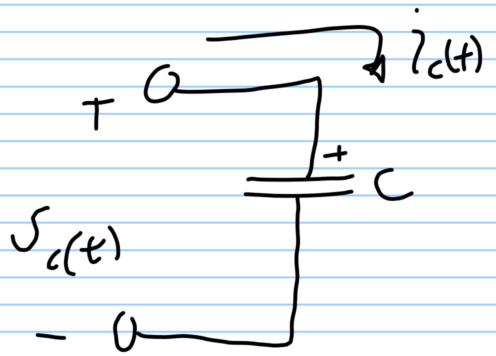
ch7

Response of first-order RL & RC circuits



$$\boxed{i_c(t) = C \frac{d}{dt} v_c(t)}$$

$\therefore \; v_c(t) = \frac{1}{C}\int_0^t i_c(t)\,dt + \underline{\underline{v_c(0^-)}}$, for $t \geq 0$

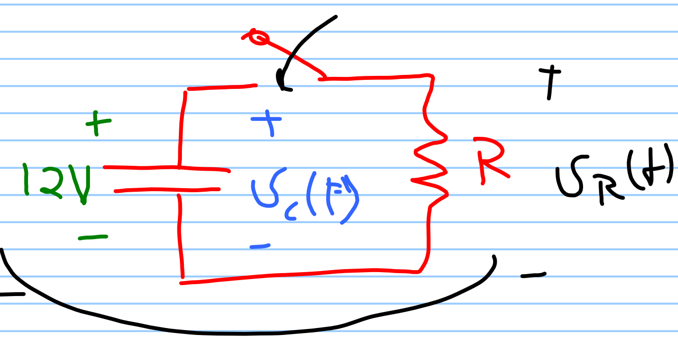
initial voltage of the capacitor

at $t = 0^+$

$$v_c(0^+) = \overbrace{\cancel{\frac{1}{C}\int_{0^-}^{0^+} i_c(t)\,dt}}^{\text{zero}} + v_c(0^-)$$

$$\boxed{v_c(0^+) = v_c(0^-)}$$

at $t = 0$



$v_c(0^-) = 12\,V$

$v_c(0^+) = 12\,V$

$v_c(\infty) = \text{zero} \rightarrow 0.0000001$

$v_R(0^-) = \text{zero}$

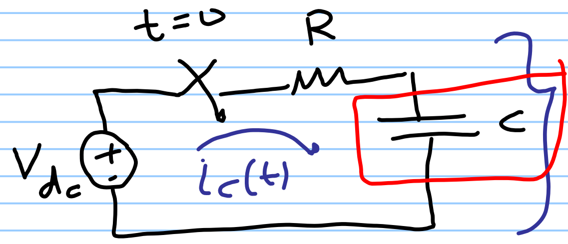
$v_R(0^+) = 12\,V$

1. The current through a capacitor is zero if the voltage across it is not changing with time.

↳ steady state

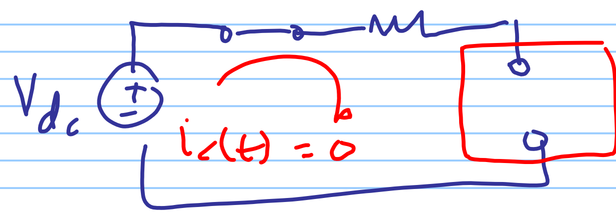
$$i_c(t) = C\frac{d}{dt}v_c(t)$$

∴ at steady state $\boxed{t \to \infty}$



$\boxed{i_c(\infty)}$ OR i_c(steady state) = zero

A capacitor is therefore an open circuit to dc!



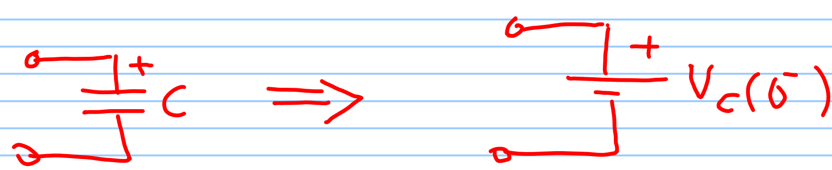
2. A finite amount of energy can be stored in a capacitor even if the current through the capacitor is zero.

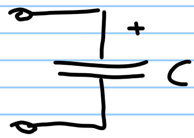
3. The capacitor never dissipate energy, but only store it.

$$v_c(0^-) = v_c(0^+)$$

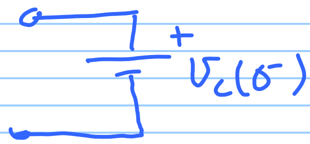
4. it is impossible to change the voltage across a capacitor by a finite amount in zero time,

5. At $t = 0^+$

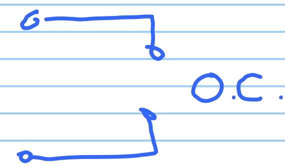




@ $t = 0^+$

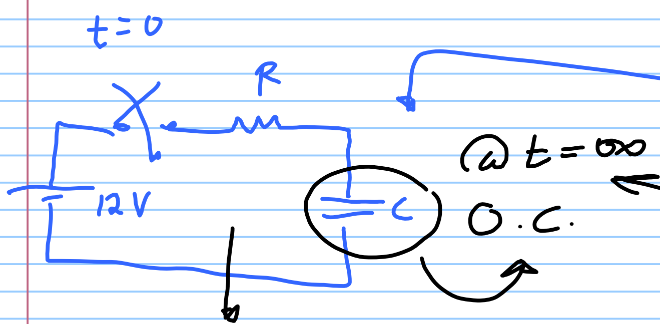


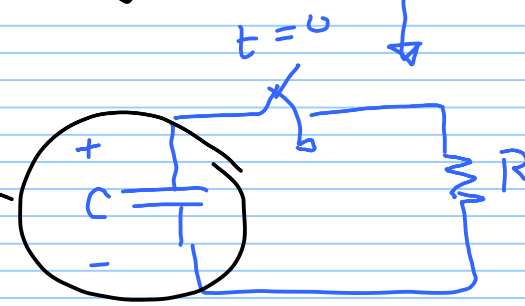
@ $t = \infty$



charging

discharging





@ $t = \infty$

$v_c(\infty) = 12V$

$i_c(\infty) = \text{zero}$

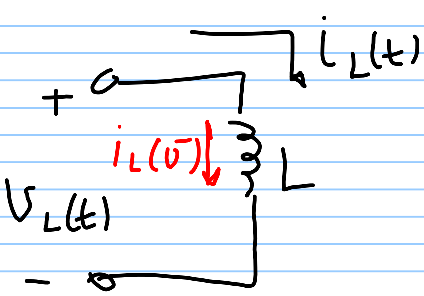
$v_c(0^-) = 12V$

$v_c(0^+) = 12V$

@ $t = \infty$

$v_c(\infty) = \text{zero}$.

