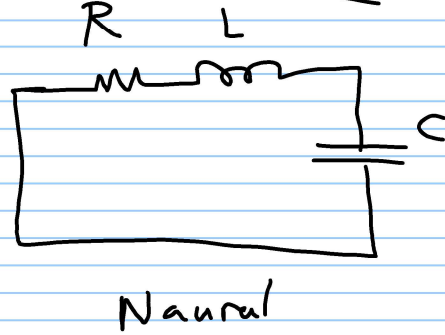
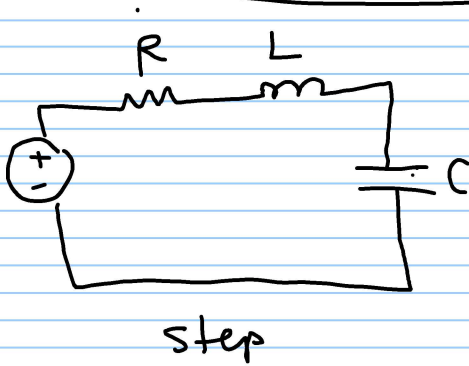


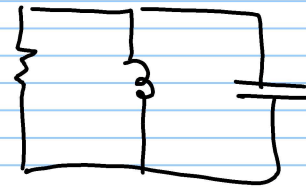
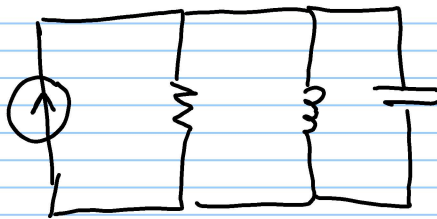
chapter 8

Natural & step Response of RLC circuits 2nd - order

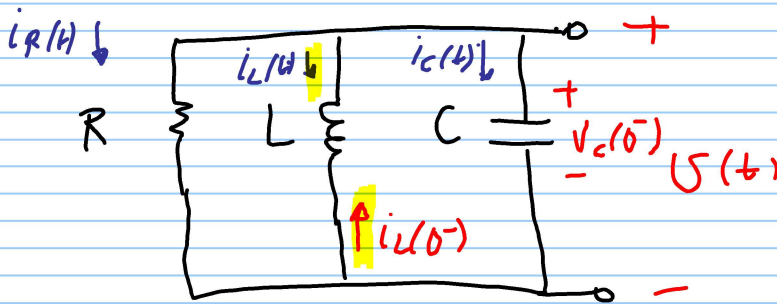
Series



Parallel



* Natural Response of parallel RLC circuits



$$v_L = L \frac{di_L}{dt}$$

$$\frac{1}{L} v_L = \frac{d}{dt} i_L$$

$$\frac{1}{L} \int v + i_C = i_L$$

let $\underline{v_C(0^-) = 0}$; $\underline{i_L(0^-) = 10 \text{ A}}$

KCL

$$\frac{d}{dt} \left(\frac{v(t)}{R} + \frac{1}{L} \int_0^t v(t) dt - i_L(0^-) + C \frac{d}{dt} v(t) = 0 \right)$$

$$\frac{1}{R} \frac{d}{dt} v(t) + \frac{1}{L} v(t) + C \frac{d^2}{dt^2} v(t) = 0$$

$$\frac{d^2}{dt^2} v(t) + \frac{1}{RC} \frac{d}{dt} v(t) + \frac{1}{LC} v(t) = 0$$

2nd-order homogeneous differential equation

$$\therefore \underline{v(t) = A e^{s_1 t} + A_2 e^{s_2 t}} \quad \text{for } t > 0$$

$$A s^2 e^{s t} + \frac{1}{RC} A s e^{s t} + \frac{1}{LC} A e^{s t} = 0$$

$$A e^{s t} \left[s^2 + \frac{1}{RC} s + \frac{1}{LC} \right] = 0$$

$$s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0 \quad \text{characteristic equation}$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s^2 + \underbrace{2\zeta \omega_n}_{2\zeta \omega_n} s + \underbrace{\omega_n^2}_{\omega_n^2}$$

