Sinusoidal Steady-state Analysis

The Sinusoidal Source

$v_s(t) = V_m \sin \omega t$

$V_m \equiv$ Amplitude of the Sinusoid

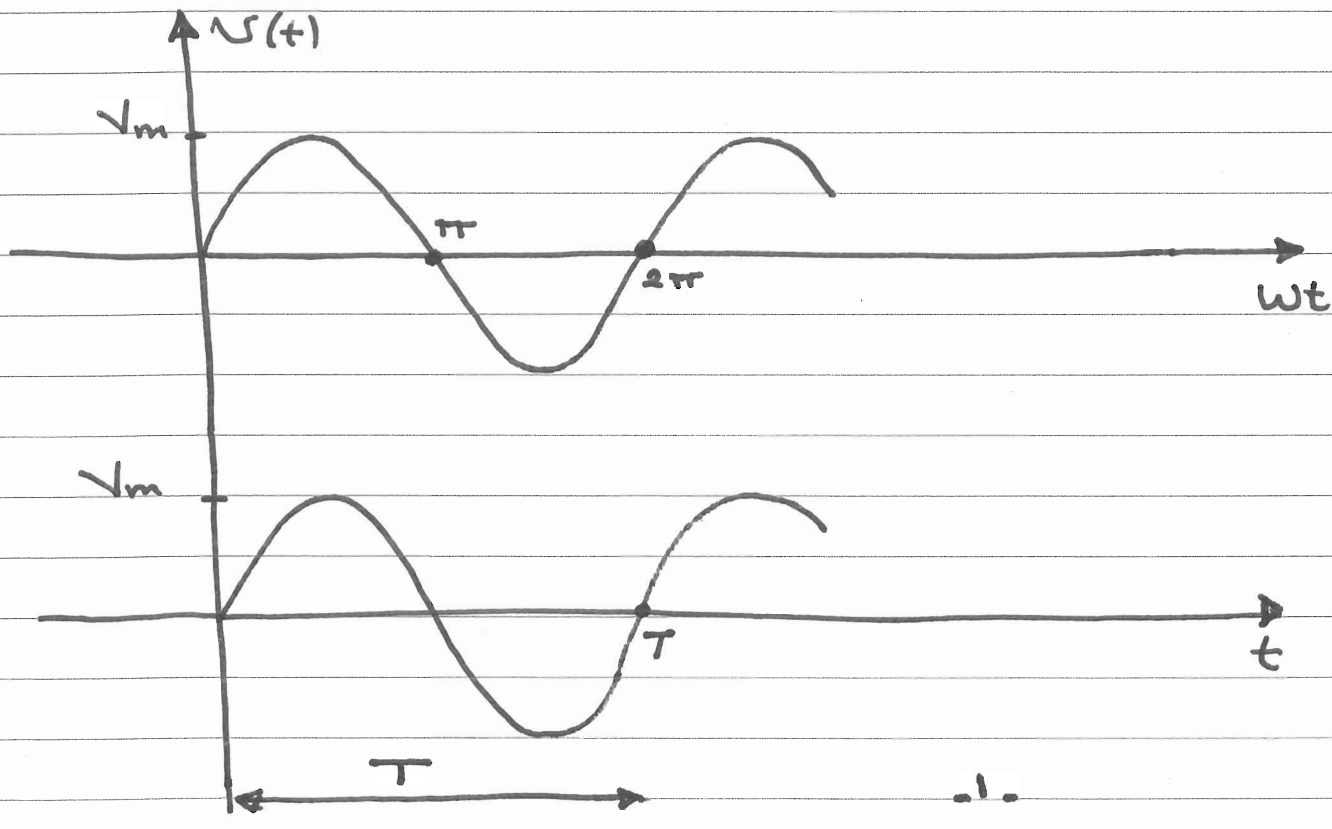
$\omega \equiv$ Angular frequency in radian/s