$i_c(\infty) = \text{zero}$



$$\boxed{v_L(t) = L \frac{d}{dt} i_L(t)}$$

$$i_L(t) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} v_L(t)\, dt, \text{ for } t \geq 0$$

1. There is no voltage across an inductor if the current through it is not changing with time.

An Inductor is therefore a short circuit to dc.

@ $t=\infty$

$$v_L(\infty) = L\frac{d}{dt}\, i_L(\infty)$$

const.

$= \text{zero}$

2. A finite amount of energy can be stored in an Inductor even if the voltage across the inductor is zero.

3. The inductor never dissipate energy, but only store it.

4. $$i_L(t) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{0^+} v_L(t)\,dt\,, \quad t \geq 0$$

@ $t = 0^+$

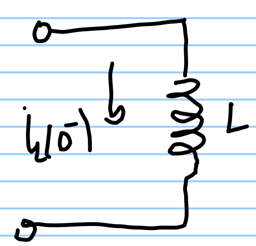
$$i_L(0^+) = i_L(0^-) + \cancel{\frac{1}{L}\int_{0^-}^{0^+} v_L(t)\,dt}$$

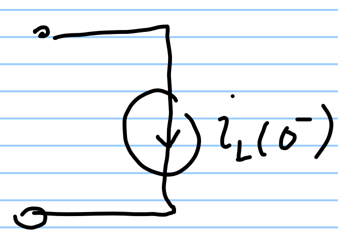
zero

$$\boxed{i_L(0^+) = i_L(0^-)}$$

it is impossible to change the current through an inductor by a finite amount in zero time, for this requires an infinite voltage across the inductor.

@ $\boxed{t = 0^+}$

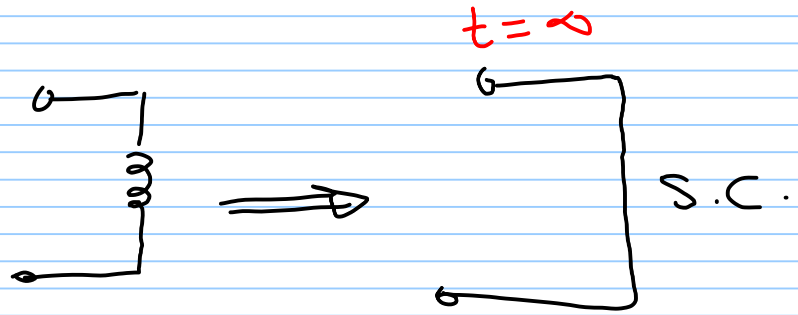




$\boxed{t=\infty}$

$$v_L(t) = L\frac{d}{dt}i_L(t)$$

An inductor is therfore a short circuit to dc.

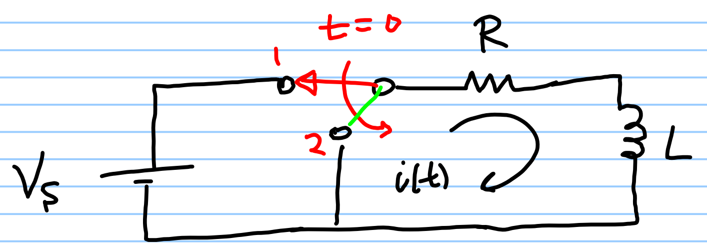


First Order Circuit

$\left.\begin{array}{l} R + L \\ \text{OR} \quad R + C \end{array}\right\}$ RL, RC circuits.

Natural Response of 1^{st} order circuit

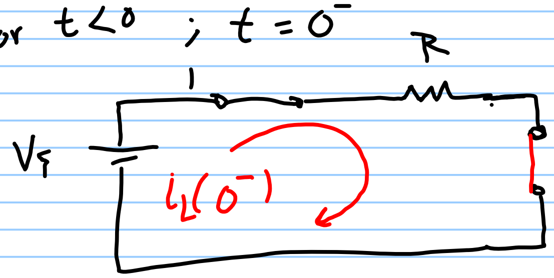
↳ discharging.



