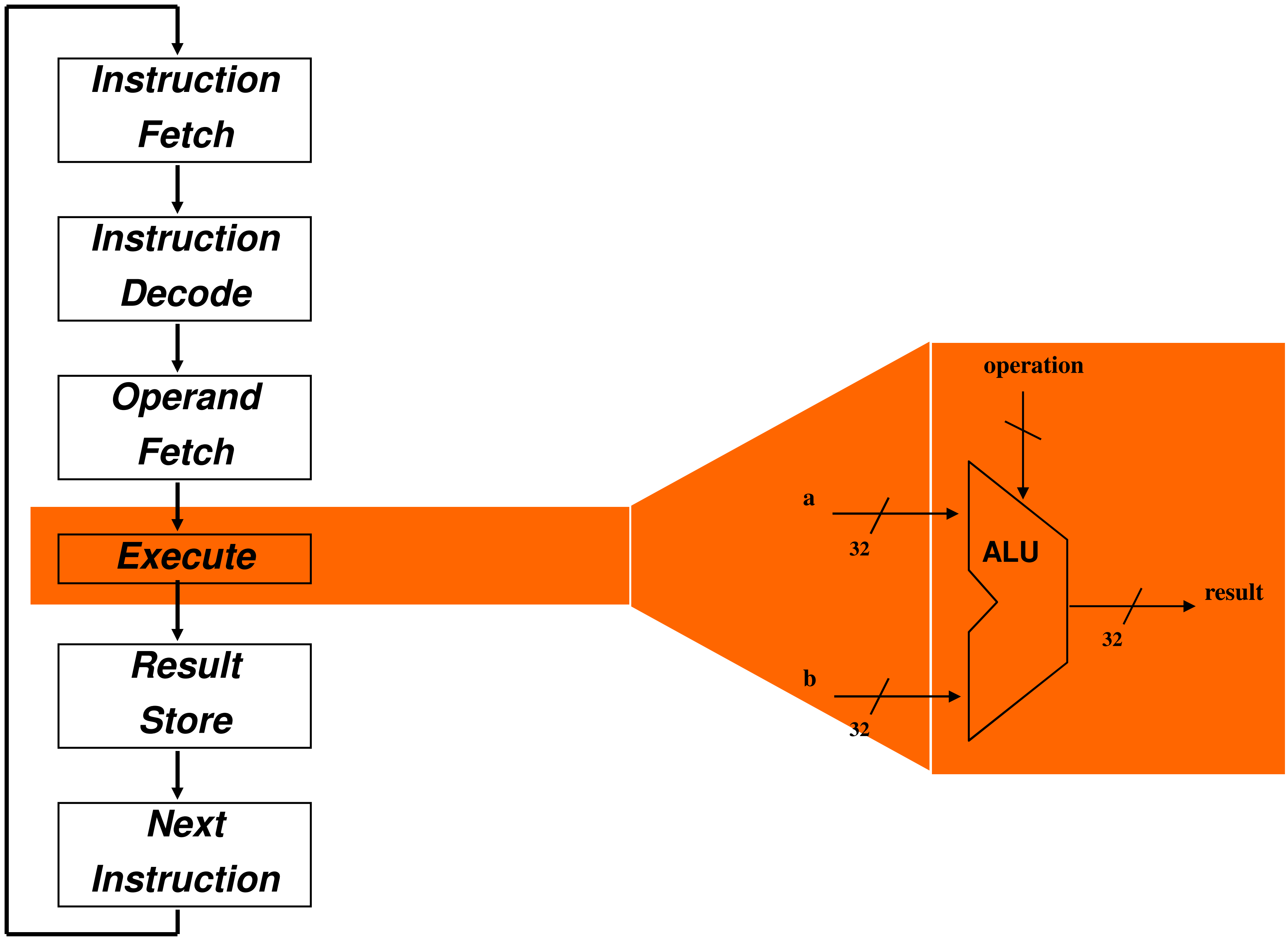
Computer Organization

Computer Arithmetic

Chapter 9

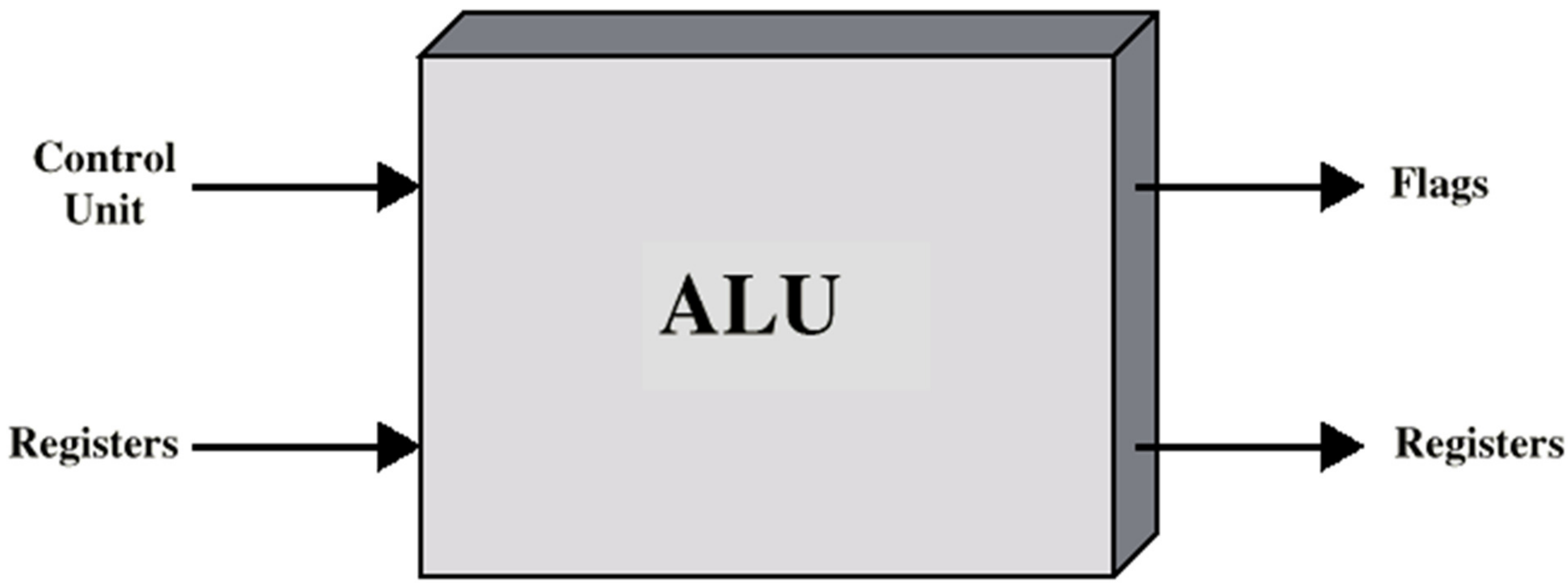
Arithmetic -- The heart of instruction execution



Arithmetic & Logic Unit

- Does the calculations
- Everything else in the computer is there to service this unit
- Handles integers
- May handle floating point (real) numbers
- May be separate FPU (maths co-processor)
- May be on chip separate FPU (486DX +)

ALU Inputs and Outputs



Positional Number Systems

Different Representations of Natural Numbers

XXVII	Roman numerals (not positional)
27	Radix-10 or decimal number (positional)
11011_2	Radix-2 or binary number (also positional)

Fixed-radix positional representation with *k* digits

Number N in radix $r = (d_{k-1} d_{k-2} \ldots d_1 d_0)_r$

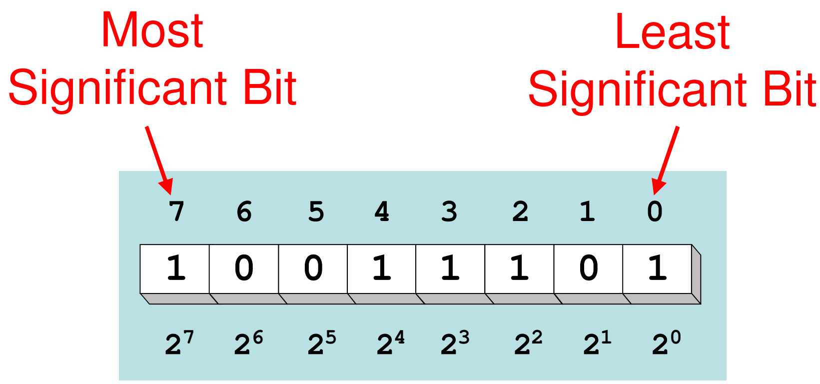
Value $= \mathrm{d}_{k-1} \times r^{k-1} + \mathrm{d}_{k-2} \times r^{k-2} + \ldots + \mathrm{d}_1 \times r + \mathrm{d}_0$

Examples: $(11011)_2 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2 + 1 = 27$

$(2103)_4 = 2 \times 4^3 + 1 \times 4^2 + 0 \times 4 + 3 = 147$

Binary Numbers

- Each binary digit (called bit) is either 1 or 0
- Bits have no inherent meaning, can represent
  - Unsigned and signed integers
  - Characters
  - Floating-point numbers
  - Images, sound, etc.



