0.9\right)\left(z^{-1} + 0.9\right)\left(1 - j0.7 z^{-1}\right)\left(1 + j0.7 z^{-1}\right)$$

$$\cdot \frac{(1 - 0.9z^{-1})}{(1 - 0.9z^{-1})} \cdot \frac{(1 + 0.9z^{-1})}{(1 + 0.9z^{-1})}$$

$$\Rightarrow H_{ap}(z) = \frac{(z^{-1} - 0.9)}{1 - 0.9z^{-1}} \cdot \frac{(z^{-1} + 0.9)}{(1 + 0.9z^{-1})}$$

$$H_{min} = -\frac{1}{0.81}\left(1 - 0.9z^{-1}\right)\left(1 + 0.9z^{-1}\right)\left(1 - j0.7z^{-1}\right)\left(1 + j0.7z^{-1}\right)$$

Example: Consider the sequence of $H(z)$ is given by

$$H(z) = \frac{1 + 5z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

Decompose $H(z)$ into H_{min} and H_{ap}.

Ans:

$$H(z) = \frac{1 + 5z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

$$= 5\,\frac{z^{-1} + 1/5}{1 + \frac{1}{2}z^{-1}} \cdot \frac{1 + \frac{1}{5}z^{-1}}{1 + \frac{1}{5}z^{-1}}$$

$$= \frac{5\,(1 + 1/5\, z^{-1})}{(1 + 1/2\, z^{-1})} \cdot \frac{z^{-1} + 1/5}{1 + \frac{1}{5}z^{-1}}$$

$$= \underbrace{}_{H_{min}(z)} \quad \underbrace{}_{H_{ap}(z)}$$