$$\rightarrow s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

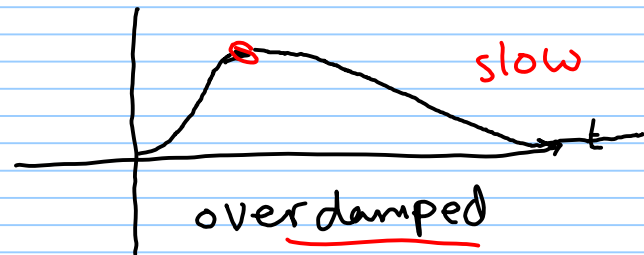
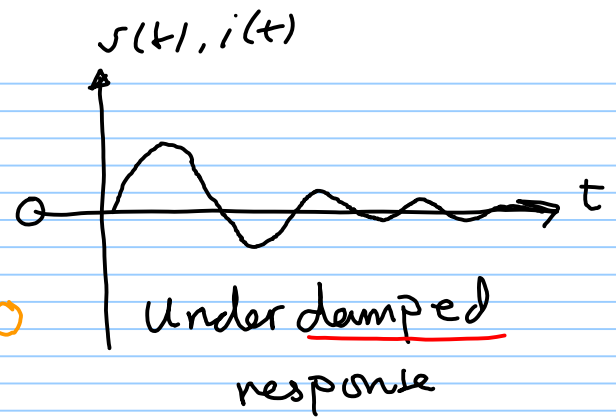
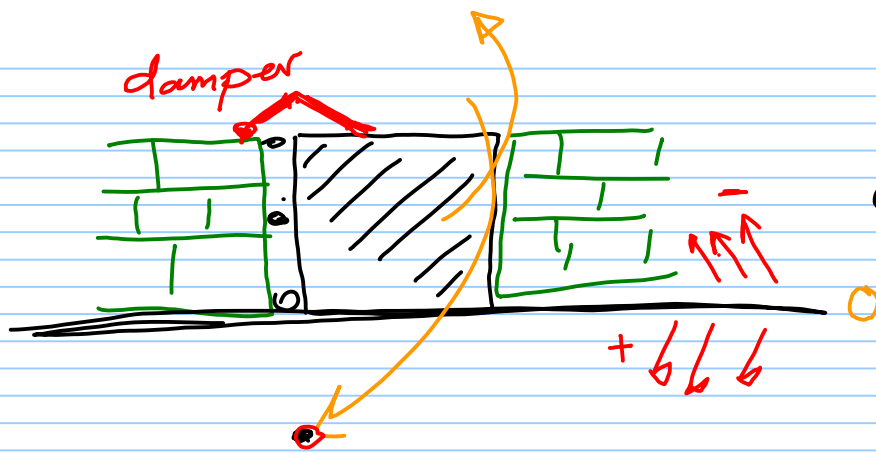
$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

let $\omega_0 = \frac{1}{\sqrt{LC}}$ resonant frequency

or $\alpha = \frac{1}{2RC}$ damping coefficient

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$



$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

- 1) if $\alpha > \omega_0$, the solutions are real, unequal & the response is **overdamped**
- 2) if $\alpha < \omega_0$, the solutions are complex conjugates & the response is termed **underdamped**
- 3) if $\alpha = \omega_0$, the solutions are real and equal, & the response is termed **critically damped**.

1] overdamped case

$$\alpha > \omega_0$$

$$\alpha = \frac{1}{2RC}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad -ve$$

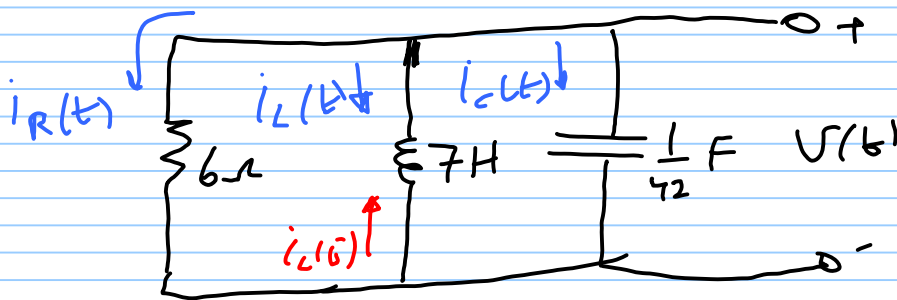
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad -ve$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

real & unequal (real & distinct)

$$s.o. \quad V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad t > 0$$

EX] overdamped parallel RLC



if $i_L(0^-) = 10A$ & $V_C(0^-) = 0$. find $V(t)$ for $t > 0$

$$\textcircled{1} \quad \alpha = \frac{1}{2RC} = 3.5, \quad \omega_0 = \frac{1}{\sqrt{LC}} = 2.45$$