- Bit Numbering
  - Least significant bit (LSB) is rightmost (bit 0)
  - Most significant bit (MSB) is leftmost (bit 7 in an 8-bit number)

Hexadecimal Integers

- ❖ 16 Hexadecimal Digits: 0 – 9, A – F
- ❖ More convenient to use than binary numbers

Binary, Decimal, and Hexadecimal Equivalents

Binary	Decimal	Hexadecimal	Binary	Decimal	Hexadecimal
0000	0	0	1000	8	8
0001	1	1	1001	9	9
0010	2	2	1010	10	A
0011	3	3	1011	11	B
0100	4	4	1100	12	C
0101	5	5	1101	13	D
0110	6	6	1110	14	E
0111	7	7	1111	15	F

Converting Binary to Hexadecimal

- Each hexadecimal digit corresponds to 4 binary bits
- Example:

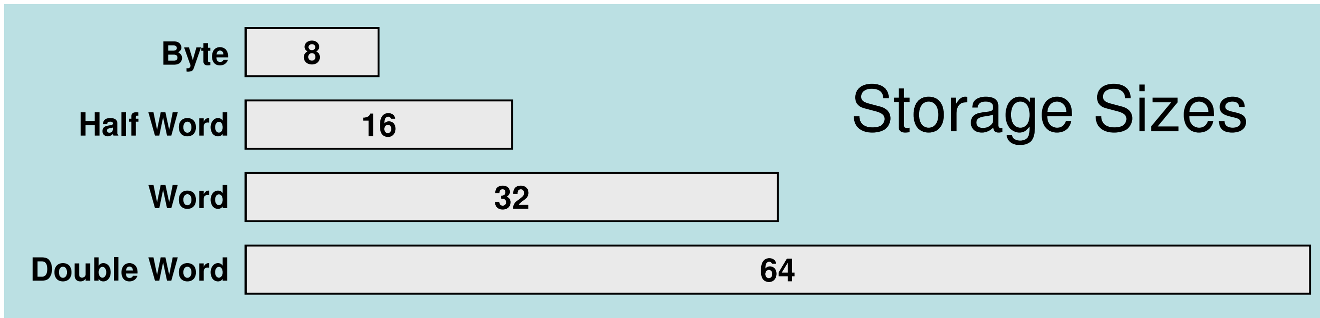
  Convert the 32-bit binary number to hexadecimal

  1110 1011 0001 0110 1010 0111 1001 0100

- Solution:

E	**B**	**1**	**6**	**A**	**7**	**9**	**4**
1110	**1011**	**0001**	**0110**	**1010**	**0111**	**1001**	**0100**

Integer Storage Sizes



Storage Type	Unsigned Range	Powers of 2
Byte	0 to 255	0 to $(2^8 - 1)$
Half Word	0 to 65,535	0 to $(2^{16} - 1)$
Word	0 to 4,294,967,295	0 to $(2^{32} - 1)$
Double Word	0 to 18,446,744,073,709,551,615	0 to $(2^{64} - 1)$

What is the largest 20-bit unsigned integer?

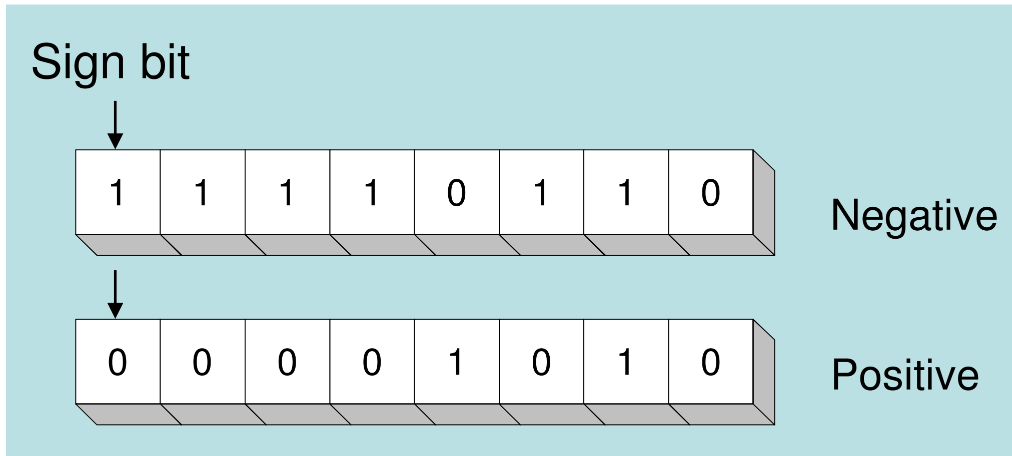
Answer: $2^{20} - 1 = 1{,}048{,}575$

Signed Integers

- Several ways to represent a signed number
  - Sign-Magnitude
  - Biased
  - 1's complement
  - 2's complement
- Divide the range of values into 2 equal parts
  - First part corresponds to the positive numbers (≥ 0)
  - Second part correspond to the negative numbers (< 0)
- Focus will be on the 2's complement representation
  - Has many advantages over other representations
  - Used widely in processors to represent signed integers

Sign Bit

- Highest bit indicates the sign
- 1 = negative
- 0 = positive



- For Hexadecimal Numbers, check most significant digit
- If highest digit is > 7, then value is negative
- Examples: 8A and C5 are negative bytes
- B1C42A00 is a negative word (32-bit signed integer)
- Problems
  - Need to consider both sign and magnitude in arithmetic
  - Two representations of zero (+0 and -0)

Two's Complement Representation

- Positive numbers
  - Signed value = Unsigned value
- Negative numbers
  - Signed value = Unsigned value – 2^n
  - n = number of bits
- Negative weight for MSB
  - Another way to obtain the signed value is to assign a negative weight to most-significant bit

1	0	1	1	0	1	0	0
-128	64	32	16	8	4	2	1

= -128 + 32 + 16 + 4 = -76

8-bit Binary value	Unsigned value	Signed value
00000000	0	0
00000001	1	+1
00000010	2	+2
. . .	. . .	. . .
01111110	126	+126
01111111	127	+127
10000000	128	-128
10000001	129	-127
. . .	. . .	. . .
11111110	254	-2
11111111	255	-1

Forming the Two's Complement

starting value	`00100100 = +36`
step1: reverse the bits (1's complement)	`11011011`
step 2: add 1 to the value from step 1	`+        1`
sum = 2's complement representation	`11011100 = -36`

Sum of an integer and its 2's complement must be zero:

00100100 + 11011100 = 00000000 (8-bit sum) $\Rightarrow$ Ignore Carry

Another way to obtain the 2's complement:

Start at the least significant 1

Leave all the 0s to its right unchanged

Complement all the bits to its left

Binary Value

= 00100 $\boxed{1}$ 00 least significant 1

2's Complement

= 11011 $\boxed{1}$ 00

Sign Extension

Step 1: Move the number into the lower-significant bits

Step 2: Fill all the remaining higher bits with the sign bit

- This will ensure that both magnitude and sign are correct
- Examples
  - Sign-Extend 10110011 to 16 bits

    10110011 = -77 ⇒ **$\boxed{11111111}$ 10110011 = -77**

  - Sign-Extend 01100010 to 16 bits

    01100010 = +98 ⇒ **$\boxed{00000000}$ 01100010 = +98**

- Infinite 0s can be added to the left of a positive number
- Infinite 1s can be added to the left of a negative number

