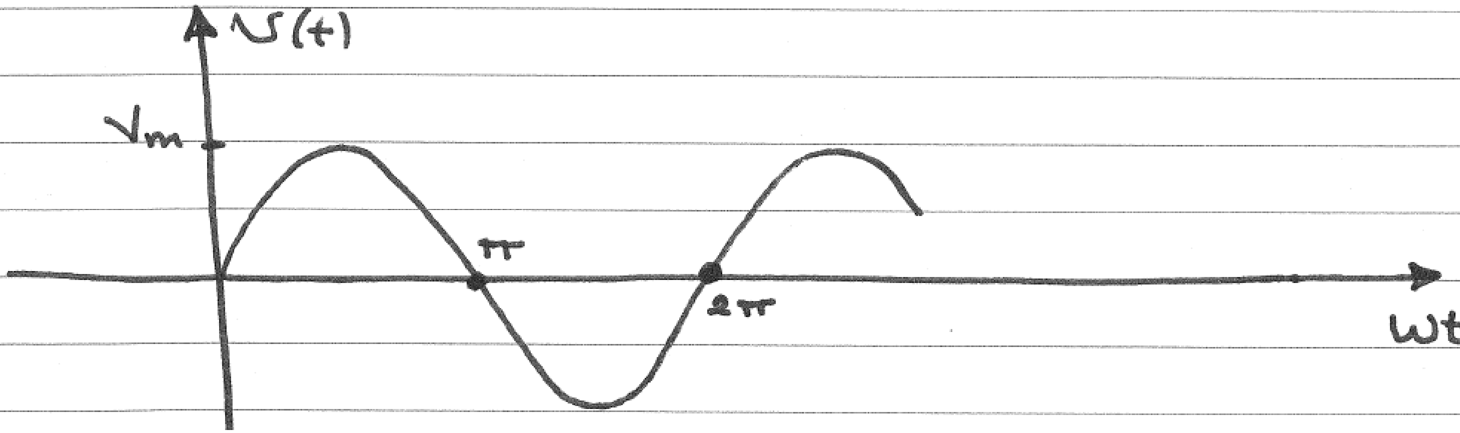
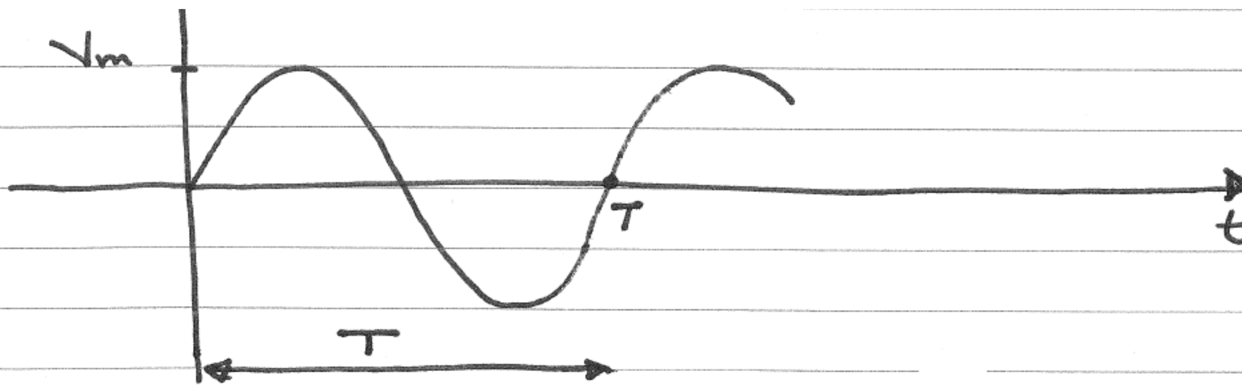
ENEE2301

