Chapter 8

Natural and Step Response of RLC Circuits

ENEE2301

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Reading Assignment: Chapter 8 in Electric Circuits, 10^{th} Ed. by Nilsson

- 2^{nd}-order circuits have 2 independent energy storage elements (inductors and/or capacitors)
- Analysis of a 2^{nd}-order circuit yields a 2^{nd}-order differential equation (DE)
- A 2nd-order differential equation has the form:

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_o x(t) = f(t)$$

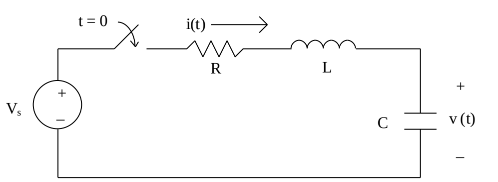
- Solution of a 2^{nd}-order differential equation requires two initial conditions: x(0) and x'(0)
- All higher order circuits (3^{rd}, 4^{th}, etc) have the same types of responses as seen in 1^{st}-order and 2^{nd}-order circuits

Series RLC and Parallel RLC Circuits

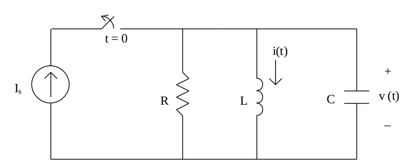
Since 2nd-order circuits have two energy-storage types, it is characterized by a second order differential equation, the circuits can have the following forms:

1) Two capacitors
2) Two inductors
3) One capacitor and one inductor
   A) Series RLC circuit *
   B) Parallel RLC circuit *
   C) Others

* The textbook focuses on these two types of 2nd-order circuits



Series RLC circuit



Parallel RLC circuit

Form of the solution to differential equations

As seen with 1st-order circuits in Chapter 7, the general solution to a differential equation has two parts:

$x(t) = x_h + x_p$ = homogeneous solution + particular solution

or $x(t) = x_n + x_f$ = natural solution + forced solution

where x_h or x_n is due to the initial conditions in the circuit

and x_p or x_f is due to the forcing functions (independent voltage and current sources for t > 0).

The forced response

The forced response is due to the independent sources in the circuit for t > 0. Since the natural response will die out once the circuit reaches steady-state, the forced response can be found by analyzing the circuit at $t = \infty$. In particular,

$$\mathbf{x_f = x(\infty)}$$

The natural response

A 2nd-order differential equation has the form:

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_o x(t) = f(t)$$

where x(t) is a voltage v(t) or a current i(t).

To find the natural response, set the forcing function f(t) (the right-hand side of the DE) to zero.
$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_o x(t) = 0$$

Substituting the general form of the solution Ae^{st} yields the ***characteristic equation***:

$$s^2 + a_1 s + a_o = 0$$

Finding the roots of this quadratic (called the ***characteristic roots*** or ***natural frequencies***) yields:

$$s_1\ ,\ s_2 = \frac{-a_1 \pm \sqrt{(a_1)^2 - 4a_o}}{2}$$

Natural Response of parallel RLC circuits:

For $t > 0$: Given

$\boldsymbol{V_c(0^-) = 0}$ **Volt**

$\boldsymbol{i_L(0^-) = 10\ A}$

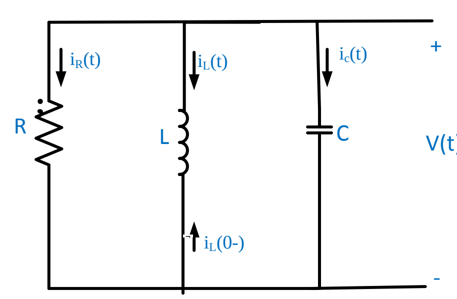


Figure3:Natural Response of parallel RLC circuits:

KCL:

$$i_R(t) + i_L(t) + i_C(t) = 0$$

$$\frac{V(t)}{R} + \frac{1}{L}\int_0^t v(t)dt - i_L(0^-) + C\frac{dv(t)}{dt} = 0$$

$$\frac{V(t)}{R} + \frac{1}{L}\int_{0^-}^t v(t)dt + C\frac{dv(t)}{dt} = i_L(0^-) \text{ ----------- } (1)$$

Differentiate (1):

$$C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$$

Second order homogeneous differential equation

$$C\frac{d^2v(t)}{dt^2}+\frac{1}{R}\frac{dv(t)}{dt}+\frac{1}{L}v(t)=0$$

❖ Solution of the following form V(t) = $A\,e^{st}$ for t>0

$$CA\,s^2e^{st}+\frac{1}{R}\,sAe^{st}+\frac{1}{L}\,Ae^{st}=0$$

$$Ae^{st}\left(Cs^2+\frac{1}{R}s+\frac{1}{L}\right)=0$$

❖ $Cs^2+\frac{1}{R}s+\frac{1}{L}=0$

Characteristic equation roots:

$$S_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$S_1=-\frac{1}{2RC}+\sqrt{(\frac{1}{2RC})^2-\frac{1}{LC}}$$

$$S_2=-\frac{1}{2RC}-\sqrt{(\frac{1}{2RC})^2-\frac{1}{LC}}$$

Resonant frequency

$$\omega_o=\frac{1}{\sqrt{LC}}$$

Damping coefficient

$$\alpha=\frac{1}{2RC}$$

❖ $S_1=-\alpha+\sqrt{\alpha^2-\omega_o{}^2}$

❖ $S_2=-\alpha-\sqrt{\alpha^2-\omega_o{}^2}$

Characteristic Roots

The roots of the characteristic equation may be real and distinct, repeated, or complex. Thus, the natural response to a 2nd-order circuit has 3 possible forms:

1) **Overdamped response**

For [$(\alpha)^2 > (\omega_o)^2$]

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_o{}^2} < 0$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_o{}^2} < 0$$

Roots are real and distinct (unequal)

Solution has the form:

$$X_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

2) **Critically damped response**

For [$(\alpha)^2 = (\omega_o)^2$]

$$S_1 = S_2 = -\alpha$$

- $S_1 = -\alpha + \sqrt{\alpha^2 - \omega_o{}^2}$
- $S_2 = -\alpha - \sqrt{\alpha^2 - \omega_o{}^2}$

Roots are real and equal Solution has the form:

$$x_n = (A_1 t + A_2) e^{st}$$

$$X_n = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$$

3) Underdamped response

For[$(\alpha)^2 < (\omega_o)^2$]

$\alpha^2 - \omega_o^2 < 0$

$\sqrt{(\alpha^2 - \omega_o^2)} = \sqrt{(-1)(\omega_o^2 - \alpha^2)}$

$= j\omega_d$

ω_d=damped radian frequency

$S_{1,2} = -\alpha \pm j\omega_d$

- $S_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$
- $S_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$
- $wd = \sqrt{\omega_o^2 - \alpha^2}$

Roots are complex conjugate

The solution has the form: $X_n = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$

$e^{j\omega dt} = \cos \omega dt + j \sin \omega dt$

$e^{-j\omega dt} = \cos \omega dt - j \sin \omega dt$

$X_n(t) =$ V(t)= $e^{-\alpha t}\left[(A_1 + A_2)\cos \omega dt + j(A_1 - A_2)\sin \omega dt\right]$

$X_n(t) =$ **V(t)=** $\boldsymbol{e^{-\alpha t}\left[\beta_1 \cos \omega dt + \beta_2 \sin \omega dt\right]}$

1) Overdamped response

For [$(\alpha)^2 > (\omega_o)^2$]

Roots are real and distinct (unequal)

$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$

$X_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

2) Critically damped response

For [$(\alpha)^2 = (\omega_o)^2$]

Roots are real and equal

$S_1 = S_2 = -\alpha$

$X_n = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$

3) Underdamped response

For[$(\alpha)^2 < (\omega_o)^2$]

Roots are complex conjugate $S_{1,2} = -\alpha \pm j\omega_d$

$X_n = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$

$X_n(t) =$ V(t)= $e^{-\alpha t}\left[(A_1 + A_2)\cos \omega dt + j(A_1 - A_2)\sin \omega dt\right]$

$X_n(t) =$ **V(t)=** $\boldsymbol{e^{-\alpha t}\left[\beta_1 \cos \omega dt + \beta_2 \sin \omega dt\right]}$

1) Over damped parallel RLC Example:

Find V(t) for t › 0

Given

$\boldsymbol{V_c(0^-)}$**= 0 Volt ;** $\boldsymbol{i_L(0^-)}$**=10 A**

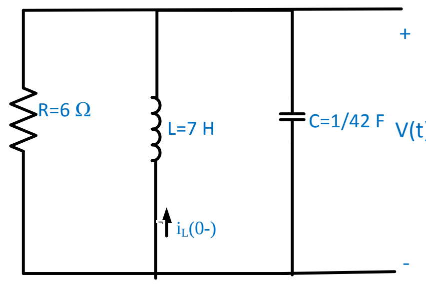


Figure4:Over damped parallel RLC

$$\alpha = \frac{1}{2RC} = 3.5$$

$$\omega_o = \frac{1}{\sqrt{LC}} = 2.45$$

α> ωo over-damped case

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0{}^2} = -1$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0{}^2} = -6$$

❖ V(t)= $A_1 e^{s_1 t} + A_2 e^{s_2 t}$; t > 0

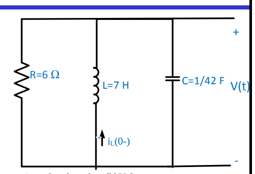


Figure4:Over damped parallel RLC

To find A_1 & A_2, we need to find $V(0^+)$ and $\frac{dv(0^+)}{dt}$

For t > 0

$$\frac{V(t)}{R} + \frac{1}{L}\int_{0^-}^{t} v(t)dt + C\frac{dv(t)}{dt} = i_L(0^-)$$

At (t=0⁺)

$$\frac{V(0^+)}{R} + \frac{1}{L}\int_{0^-}^{0^+} v(t)dt + C\frac{dv(0^+)}{dt} = i_L(0^-)$$

$$V(0^+) = V_c(0^+) = V_c(0^-) = 0$$

$$\int_{0^-}^{0^+} v(t)dt = 0$$

❖ $i_L(0^-) = C\frac{dv(0^+)}{dt}$

$$\frac{dv(0^+)}{dt} = \frac{i_L(0^-)}{C} = 420$$

Also $V(0^+) = 0 = V_c(0^-)$

V(t)= $A_1e^{-t} + A_2e^{-6t}$; t>0

$V(0^+) = A_1 + A_2 = V_c(0^-) = 0$

❖ $A_1 + A_2 = 0$ ....(2)

V(t)= $A_1e^{-t} + A_2e^{-6t}$

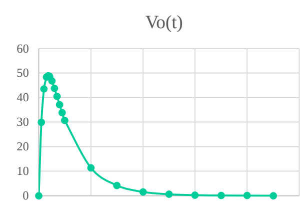
$$\frac{dV(t)}{dt} = -A_1e^{-t} - 6A_2e^{-6t}$$

$$\frac{dv(0^+)}{dt} = -A_1 - 6A_2 = 420$$ ....(3)