$\alpha > \omega_0 \rightarrow$ overdamped

$$\textcircled{2} \quad s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -1$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -6$$

$$s.o. \quad V(t) = A_1 e^{-t} + A_2 e^{-6t}, \quad t > 0$$

$\textcircled{3}$ to find $A_1, A_2 \rightarrow$ from the initial conditions

$V(0^+) & \frac{d}{dt} V(0^+)$

$$\star \quad v(t) = A_1 e^{-t} + A_2 e^{-6t}$$

math

$$\rightarrow v(0^+) = A_1 + A_2$$

$$\rightarrow \frac{d}{dt} v(0^+) = -A_1 - 6A_2$$

$\stackrel{?}{=} \epsilon_1 A_1 + \epsilon_2 A_2$

circuit

$v(0^+) \rightarrow$ The initial voltage on the capacitor

$\frac{d}{dt} v(0^+) \rightarrow ??$

KCL $i_c(t) + i_R(t) + i_L(t) = 0$

$$i_c(0^+) \quad c \frac{d}{dt} v(t) + \frac{v(t)}{R} + \frac{1}{L} \int_0^t v(t) dt \pm I_0 = 0$$

$t \rightarrow 0^+$

$$i_c(0^+) \quad c \frac{d}{dt} v(0^+) + \frac{v(0^+)}{R} \pm I_0 = 0 \quad v(0^+) = V_0$$

$$\therefore \frac{d}{dt} v(0^+) = \left(\pm I_0 - \frac{V_0}{R} \right) \frac{1}{c}$$

$$\boxed{\frac{d}{dt} v(0^+) = \frac{i_c(0^+)}{c} = \frac{\pm I_0 - \frac{V_0}{R}}{c}}$$

$$\rightarrow \frac{d}{dt} v(0^+) = \frac{-(-10)}{1/42} = 420$$

$$\begin{cases}
 v(0^+) = A_1 + A_2 \\
 \frac{d}{dt} v(0^+) = -A_1 - 6A_2
 \end{cases}
 \begin{cases}
 A_1 + A_2 = 0 & \text{--- (1)} \\
 -A_1 - 6A_2 = 420 & \text{--- (2)}
 \end{cases}$$

Solving (1) & (2) $A_1 = 84, A_2 = -84$

$$\begin{aligned}
 v(t) &= 84 e^{-t} - 84 e^{-6t} \quad V, \text{ for } t > 0 \\
 &= 84 \left(e^{-t} - e^{-6t} \right) V
 \end{aligned}$$

2] critical damping case

$$\alpha = \omega_0$$

$$s_{1,2} = -\alpha = -\frac{1}{2RC}$$

$s_1 = s_2$ real & equal

$$v(t) = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$$

$$A_1 \& A_2 \Rightarrow v(0^+) \& \frac{d}{dt} v(0^+)$$

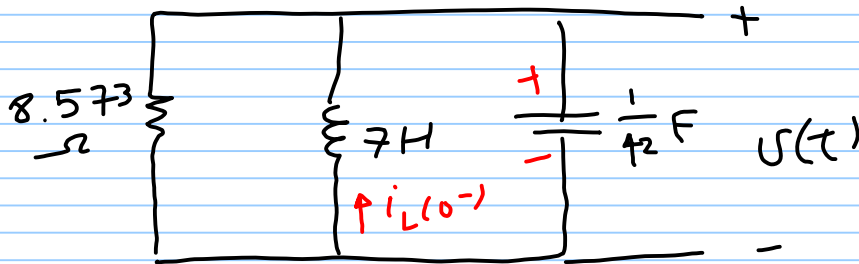
$$v(0^+) = A_2 = V_0$$

$$\frac{d}{dt} v(t) = (A_1 t)(-\alpha e^{-\alpha t}) + A_1 e^{-\alpha t} - \alpha A_2 e^{-\alpha t}$$

$$\begin{aligned}
 \frac{d}{dt} v(0^+) &= A_1 - \alpha A_2 \\
 &= \frac{i_c(0^+)}{C}
 \end{aligned}$$

$$\begin{aligned}
 i_c(0^+) &= -i_L(0^+) - \frac{v(0^+)}{R} \\
 &= -I_0 - \frac{V_0}{R}
 \end{aligned}$$