SINUSOIDAL STEADY-STATE ANALYSIS

CH 9

Sinusoidal Steady – State Analysis



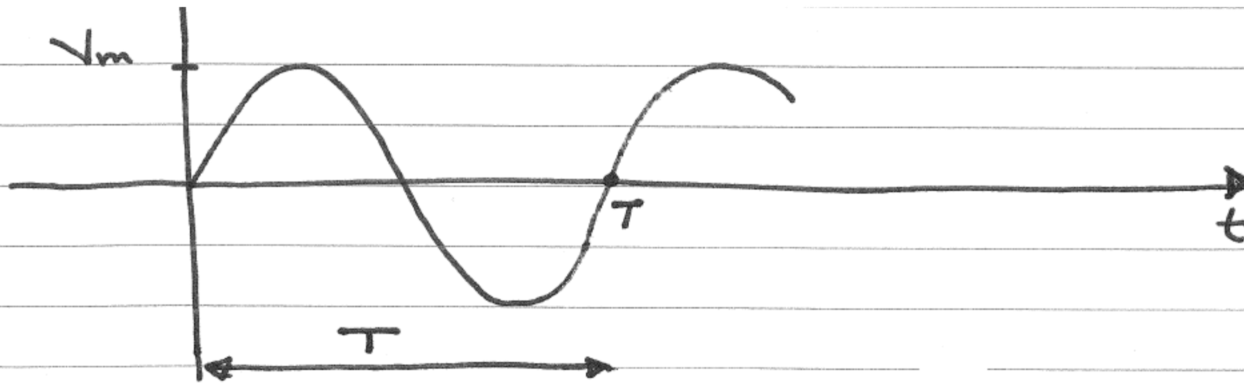


$v_s(t) = V_m \sin \omega t$

$V_m \equiv$ Amplitude of the sinusoid

$\omega \equiv$ Angular frequency in radian/s

Sinusoidal Steady – State Analysis



$$\omega = 2\pi f$$

$$f = \frac{1}{T}$$

$f \equiv$ frequency in Hertz

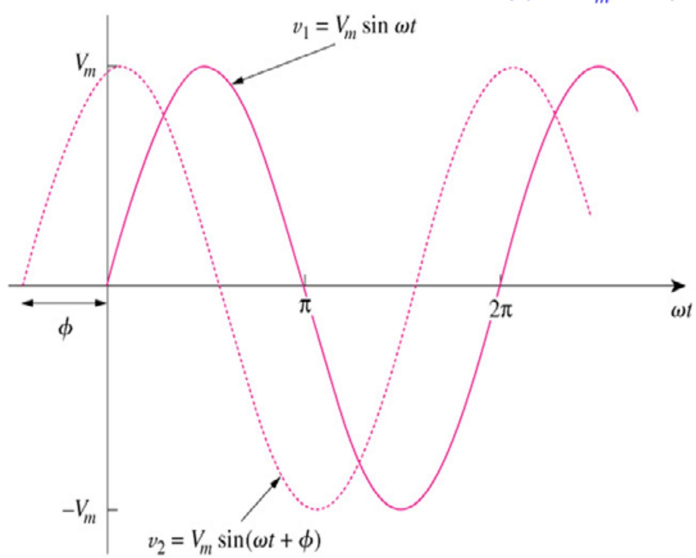
$T \equiv$ Period in seconds

PHASE OF SINUSOIDS

- The term lead and lag are used to indicate the relationship between two sinusoidal wave forms of the same frequency plotted on the same set of axes.

$$V1 = Vm \sin \omega t$$
$$V2 = Vm \sin(\omega t + \theta)$$
$$\therefore V2\ leads\ V1\ by\ \theta\ or\ V1\ lags\ V2\ by\ \theta$$

- Consider the sinusoidal voltage having phase φ, $v(t) = V_m \sin(\omega t + \phi)$



- v_2 LEADS v_1 by phase φ.
- v_1 LAGS v_2 by phase φ.
- v_1 and v_2 are out of phase.

