Impulse Response for Rational System Functions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$\Downarrow$ DFE

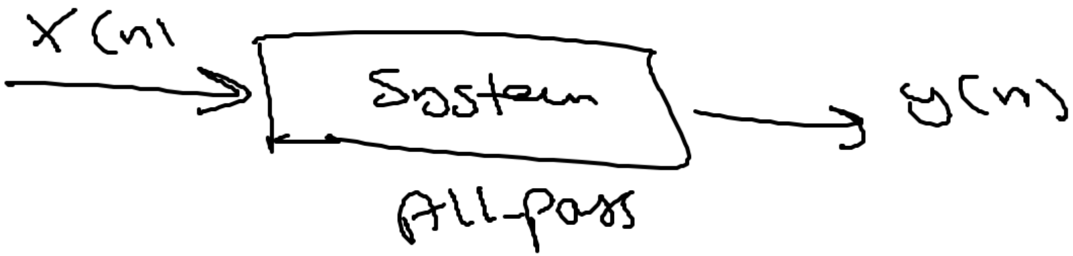
$$H_2(z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} = IIR$$

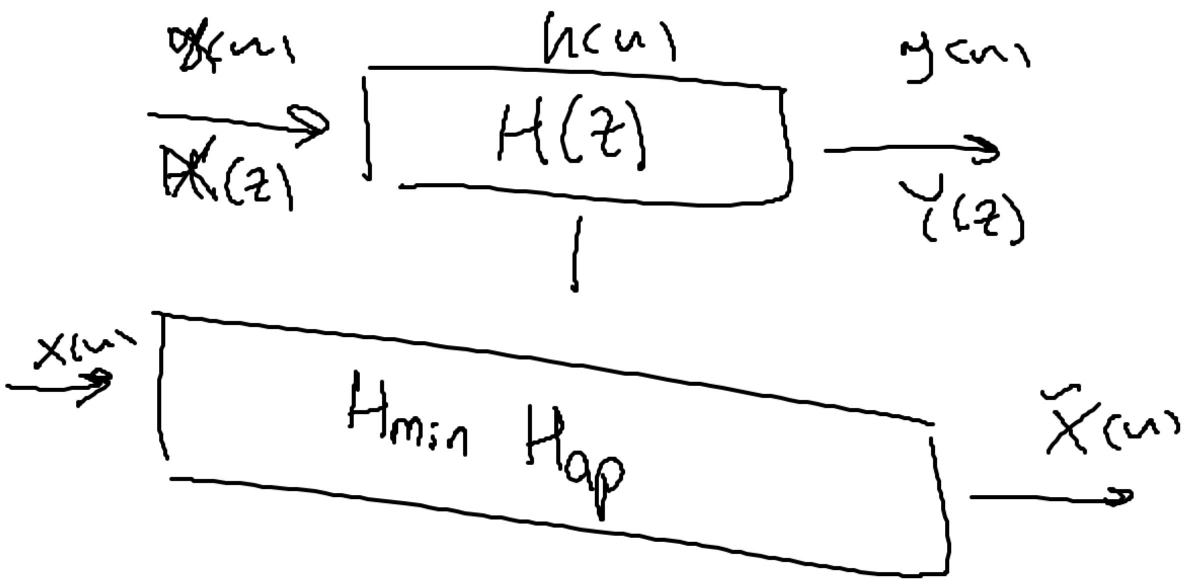
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \Rightarrow h(n) = \sum_{r=0}^{M-N} B_r \delta(n-r) + \sum_{k=1}^{N} A_k (d_k)^n u[n]$$

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} = FIR$$

⇒ All-Pass System





$$H_{ap}(z) = \frac{z^{-1} - c^*}{1 - c z^{-1}}$$

$$|H_{ap}(z)| = 1 \Rightarrow |H(e^{j\omega})| = 1$$

Ex: $$H(z) = \Big(1 - \overset{>1}{\frac{1}{0.9}} z^{-1}\Big)\Big(1 + \overset{>1}{\frac{1}{0.9}} z^{-1}\Big)\Big(1 - \overset{<1}{j0.7} z^{-1}\Big)\Big(1 + \overset{<1}{j0.7} z^{-1}\Big)$$

Decompose $H(z)$ into min-phase and All-pass

Ans.