Ranges of Signed Integers

For n-bit signed integers: Range is -2^{n-1} to $(2^{n-1} - 1)$

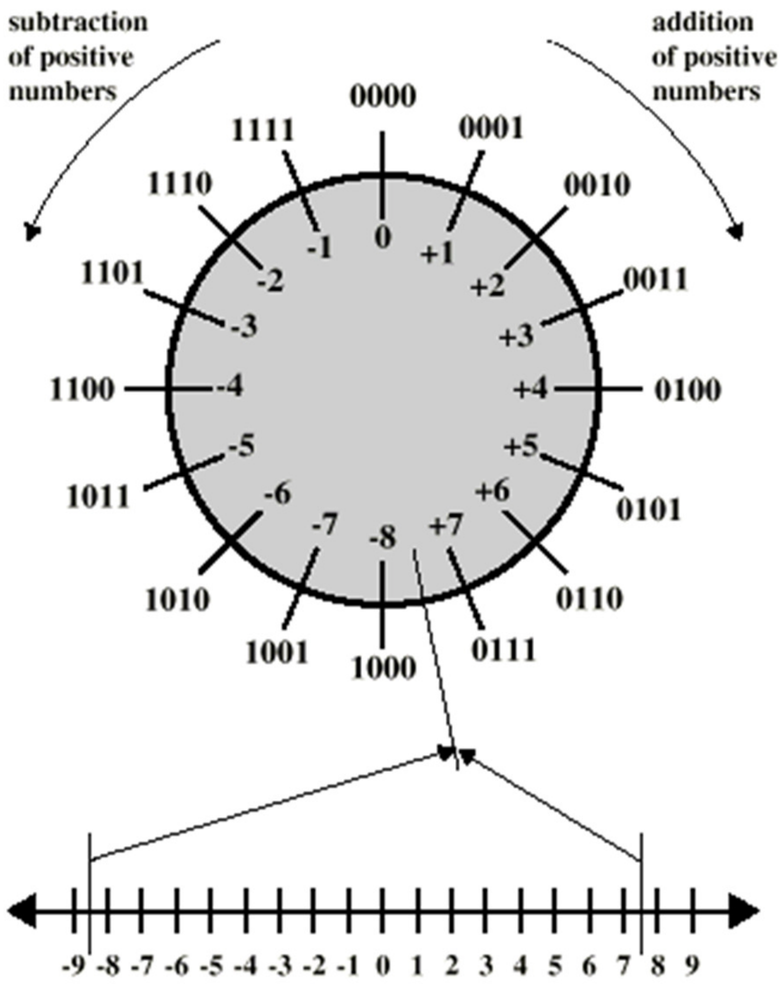
Positive range: 0 to $2^{n-1} - 1$

Negative range: -2^{n-1} to -1

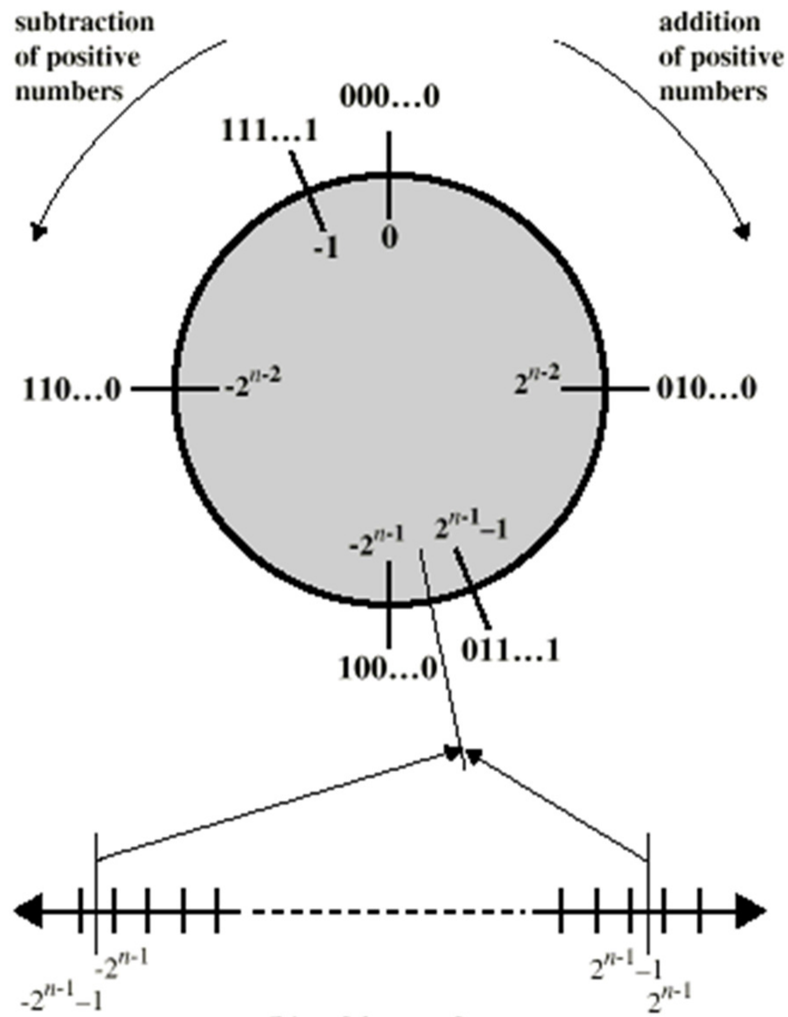
Storage Type	Signed Range	Powers of 2
Byte	–128 to +127	-2^7 to $(2^7 - 1)$
Half Word	–32,768 to +32,767	-2^{15} to $(2^{15} - 1)$
Word	–2,147,483,648 to +2,147,483,647	-2^{31} to $(2^{31} - 1)$
Double Word	–9,223,372,036,854,775,808 to +9,223,372,036,854,775,807	-2^{63} to $(2^{63} - 1)$

Practice: What is the range of signed values that may be stored in 20 bits?

Geometric Depiction of Twos Complement Integers



(a) 4-bit numbers



(b) n-bit numbers

Two's Complement Special Cases

- **Case 1**
  - 0 = 00000000
  - Bitwise not 11111111
  - Add 1 to LSB +1
  - Result 1 00000000
  - Overflow is ignored, so:
  - - 0 = 0 √
- **-128 = 10000000**
  - bitwise not 01111111
  - Add 1 to LSB +1
  - Result 10000000
  - Monitor MSB (sign bit)
  - It should change during negation

Two's Compliment - Summery

Range	-2^{n-1} through $2^{n-1} - 1$
Number of Representations of Zero	One
Negation	Take the Boolean complement of each bit of the corresponding positive number, then add 1 to the resulting bit pattern viewed as an unsigned integer.
Expansion of Bit Length	Add additional bit positions to the left and fill in with the value of the original sign bit.
Overflow Rule	If two numbers with the same sign (both positive or both negative) are added, then overflow occurs if and only if the result has the opposite sign.
Subtraction Rule	To subtract B from A, take the twos complement of B and add it to A.

Benefits:

- **One representation of zero**
- **Arithmetic works easily (see later)**

Character Storage

- Character sets
  - Standard ASCII: 7-bit character codes (0 – 127)
  - Extended ASCII: 8-bit character codes (0 – 255)
  - Unicode: 16-bit character codes (0 – 65,535)
  - Unicode standard represents a universal character set
    - Defines codes for characters used in all major languages
    - Used in Windows-XP: each character is encoded as 16 bits
  - UTF-8: variable-length encoding used in HTML
    - Encodes all Unicode characters
    - Uses 1 byte for ASCII, but multiple bytes for other characters
- Null-terminated String
  - Array of characters followed by a NULL character

Binary Addition

- Start with the least significant bit (rightmost bit)
- Add each pair of bits
- Include the carry in the addition, if present

carry		1	1	1	1				
	0	0	1	1	0	1	1	0	(54)
+	0	0	0	1	1	1	0	1	(29)
	0	1	0	1	0	0	1	1	(83)
bit position:	7	6	5	4	3	2	1	0	

Binary Subtraction

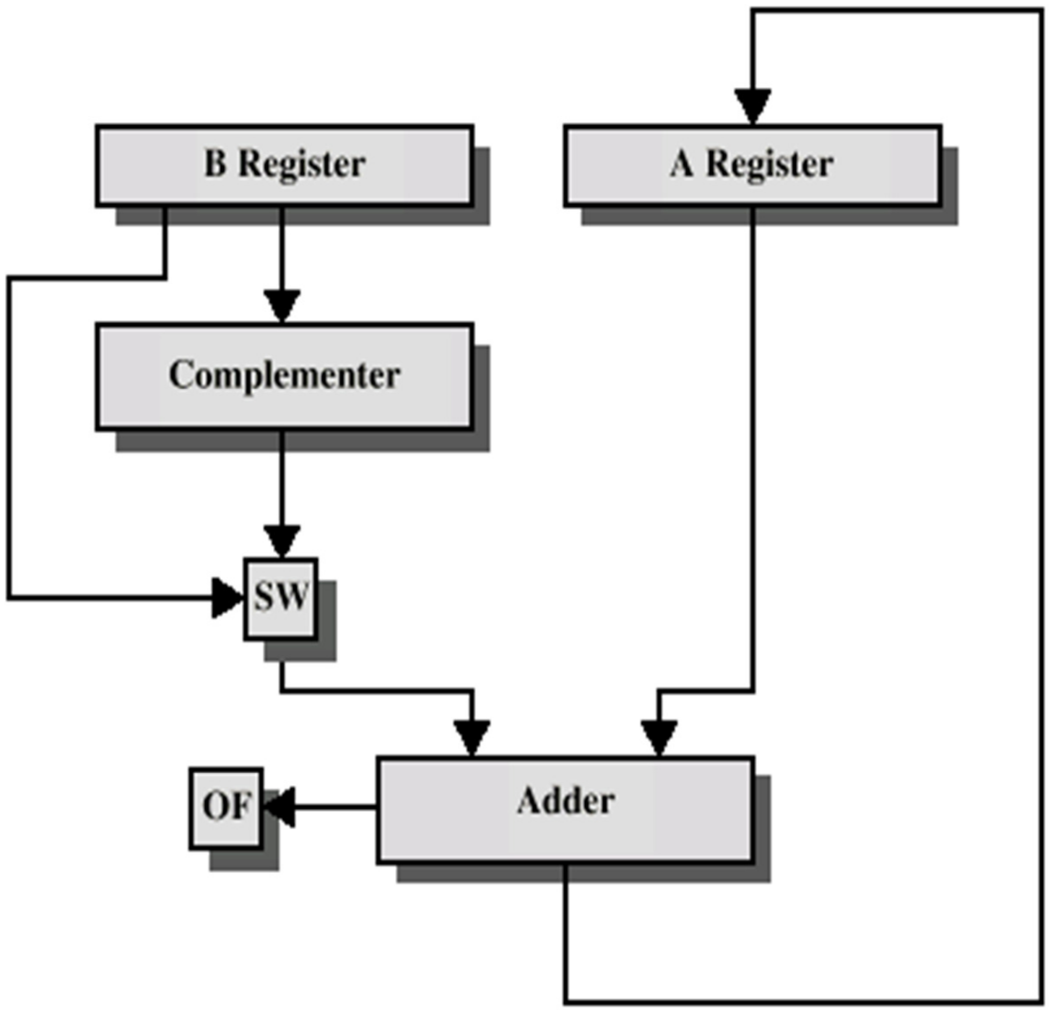
- When subtracting A – B, convert B to its 2's complement
- Add A to (–B)

borrow:		1	1			1		
	0	1	0	0	1	1	0	1
–	0	0	1	1	1	0	1	0
	0	0	0	1	0	0	1	1

carry:	1	1			1	1				
		0	1	0	0	1	1	0	1	
+		1	1	0	0	0	1	1	0	(2's complement)
		0	0	0	1	0	0	1	1	(same result)

- Final carry is ignored, because
  - Negative number is sign-extended with 1's
  - You can imagine infinite 1's to the left of a negative number
  - Adding the carry to the extended 1's produces extended zeros

Hardware for Addition and Subtraction



OF = overflow bit
SW = Switch (select addition or subtraction)

Carry and Overflow

- Carry is important when …
  - Adding or subtracting unsigned integers
  - Indicates that the unsigned sum is out of range
  - Either < 0 or >maximum unsigned *n*-bit value
- Overflow is important when …
  - Adding or subtracting signed integers
  - Indicates that the signed sum is out of range
- Overflow occurs when
  - Adding two positive numbers and the sum is negative
  - Adding two negative numbers and the sum is positive
  - Can happen because of the fixed number of sum bits

Carry and Overflow Examples

- We can have carry without overflow and vice-versa
- Four cases are possible (Examples are 8-bit numbers)

					1				
		0	0	0	0	1	1	1	1	15
+		0	0	0	0	1	0	0	0	8
		0	0	0	1	0	1	1	1	23

Carry = 0 Overflow = 0

	1	1	1	1	1					
		0	0	0	0	1	1	1	1	15
+		1	1	1	1	1	0	0	0	248 (-8)
		0	0	0	0	0	1	1	1	7

Carry = 1 Overflow = 0

		1								
		0	1	0	0	1	1	1	1	79
+		0	1	0	0	0	0	0	0	64
		1	0	0	0	1	1	1	1	143 (-113)

Carry = 0 Overflow = 1

	1			1	1					
		1	1	0	1	1	0	1	0	218 (-38)
+		1	0	0	1	1	1	0	1	157 (-99)
		0	1	1	1	0	1	1	1	119