Trigonometric identities :

$$\sin(A \mp B) = sinA \cos B \mp \cos A \sin B$$

$$\cos(A \mp B) = \cos A \cos B \pm \sin A \sin B$$

$$\sin(\omega t \mp 180°) = -\sin \omega t$$

$$\cos(\omega t \mp 180°) = -\cos \omega t$$

$$\sin(\omega t \mp 90°) = \mp \cos \omega t$$

$$\cos(\omega t \mp 90) = \pm \sin \omega t$$

$$A \cos \omega t + Bsin\ \omega t = C \cos(\omega t - \theta)$$

$where$

$$C = \sqrt{A^2 + B^2} \quad And \quad \theta = \tan^{-1} \frac{B}{A}$$

Phase and Phase Angle

*** We usually use *cosine function* to model a sinusoidal signal.**

In case there is a sine function, we can use the following conversion:

$$\sin(z) = \cos(z - 90)$$

For example:

$$v_x(t) = 10\sin(200t + 30°)$$

$$v_x(t) = 10\cos(200t + 30° - 90°)$$
$$= 10\cos(200t - 60°)$$

and thus we can say that the phase angle of $v_x(t)$ is -60°

Examples :

1) let

$V1(t) = 10\sin(5t - 30°)$, $V2(t) = 15\sin(5t + 10°)$

$\therefore V2(t)\ leads\ V1(t) by\ 40°$

2) let

$$i1(t) = 2\sin(377t + 45°)$$

$$i2(t) = 0.5\cos(377t + 10°)$$

$$\cos\alpha = \sin(\alpha + 90°)$$

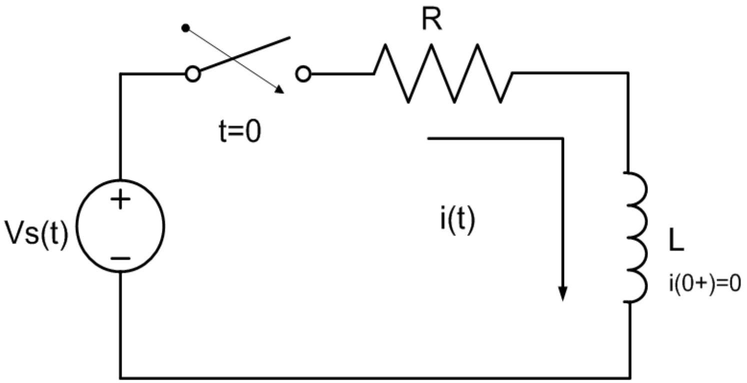
$$0.5\cos(377t + 10°) = 0.5\sin(377t + 100°)$$

$$\therefore i2(t) leads\ i1(t) by\ 55°$$

The sinusoidal response :

find i(t) for t>0
given

$Vs(t) = Vm\cos\omega t \;\; volt.$



Solution

KVL:

$$Vs(t) = Ri(t) + L\frac{di(t)}{dt}$$

$$Vm\cos\omega t = Ri(t) + L\frac{di(t)}{dt}$$

first order non homogenous differential equation .

$$\therefore i(t) = in(t) + if(t)$$

$$i(t) = Ae^{-\frac{t}{\tau}} + if(t)$$

$$if(t) = I1\cos\omega t + I2\sin\omega t$$

$\text{To } find\ I1\ and\ I2:$

$$Vm\ cos\omega t = Ri(t) + L\frac{di(t)}{dt}$$

$$Vm\cos\omega t = R[I1\cos\omega t + I2\sin\omega t] + L\omega[-I1\sin\omega t + I2\cos\omega t]$$

$collect\ the\ sine\ and\ cosine\ terms$

$$0 = [-L\omega I1 + RI2]\sin\omega t + [L\omega I2 + RI1 - Vm]\cos\omega t$$
$$-\omega LI1 + RI2 = 0$$
$$\omega LI2 + RI1 - Vm = 0$$

$$I1 = \frac{RVm}{R^2 + \omega^2 L^2}$$

$$I2 = \frac{\omega LVm}{R^2 + \omega^2 L^2}$$

$$if(t) = \frac{RVm}{R^2 + \omega^2 L^2} \cos \omega t + \frac{\omega LVm}{R^2 + \omega^2 L^2} \sin \omega t$$