Solving (2) and (3) ,we get

A_1=84 and A_2= -84

V(t)= $84\left(e^{-t} - e^{-6t}\right)$ *volt* ,for t>0



2. Critical Damping case: (example)

Given

$\boldsymbol{V_c(0^-)}$**= 0 Volt ;** $\boldsymbol{i_L(0^-)}$**=10 A**

Find V(t) for t › 0

$\alpha = \frac{1}{2RC} = \sqrt{6}$

$\omega_o = \frac{1}{\sqrt{LC}} = \sqrt{6}$

$\alpha = \omega_o$

$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha$

$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -\alpha$

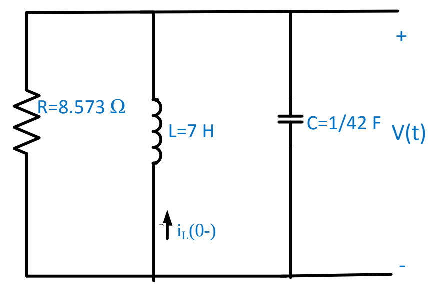


Figure6:critical damped parallel RLC

$S_1 = S_2 = -\sqrt{6}$ ⟹ V(t)= $A_1 t e^{-\sqrt{6}\,t} + A_2 e^{-\sqrt{6}\,t}$; t>0

$V_c(0^+) = 0$

$\frac{dv(0^+)}{dt} = 420$

V(t)= $A_1 t e^{-\sqrt{6}\,t} + A_2 e^{-\sqrt{6}\,t}$; t>0

$V(0^+) = A_2 = 0$

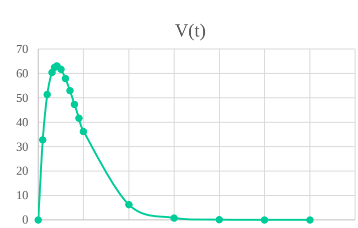
V(t)= $A_1 t e^{-\sqrt{6}\,t}$

$\frac{dV(t)}{dt} = (A_1 t)(-\sqrt{6})e^{-\sqrt{6}t} + A_1 e^{-\sqrt{6}t}$

$\frac{dv(0^+)}{dt} = 0 + A_1 = 420$

$A_1 = 420$

V(t)= $\boldsymbol{420 t e^{-\sqrt{6}\,t}}$ ***volt*** **; t>0**



3) Underdamped parallel RLC :

Given

$V_c(0^-)$= **0 Volt;** $i_L(0^-)$=**10 A**

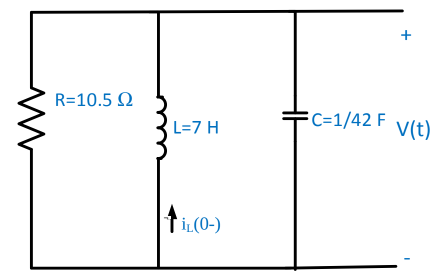


Figure8:underdamped parallel RLC

Find V(t) for t › 0

$\alpha = \frac{1}{2RC} = 2$

$\omega_o = \frac{1}{\sqrt{LC}} = \sqrt{6}$

$\alpha < \omega_o$

$\omega_d = \sqrt{(\omega_o^2 - \alpha^2)} = \sqrt{2}$

V(t)= $e^{-\alpha t}\left[\beta_1 \cos \omega_d t + \beta_2 \sin \omega_d t\right]$ volt For t>0

V(t)= $e^{-2t}\left[\beta_1 \cos \sqrt{2}t + \beta_2 \sin \sqrt{2}t\right]$ volt For t>0

$V_c(0^+)$= 0 $\frac{dv(0^+)}{dt} = 420$

V(t)= $e^{-2t}\left[\beta_1 \cos \sqrt{2}t + \beta_2 \sin \sqrt{2}t\right]$ volt

$V_c(0^+) = \beta_1 \Rightarrow \beta_1 = 0$

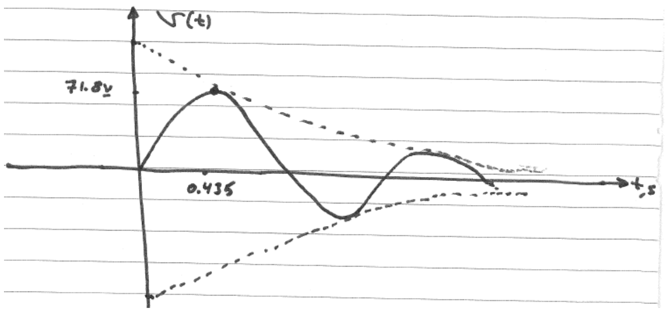
V(t)= $e^{-2t}\left[\beta_2 \sin \sqrt{2}t\right]$ volt For t>0

$$\frac{dV(t)}{dt} = (\beta_2\, e^{-2t})\left(\sqrt{2}\cos\sqrt{2}t\right) + (\sin\sqrt{2}t)(-2\beta_2 e^{-2t})$$

$$\frac{dv(0^+)}{dt} = \sqrt{2}\beta_2 + 0 = 420 \Rightarrow \beta_2 = \frac{420}{\sqrt{2}}$$