$\omega = 2\pi f$

$f \equiv$ frequency in Hertz

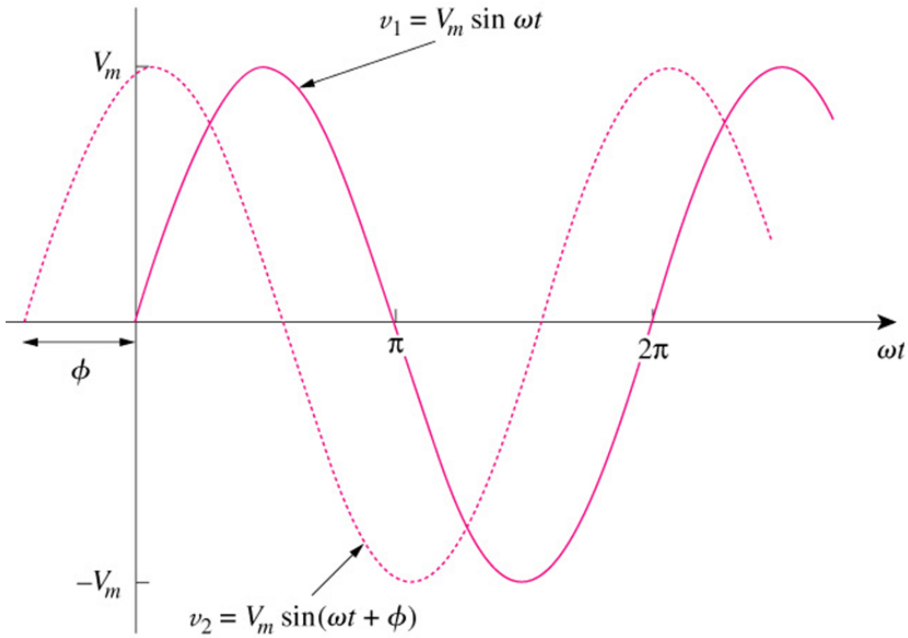
$f = \frac{1}{T}$

$T \equiv$ Period in seconds



Phase of Sinusoids

➢ Consider the sinusoidal voltage having phase φ, $v(t) = V_m \sin(\omega t + \phi)$



- v_2 LEADS v_1 by phase φ.
- v_1 LAGS v_2 by phase φ.
- v_1 and v_2 are out of phase.

Phase of Sinusoids

The terms Lead and Lag are used to indicate the relationship between two sinusoidal wave forms of the same frequency plotted on the same set of axes.

$v_1(t) = V_m \sin \omega t$

$v_2(t) = V_m \sin(\omega t + \theta)$

$\therefore$ $v_2(t)$ Leads $v_1(t)$ by θ

or

$v_1(t)$ Lags $v_2(t)$ by θ

Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$

$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$

$$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$$

$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$$

$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta)$$

where

$$C = \sqrt{A^2 + B^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{B}{A}$$

Let $v_1(t) = 10 \sin(5t - 30^\circ)$

$v_2(t) = 15 \sin(5t + 10^\circ)$

$\therefore$ $v_2(t)$ Leads $v_1(t)$ by 40°

Let $i_1(t) = 2 \sin(377t + 45^\circ)$

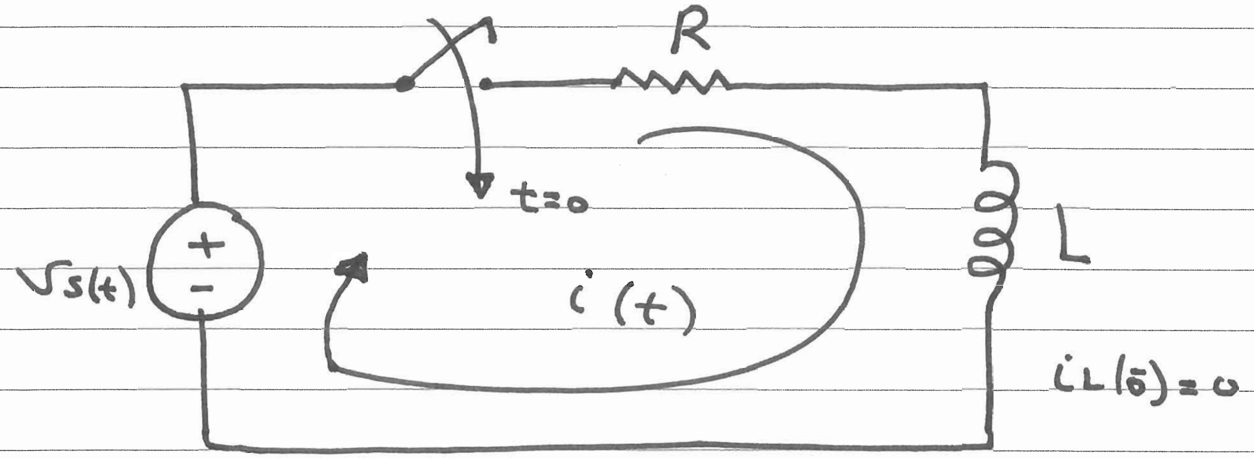
$i_2(t) = 0.5 \cos(377t + 10^\circ)$

$\cos \alpha = \sin(\alpha + 90^\circ)$

$0.5 \cos(377t + 10^\circ) = 0.5 \sin(377t + 100^\circ)$

$\therefore$ $i_2(t)$ leads $i_1(t)$ by 55°

The sinusoidal Response



Find $i(t)$ for $t > 0$

given $v_s(t) = V_m \cos \omega t$ V

KVL :