find $i(t)$ for $t>0$

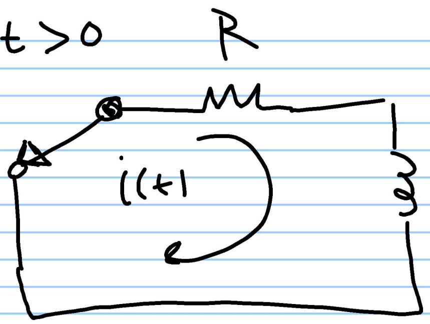
$$V_L = L\frac{di}{dt}$$

1) for $t<0$; $t=0^-$



$$\therefore i_L(0^-) = \frac{V_s}{R}$$

2) for $t>0$



$\downarrow i_L(0^-) = i_L(0^+)$

→ KVL

$$R\, i(t) + L \frac{d}{dt} i(t) = 0$$

$$i(t) = A e^{st} \quad \text{for } t>0$$

$$R A e^{st} + L A s e^{st} = 0$$

$$A e^{st} \left(\underline{R + Ls} \right) = 0$$

$$\therefore \boxed{s = -\frac{R}{L}}$$

How to find A

From initial conditions

$$i(t) = A e^{st} \qquad t>0$$

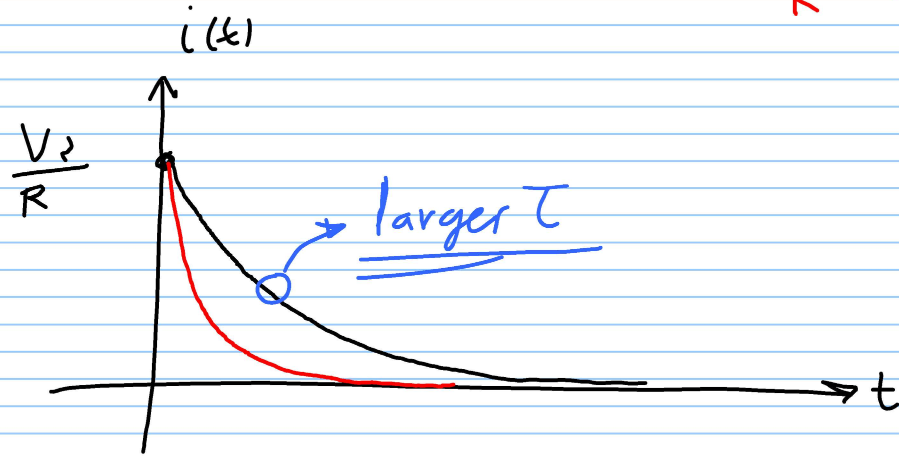
$$i(0^+) = A$$

$$i(0^+) = i_L(0^+) = i_L(0^-) = \frac{V_s}{R}$$

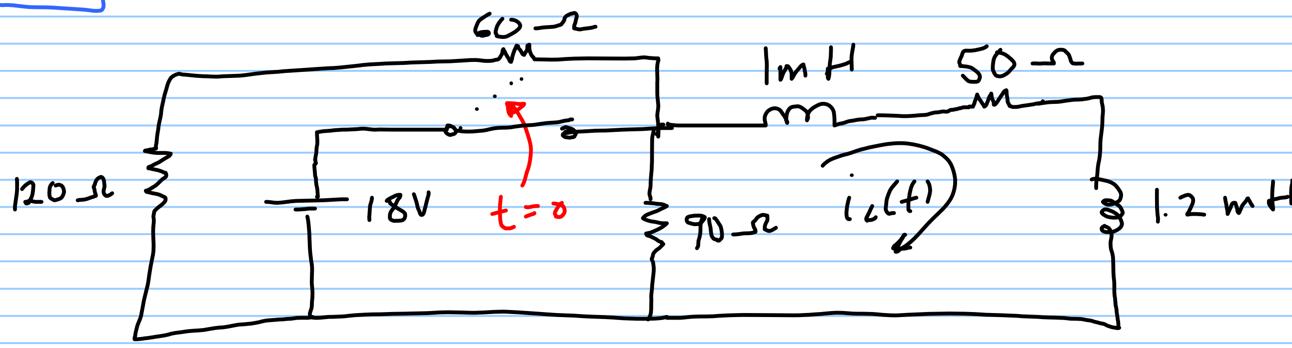
$$\therefore \boxed{i(t) = \frac{V_s}{R} e^{-\frac{R}{L} t}} \qquad t>0$$

$\tau = \frac{L}{R}$, Time constant

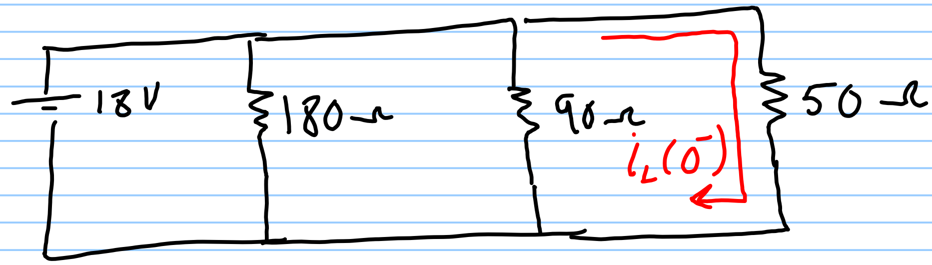
$$i(t) = \frac{V_s}{R} e^{-t/\tau}$$



EX) find $i_L(t)$ for $t>0$



1) for $t<0$, $t=0^-$

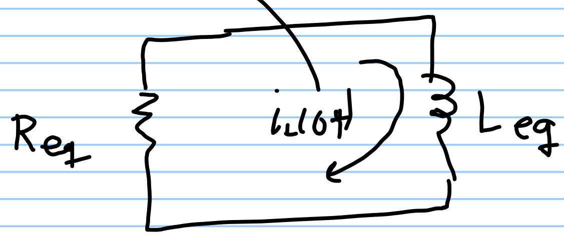


$$i_L(0^-) = \frac{18}{50} = 0.36\ A$$

$i_L(0^-) = i_L(0^+)$ ✓

$V(0^-) \neq V(0^+)$

2) for $t>0$



$R_{eq} = (180 // 90) + 50 = 110\ \Omega$

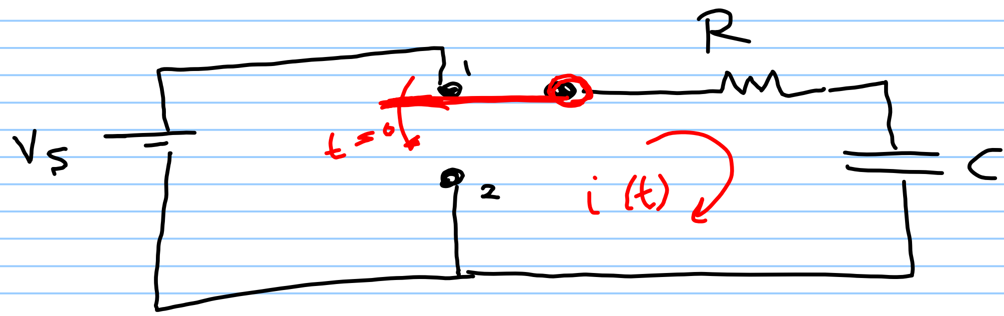
$L_{eq} = 1mH + 1.2mH = 2.2mH$

$$\tau = \frac{L_{eq}}{R_{eq}} = 20\ \mu\ sec$$

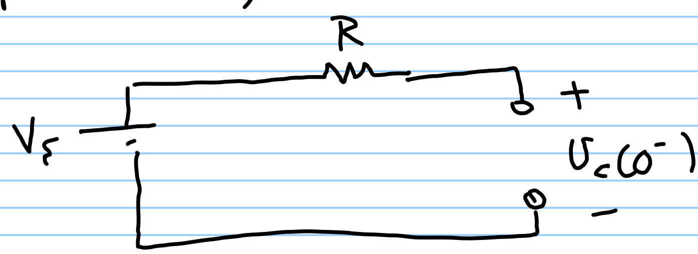
$$i_L(t) = A e^{-t/\tau} \quad \text{for } t>0$$