V(t)= $e^{-2t}\left[\beta_2 \sin \sqrt{2}t\right]$ volt For t>0

V(t)= $\frac{420}{\sqrt{2}} e^{-2t}\left[\sin \sqrt{2}t\right]$ volt For t>0



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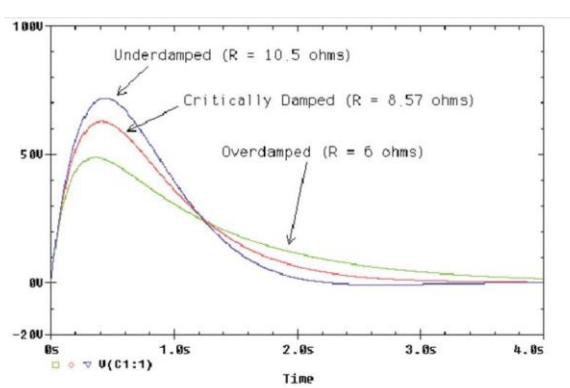
Comparing the responses



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The lossless LC circuit

L=7H , C=$\frac{1}{42}$F , R=∞

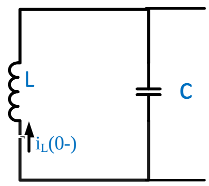


Figure9:LC circuit

$$\alpha = \frac{1}{2RC} = 0$$

$$\omega_o = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$$\alpha < \omega_o$$

$$\omega d = \sqrt{(\omega o^2 - \alpha^2)} = \sqrt{6}$$

V(t)= $[\beta_1 \cos\sqrt{6}t + \beta_2 \sin\sqrt{6}t]$ volt **For t>0**

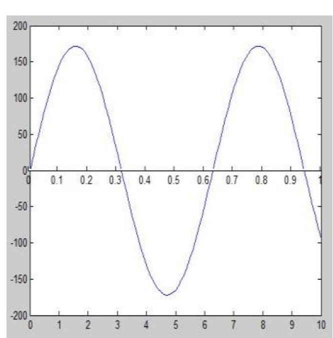
$$V(0^+) = 0 \quad , \quad \frac{dv(0^+)}{dt} = 420$$

$$V(0^+) = \beta_1 = 0$$

$$\frac{dv(0^+)}{dt} = \sqrt{6}\,\beta_2 = 420$$

$$\beta_2 = \frac{420}{\sqrt{6}}$$

V(t)= $\frac{420}{\sqrt{6}}[\sin\sqrt{6}t]$ volt **For t>0**



Step response of parallel RLC:

Given

$V_c(0^-)$= **0 Volt** ;

$i_L(0^-)$= **0 A**

Find v(t) for t>0

Find $i_L(t)$ for t>0

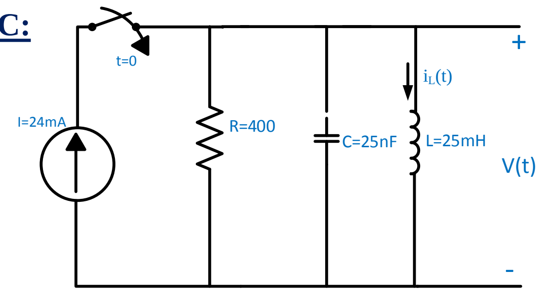


Figure16:step response of parallel RLC circuit

For t>0

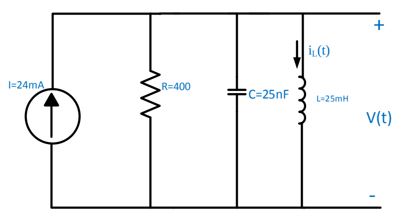


Figure17:step response of parallel RLC circuit switch is closed

KCL:

$$I = i_R(t) + i_L(t) + i_C(t)$$

$$I = \frac{V(t)}{R} + i_L(t) + C\frac{dv(t)}{dt}$$

$$v_C(t) = v_L(t) = L\frac{d_{iL(t)}}{dt}$$

$$I = LC\frac{d^2 i_L(t)}{dt^2} + \frac{L}{R}\frac{di_L(t)}{dt} + i_L(t)$$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{Lc}i_L(t) = \frac{I}{LC}$$

Second order nonhomogeneous differential equation.

$$i_L(t) = i_n(t) + i_f(t)$$

$i_n(t)$: Natural response. <== was found earlier

$i_f(t)$: forced response.

To find $i_f(t)$:
$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{Lc}i_L(t) = \frac{I}{LC}$$

Let $i_f(t) = k$; k is constant.

$$0 + 0 + \frac{1}{LC} i_f(t) = \frac{I}{LC}$$

$$i_f(t) = I = k$$

To find $i_n(t)$:

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{Lc} i_L(t) = 0$$

$$S^2 + \frac{1}{RC}S + \frac{1}{LC} = 0$$

$$S_{1,2} = -\frac{1}{2RC} \pm \sqrt{(\frac{1}{2RC})^2 - \frac{1}{LC}}$$

S_1= -20000

S_2=-80000

Since S_1 &S_2 are real and unequal → We have over damped case.

$$i_{Ln}(t) = A_1 e^{-20000t} + A_2 e^{-80000t}$$

$$i_L(t) = i_{Ln}(t) + i_{Lf}(t)$$

$$\boldsymbol{i_L(t) = 24mA + A_1 e^{-20000t} + A_2 e^{-80000t}} \qquad \boldsymbol{t>0}$$

To find A1 & A2 , we need:

$i_L(0^+) = 0$ And $\frac{di_L(0^+)}{dt}$

$i_L(0^+) = i_L(0^-)$ =0

$$v_C(t) = v_L(t) = L\frac{d_{iL(t)}}{dt}$$

$$v_C(0^+) = L\frac{d_{iL(0^+)}}{dt} = v_C(0^-) = 0 \quad \Rightarrow \quad \frac{d_{iL(0^+)}}{dt} = 0$$

$$\boldsymbol{i_L(t) = 24mA + A_1 e^{-20000t} + A_2 e^{-80000t}} \qquad \textbf{\textit{t>0}}$$

$$i_L(0^+) = 24mA + A_1 + A_2$$

$$A_1 + A_2 = -24mA \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{d_{iL(0^+)}}{dt} = -20000A_1 - 80000A_2 = 0 \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