$$v_s(t) = R\,i(t) + L\frac{di(t)}{dt}$$

$$V_m \cos \omega t = R\,i(t) + L\frac{di(t)}{dt}$$

First order nonhomogenouse differential equation

$$\therefore \quad i(t) = i_n(t) + i_f(t)$$

$$i_n(t) = A e^{-t/\tau} + i_f(t)$$

$$i_f(t) = I_1 \cos \omega t + I_2 \sin \omega t$$

To find I_1 and I_2

$$V_m \cos wt = R\,i(t) + L\frac{di(t)}{dt}$$

$$V_m \cos wt = R\left[I_1 \cos wt + I_2 \sin wt\right]$$

$$+ Lw\left[-I_1 \sin wt + I_2 \cos wt\right]$$

Collect the cosine and sine terms

$$0 = \left(-LI_1 w + RI_2\right)\sin wt + \left(LI_2 w + RI_1 - V_m\right)\cos wt$$

$\therefore$

$$-wLI_1 + RI_2 = 0$$

$$wLI_2 + RI_1 - V_m = 0$$

$$I_1 = \frac{RV_m}{R^2 + w^2L^2}$$

$$I_2 = \frac{wLV_m}{R^2 + w^2L^2}$$

$$\therefore i_f(t) = \frac{RV_m}{R^2 + w^2L^2}\cos wt + \frac{wLV_m}{R^2 + w^2L^2}\sin wt$$

$$i_f(t) = I_1 \cos \omega t + I_2 \sin \omega t$$

$$i_f(t) = C \cos(\omega t - \phi)$$

$$i_f(t) = C \cos \omega t \cos \phi + C \sin \omega t \sin \phi$$

$$\therefore \quad I_1 = C \cos \phi$$

$$I_2 = C \sin \phi$$

$$\frac{I_2}{I_1} = \tan \phi$$

$$\therefore \quad \phi = \tan^{-1} \frac{I_2}{I_1} = \tan^{-1} \frac{\omega L}{R} \qquad \text{——— (1)}$$

$$I_1^2 + I_2^2 = C^2 \cos^2 \phi + C^2 \sin^2 \phi$$

$$I_1^2 + I_2^2 = C^2$$

$$\therefore \quad C = \sqrt{I_1^2 + I_2^2}$$

$$C = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \qquad \text{——— (2)}$$

$$\therefore \quad i_f(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R}\right)$$

$$\therefore\ i(t) = A e^{-t/\tau} + \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

$$i(0^+) = A + \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cos\left(-\tan^{-1}\frac{\omega L}{R}\right) = 0$$

$$\therefore\ A = -\frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cos\left(-\tan^{-1}\frac{\omega L}{R}\right)$$

$$\therefore\ i(t) = i_n(t) + i_f(t)$$

$i(t)$ = transient Componet + Steady-state Component

* The steady-state solution is a sinusoidal function with the same frequency as the source signal.