$$i_L(t) = 0.36\ e^{-50\,000\,t}, \quad t>0$$

RC circuit

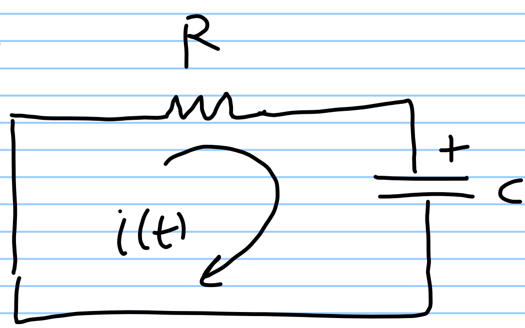


1) for $t<0$; $t=0^-$



$v_c(0^-) = V_s$

2) for $t>0$



KVL

$$R\,i(t) + v_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)\,dt = 0 \quad , \; t>0$$

$$R\frac{d}{dt}i(t) + \frac{1}{C}i(t) = 0$$

$$\therefore \boxed{i(t) = A e^{st}} \quad , \; t>0$$

$$RAs\,e^{st} + \frac{1}{C}A\,e^{st} = 0$$

$$A e^{st}\left(Rs + \frac{1}{C}\right) = 0$$

$$\therefore \boxed{s = -\frac{1}{RC}}$$

s ✓

$$\therefore i(t) = A\, e^{-\frac{1}{RC}t} \qquad t>0$$

$$\therefore \tau = RC \quad \text{(time constant)}$$

$$i(t) = A e^{-t/\tau} \qquad t>0$$

to find A

$$i(t) = A e^{-t/\tau}$$

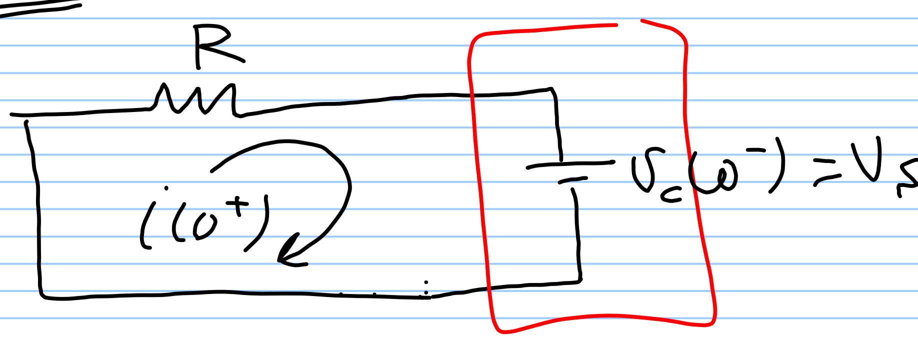
$$\boxed{i(0^+) = A}$$

$\to i(0^-)$ ✗

$i(0^-) \neq i(0^+)$!!

$V_c(0^-) = V_c(0^+)$ ✓

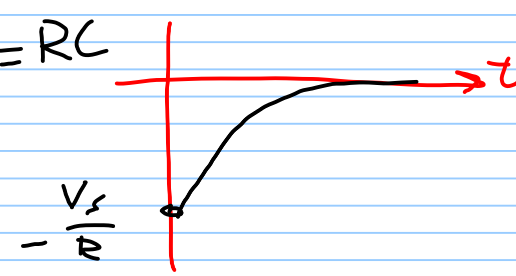
to find $i(0^+)$



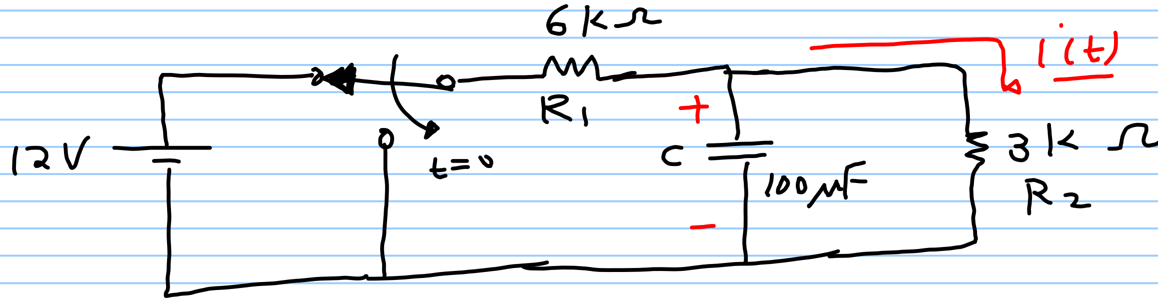
$$i(0^+) = -\frac{V_c(0^-)}{R} = -\frac{V_s}{R} = A \quad ✓$$