$$H(z) = \left(1 - \frac{1}{0.9} z^{-1}\right)\left(1 + \frac{1}{0.9} z^{-1}\right)\left(1 - j0.7 z^{-1}\right)\left(1 + j0.7 z^{-1}\right)$$

$$= \left(-\frac{1}{0.9}\right)\left(z^{-1} - 0.9\right)\left(\frac{1}{0.9}\right)\left(z^{-1} + 0.9\right)\left(1 - j0.7 z^{-1}\right)\left(1 + j0.7 z^{-1}\right)$$

$$= \left(-\frac{1}{0.81}\right) \frac{\left(z^{-1} - 0.9\right) \cdot \left(1 - 0.9^{*} z^{-1}\right)}{\left(1 - 0.9^{*} z^{-1}\right)} \frac{\left(z^{-1} + 0.9\right) \cdot \left(1 + 0.9 z^{-1}\right)}{1 + 0.9 z^{-1}}$$

$$\left(1 - j0.7 z^{-1}\right)\left(1 + j0.7 z^{-1}\right)$$

$$H(z) = \left(\frac{-1}{0.81}\right)\left(\frac{(z^{-1}-0.9)}{1-0.9z^{-1}}\right)\left(\frac{z^{-1}+0.9}{1+0.9z^{-1}}\right)(1-0.9z^{-1})(1+0.9z^{-1})(1-j0.7z^{-1}) \cdot (1+j0.7z^{-1})$$

$$\Rightarrow H_{ap}(z) = \left(\frac{z^{-1}-0.9}{1-0.9z^{-1}}\right)\left(\frac{z^{-1}+0.9}{1+0.9z^{-1}}\right)$$

$$H_{min} = \left(\frac{-1}{0.81}\right)(1-0.9z^{-1})(1+0.9z^{-1})(1-j0.7z^{-1})(1+j0.7z^{-1})$$

$$\xrightarrow{x(n)} \boxed{\underset{= H_{min} H_{ap}}{H(z)}} \rightarrow \boxed{1/H_{min}} \xrightarrow{y(n) \simeq} x(n)$$

Example: Consider the sequence of $H(z)$ is given by

$$H(z) = \frac{1+5z^{-1}}{1+\frac{1}{2}z^{-1}}$$

Decompose $H(z)$ into H_{min} and H_{ap}

$$H(z) = \frac{1+5z^{-1}}{1+\frac{1}{2}z^{-1}} = \frac{5\left(z^{-1}+1/5\right)}{1+\frac{1}{2}z^{-1}} \cdot \frac{\left(1+\frac{1}{5}^{*}z^{-1}\right)}{1+\frac{1}{5}^{*}z^{-1}}$$

$$= \underbrace{\frac{5\left(1+\frac{1}{5}z^{-1}\right)}{1+\frac{1}{2}z^{-1}}}_{H_{min}(z)} \cdot \underbrace{\left(\frac{\left(z^{-1}+1/5\right)}{1+\frac{1}{5}z^{-1}}\right)}_{H_{ap}(z)}$$

Now, let

$$H(z) = \frac{1 + j5\,z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{j5\left(z^{-1} + \frac{1}{j5}\right)}{1 + \frac{1}{2}z^{-1}} \cdot \frac{1 + \left(\frac{1}{j5}\right)^{*} z^{-1}}{1 + \left(\frac{1}{j5}\right)^{*} z^{-1}}$$

$$= \frac{j5\left(z^{-1} - j/5\right)}{1 + (1/2)z^{-1}} \cdot \frac{1 + \left(\frac{-j1}{5}\right)^{*} z^{-1}}{1 + \left(\frac{-j}{5}\right)^{*} z^{-1}}$$

$$= \frac{j5\left(z^{-1} - j/5\right)}{1 + (\frac{1}{2})z^{-1}} \cdot \frac{1 + (j/5)z^{-1}}{1 + (j/5)z^{-1}}$$