Carry = 1 Overflow = 1

Addition of Numbers in Twos Complement Representation

1001 = −7 +0101 = 5 1110 = −2 (a) (−7) + (+5)	1100 = −4 +0100 = 4 10000 = 0 (b) (−4) + (+4)
0011 = 3 +0100 = 4 0111 = 7 (c) (+3) + (+4)	1100 = −4 +1111 = −1 11011 = −5 (d) (−4) + (−1)
0101 = 5 +0100 = 4 1001 = Overflow (e) (+5) + (+4)	1001 = −7 +1010 = −6 10011 = Overflow (f) (−7) + (−6)

Subtraction of Numbers in Twos Complement Representation (M – S)

```
  0010 = 2
 +1001 = –7
  1011 = –5

(a) M = 2 = 0010
    S = 7 = 0111
   –S =     1001
```

```
  0101 = 5
 +1110 = –2
 10011 = 3

(b) M = 5 = 0101
    S = 2 = 0010
   –S =     1110
```

```
  1011 = –5
 +1110 = –2
 11001 = –7

(c) M =–5 = 1011
    S = 2 = 0010
   –S =     1110
```

```
  0101 = 5
 +0010 = 2
  0111 = 7

(d) M = 5 = 0101
    S =–2 = 1110
   –S =     0010
```

```
  0111 = 7
 +0111 = 7
  1110 = Overflow

(e) M =  7 = 0111
    S = –7 = 1001
   –S =      0111
```

```
  1010 = –6
 +1100 = –4
 10110 = Overflow

(f) M = –6 = 1010
    S =  4 = 0100
   –S =      1100
```

Unsigned Multiplication

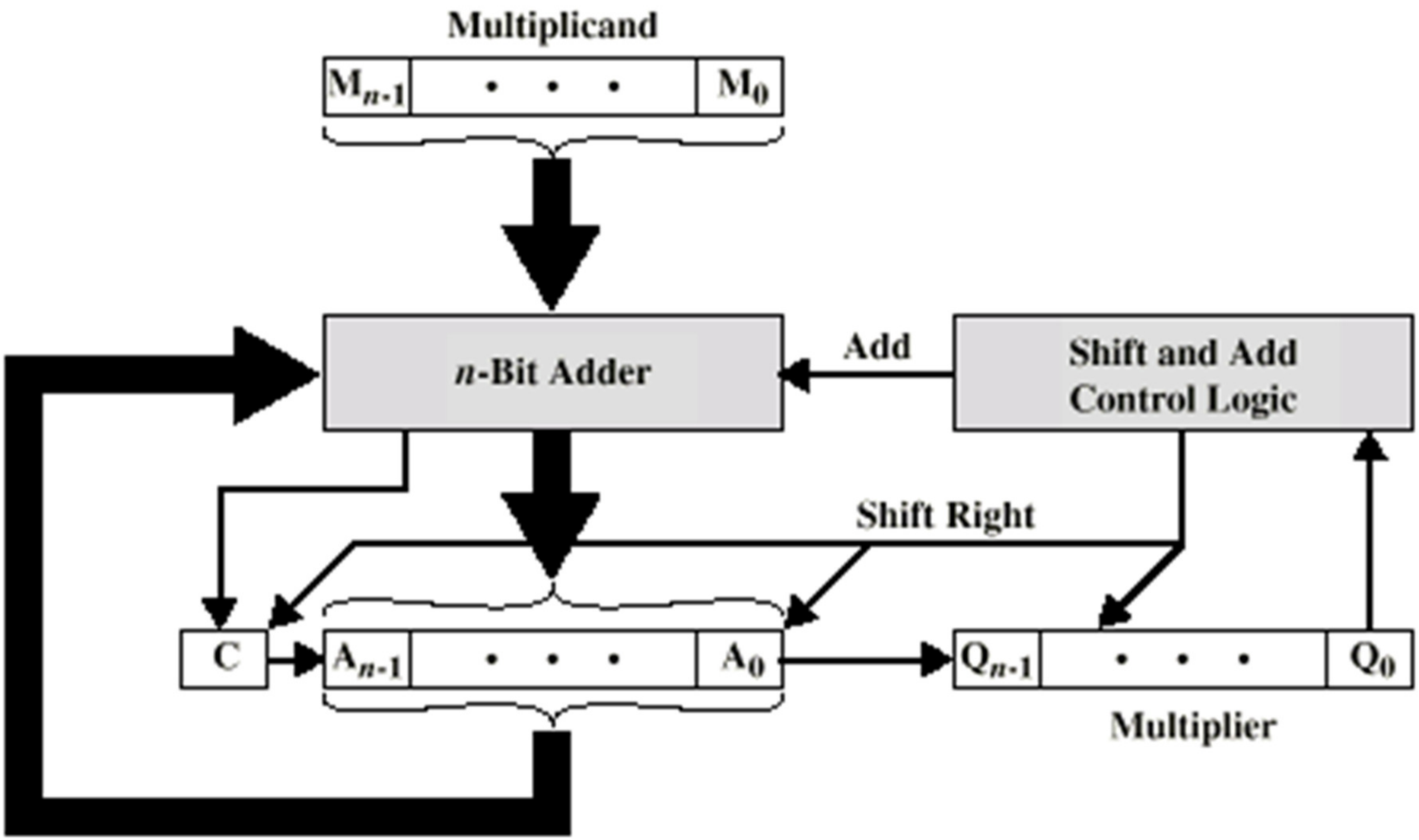
- Paper and Pencil Example:

Multiplicand		$1100_2 = 12$
Multiplier	$\times$	$1101_2 = 13$
		1100
		0000
		1100
		1100
Product		$10011100_2 = 156$

Binary multiplication is easy
0 × multiplicand = 0
1 × multiplicand = multiplicand

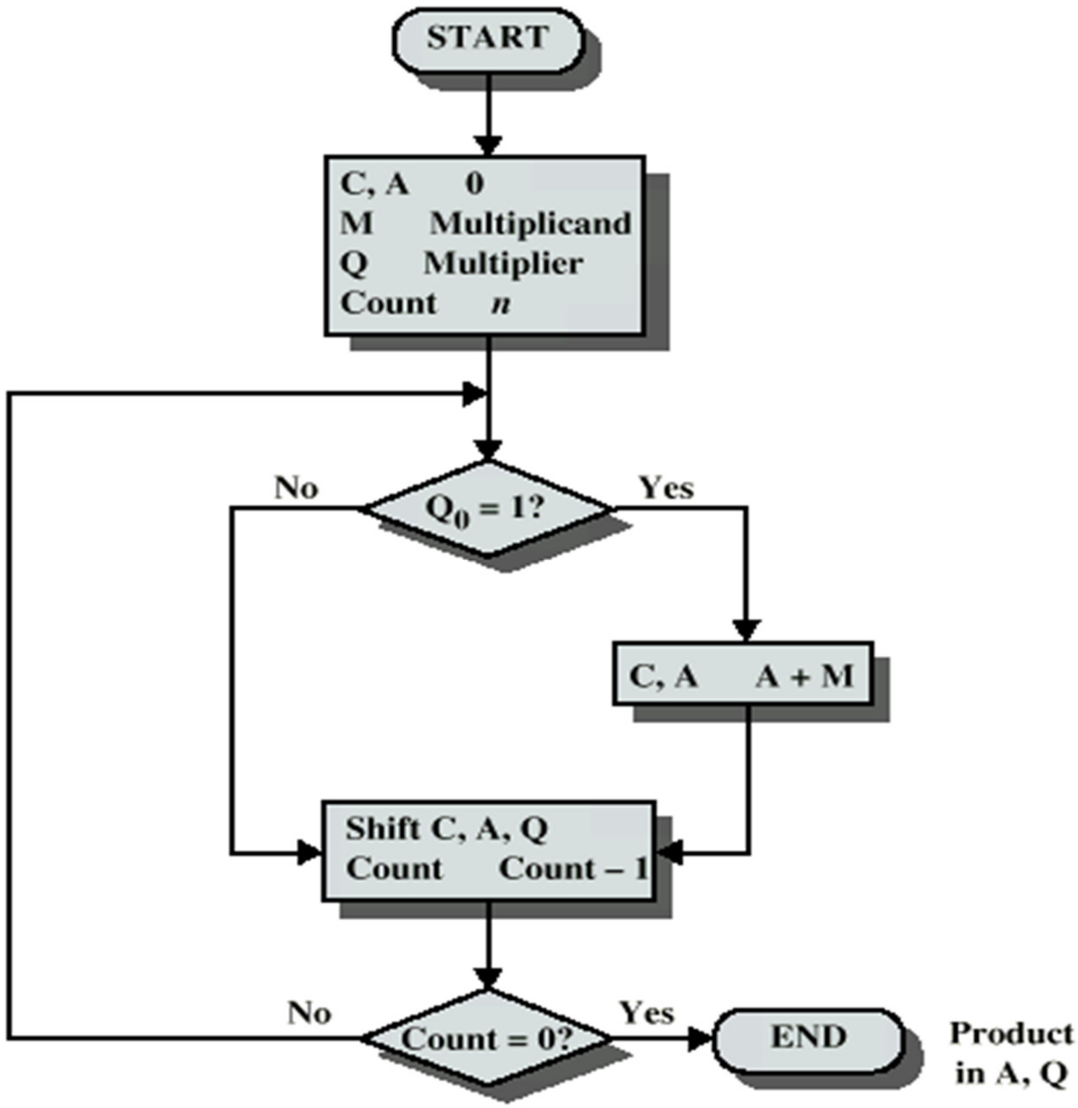
- m-bit multiplicand × n-bit multiplier = (m+n)-bit product
- Accomplished via shifting and addition
- Consumes more time and more chip area