EX1 critical damping Parallel RLC



$$v_c(0^-) = 0, \quad i_L(0^-) = 10A$$

$$\alpha = \frac{1}{2RC} = \sqrt{6} \quad \omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$$s_1 = s_2 = -\sqrt{6}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$v(t) = A_1 t e^{-\sqrt{6}t} + A_2 e^{-\sqrt{6}t} \quad V, \text{ for } t > 0$$

$$v(0^+) = 0 \quad \& \quad \frac{d}{dt} v(0^+) = 420$$

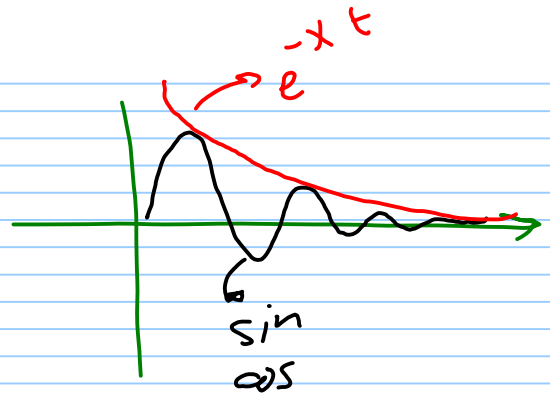
$$\boxed{v(0^+) = A_2 = 0}$$

$$\frac{d}{dt} v(0^+) = A_1 - \alpha A_2 = \boxed{A_1 = 420}$$

$$v(t) = 420 t e^{-\sqrt{6}t} \quad V, \text{ for } t > 0$$

3 underdamped case

$$\alpha < \omega_0$$
$$\alpha^2 - \omega_0^2 < 0 \quad (-ve)$$



$$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{(-1)(\omega_0^2 - \alpha^2)}$$
$$= j \sqrt{\omega_0^2 - \alpha^2}$$
$$= j \omega_d$$

$\omega_d \equiv$ damped radian frequency

$$s_{1,2} = -\alpha \pm j \sqrt{\omega_0^2 - \alpha^2}$$
$$= -\alpha \pm j \omega_d \quad \text{Complex conjugate}$$

$$v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$

$$e^{j\omega_d t} = \cos \omega_d t + j \sin \omega_d t$$

$$e^{-j\omega_d t} = \cos \omega_d t - j \sin \omega_d t$$

$$v(t) = e^{-\alpha t} \left[\underbrace{(A_1 + A_2)}_{B_1} \cos \omega_d t + j \underbrace{(A_1 - A_2)}_{B_2} \sin \omega_d t \right]$$

$$v(t) = e^{-\alpha t} \left[B_1 \cos \omega_d t + B_2 \sin \omega_d t \right]$$

NOTE

B_1 & B_2 are real #
Not complex

$$B_2 = j(A_1 - A_2) \text{ (real)}$$

$$= j([x+jy] - [x-jy])$$

$$= j(2jy)$$

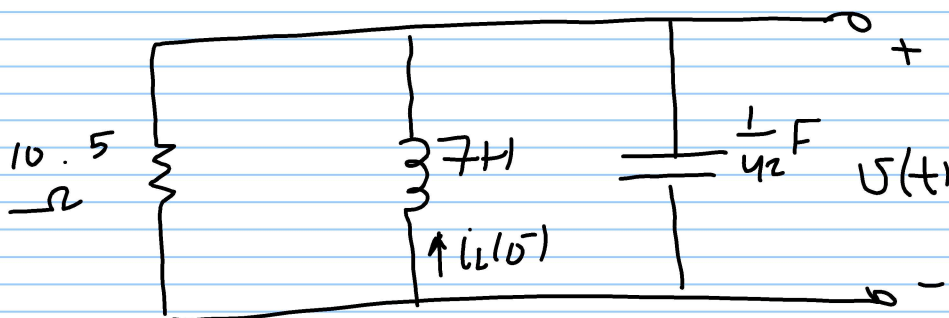
$$= -2y \text{ real } \neq$$

$$v(t) = e^{-\alpha t} [B_1 \cos \omega_d t + B_2 \sin \omega_d t]$$

$$B_1 \& B_2 \rightarrow v(0^+) \& \frac{d}{dt} v(0^+)$$

$$\begin{cases} v(0^+) = B_1 = V_0 \\ \frac{d}{dt} v(0^+) = \frac{i_C(0^+)}{C} = \frac{-I_0 - \frac{V_0}{R}}{C} = -\alpha B_1 + \omega_d B_2 \end{cases}$$