$$\boxed{\therefore i(t) = -\frac{V_s}{R}\, e^{-t/\tau}, \quad t>0}$$

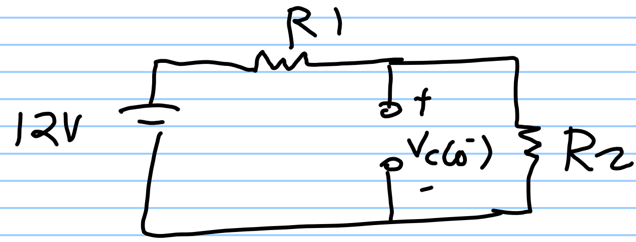
$\tau = RC$



EX) calculate $v_c(t)$ & $i(t)$ for $t>0$



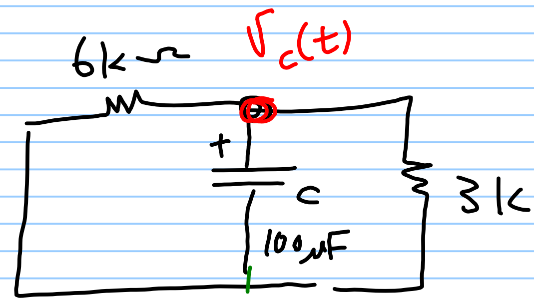
1) for $t<0$; $t=0^-$



$$v_c(0^-) = \frac{R_2}{R_2+R_1} 12 = \frac{3}{3+6} \times 12 = 4V$$

$$\boxed{v_c(0^-) = 4V}$$

2) for $t>0$

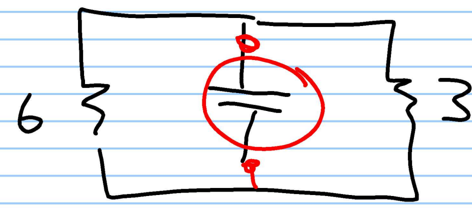


$$\frac{v_c(t)}{6k} + \frac{v_c(t)}{3k} + C\frac{d}{dt}v_c(t) = 0$$

$$\frac{d}{dt}V_c(t) + 5V_c(t) = 0$$

$$v_c(t) = A e^{-t/\tau} \quad t>0 \quad , \quad \tau = R_{eq} C$$

seen by C



$$R_{eq} = 6k // 3k$$

$$= 2\,k\Omega$$

$$\tau = R_{eq} C = 0.2 \text{ sec.}$$

$$v_c(t) = A e^{-5t} \quad t>0$$

<u>to find A</u>

$$v_c(0^+) = A = V_c(0^-) = 4 \text{ Volt.}$$

$$\therefore v_c(t) = 4e^{-5t} \text{ V}, \quad t>0$$

$$\therefore i(t) = \frac{v_c(t)}{3k} = \frac{4}{3} e^{-5t} \text{ mA} \quad t>0.$$

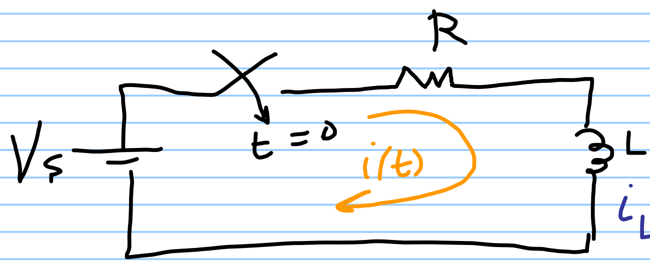
find $i_c(t)$

$$i_c(t) = C \frac{d}{dt} v_c(t)$$

The step Response of RC & RL circuits

→ The response of a circuit to the sudden application of a const. Voltage or current source.

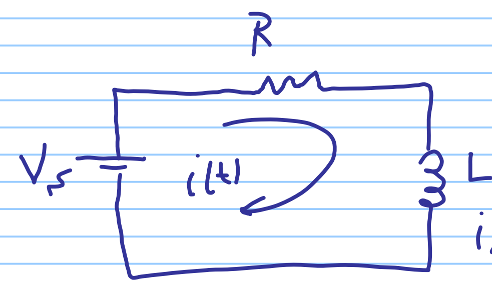
* **The step response of RL circuits**



find $i(t)$ for $t>0$

$i_L(0^-)=0$

for $t>0$



$i_L(0^-)=0$

KVL

$$V_s = R\,i(t) + L\frac{d}{dt}i(t) \qquad t>0$$

non homogenous differential equation

$$\boxed{i(t) = i_n(t) + i_f(t)} \quad *$$

$i_n(t)$ = Natural response $(e^{\wedge})$

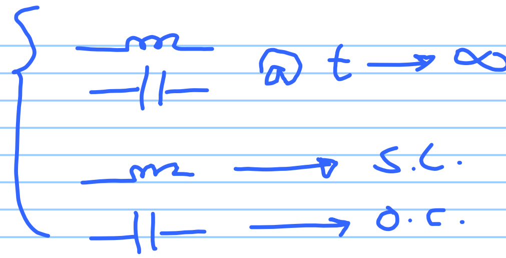
$i_f(t)$ = forced response. [final value]

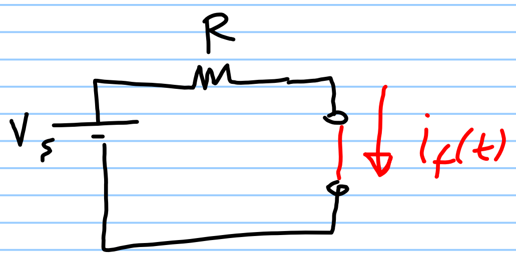
→ to find $i_f(t)$

let $i_f(t) = K$

$$\left.\begin{aligned} &V_s = R\,i(t) + L\frac{d}{dt}i(t) \\ &V_s = R\,K + L[0] \end{aligned}\right\} \Rightarrow K = \frac{V_s}{R} = i_f(t)$$