Unsigned Binary Multiplication



(a) Block Diagram

Flowchart for Unsigned Binary Multiplication



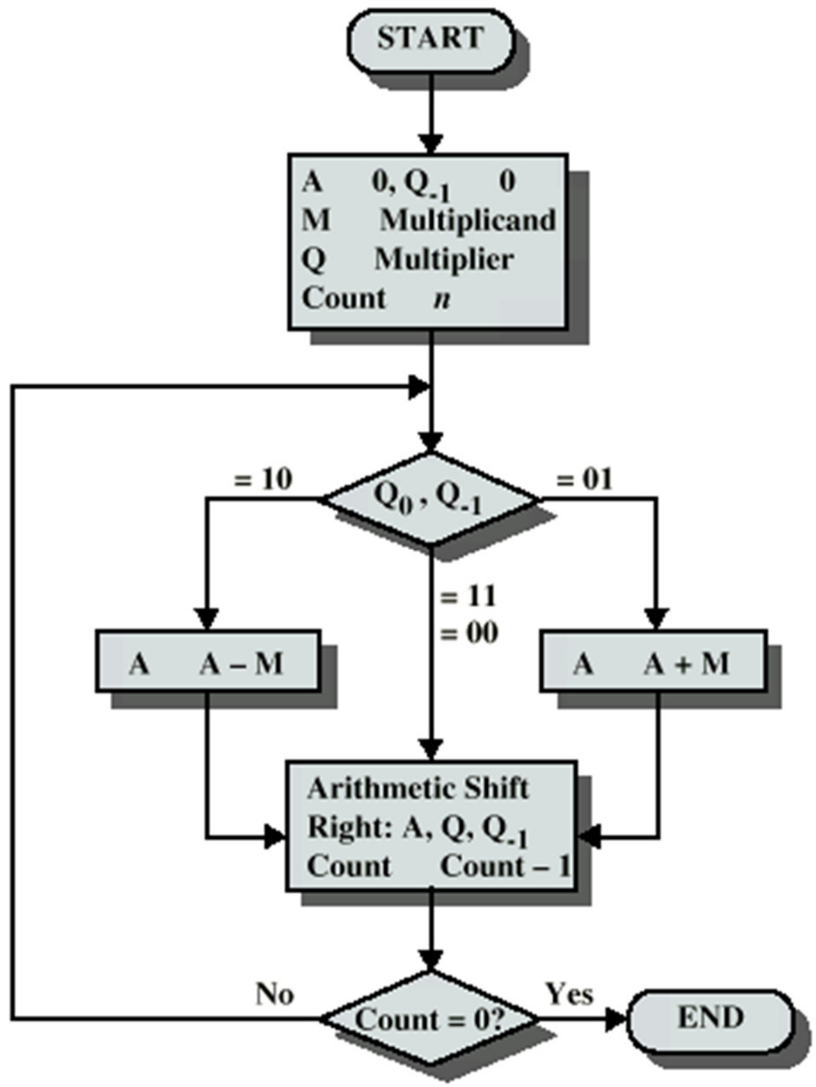
Execution of Example

C	A	Q	M		
0	0000	1101	1011	Initial Values	
0	1011	1101	1011	Add	First Cycle
0	0101	1110	1011	Shift	
0	0010	1111	1011	Shift	Second Cycle
0	1101	1111	1011	Add	Third Cycle
0	0110	1111	1011	Shift	
1	0001	1111	1011	Add	Fourth Cycle
0	1000	1111	1011	Shift	

Multiplying Negative Numbers

- This does not work!
- Solution 1
  - Convert to positive if required
  - Multiply as above
  - If signs were different, negate answer
- Solution 2
  - Booth's algorithm

Booth's Algorithm



Example of Booth's Algorithm

A	Q	Q_{-1}	M		
0000	0011	0	0111	Initial Values	
1001	0011	0	0111	A A − M	First Cycle
1100	1001	1	0111	Shift	
1110	0100	1	0111	Shift	Second Cycle
0101	0100	1	0111	A A + M	Third Cycle
0010	1010	0	0111	Shift	
0001	0101	0	0111	Shift	Fourth Cycle

Examples Using Booth's Algorithm

```
    0111
   ×0011     (0)
--------
11111001     1–0
0000000      1–1
000111       0–1
--------
00010101     (21)
```

(a) (7) × (3) = (21)

```
    0111
   ×1101     (0)
--------
11111001     1–0
0000111      0–1
111001       1–0
--------
11101011     (–21)
```

(b) (7) × (–3) = (–21)

```
    1001
   ×0011     (0)
--------
00000111     1–0
0000000      1–1
111001       0–1
--------
11101011     (–21)
```

(c) (–7) × (3) = (–21)

```
    1001
   ×1101     (0)
--------
00000111     1–0
1111001      0–1
000111       1–0
--------
00010101     (21)
```

(d) (–7) × (–3) = (21)

How it works

- Consider a positive multiplier consisting of a **block of 1s** surrounded by 0s. For example, 00111110. The product is given by :

$$M \times "0\,0\,1\,1\,1\,1\,1\,0" = M \times (2^5 + 2^4 + 2^3 + 2^2 + 2^1) = M \times 62$$

- where M is the multiplicand.
- The number of operations can be **reduced** to **two** by rewriting the same as

$$M \times "0\,1\,0\,0\,0\,0\,0\,\text{-}1\,0" = M \times (2^6 - 2^1) = M \times 62.$$

- Note that:

$2^n + 2^{n-1} + \ldots + 2^{n-k} = 2^{n+1} - 2^{n-k}$

How it works

- So, the product can be generated by one addition and one subtraction

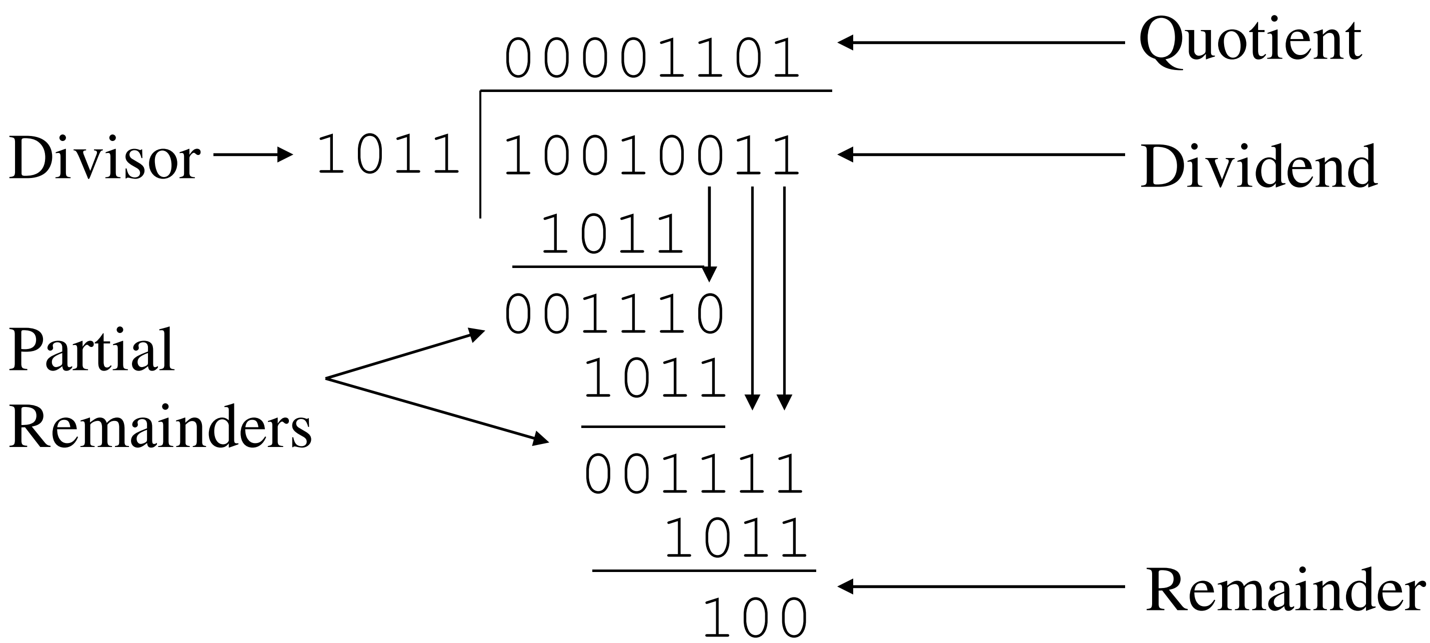
- In Booth’s algorithm
  - perform subtraction when the first 1 of the block is encountered (1 - 0)
  - perform addition when the last 1 of the block is encountered (0 - 1)

- (1 - 0) and (0 - 1) are observed from $Q_0 - Q_{-1}$ (see previous example)

Division

- More complex than multiplication
- Negative numbers are really bad!
- Based on long division

Division of Unsigned Binary Integers



Real Numbers

- Numbers with fractions
- Could be done in pure binary
  - $1001.1010 = 2^3 + 2^0 + 2^{-1} + 2^{-3} = 9.625$
- Where is the binary point?
- Fixed?
  - Very limited
- Moving?
  - How do you show where it is?