Solving and we get :

A_1 = -32mA **A_2 =8mA**

$$\boldsymbol{i_L(t) = (24 - 32e^{-20000t} + 8e^{-80000t})mA} \qquad \textbf{\textit{t>0}}$$

Natural response of series RLC circuits:

For t>0

$V_c(0^-) = V_o$, And $i_L(0^-) = I_o$

Find $i(t)$ for t>0

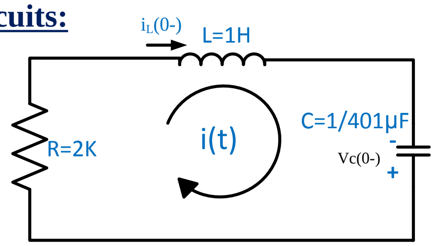


Figure19 : RLC series circuit.

$$L\frac{d_i(t)}{dt} + R_i(t) - V_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)dt = 0$$

$$L\frac{d_i(t)}{dt} + R_i(t) + \frac{1}{C}\int_{0^-}^{t} i(t)dt = V_c(0^-) \qquad \text{-------- (1)}$$

Differentiation of (1)

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$$

Second order homogenous differential equation.

$$LS^2 + RS + \frac{1}{C} = 0$$

$$S_1 = -\frac{R}{2L} + \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}$$

$$S_2 = -\frac{R}{2L} - \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}$$

Let $\omega_0 = \frac{1}{\sqrt{LC}}$ ω_0 = response frequency.

And

$\alpha = \frac{R}{2L}$ α =damping coefficient.

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

Let $V_c(0^-) = V_o = 2\ volt$

$i_L(0^-) = I_o = 2mA$

$$\omega_o = \frac{1}{\sqrt{LC}} = 20025$$

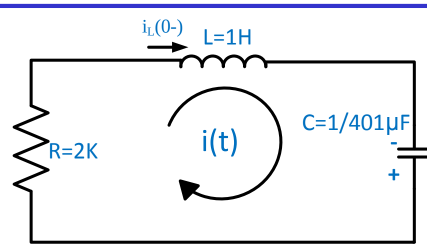


Figure19 : RLC series circuit.

$$\alpha = \frac{R}{2L} = 1000$$

$\alpha < \omega_o \rightarrow$ We have under damped case:

$$\omega_d = \sqrt{(\omega_0^2 - \alpha^2)} = 20000$$

$i(t) = e^{-\alpha t}\left[\beta_1 \cos \omega_d t + \beta_2 \sin \omega_d t\right]$ For t>0

$i(t) = e^{-1000t}\,[\beta_1 \cos 20000t + \beta_2 \sin 20000t]$

To find β_1 and β_2 , we need to have :

$i(0^+)$ And $\dfrac{di(0^+)}{dt}$

$$i(0^+) = i_L(0^+) = i_L(0^-) = 2mA$$

At $t = 0^+$

$$L\frac{d_i(0^+)}{dt} + R_i(0^+) - V_c(0^-) + 0 = 0$$

$$\frac{di(0^+)}{dt} = \frac{V_c(0^-) - R_i(0^+)}{L} = -2$$

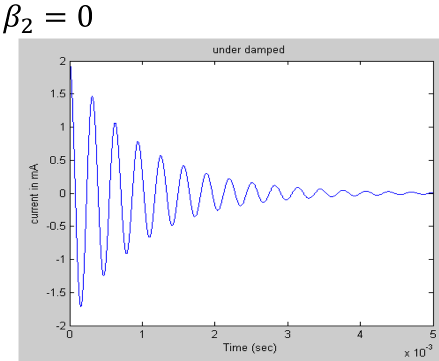
$i(t) = e^{-1000t}\,[\beta_1 \cos 20000t + \beta_2 \sin 20000t]$ For t>0

$$i(0^+) = \beta_1 = 2mA$$

$$\frac{di(0^+)}{dt} = 20000\beta_2 - 2 * 10^{-3}(1000)$$

$$\frac{di(0^+)}{dt} = 20000\beta_2 - 2 = -2 \Rightarrow \beta_2 = 0$$

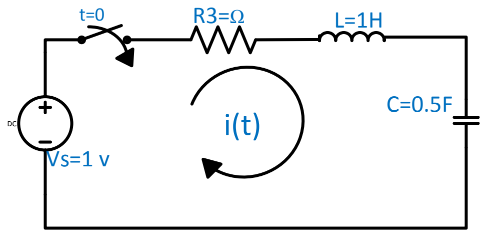
$$i(t) = 2 * e^{-1000t}\,[\cos 20000t]mA$$



Step response of series RLC circuit:

$V_c(0^-) = 0 \quad , \; i_L(0^-) = 0$

Find $i(t)$ for t >0



KVL:

$$Vs = R_i(t) + L\frac{d_i(t)}{dt} + V_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)dt$$

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0 \quad \Rightarrow \quad i(t) = i_n(t)$$

$$LS^2 + RS + \frac{1}{C} = 0 \qquad \rightarrow S^2 + 3S + 2 = 0$$

$S_1 = -1 \quad , \quad S_2 = -2$

S_1 and S_2 are real and unequal
Over-damped case

$$\boldsymbol{i(t) = A_1e^{-t} + A_2e^{-2t}} \qquad ; \qquad \textbf{t>0}$$

OR

$$\omega_o = \frac{1}{\sqrt{LC}} = \sqrt{2}$$

$$\alpha = \frac{R}{2L} = 1.5$$

$$\alpha > \omega_o \quad \rightarrow overdamped\ case$$

To find A_1 & A_2

$i(0^+) = i(0^-) = 0$

$$Vs = R_i(t) + L\frac{d_i(t)}{dt} + V_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)dt$$