Structures For Discrete time

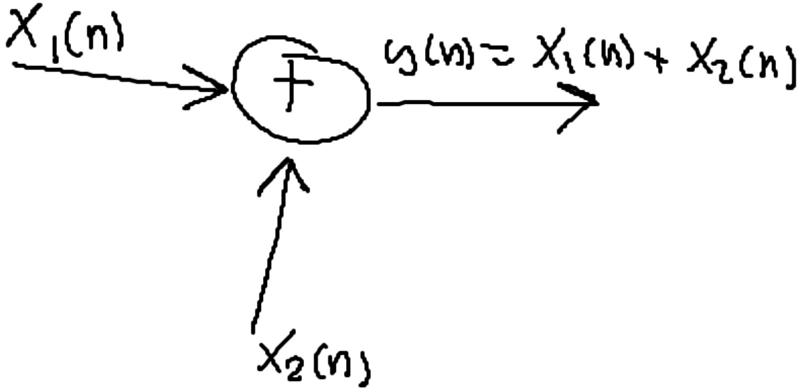
$$y[n] = \sum_{k=0}^{n} x[k]$$

$$= x[n] + \underbrace{\sum_{k=0}^{n-1} x[k]}_{y(n-1)}$$

$$y[n] = x[n] + y[n-1]$$

$$y(n) - y(n-1) = x(n) \Longrightarrow Y(z) - z^{-1} Y(z) = X(z)$$

$$[1 - z^{-1}] Y(z) = X(z) \Rightarrow H(z) = \frac{1}{1 - z^{-1}}$$



In general

$$Y(z) - \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\left[1 - \sum_{k=1}^{N} a_k z^{-k}\right] Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k} \Big/ \left(1 - \sum_{k=1}^{N} a_k z^{-k}\right)$$

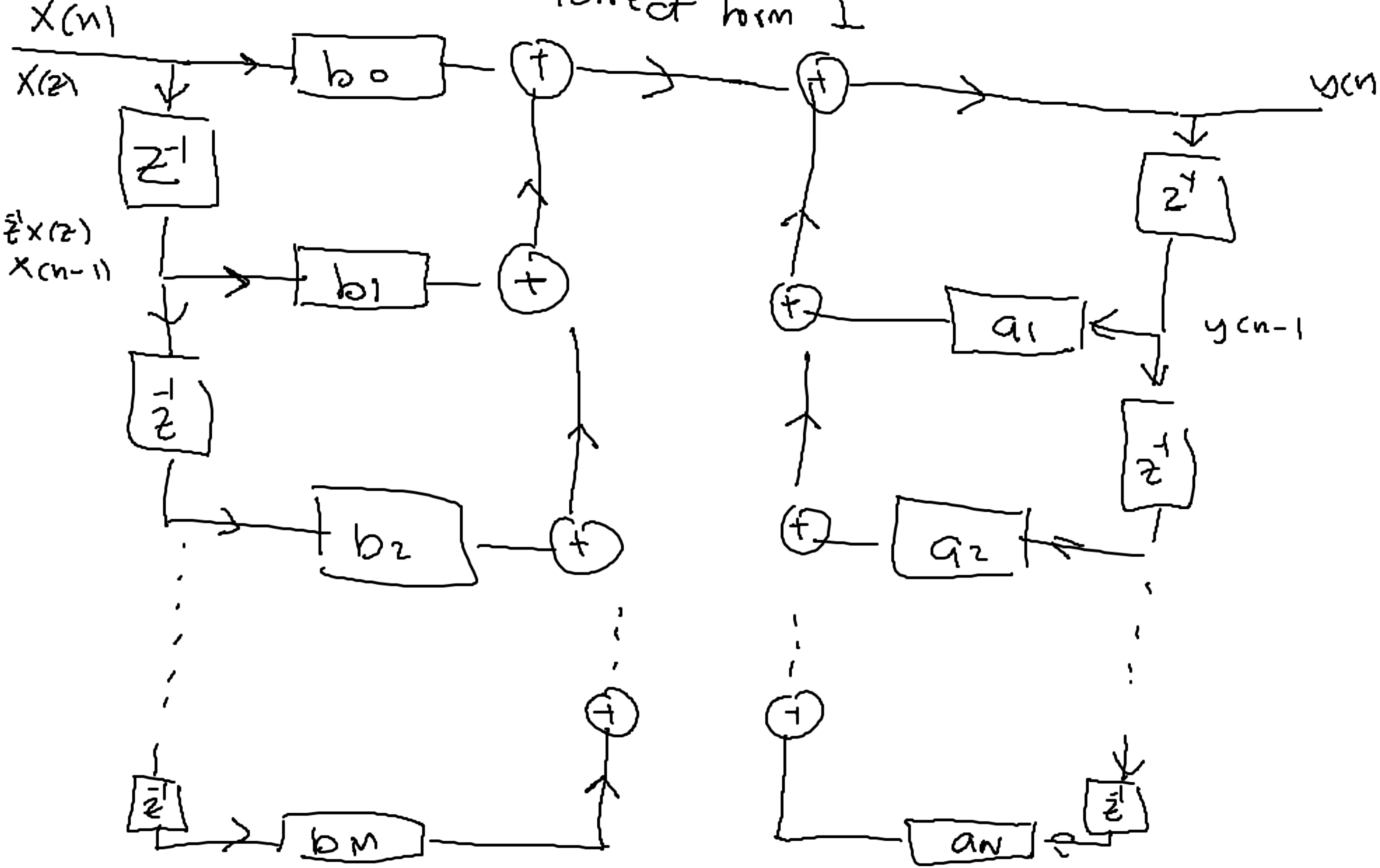
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_n z^{-k}}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_n z^{-k}}{1 - \sum_{k=1}^{N} a_n z^{-k}} \Rightarrow Y(z)\left[1 - \sum_{k=1}^{N} a_n z^{-k}\right] = \sum_{k=0}^{M} b_n z^{-k} X(z)$$