Complex Numbers

A complex number may be written in three forms

1) Rectangular Form

$$Z = x + j y$$

$j = \sqrt{-1}$, $x = Re(Z)$, $y = Im(Z)$

2) Exponential Form

$$Z = |Z| e^{j\theta}$$

$|Z|$ = Magnitude , θ = angle

3) Polar Form

$$Z = |Z| \underline{\angle \theta}$$

Euler's Law

$$e^{j\theta} = \cos\theta + j\sin\theta$$

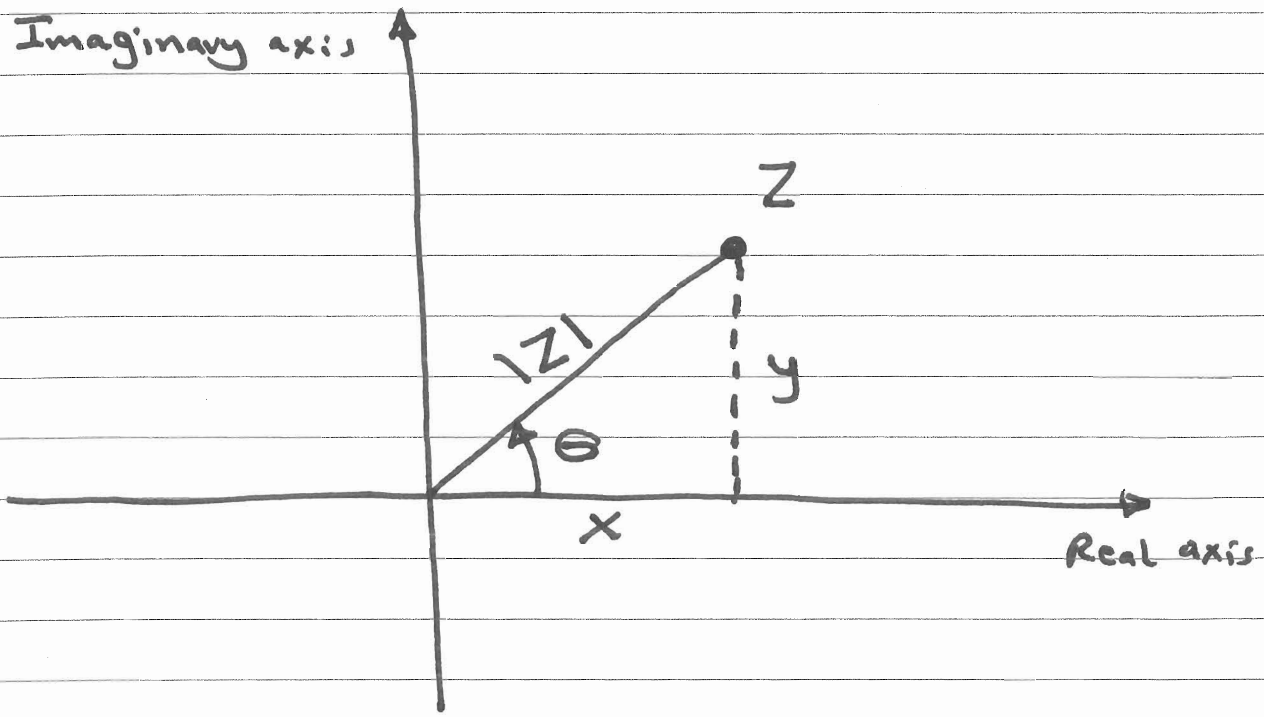
$$Z = |Z|\, e^{j\theta}$$

$$Z = |Z|\cos\theta + j|Z|\sin\theta$$

$$Z = x + jy$$

$$\therefore\ x = |Z|\cos\theta$$

$$y = |Z|\sin\theta$$



Mathematical Operations of Complex numbers

Addition : $Z_1 + Z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Subtraction : $Z_1 - Z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication : $Z_1 Z_2 = |Z_1||Z_2| \underline{\left|\theta_1 + \theta_2\right.}$

Division : $\dfrac{Z_1}{Z_2} = \dfrac{|Z_1|}{|Z_2|} \underline{\left|\theta_1 - \theta_2\right.}$

Complex Conjugate : $Z^* = x - jy$

$= |Z| \underline{\left|-\theta\right.}$

$$x = |Z| \cos\theta$$

$$y = |Z| \sin\theta$$

$$x^2 + y^2 = |Z|^2 \cos^2\theta + |Z| \sin^2\theta$$

$$x^2 + y^2 = |Z|^2 \left(\cos^2\theta + \sin^2\theta \right)$$

$$x^2 + y^2 = |Z|^2$$

$$\therefore |Z| = \sqrt{x^2 + y^2}$$

$$\frac{y}{x} = \frac{|Z| \sin\theta}{|Z| \cos\theta} = \tan\theta$$

$$\therefore \theta = \tan^{-1} \frac{y}{x}$$

$$Z_1 = 4 + j3 = 5\angle 36.9^\circ$$

$$Z_2 = 3 + j4 = 5\angle 53.1^\circ$$

$$Z_1 + Z_2 = 7 + j7$$

$$Z_1 - Z_2 = 1 - j1$$

$$Z_1 Z_2 = 5\angle 36.9^\circ \cdot 5\angle 53.1^\circ = 25\angle 90^\circ$$

$$\frac{Z_1}{Z_2} = \frac{5\angle 36.9^\circ}{5\angle 53.1^\circ} = 1\angle -16.2^\circ$$