OR to find $i_f(t)$





$\therefore \; i_f(t) = V_s/R$ ✓

$\therefore$ Now

$$i(t) = i_n(t) + \frac{V_s}{R}$$

$$= A e^{-t/\tau} + \frac{V_s}{R}$$

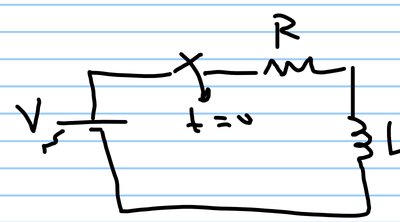
$$\tau = L/R_{eq} = L/R \qquad t>0$$

→ to find A

$$i(t) = \frac{V_s}{R} + A e^{-t/\tau}$$

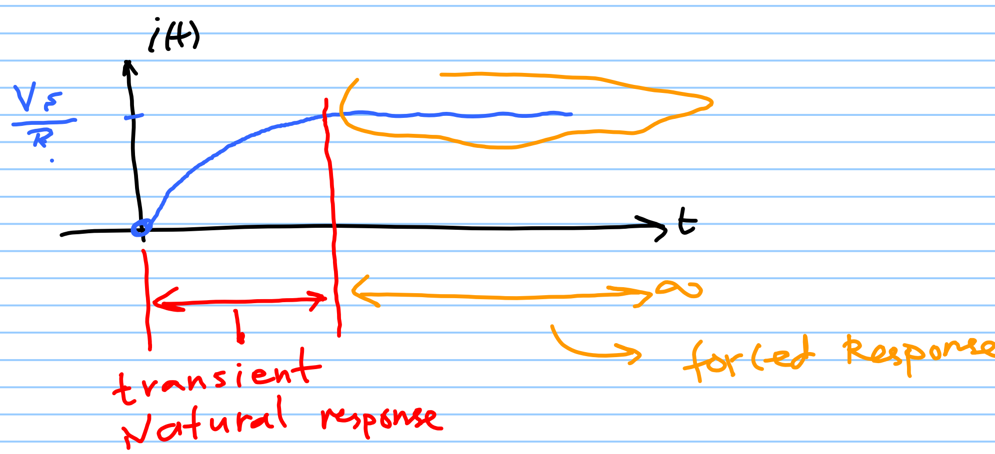
$$i(0^+) = \frac{V_s}{R} + A$$

But $i(0^+) = i_L(0^+) = i_L(0^-) = 0$

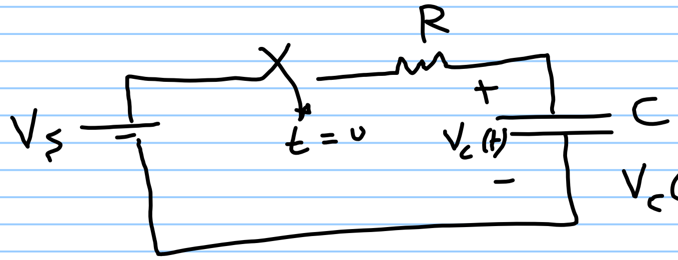


$$\therefore \; A = -\frac{V_s}{R}$$

$$i(t) = \frac{V_s}{R} - \frac{V_s}{R} e^{-t/\tau} \qquad t \geq 0$$

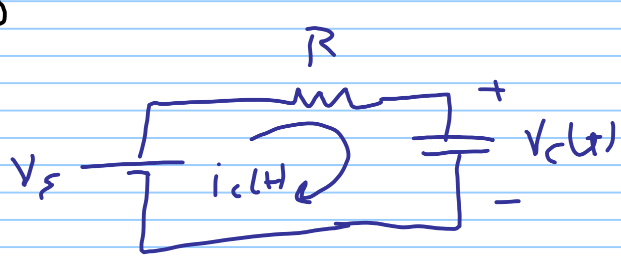


The step response of RC circuits



$V_c(0^-) = 0$

find $v_c(t)$ for $t>0$



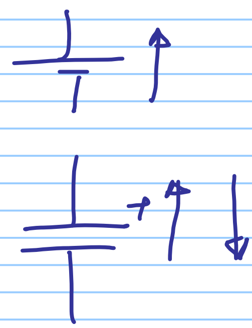
for $t>0$

$$-V_s + V_R + V_c(t) = 0$$

$$V_s = R\, i_c(t) + V_c(t)$$

$$V_s = RC \frac{d}{dt} V_c(t) + V_c(t)$$

$$\frac{V_s}{R} = C \frac{d}{dt} V_c(t) + \frac{V_c(t)}{R}$$

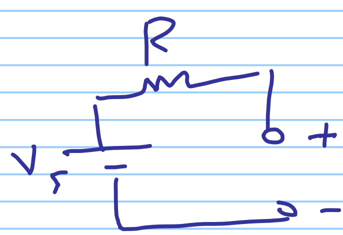


$$v_c(t) = v_{cn}(t) + v_{cf}(t) \qquad t>0$$

$$= A e^{-t/\tau} + K$$

to find K

$t \to \infty \rightarrow c$ is O.C.



$$\boxed{K = V_s}$$

$$\therefore\ v_c(t) = A e^{-t/\tau} + V_s \qquad t>0$$

$$\tau = R_{eq} C = RC$$

to find A

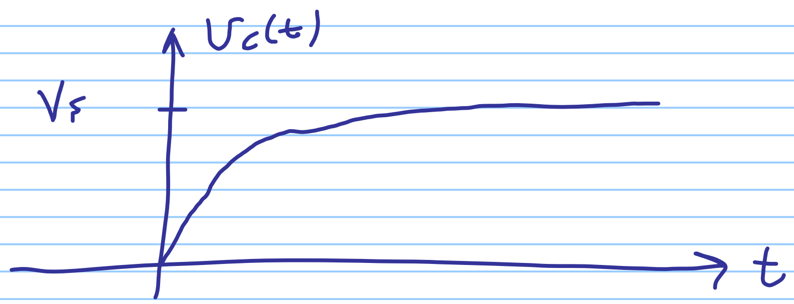
$v_c(0^+) = V_c(0^-) = 0$

$v_c(0^+) = \underline{A[1] + V_s = 0}$

$A = -V_s$

$v_c(t) = V_s - V_s e^{-t/\tau} \qquad t \geq 0$

$= V_s(1 - e^{-t/\tau}) \qquad t \geq 0$



7.4 A General Solution for Step and Natural Responses

General solution for natural and step responses of *RL* and *RC* circuits ▶

$$x(t) = x_f + [x(t_0) - x_f]e^{-(t-t_0)/\tau}. \quad (7.59)$$

The importance of Eq. 7.59 becomes apparent if we write it out in words:

$$\begin{matrix}\text{the unknown} \\ \text{variable as a} \\ \text{function of time}\end{matrix} = \begin{matrix}\text{the final} \\ \text{value of the} \\ \text{variable}\end{matrix} + \left[\begin{matrix}\text{the initial} \\ \text{value of the} \\ \text{variable}\end{matrix} - \begin{matrix}\text{the final} \\ \text{value of the} \\ \text{variable}\end{matrix}\right] \times e^{\frac{-[t-(\text{time of switching})]}{(\text{time constant})}} \quad (7.60)$$