$$Y(z) = \sum_{k=0}^{M} b_n z^{-k} X(z) + \sum_{k=1}^{N} a_n z^{-k} Y(z)$$

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) + \sum_{k=1}^{N} a_k y(n-k)$$

Direct Form I



$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) \left(\cdot \sum_{k=0}^{M} b_k z^{-k} \right)$$

$$\frac{Y(z)}{X(z)} = H_1(z) \quad H_2(z)$$

$$\Rightarrow H_1(z) = \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \quad \text{and} \quad H_2(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$W(z) = H_1(z)\, X(z)$$

$$Y(z) = H_2(z)\, W(z)$$

$$; \quad \frac{Y(z)}{X(z)} = H_1(z) \quad H_2(z) \Rightarrow Y(z) = H_1(z)\, X(z)\, H_2(z)$$

$$= W(z)\, H_2(z)$$

Since

$$H_1(z) = \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$W(z) = \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} X(z)$$

$$X(z) = \left[1 - \sum_{k=1}^{N} a_k z^{-k}\right] W(z)$$

$$W(z) = X(z) + \sum_{k=1}^{N} a_k z^{-k} W(z) \Rightarrow w(n) = x(n) + \sum_{k=1}^{N} a_k w(n-k)$$

$$\longrightarrow (1)$$

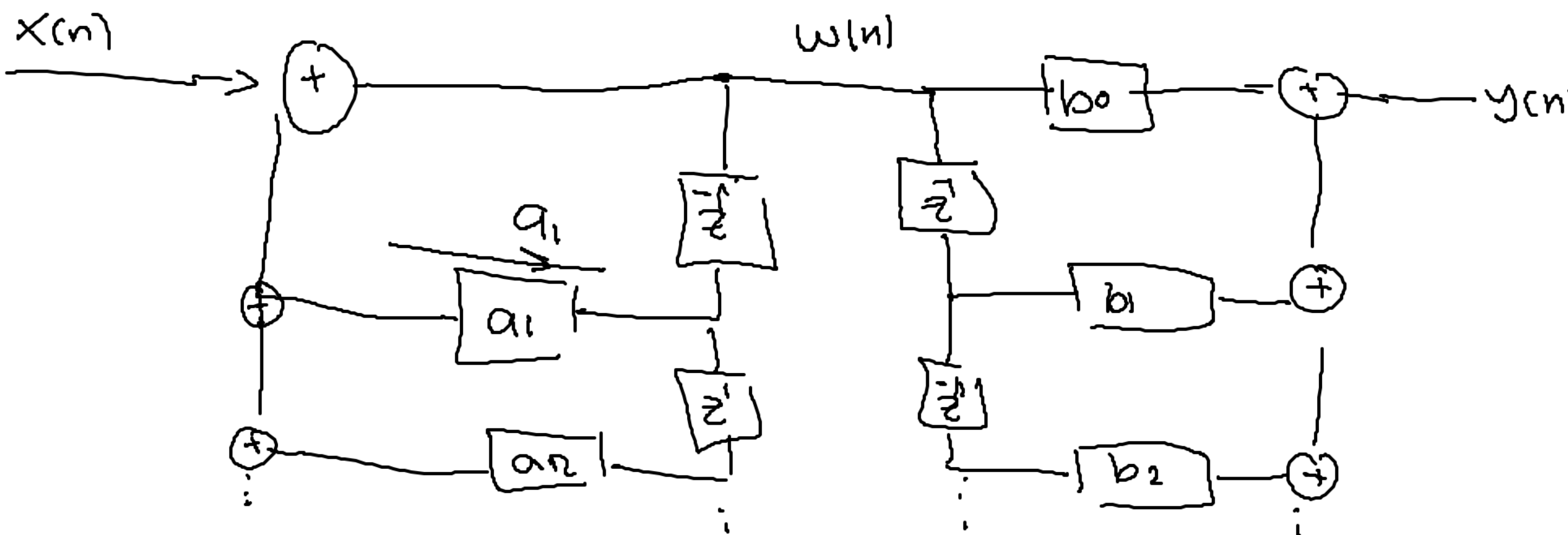
$$Y(z) = H_2(z)\,W(z)$$

$$= \sum_{k=0}^{M} b_k z^{-k}\, W(z)$$

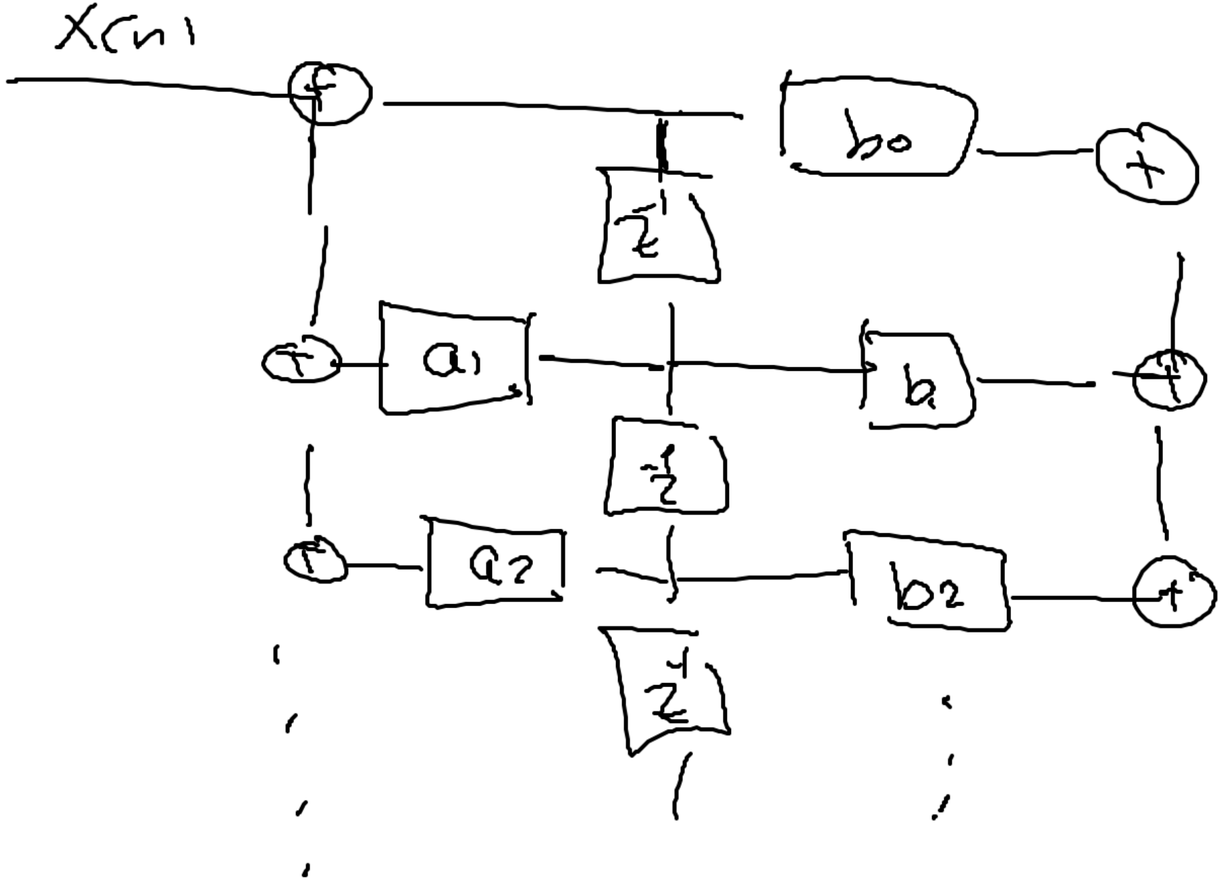
$$y(n) = \sum_{k=0}^{M} b_k\, w(n-k)$$

$$w(n) = x(n) + \sum_{k=1}^{N} a_k\, w(n-k)$$

$$y(n) = \sum_{k=0}^{M} b_k\, w(n-k)$$



⇒



Direct Form II

Canonical Form

Example: Consider the following transfer function of the system

$$H(z) = \frac{1 + 8z^{-1}}{1 - 1.5z^{-1} + 0.9z^{-2}}$$

Draw

① Direct Form I

② Direct Form II

Ans:

To draw direct form I

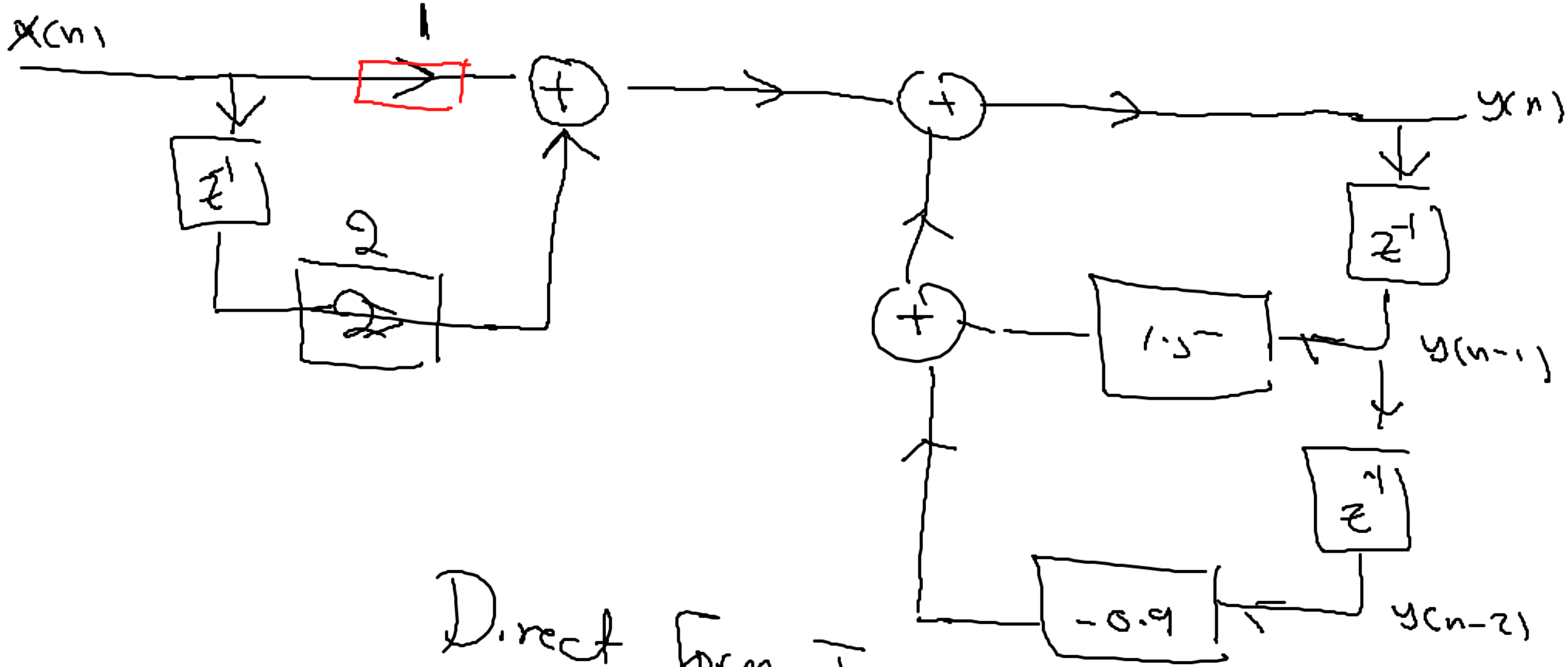
- $H(z) = \dfrac{1+2z^{-1}}{1-1.5z^{-1}+0.9z^{-2}}$

- $\dfrac{Y(z)}{X(z)} = \dfrac{1+2z^{-1}}{1-1.5z^{-1}+0.9z^{-2}}$

- $Y(z)\left[1-1.5z^{-1}+0.9z^{-2}\right] = \left[1+2z^{-1}\right]X(z)$

- $Y(z) = \left[1+2z^{-1}\right]X(z) + \left[1.5z^{-1}-0.9z^{-2}\right]Y(z)$

- $y(n) = x(n) + 2x(n-1) + 1.5y(n-1) - 0.9y(n-2)$



Direct Form I