$$if(t) = I1 \cos \omega t + I2 \sin \omega t$$

$$if(t) = C \cos(\omega t - \emptyset)$$

$$if(t) = C \cos \omega t \cos \emptyset + C \sin \omega t \, sin\emptyset$$

$$\therefore I1 = C \cos \emptyset$$

$$I2 = C \sin \emptyset$$

$$\frac{I2}{I1} = \tan \emptyset$$

$$\emptyset = \tan^{-1}\frac{I2}{I1} = \tan^{-1}\frac{\omega L}{R} \rightarrow \rightarrow (1)$$

$$I1^2 + I2^2 = C^2 cos^2\emptyset + C^2 sin^2\emptyset$$

$$I1^2 + I2^2 = C^2$$

$$C = \sqrt{I1^2 + I2^2}$$

$$C = \frac{Vm}{\sqrt{R^2 + \omega^2 L^2}} \rightarrow \rightarrow (2)$$

$$if(t) = \frac{Vm}{\sqrt{R^2 + \boldsymbol{\omega}^2 L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

$$I1 = C\cos\emptyset$$

$$I2 = C\sin\emptyset$$

$$\frac{I2}{I1} = \tan\emptyset$$

$$I1 = \frac{RVm}{R^2 + \omega^2 L^2}$$

$$I2 = \frac{\omega L Vm}{R^2 + \omega^2 L^2}$$

$$\therefore i(t) = Ae^{-\frac{t}{\tau}} + \frac{Vm}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

$$i(0^+) = A + \frac{Vm}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(-\tan^{-1}\frac{\omega L}{R}\right) = 0$$

$$\therefore A = -\frac{Vm}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(-\tan^{-1}\frac{\omega L}{R}\right)$$

$$i(t) = in(t) + if(t)$$

$$i(t) = \ transient\ component + steady - state\ component$$

$$\boldsymbol{the\ steady\ state\ solution\ is\ a\ sinusoidal\ function\ with\ the\ same\ frequency\ as\ the\ source\ signal.}$$

R-L Circuit

If Vs(t) = $V_m \cos(\omega t + \phi)$

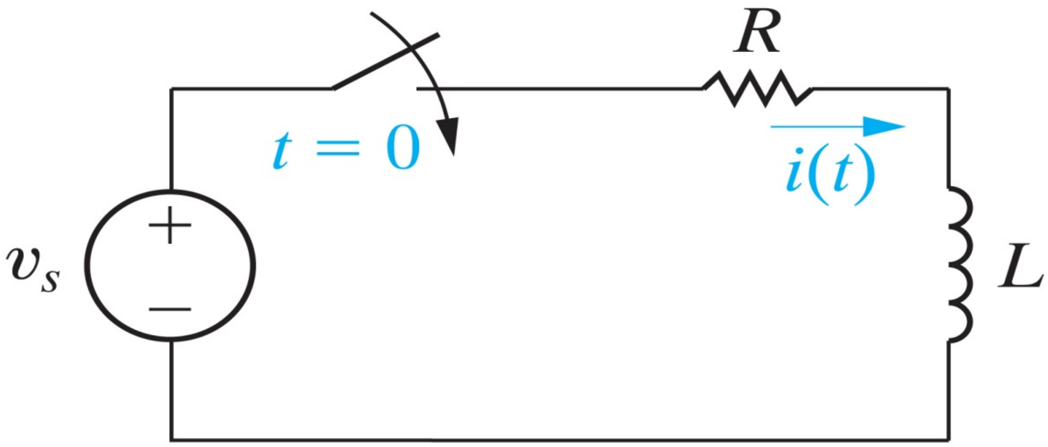


Figure: 09-05

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

$$i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$

$$\theta = \tan^{-1}\frac{\omega L}{R}$$

Transient

Steady-state

Complex Numbers:

a complex number may be written in three forms :