or

$$Z_1 Z_2 = (4 + j3)(3 + j4)$$

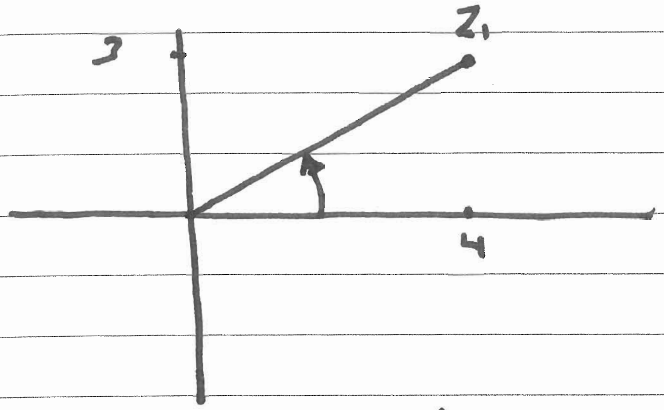
$$= 12 + j16 + j9 - 12$$

$$Z_1 Z_2 = j25$$

$$\frac{Z_1}{Z_2} = \frac{4 + j3}{3 + j4} \cdot \frac{3 - j4}{3 - j4} = \frac{12 - j16 + j9 + 12}{25}$$

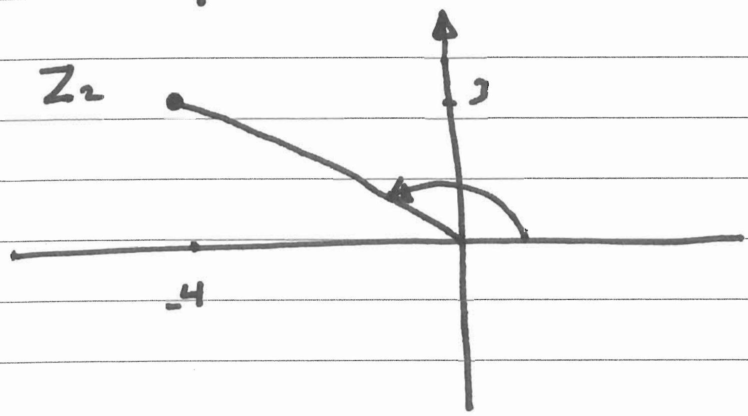
$$= \frac{24 - j5}{25} = \frac{24}{25} - j\frac{5}{25}$$