EX underdamped Parallel RLC



$$v_C(0^-) = 0, \quad i_L(0^-) = 10A$$

$$\alpha = \frac{1}{2RC} = 2, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6} = 2.45$$

$$\alpha < \omega_0$$

$$\rightarrow \omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{2}$$

$$\rightarrow v(t) = e^{-\alpha t} (\beta_1 \cos \omega_d t + \beta_2 \sin \omega_d t)$$

$$= e^{-2t} (\beta_1 \cos \sqrt{2} t + \beta_2 \sin \sqrt{2} t)$$

$$\rightarrow v(0^+) = 0$$

$$\frac{d}{dt} v(0^+) = 420$$

$$\rightarrow v(0^+) = \beta_1 = 0$$

$$\therefore v(t) = e^{-2t} \beta_2 \sin \sqrt{2} t \quad \forall t > 0$$

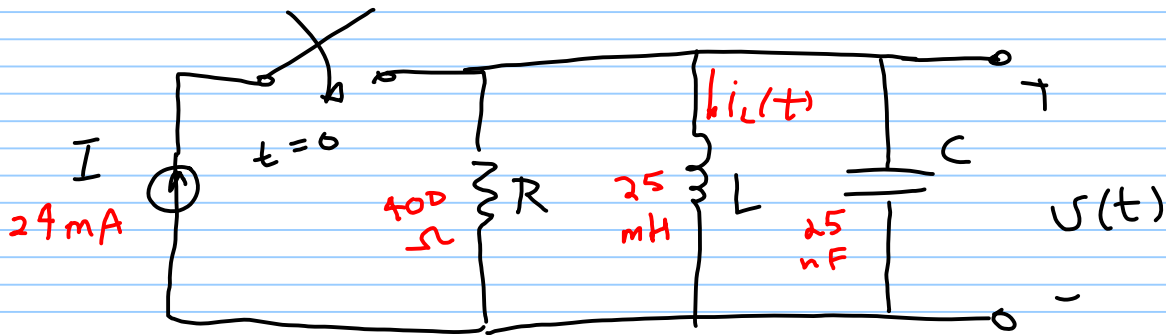
$$\frac{d}{dt} v(t) = (\beta_2 e^{-2t}) (\sqrt{2} \cos \sqrt{2} t) + (\sin \sqrt{2} t) (-2e^{-2t} \beta_2)$$

$$\frac{d}{dt} v(0^+) = 420 = \sqrt{2} \beta_2 \Rightarrow \beta_2 = \frac{420}{\sqrt{2}}$$

$$\therefore v(t) = \frac{420}{\sqrt{2}} e^{-2t} \sin \sqrt{2} t \quad \forall t > 0$$

$$s_{1,2} = -2 \pm j\sqrt{2}$$

Step Response of parallel RLC circuits



if $i_L(0^-) = 0$ & $v_C(0^-) = 0$
 find $v(t)$ & $i_L(t)$ for $t > 0$

→ $i_L(t)$

KCL

$$I = i_R(t) + i_L(t) + i_C(t)$$

$$I = \frac{v(t)}{R} + i_L(t) + C \frac{d}{dt} v(t)$$

But $v_L(t) = v(t) = L \frac{d}{dt} i_L(t)$

$$I = \frac{L}{R} \frac{d}{dt} i_L(t) + i_L(t) + LC \frac{d^2}{dt^2} i_L(t)$$

$$\frac{d^2}{dt^2} i_L(t) + \frac{1}{RC} \frac{d}{dt} i_L(t) + \frac{1}{LC} i_L(t) = \frac{I}{LC}$$

$$i_L(t) = i_n(t) + i_f(t)$$

$i_n(t) \rightarrow$ natural response

$i_f(t) \rightarrow$ forced response

→ to find $i_f(t)$

let $i_L(t) = k$

$$0 + 0 + \frac{1}{LC} k = \frac{I}{LC}$$

$$i_f(t) = I = 24 \text{ mA}$$

→ to find $i_L(t)$

$$\frac{d^2}{dt^2} i_L(t) + \frac{1}{RC} \frac{d}{dt} i_L(t) + \frac{1}{LC} i_L(t) = 0$$

$$s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \left(\frac{1}{LC}\right)^2}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \left(\frac{1}{LC}\right)^2}$$

$$-\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$s_1 = -20000$, $s_2 = -80000$