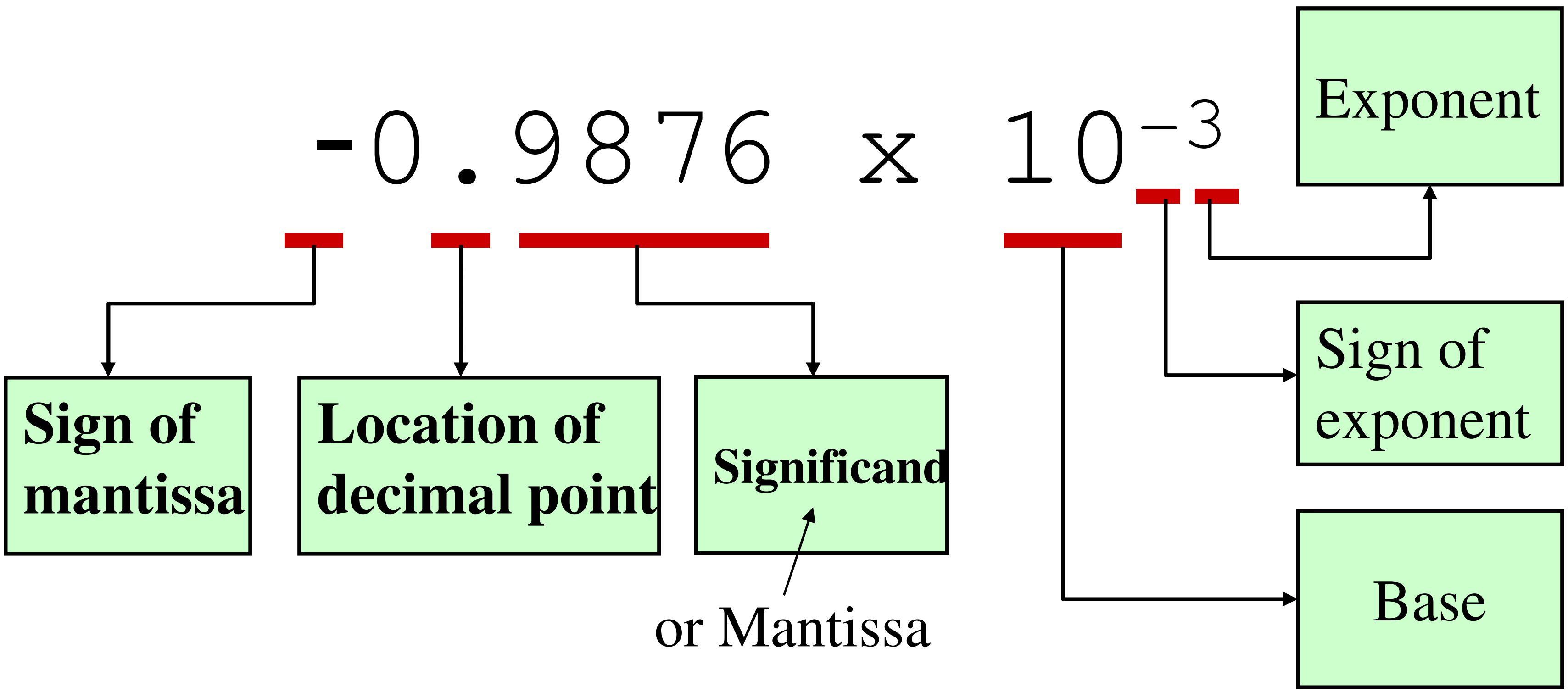
Exponential Notation

- The following are equivalent representations of 1,234

123,400.0	x	10^{-2}
12,340.0	x	10^{-1}
1,234.0	x	10^{0}
123.4	x	10^{1}
12.34	x	10^{2}
1.234	x	10^{3}
0.1234	x	10^{4}

The representations differ in that the decimal place – the “point” -- “floats” to the left or right (with the appropriate adjustment in the exponent).

Parts of a Floating Point Number

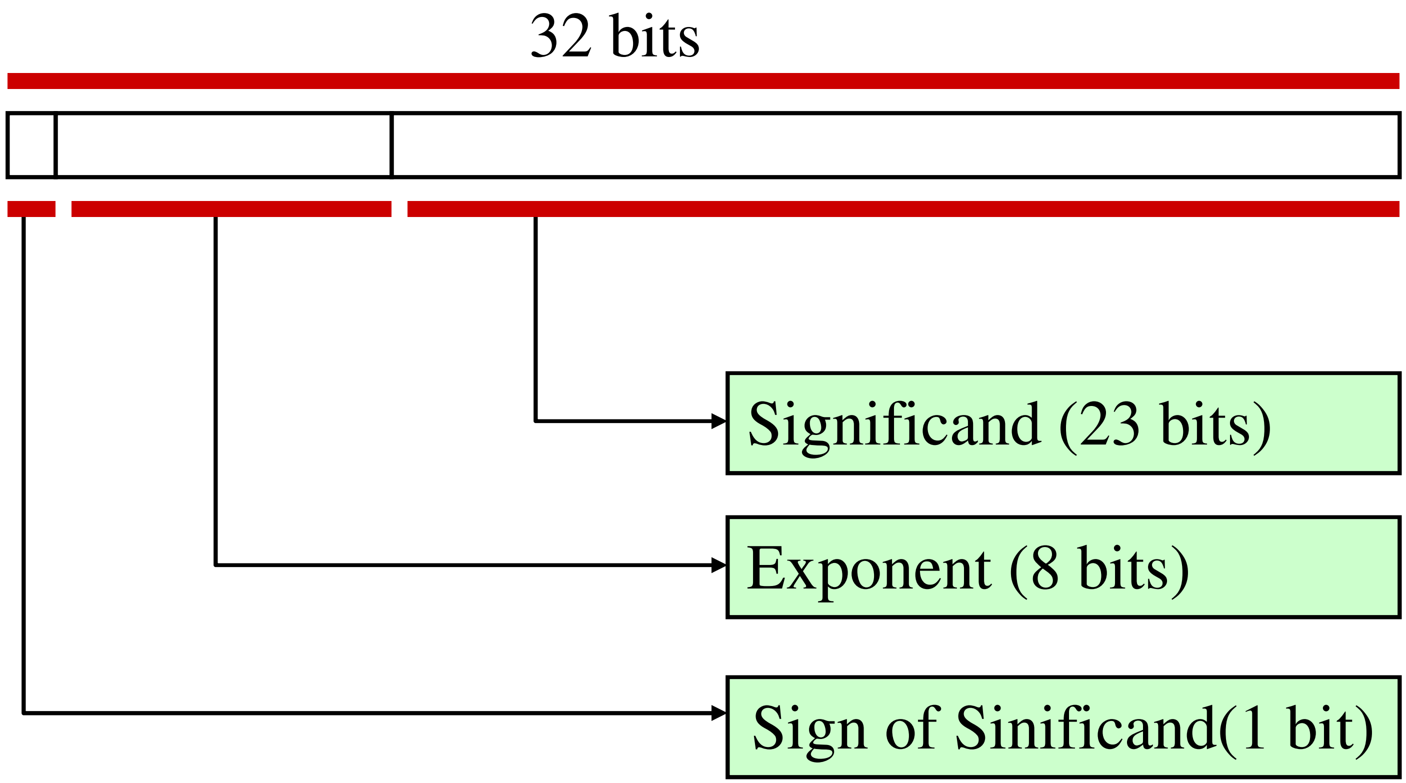


In binary, the significand is represented by 1s and 0's, and the Base = 2. E.g. -1.1111011×2^3

Biased Representation

- Other type of binary number representations
- A fixed value called Bias is added for the binary value
- Typically, the bias equals $(2^{k-1}-1)$, where K is the number of bits in the binary number.
- E.g for 4 bit representation,
  - The bias value= $2^{4-1}-1= 7$
  - Representation of +8 = 1111
  - Representation of -7 = 0000

Representation Format

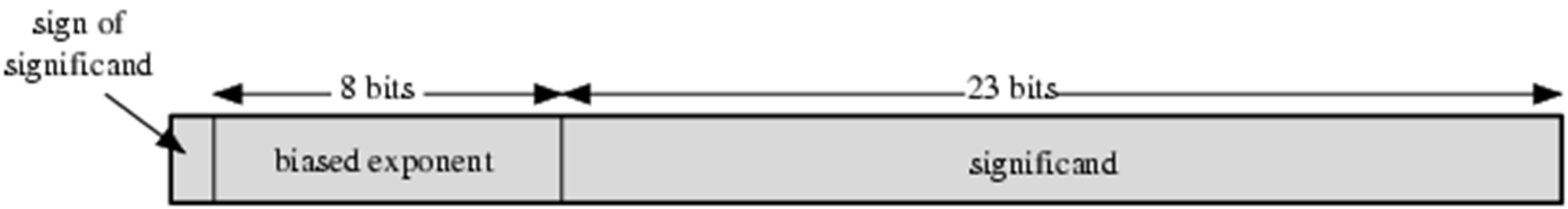


Floating Point

(a) Format

- +/- .significand x $2^{exponent}$
- Misnomer
- Point is actually fixed between sign bit and body of mantissa
- Exponent indicates place value (point position)

Floating Point Examples



(a) Format

$$
\begin{array}{rcl}
1.1010001 \times 2^{10100} & = 0\ 10010011\ 10100010000000000000000 = & 1.638125 \times 2^{20} \\
-1.1010001 \times 2^{10100} & = 1\ 10010011\ 10100010000000000000000 = & -1.638125 \times 2^{20} \\
1.1010001 \times 2^{-10100} & = 0\ 01101011\ 10100010000000000000000 = & 1.638125 \times 2^{-20} \\
-1.1010001 \times 2^{-10100} & = 1\ 01101011\ 10100010000000000000000 = & -1.638125 \times 2^{-20}
\end{array}
$$

(b) Examples

Signs for Floating Point

- Mantissa is stored in 2s compliment
- Exponent is in excess or **biased notation**
  - e.g. Excess (bias) 128 means
  - 8 bit exponent field
  - Pure value range 0-255 (8-bit)
  - Subtract 127 to get correct value
    - **Bias= 2^{8-1}-1= 127**
  - Range of exponent values: -127 to +128
    - For representation: bias must be added for any value
    - Exponent value -127 is represented as -127+127 = 0 (00000000:Min value)
    - Exponent value +128 is represented as 128+127 = 255 (11111111:Max value)

Normalization

- FP numbers are usually normalized
- i.e. exponent is adjusted so that leading **bit (MSB) of Significand is 1**
- Since it is **always 1** there is no need to store it
- (c.f. Scientific notation where numbers are normalized to give a single digit before the decimal point
- E.g.,
  - Significand $\rightarrow$ **10100000000000000000000**
  - Represents… $1.101_2 = 1.625_{10}$

Converting <u>from</u> Floating Point

- E.g., What decimal value is represented by the following 32-bit floating point number?

$$\mathtt{C17B0000}_{16}$$

- Step 1
  - Express in binary and find S, E, and M

$$C17B0000_{16} =$$

$$\underbrace{1}_{S}\ \underbrace{10000010}_{E}\ \underbrace{11110110000000000000000}_{M}{}_{2}$$

1 = negative
0 = positive

- Step 2
  - Find “real” exponent, n
  - $n = E - 127$

$$
\begin{aligned}
n &= E - 127 \\
  &= 10000010_2 - 127 \\
  &= 130 - 127 \\
  &= 3
\end{aligned}
$$

- Step 3
  - —Put S, M, and *n* together to form binary result
  - —(Don’t forget the implied “1.” on the left of the mantissa.)