The graphical Representation



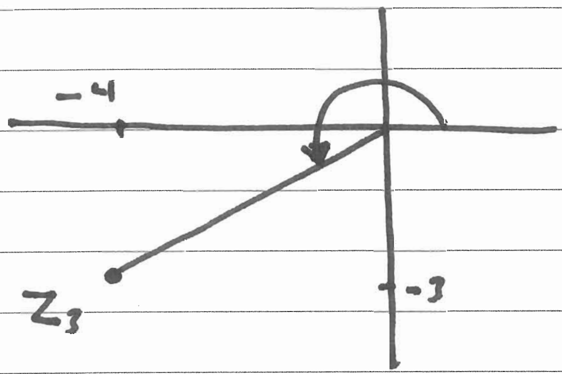
$$Z_1 = 4 + j3$$

$$Z_1 = 5\angle 36.9^\circ$$



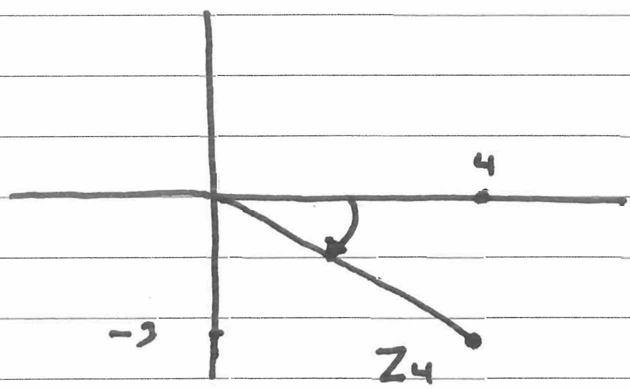
$$Z_2 = -4 + j3$$

$$Z_2 = 5\angle 143.1$$



$$Z_3 = -4 - j3$$

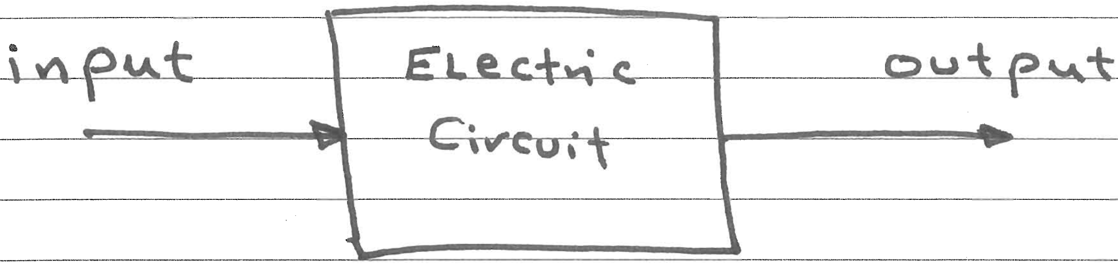
$$Z_3 = 5\angle 216.9$$



$$Z_4 = 4 - j3$$

$$Z_4 = 5\angle -36.9^\circ$$

The phasor Concept



$$V_m \cos(\omega t+\theta) \longrightarrow I_m \cos(\omega t+\phi)$$

$$V_m \sin(\omega t+\theta) \longrightarrow I_m \sin(\omega t+\phi)$$

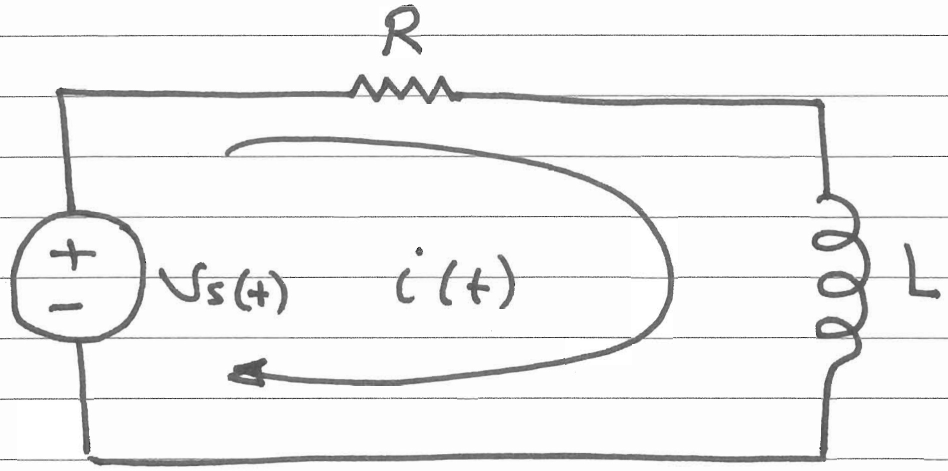
$$j V_m \sin(\omega t+\theta) \longrightarrow j I_m \sin(\omega t+\phi)$$

$$\begin{array}{c} V_m \cos(\omega t+\theta) \\ + \\ j V_m \sin(\omega t+\theta) \end{array} \Longrightarrow \begin{array}{c} I_m \cos(\omega t+\phi) \\ + \\ j I_m \sin(\omega t+\phi) \end{array}$$

$$V_m e^{j(\omega t+\theta)} \longrightarrow I_m e^{j(\omega t+\phi)}$$

Instead of Applying a real forcing function to obtain the desired real response, we apply a Complex forcing function whose real part is the given real forcing function.
We obtain a Complex response whose real part is the desired real response.

Sinusoidal and Complex forcing function



$$v_s(t) = V_m \cos \omega t$$

$$\therefore \quad i(t) = I_m \cos(\omega t + \phi)$$

$$v_s(t) \longrightarrow V_m e^{j\omega t}$$

$$i(t) \longrightarrow I_m e^{j(\omega t + \phi)}$$

KVL :

$$v_s(t) = R\, i(t) + L \frac{d\, i(t)}{dt}$$

$$V_m e^{j\omega t} = R I_m e^{j(\omega t+\phi)} + j\omega L I_m e^{j(\omega t + \phi)}$$

a Complex algebraic equation

To find I_m and ϕ ; devide by $e^{j\omega t}$

$$V_m = R I_m e^{j\phi} + j\omega L I_m e^{j\phi}$$

$$V_m = I_m e^{j\phi} (R + j\omega L)$$

$$\therefore I_m e^{j\phi} = \frac{V_m}{R + j\omega L}$$

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2} \; e^{j \tan^{-1} \frac{\omega L}{R}}}$$

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{-j \tan^{-1} \frac{\omega L}{R}}$$

$$\therefore I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\phi = -\tan^{-1} \frac{\omega L}{R}$$

$$\therefore i(t) = I_m \cos(\omega t + \phi)$$

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi)$$