Real & distinct → overdamped response

$$i_L(t) = A_1 e^{-20000t} + A_2 e^{-80000t}$$

$$i_L(t) = i_{L_f}(t) + i_{L_n}(t)$$

$$i_L(t) = 24\text{mA} + A_1 e^{-20000t} + A_2 e^{-80000t}$$

⇒ to find A_1 & A_2 , we need $i_L(0^+)$ & $\frac{d}{dt} i_L(0^+)$

$$i_L(0^+) = i_L(0^-) = 0$$

$$v_c(0^+) = v_L(0^+) = L \frac{d}{dt} i_L(0^+)$$

$$v_c(0^+) = v_c(0^-) = 0$$

$$\therefore \frac{d}{dt} i_L(0^+) = 0$$

$$i_L(0^+) = 24\text{mA} + A_1 + A_2$$

$$0 = 24\text{mA} + A_1 + A_2$$

$$A_1 + A_2 = -24\text{mA} \quad \text{--- ①}$$

$$\frac{d}{dt} i_L(t) = 0 - 20000 A_1 e^{-20000t} - 80000 A_2 e^{-80000t}$$

$$\frac{d}{dt} i_L(0^+) = -20000 A_1 - 80000 A_2 = 0 \quad \text{--- (2)}$$

from (1) & (2) $A_1 = -32 \text{ mA}, A_2 = 8 \text{ mA}$

$$i_L(t) = 24 - 32 e^{-20000t} + 8 e^{-80000t} \text{ mA, for } t > 0$$

$$V(t) = L \frac{d}{dt} i_L(t)$$

$$= 16 e^{-20000t} - 16 e^{-80000t} \text{ V, for } t > 0$$

Example 8.10 Finding Step Response of a Parallel RLC Circuit with Initial Stored Energy

Energy is stored in the circuit in Example 8.8 (Fig. 8.12, with $R = 500 \Omega$) at the instant the dc current source is applied. The initial current in the inductor is 29 mA, and the initial voltage across the capacitor is 50 V. Find (a) $i_L(0)$; (b) $di_L(0)/dt$; (c) $i_L(t)$ for $t \geq 0$; (d) $v(t)$ for $t \geq 0$.

a dc current source of 24 mA

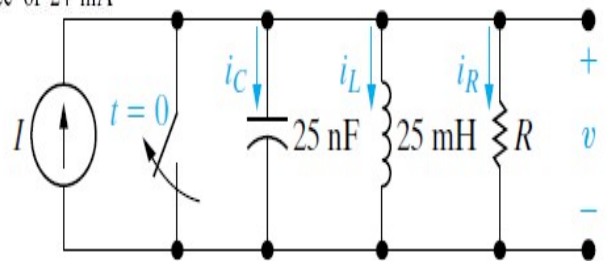


Figure 8.12 \blacktriangle The circuit for Example 8.6.

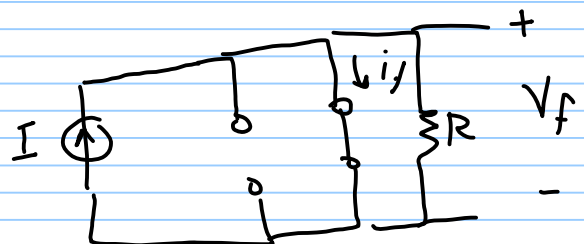
(a) $i_L(0) = i_L(0^+) = i_L(0^-) = 29 \text{ mA}$

(b) $v(t) = L \frac{d}{dt} i_L(t)$

$$\frac{d}{dt} i_L(t) = \frac{v(t)}{L} \Rightarrow \frac{d}{dt} i_L(0) = \frac{v(0)}{L} = \frac{50}{25\text{m}} = 2000 \text{ A/s}$$