$$-1.1111011_2 \text{ x } 2^n =$$

$$-1.1111011_2 \text{ x } 2^3 =$$

$$-1111.1011_2$$

- Step 4
  - Express result in decimal

$$-1111.1011_2$$

-15

$$
\begin{aligned}
2^{-1} &= 0.5\\
2^{-3} &= 0.125\\
2^{-4} &= \underline{0.0625}\\
&\ 0.6875
\end{aligned}
$$

Answer: -15.6875

Converting <u>to</u> Floating Point

- E.g., Express 36.5625_{10} as a 32-bit floating point number (in hexadecimal)

- Step 1
  - Express original value in binary

$36.5625_{10} =$

100100.1001_2

- Step 2
  - Normalize

$100100.1001_2 =$

$1.001001001_2 \times 2^5$

- Step 3
  - Determine S, E, and M

$$\underbrace{+}_{S}1.\underbrace{001001001}_{M}{}_{2} \times 2^{\underbrace{5}_{n}}$$

$$\begin{aligned} E &= n + 127 \\ &= 5 + 127 \\ &= 132 \\ &= 10000100_2 \end{aligned}$$

S = 0 (because the value is positive)

- Step 4
  - —Put S, E, and M together to form 32-bit binary result

$$\underbrace{0}_{S}\ \underbrace{10000100}_{E}\ \underbrace{00100100100000000000000}_{M}{}_2$$

- Step 5
  - Express in hexadecimal

$$0\ 10000100\ 00100100100000000000000_2 =$$

$$0100\ 0010\ 0001\ 0010\ 0100\ 0000\ 0000\ 0000_2 =$$

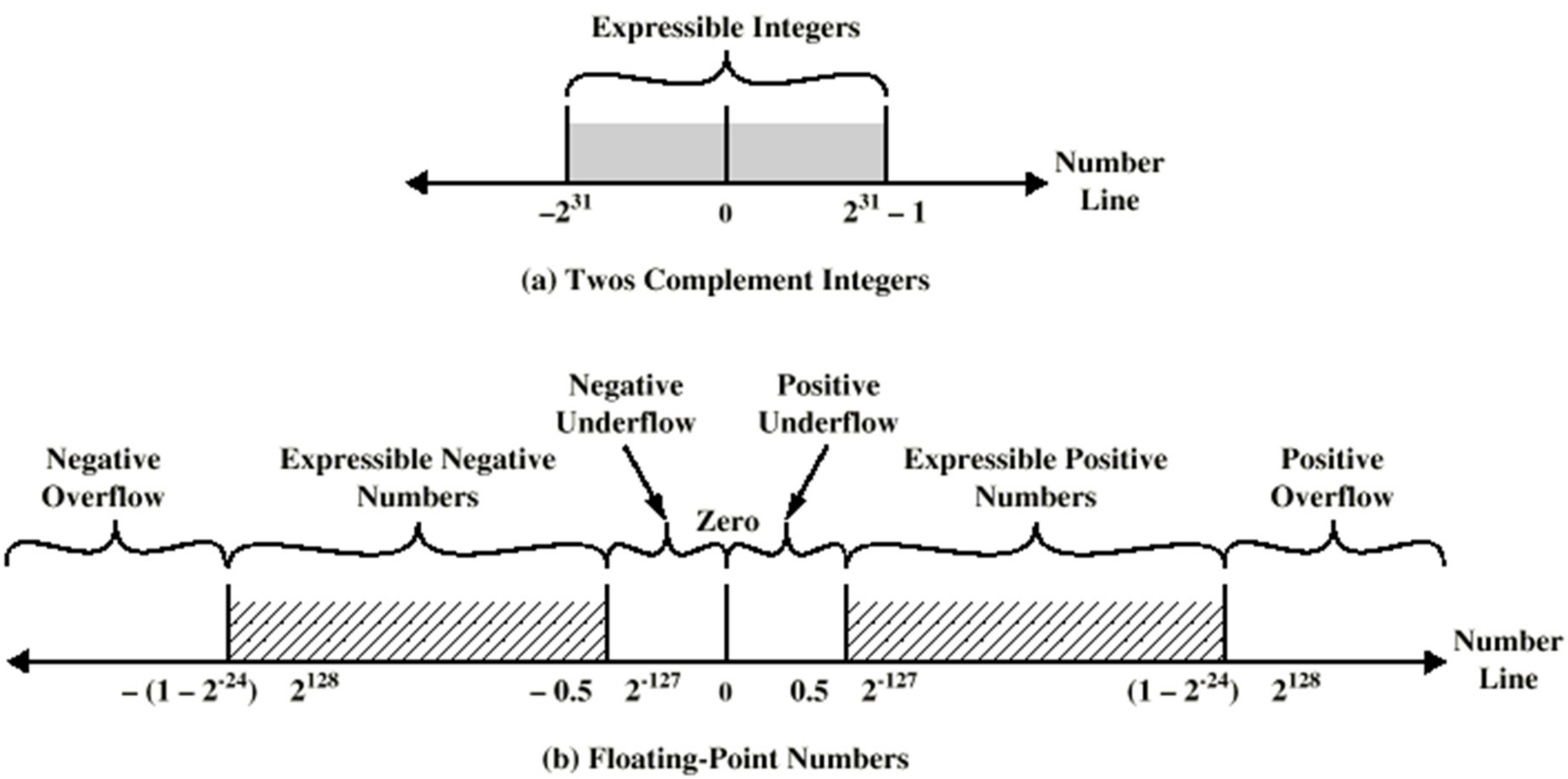
$$4\quad 2\quad 1\quad 2\quad 4\quad 0\quad 0\quad 0_{16}$$

Answer: 42124000_{16}

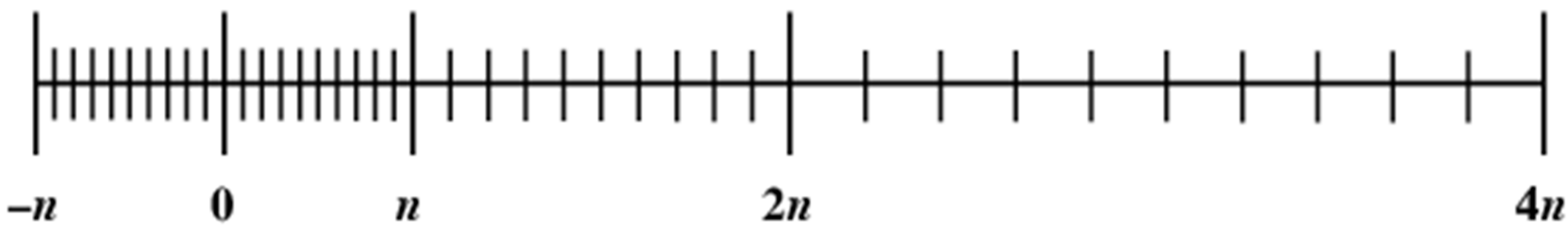
FP Ranges

- For a 32 bit number
  - 8 bit exponent
  - +/- $2^{256} \approx 1.5 \times 10^{77}$
- Accuracy
  - The effect of changing lsb of mantissa
  - 23 bit mantissa $2^{-23} \approx 1.2 \times 10^{-7}$
  - About 6 decimal places

Expressible Numbers



Density of Floating Point Numbers

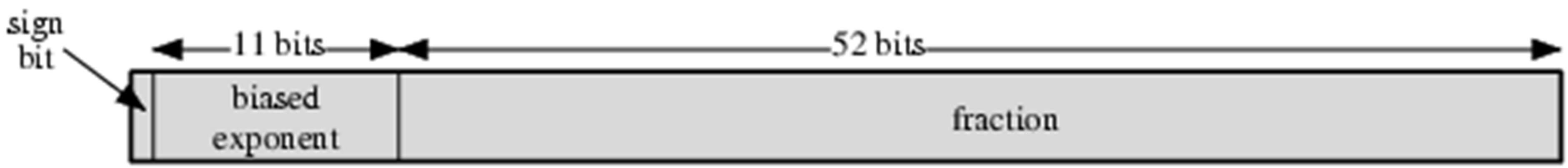


IEEE 754

- Standard for floating point storage
- 32 and 64 bit standards
- 8 and 11 bit exponent respectively
- Extended formats (both mantissa and exponent) for intermediate results