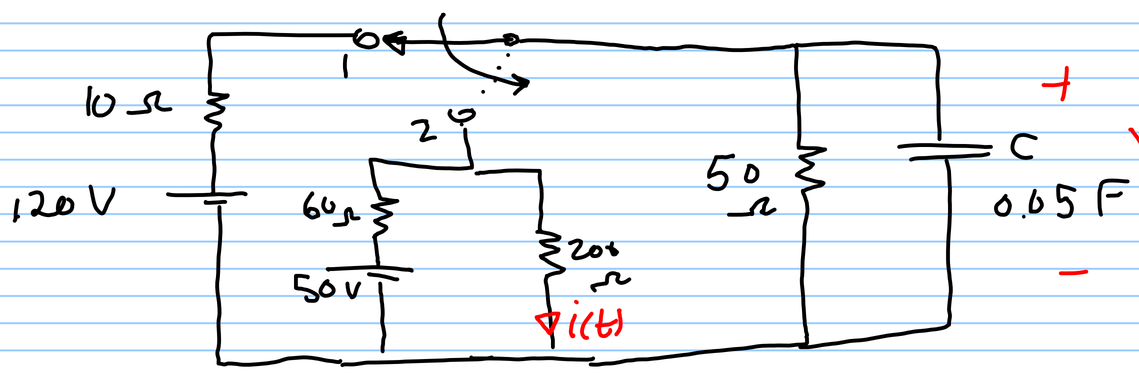
In many cases, the time of switching—that is, t_0—is zero.

When computing the step and natural responses of circuits, it may help to follow these steps:

Calculating the natural or step response of *RL* or *RC* circuits ▶

1. Identify the variable of interest for the circuit. For *RC* circuits, it is most convenient to choose the capacitive voltage; for *RL* circuits, it is best to choose the inductive current.
2. Determine the initial value of the variable, which is its value at t_0. Note that if you choose capacitive voltage or inductive current as your variable of interest, it is not necessary to distinguish between $t = t_0^-$ and $t = t_0^+$.[2] This is because they both are continuous variables. If you choose another variable, you need to remember that its initial value is defined at $t = t_0^+$.
3. Calculate the final value of the variable, which is its value as $t \to \infty$.
4. Calculate the time constant for the circuit.

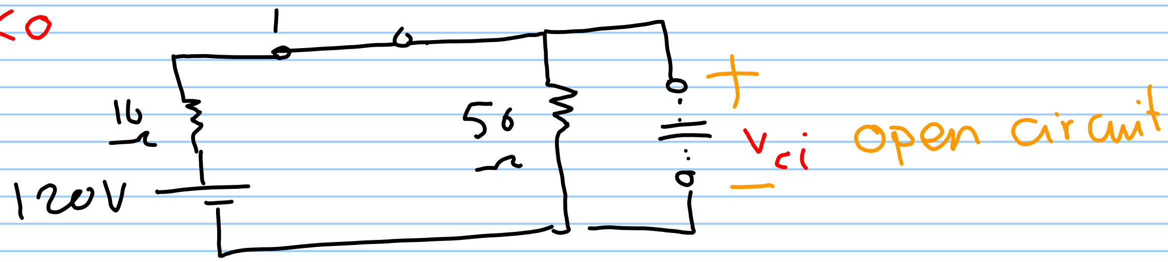
EX) find $i(t)$ for $t>0$



1st we will find $V_c(t)$, then $i(t)$

$$V_c(t) = V_{cf} + [V_{ci} - V_{cf}]\,e^{-t/\tau}$$

$\boxed{V_{ci}}$ $t<0$



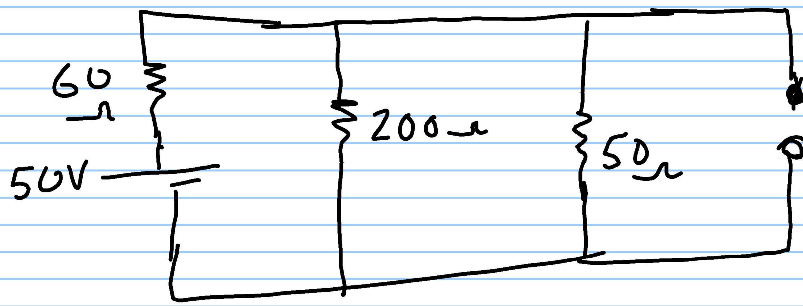
$$V_{ci} = \frac{50}{50+10} * 120$$

$$= 100\ V$$

$\boxed{V_{cf}}$ $t \to \infty$

$t \to \infty$

$C \to$ o.c.



$$V_{cf} = \frac{[50 // 200]}{[50//200] + 60} \times 50 = 20\ V$$

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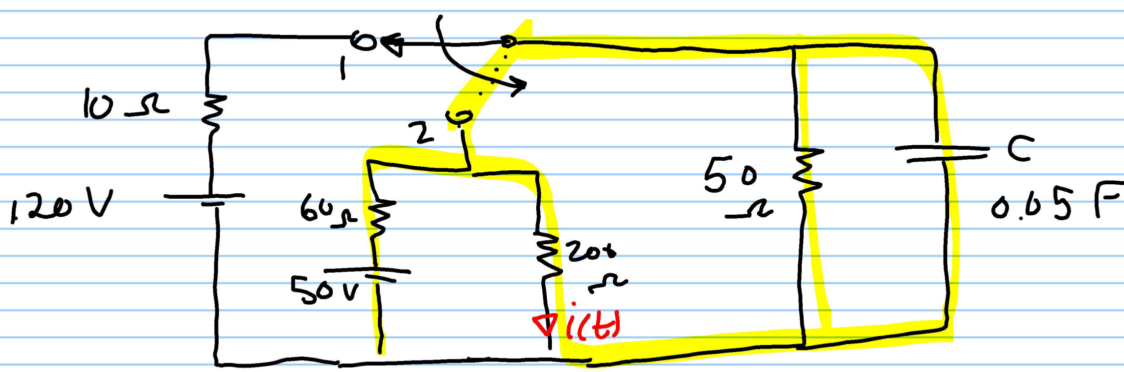
to find τ