1) rectangular form : $Z = X + jY$

$$j = \sqrt{-1} \quad ,$$

$$X = RE\{Z\} \quad ,$$

$$Y = Im\{Z\}$$

2) Exponential form : $Z = |Z|e^{j\theta}$

$$|Z| = magnitude \quad ,\theta = angle$$

3) Polar form $Z = |Z|\angle\theta$

Euler's law: $e^{j\theta} = \cos\theta + j\sin\theta$

1 - Conversion from polar to rectangular form

$$Z = |Z|e^{j\theta}$$

$$Z = |Z|\cos\theta + j|Z|\sin\theta$$

$$Z = X + jY$$

$$X = |Z|\cos\theta$$

$$Y = |Z|\sin\theta$$

2 - Conversion from rectangular to polar form

$$X^2 + Y^2 = |Z|^2 cos^2\theta + |Z|^2 sin^2\theta$$

$$X^2 + Y^2 = |Z|^2[cos^2\theta + sin^2\theta]$$

$$X^2 + Y^2 = |Z|^2$$

$$\therefore |Z| = \sqrt{X^2 + Y^2}$$

$$\frac{Y}{X} = \frac{|Z|\sin\theta}{|Z|\cos\theta} = \tan\theta \Longrightarrow \therefore \theta = \tan^{-1}\frac{Y}{X}$$

Mathematical operation of complex numbers:

1) Addition : $Z1 + Z2 = (X1 + X2) + j(Y1 + Y2)$

2) Subtraction: $Z1 - Z2 = (X1 - X2) + j(Y1 - Y2)$

3) Multiplication: $Z1\,Z2 = |Z1||Z2|\angle\theta 1 + \theta 2$

4) Division: $\frac{Z1}{Z2} = \frac{|Z1|}{|Z2|}\angle\theta 1 - \theta 2$

5) Complex Conjugate:

If Z = X+jY = $|Z|\angle + \theta$

$Z^{*} = X - jY \quad OR \quad Z^{*} = |Z|\angle - \theta$

Example

$$Z1 = 4 + j3 = 5\angle 36.9°$$

$$Z2 = 3 + j4 = 5\angle 53.1°$$

$$Z1 + Z2 = 7 + j7$$

$$Z1 - Z2 = 1 - j1$$

$$Z1\,Z2 = 5\angle 36.9° \,.\, 5\angle 53.1° = 25\angle 90°$$

$$\frac{Z1}{Z2} = \frac{5\angle 36.9°}{5\angle 53.1°} = 1\angle - 16.2°$$

OR

$$Z1\,Z2 = (4 + j3)(3 + j4)$$

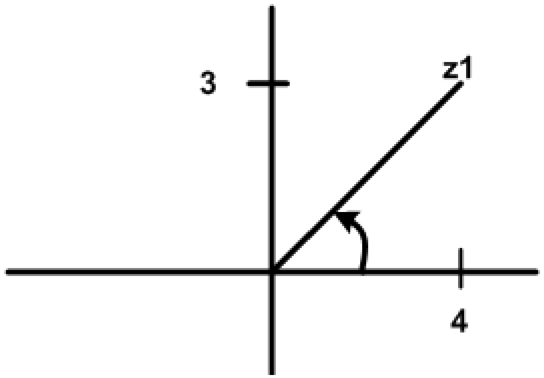
$$= 12 + j16 + j9 - 12$$

$$Z1\,Z2 = j25$$

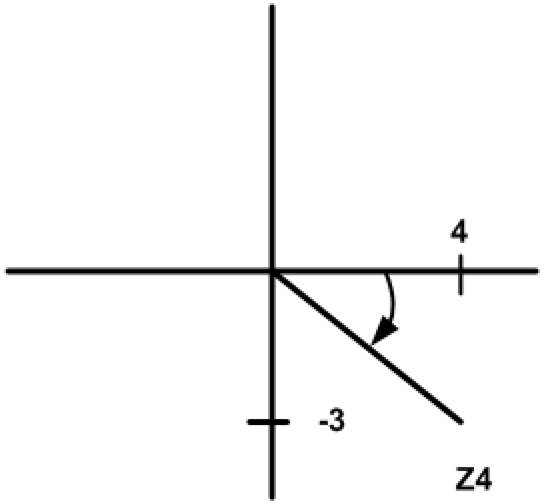
$$\frac{Z1}{Z2} = \frac{4 + j3}{3 + j4} \cdot \frac{3 - j4}{3 - j4} = \frac{12 - j16 + j9 + 12}{25} = \frac{24 - j7}{25}$$