(c) $i_L(t) = i_n(t) + i_f(t)$

$$i_f(t) = I = 24 \text{ mA}$$



α, ω

$$\alpha = \frac{1}{2RC}$$

$$= 40000$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$= 40000$$

$\alpha = \omega_0 \Rightarrow$ critically damped.

$$\therefore i_L(t) = i_f(t) + i_n(t)$$

$$= 24\text{mA} + A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$$

$$i_L(0) = 24\text{mA} + A_2 = 29\text{mA}$$

$$A_2 = 5\text{mA}$$

$$\rightarrow \frac{d}{dt} i_L(0^+) = \frac{V_C(0^-) \cdot 50}{L \cdot 25\text{m}} = 2000$$

$$\underline{V_L} = L \frac{d}{dt} i = V_C$$
$$\frac{d}{dt} i(0^+) = \frac{V_C(0^+)}{L}$$

$$\rightarrow \frac{d}{dt} i_L(t) = -A_1 t \alpha e^{-\alpha t} + A_1 e^{-\alpha t} - \alpha A_2 e^{-\alpha t}$$

$$\frac{d}{dt} i_L(0) = A_1 - \alpha A_2 = 2000$$

$$A_1 - (40000)(5\text{m}) = 2000$$

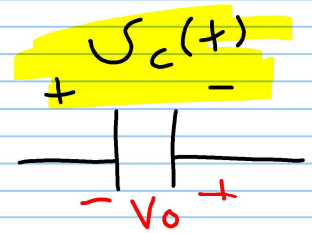
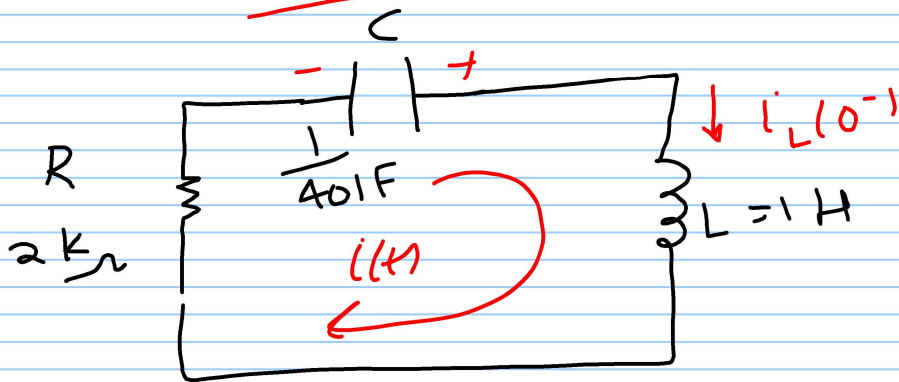
$$\boxed{A_1 = 2200}$$

$$-40000t \quad -40000t$$

$$\therefore i_L(t) = 24\text{mA} + 2200t e^{-40000t} + 5\text{mA} e^{-40000t} \quad A$$

for $t > 0$

Natural Response of Series RLC circuits



if $v_c(0^-) = v_0$ & $i_L(0^-) = I_0$
 find $i(t)$ for $t > 0$

$$i_c = C \frac{d}{dt} v_c(t)$$

$$v_c = \frac{1}{C} \int i_c dt + v_c(0^-)$$

KVL $R i(t) + v_c(t) + L \frac{d}{dt} i(t) = 0$

$$R i(t) + \frac{1}{C} \int_0^t i(t) dt - v_c(0^-) + L \frac{d}{dt} i(t) = 0$$

$$L \frac{d^2}{dt^2} i(t) + R \frac{d}{dt} i(t) + \frac{1}{C} i(t) = 0$$

$$\frac{d^2}{dt^2} i(t) + \frac{R}{L} \frac{d}{dt} i(t) + \frac{1}{LC} i(t) = 0$$

$$s^2 + \left(\frac{R}{L} \right) s + \frac{1}{LC} = 0$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L} \right)^2 - \left(\frac{1}{LC} \right)^2}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L} \right)^2 - \left(\frac{1}{LC} \right)^2}$$

$$\alpha = \frac{R}{2L} \quad \text{New}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \checkmark$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