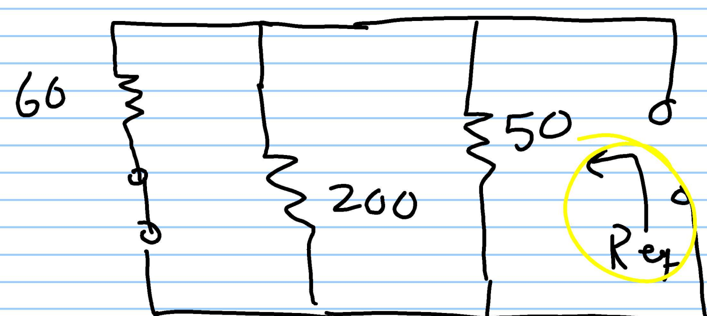
$\tau = R_{eq}\, C$

$R_{eq} = 60 // 200 // 50$

$= 24\ \Omega$ ✓

$\therefore \tau = 24 \times 0.05$

$= 1.2$ Sec.



$$V_c(t) = 20 + [100 - 20]\, e^{-t/1.2}$$

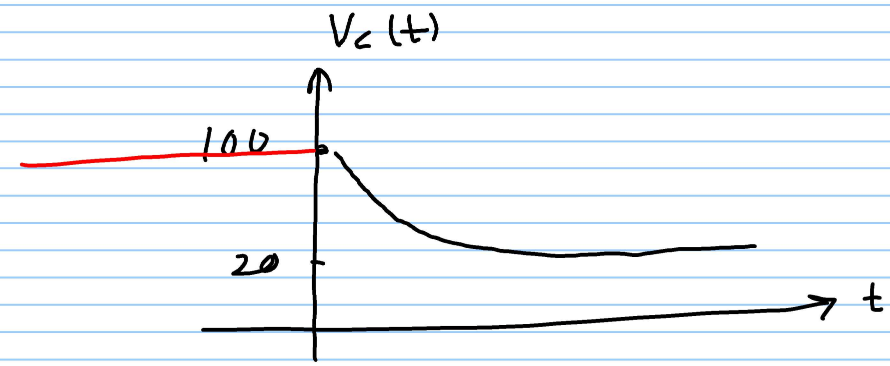
$$= 20 + 80\, e^{-t/1.2}\ V, \ t \geq 0$$

Now, $V_c(t) = V_{200\Omega}$

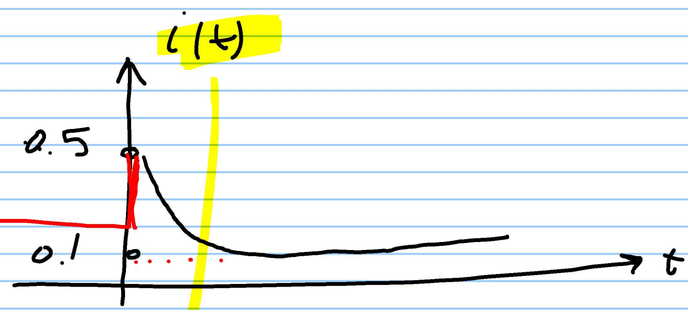
$$\therefore i(t) = \frac{V_{200}}{200} = \frac{V_c(t)}{200}$$

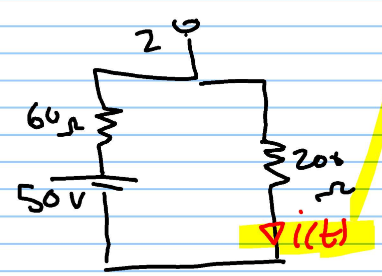
$$= 0.1 + 0.4\,e^{-t/1.2}\ \text{A}, \quad t > 0$$

$v_c(0^-) = v_c(0^+)$



sudden change of $i(t)$ for $R = 200\Omega$



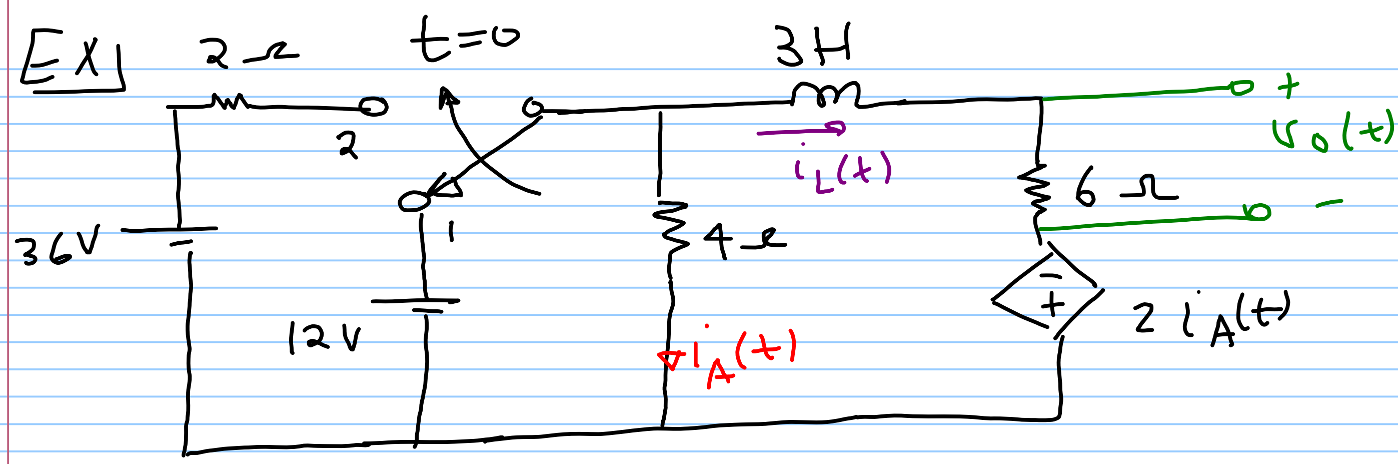


$$i(0^-) = \frac{50}{260}$$

$$= 0.1923\ \text{A}$$

HW

EX 1



find $v_o(t)$ for $t > 0$

$$i_L(t) = i_f + [i_i - i_f]\, e^{-t/\tau}$$