$$\frac{Z1}{Z2} = \frac{24}{25} - j\,\frac{7}{25}$$

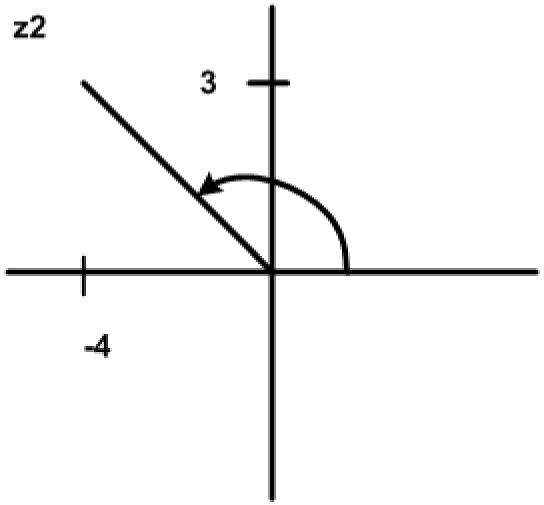
THE GRAPHICAL REPRESENTATION

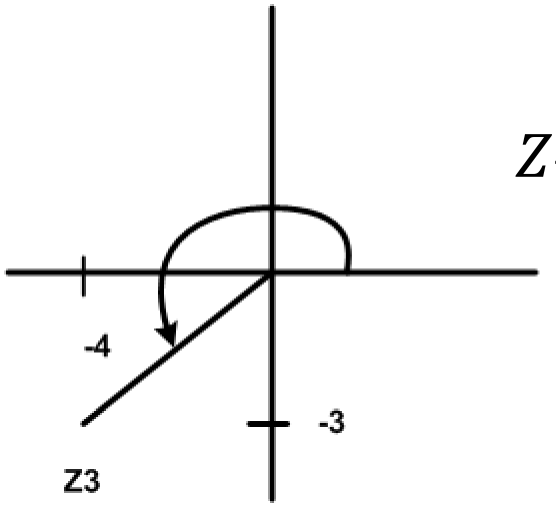


$$Z1 = 4 + j3$$
$$Z1 = 5\angle 36.9°$$



$$Z4 = 4 - j3$$
$$Z4 = 5\angle -36.9°$$

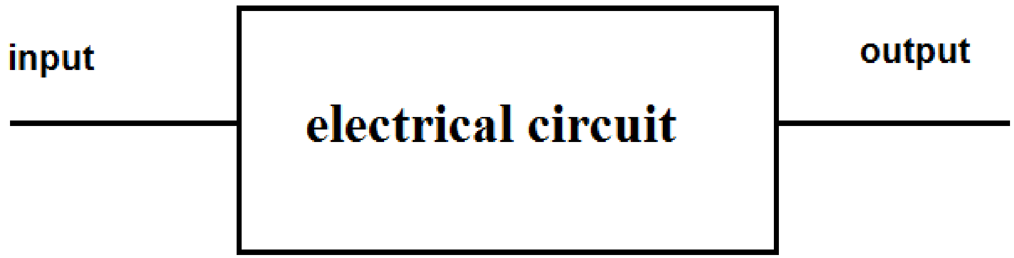




$$Z2 = -4 + j3$$
$$Z2 = 5\angle 143.1°$$

$$Z3 = -4 - j3$$
$$Z3 = 5\angle 216.9$$

The phasor concept



$$Vm\cos(\omega t+\theta) \longrightarrow Im\cos(\omega t+\emptyset)$$
$$Vm\sin(\omega t+\theta) \longrightarrow Im\sin(\omega t+\emptyset)$$

$$jVm\sin(\omega t+\theta) \longrightarrow j\,Im\sin(\omega t+\emptyset)$$
$$Vm\cos(\omega t+\theta)+jVm\sin(\omega t+\theta) \longrightarrow Im\cos(\omega t+\emptyset)\;+\mathrm{j\,Im}\sin(\omega t+\emptyset)$$

$$Vme^{j(\omega t+\theta)} \longrightarrow Im\,e^{j(\omega t+\emptyset)}$$

- Instead of applying a real forcing function to obtain the desired real response , we apply a **complex forcing function** whose real part is the given real forcing function.