<u>At $t = 0^+$:</u>

$$Vs = R_i(0^+) + L\frac{d_i(0^+)}{dt} + 0 + 0$$

$$\frac{di(0^+)}{dt} = \frac{V_c(0^-) - R_i(0^+)}{L} = \frac{Vs}{L} = \frac{1}{1} = 1$$

$$\rightarrow \boldsymbol{i(t) = A_1 e^{-t} + A_2 e^{-2t}}$$

$i(0^+) = A_1 + A_2 = 0$ --------- (1)

$\frac{di(0^+)}{dt} = -A_1 - 2A_2 = 1$ --------- (2)

Solving (1)&(2)

$\boldsymbol{A_1} = 1$ $\boldsymbol{A_2} = -1$ $\rightarrow \boldsymbol{i(t) = e^{-t} - e^{-2t}}$ **;** ***t>0***

$\boldsymbol{v_c(t) = (1 - 2e^{-t} + e^{-2t})}$ ***volt*** **;** **t>0**

Another Method To find $V_c(t)$ *directly*:

$$Vs = R_i(t) + L\frac{d_i(t)}{dt} + V_c(t)$$

$$i(t) = i_c(t) = C\frac{dV_c(t)}{dt}$$

$$Vs = RC\frac{dV_c(t)}{dt} + LC\frac{d^2V_c(t)}{dt} + V_c(t)$$

$$V_c(t) = V_{cn}(t) + V_{cf}(t)$$

$V_{cf}(t) = k$, k is constant

$$k = Vs$$

$$V_c(t) = V_s(t) + V_{cn}(t)$$

$$0 = LCS^2 + RCS + 1 \rightarrow 0 = \frac{1}{2}S^2 + \frac{3}{2}S + 1 \rightarrow \boldsymbol{S_1 = -1; S_2 = -2}$$

$V_c(t) = V_{cf}(t) + V_{cn}(t)$

$V_c(t) = A_1 e^{-t} + A_2 e^{-2t} + 1$

To find A_1,A_2

$V_c(0^+) = V_c(0^-) = 0$

$$i(t) = i_L(t) = i_c(t) = C\frac{dV_c(t)}{dt}$$

$$i_L(0^+) = i_c(0^+) = C\frac{dV_c(0^+)}{dt} = 0$$

$$\frac{dV_c(0^+)}{dt} = 0$$

$V_c(0^+) = 0$

$V_c(t) = A_1 e^{-t} + A_2 e^{-2t} + 1$

$0 = A_1 + A_2 + 1$

$A_1 + A_2 = -1$ ------- (1)

$$\frac{dV_c(t)}{dt} = -A_1 e^{-t} - 2A_2 e^{-2t}$$

$$\frac{dV_c(0^+)}{dt} = -A_1 - 2A_2 = 0 \text{ ---- (2)}$$

Solving (1),(2)

$A_1 = -2$

$A_2 = 1$

$\boldsymbol{V_c(t) = 1 - 2e^{-t} + e^{-2t}}$ V

1) find $i(t)$ for t>0

2) find Vc(t) for t>0

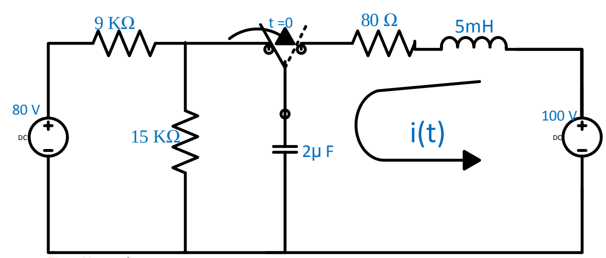


Figure 22:example

a) for t<0 , t=0-

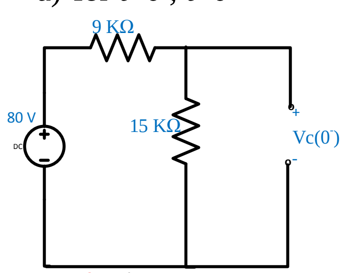


Figure 23:For t<0

$$V_c(0^-) = \frac{15k}{15k + 9k} * 80 = 50\ volt.$$

$i_L(0^-) = 0$

2) for t>0

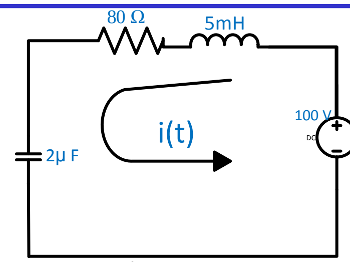


Figure 24: For t>0

KVL:

$$100 = Ri(t) + L\frac{d_i(t)}{dt} + V_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)dt$$

$$50 = Ri(t) + L\frac{d_i(t)}{dt} + \frac{1}{C}\int_{0^-}^{t} i(t)dt$$

Differentiate:

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$$

Second order homogenous differential equation

$$i(t) = i_n(t)$$

$$0 = LCS^2 + RCS + 1$$

$$0 = 10 * 10^{-9}S^2 + 10 * 160^{-6}S + 1$$

$$S_1 = -8000 + j6000$$

$$S_2 = -8000 - j6000$$

Under damped case

$$i(t) = e^{-8000t}\,[\beta_1 \cos{6000t} + \beta_2 \sin{6000t}]$$ For t>0

To find β_1& β_2:

$$i(0^+) = i(0^-) = 0$$

$$\frac{di(0^+)}{dt} = \frac{V_s - V_c(0^-)}{L} = 10000$$

$$\beta_1 = 0$$

$$\beta_2 = 1.67$$

$$\boldsymbol{i(t) = 1.67e^{-8000t}\,[\sin_{6000t}]}$$ **A** **For t>0**

2) find Vc(t) for t>0

$$Vs = R_i(t) + L\frac{di(t)}{dt} + V_c(t)$$

$$i(t) = i_c(t) = C\frac{dv_c(t)}{dt}$$

$$Vs = LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t)$$

Second order nonhomogeneous differential equation.

$$V_c(t) = V_{cf}(t) + V_{cn}(t)$$

$$V_{cf}(t) = k$$

$$Vs = 0 + 0 + k$$

$$V_{cf}(t) = k$$

To find $V_{cn}(t)$:

$$0 = LCS^2 + RCS + 1$$

$$0 = 10 * 10^{-9} S^2 + 10 * 160^{-6} S + 1$$

$$S_1 = -8000 + j6000$$
$$S_2 = -8000 - j6000$$

→ Under damped case

$$\boldsymbol{V_c(t) = 100 + e^{-8000t}\,[\beta_1 \cos 6000t + \beta_2 \sin 6000t]} \quad \textbf{For t>0}$$

To find β_1& β_2 we need: $V_c(0^+)\ and\ \dfrac{dv_c(0^+)}{dt}$

$$v_c(0^+) = v_c(0^-) = 50\ volt$$

$$i_c(0^+) = i_L(0^-) = C\frac{dv_c(0^+)}{dt} = 0 \quad \rightarrow \quad \frac{dv_c(0^+)}{dt} = 0$$

$$v_c(0^+) = 100 + \beta_1 = 50$$

$$\beta_1 = -50$$

$$\frac{dv_c(0^+)}{dt} = -8000\beta_1 + 6000\beta_2$$

$$\beta_2 = -66.67$$

$$\boldsymbol{V_c(t) = \left[100 + e^{-8000t}\,(-50 \cos 6000t - 66.67 \sin 6000t)\right]}$$

For t>0

Procedure for analyzing 2nd-order circuits (series RLC and parallel RLC)

1. Find the characteristic equation and the natural response

A) Is the circuit a series RLC or parallel RLC? (for $t > 0$ with independent sources killed)

B) Find α and ω_o and use these values in the ***characteristic equation***: $s^2 + 2\alpha s + \omega_o^2$.

$$\alpha = \frac{R}{2L} \text{ (series RLC)} \qquad \alpha = \frac{1}{2RC} \text{ (parallel RLC)} \qquad \omega_o^2 = \frac{1}{LC} \text{ (both series and parallel RLC)}$$

C) Find the roots of the characteristic equation (characteristic roots or natural frequencies).

D) Determine the form of the natural response based on the type of characteristic roots:

a) Overdamped: Real, distinct roots : $x_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

b) Underdamped: Complex roots : $x_n = e^{\alpha t}\left[A_1 \cos(\beta\, t) + A_2 \sin(\beta\, t)\right]$

c) Critically damped: Repeated roots : $x_n = (A_1 t + A_2) e^{st}$

2. Find the forced response - Analyze the circuit at $t = \infty$ to find $x_f = x(\infty)$.

3. Find the initial conditions, $x(0)$ and $x'(0)$.

A) Find $x(0)$ by analyzing the circuit at $t = 0^-$ (find all capacitor voltages and inductor currents)

B) Analyze the circuit at $t = 0^+$ (using $v_C(0)$ and $i_L(0)$ from step 3B) and find:

$$\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C} \quad \text{or} \quad \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

4. Find the complete response

A) Find the total response, $x(t) = x_n + x_f$.

B) Use the two initial conditions to solve for the two unknowns in the total response.

Important Note: In determining if a circuit is a series RLC or parallel RLC circuit, consider the circuit for $t > 0$ with all independent sources killed.

TABLE 8.2 The Response of a Second-Order Circuit is Overdamped, Underdamped, or Critically Damped

The Circuit is	When	Qualitative Nature of the Response
Overdamped	$\alpha^2 > \omega_0^2$	The voltage or current approaches its final value without oscillation
Underdamped	$\alpha^2 < \omega_0^2$	The voltage or current oscillates about its final value
Critically damped	$\alpha^2 = \omega_0^2$	The voltage or current is on the verge of oscillating about its final value

TABLE 8.3 In Determining the Natural Response of a Second-Order Circuit, We First Determine Whether it is Over-, Under-, or Critically Damped, and Then We Solve the Appropriate Equations

Damping	Natural Response Equations	Coefficient Equations
Overdamped	$x(t) = A_1e^{s_1t} + A_2e^{s_2t}$	$x(0) = A_1 + A_2$; $dx/dt(0) = A_1s_1 + A_2s_2$
Underdamped	$x(t) = (B_1 \cos \omega_d t + B_2 \sin \omega_d t)e^{-\alpha t}$	$x(0) = B_1$; $dx/dt(0) = -\alpha B_1 + \omega_d B_2$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Critically damped	$x(t) = (D_1t + D_2)e^{-\alpha t}$	$x(0) = D_2$, $dx/dt(0) = D_1 - \alpha D_2$

TABLE 8.4 In Determining the Step Response of a Second-Order Circuit, We Apply the Appropriate Equations Depending on the Damping

Damping	Step Response Equations[a]	Coefficient Equations
Overdamped	$x(t) = X_f + A_1' e^{s_1t} + A_2' e^{s_2t}$	$x(0) = X_f + A_1' + A_2'$; $dx/dt(0) = A_1' s_1 + A_2' s_2$
Underdamped	$x(t) = X_f + (B_1' \cos \omega_d t + B_2' \sin \omega_d t)e^{-\alpha t}$	$x(0) = X_f + B_1'$; $dx/dt(0) = -\alpha B_1' + \omega_d B_2'$
Critically damped	$x(t) = X_f + D_1' te^{-\alpha t} + D_2' e^{-\alpha t}$	$x(0) = X_f + D_2'$; $dx/dt(0) = D_1' - \alpha D_2'$

[a] where X_f is the final value of $x(t)$.