IEEE 754 Formats

(a) Single format

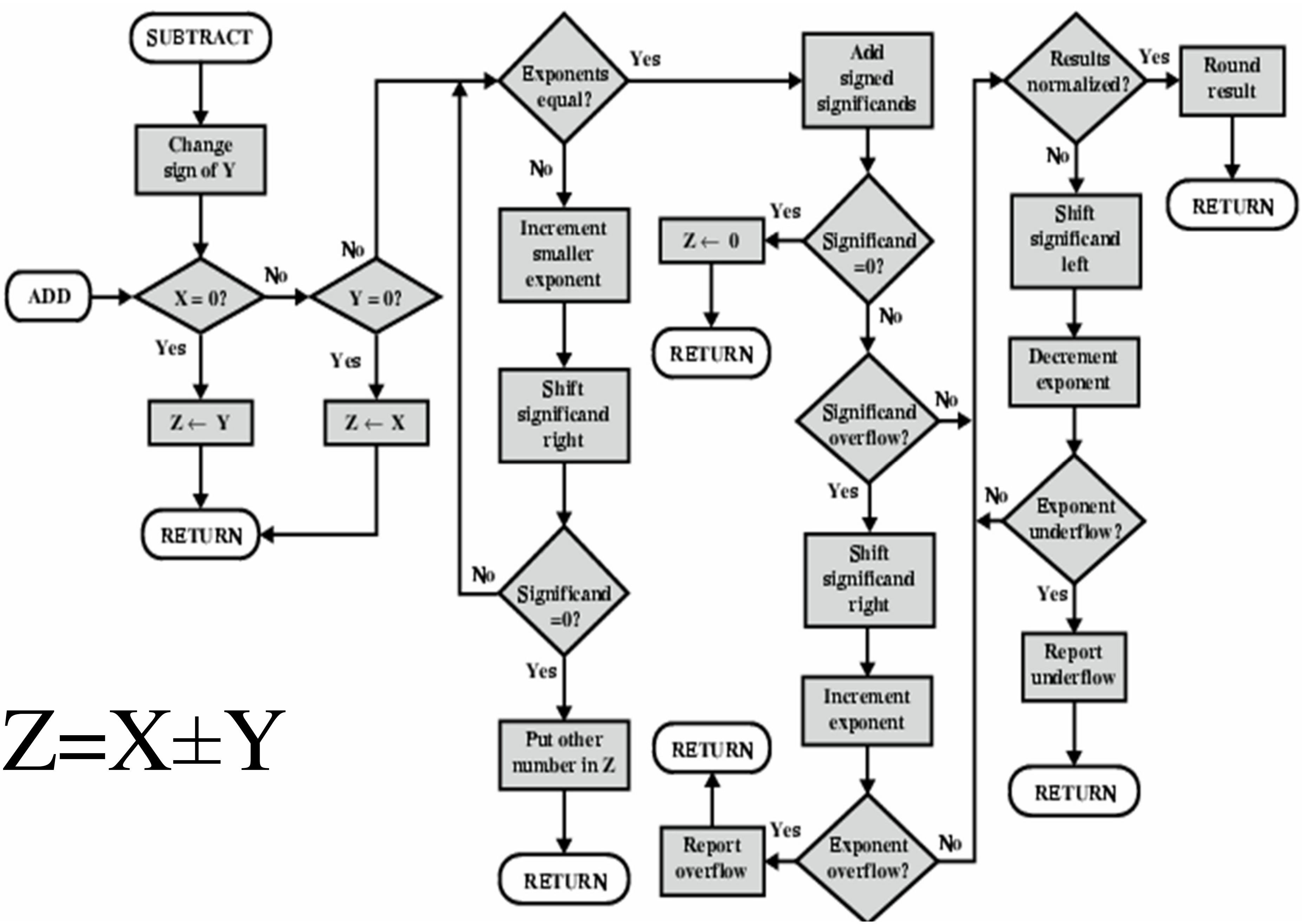


(b) Double format

FP Arithmetic +/-

- Check for zeros
- Align significands (adjusting exponents)
- Add or subtract significands
- Normalize result

FP Addition & Subtraction Flowchart

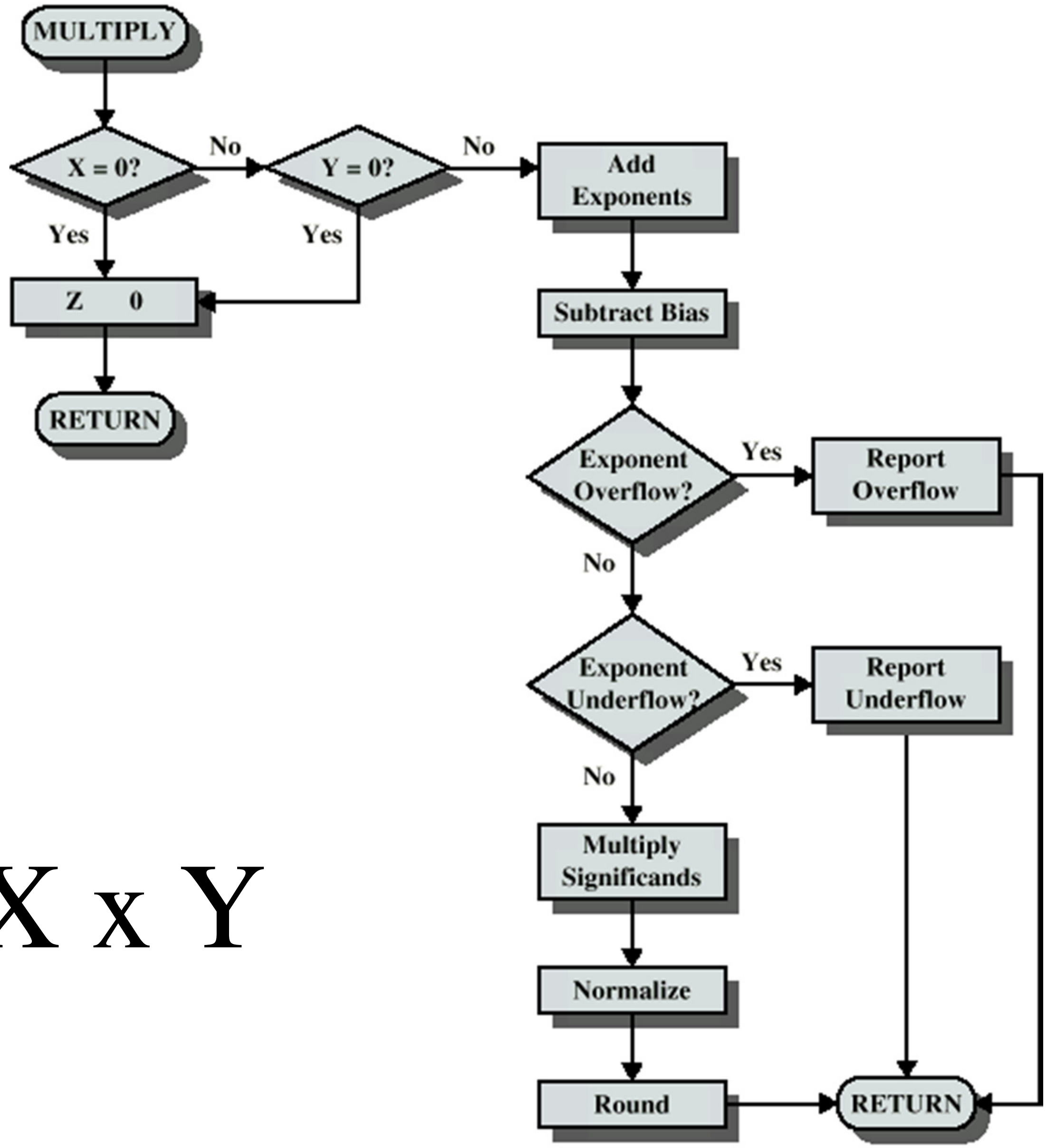


$$Z = X \pm Y$$

FP Arithmetic x/÷

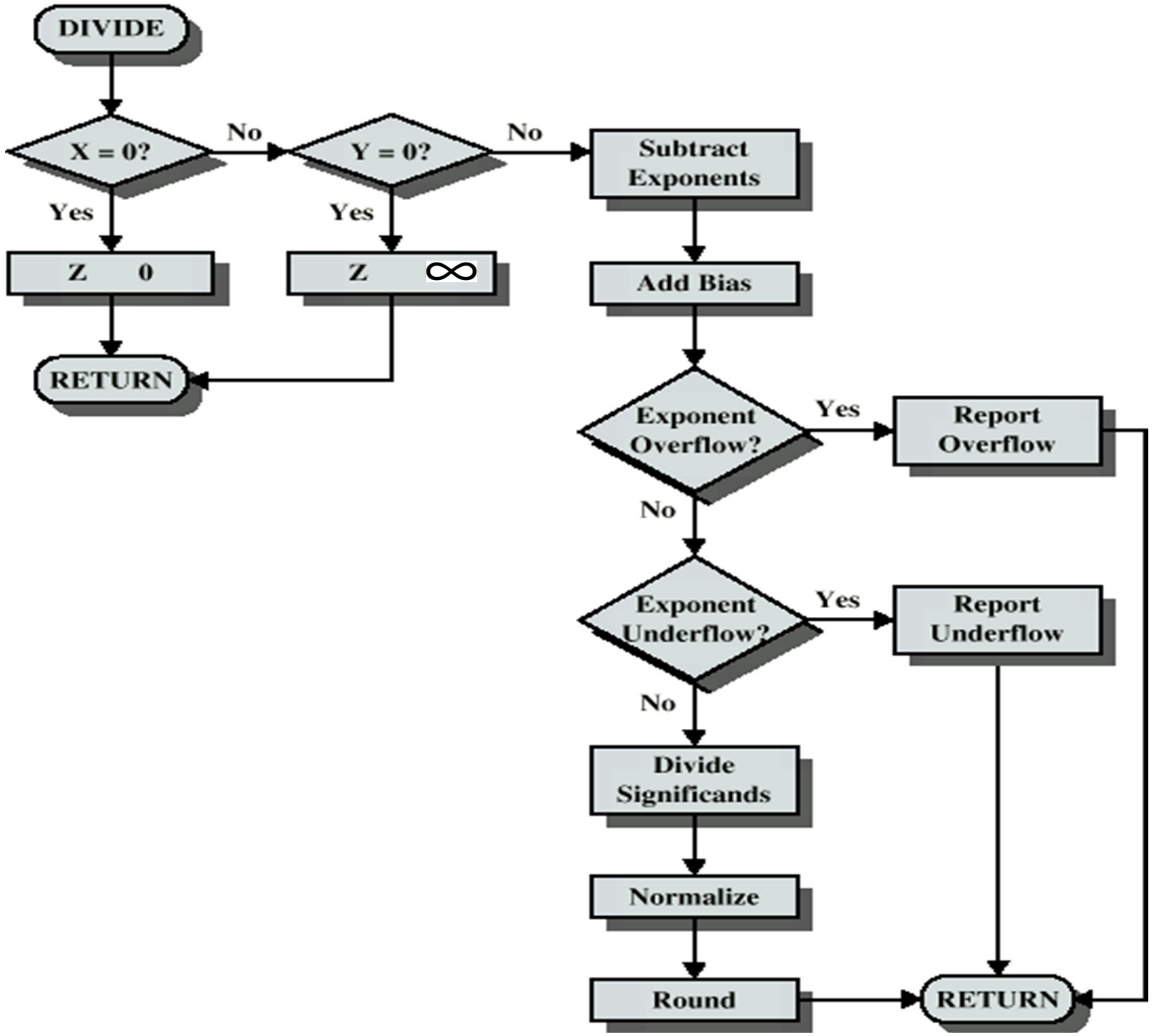
- Check for zero
- Add/subtract exponents
- Multiply/divide significands (watch sign)
- Normalize
- Round
- All intermediate results should be in double length storage

Floating Point Multiplication



Z=X x Y

Floating Point Division



Z=X/Y