8.8

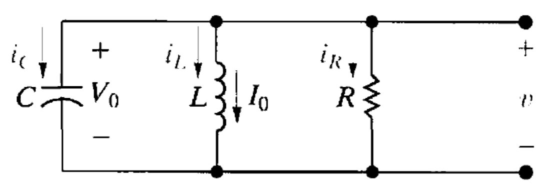


Figure 8.1 ▲ A circuit used to illustrate the natural response of a parallel *RLC* circuit.

8.8 Suppose the capacitor in the circuit shown in Fig. 8.1 has a value of 0.1 μF and an initial voltage of 24 V. The initial current in the inductor is zero. The resulting voltage response for $t \geq 0$ is

$$v(t) = -8e^{-250t} + 32e^{-1000t}\ \text{V}.$$

a) Determine the numerical values of R, L, α, and ω_0.

b) Calculate $i_R(t)$, $i_L(t)$, and $i_C(t)$ for $t \geq 0^+$.

Answer

$\alpha = 625\ \text{rad/s}$

$R = 8\ \text{k}\Omega$

$\omega_o = 500\ \text{rad/s}$

L=40 H

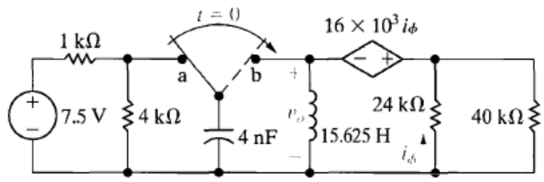
$i_R = -1e^{-250t} + 4e^{-1000t}\ \text{mA}$,

$i_L = 0.8e^{-250t} - 0.8e^{-1000t}\ \text{mA}$

$i_C = 0.2e^{-250t} - 3.2e^{-1000t}\ \text{mA}$,

8.21 PSPICE MULTISIM The switch in the circuit of Fig. P8.21 has been in position a for a long time. At $t = 0$ the switch moves instantaneously to position b. Find $v_o(t)$ for $t \geq 0$.

Figure P8.21

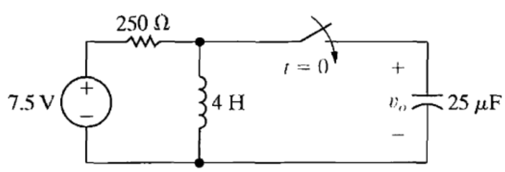


Answer

$$v_o = 8e^{-8000t} - 2e^{-2000t}\,\text{V}, \qquad t \geq 0$$

8.36 PSPICE MULTISIM The switch in the circuit in Fig. P8.36 has been open a long time before closing at $t = 0$. At the time the switch closes, the capacitor has no stored energy. Find v_o for $t \geq 0$.

Figure P8.36



Answer

$$v_o = 0 \text{ for } t \geq 0$$

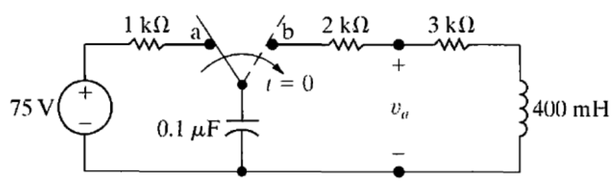
8.46 PSPICE MULTISIM The switch in the circuit in Fig. P8.46 on the next page has been in position a for a long time. At $t = 0$, the switch moves instantaneously to position b.

a) What is the initial value of v_a?

b) What is the initial value of dv_a/dt?

c) What is the numerical expression for $v_a(t)$ for $t \geq 0$?

Figure P8.46



Answer

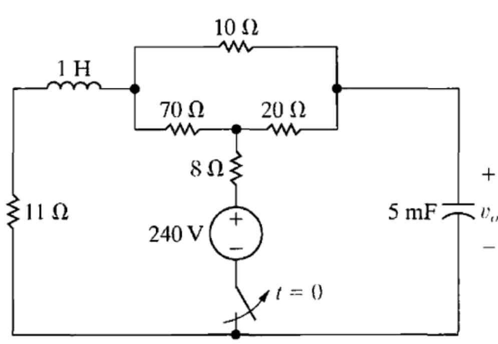
a) $$v_a(0^+) = 75\text{ V}$$

b) $$\frac{dv_a(0^+)}{dt} = -375{,}000\text{ V/s}$$

c) $$v_a = 50e^{-2500t} + 25e^{-10{,}000t}\text{ V}, \quad t \geq 0^+$$

8.48 The switch in the circuit shown in Fig. P8.48 has been closed for a long time. The switch opens at $t = 0$. Find $v_o(t)$ for $t \geq 0$.

Figure P8.48



Answer

$$v_o = 108e^{-10t}\cos 10t - 12e^{-10t}\sin 10t\text{ V}, \qquad t \geq 0$$