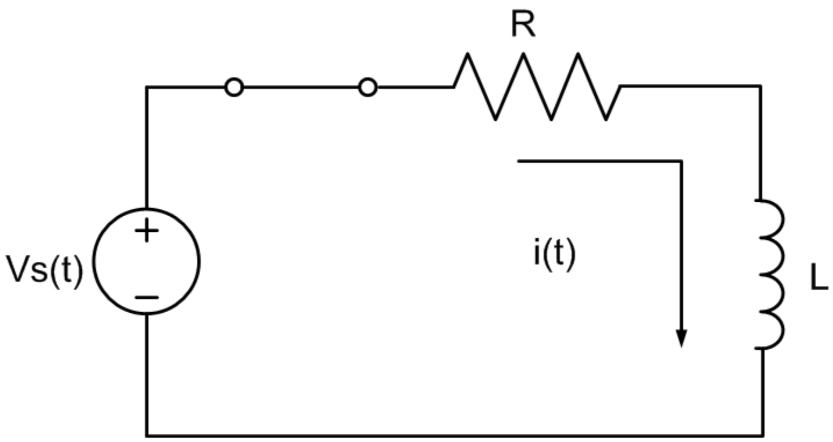
- we obtain a complex response whose real part is the desired real response.

Sinusoidal and Complex forcing function:

$$Vs(t) = Vm\cos\omega t$$

$$\therefore\ i(t) = Im\cos(\omega t + \emptyset)$$

The steady – state response



$$\mathrm{Vs}(t) \to Vme^{j\omega t}$$

$$i(t) \to Im\ e^{j(\omega t+\emptyset)}$$

$$KVL: \quad Vs(t) = Ri(t) + L\frac{di(t)}{dt}$$

$$Vm\ e^{j\omega t} = RIm\ e^{j(\omega t+\emptyset)} + j\omega L\ Im\ e^{j(\omega t+\emptyset)}$$

a complex algebraic equation .

to find Im and Ø ;devide by $e^{j\omega t}$

$$Vm = RIm\ e^{j\emptyset} + j\omega L\ Im\ e^{j\emptyset}$$

$$Vm = Im\ e^{j\emptyset}(R + j\omega L)$$

$$\therefore\ Im\ e^{j\emptyset} = \frac{Vm}{R + j\omega L}$$

$$Im\ e^{j\emptyset} = \frac{Vm}{\sqrt{R^2 + \omega^2 L^2}\, e^{j\tan^{-1}\frac{\omega L}{R}}}$$

$$Im\ e^{j\emptyset} = \frac{Vm}{\sqrt{R^2 + \omega^2 L^2}}\, e^{-j\tan^{-1}\frac{\omega L}{R}}$$

$$\therefore\ Im = \frac{Vm}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\emptyset = -\tan^{-1}\frac{\omega L}{R}$$

$$i(t) = Im\cos(\omega t + \emptyset)$$

$$i(t) = \frac{Vm}{\sqrt{R^2 + \omega^2 L^2}}\cos(\omega t + \emptyset)$$

The steady – state response

Phasors:

Given the sinusoids $i(t) = Im\cos(\omega t + \emptyset i)$

$$V(t) = Vm\cos(\omega t + \theta v)\,.$$

$we\ can\ obtain\ the\ phasor\ forms\ as$:

$$i(t) = Im\cos(\omega t + \emptyset i)$$
$$then\ \vec{I} = Im\ \angle\emptyset i$$

$$V(t) = Vm\cos(\omega t + \theta v)$$
$$then\ \vec{V} = Vm\ \angle\theta v.$$

$$V(t) = -4\sin(30t + 50°)\,V.$$

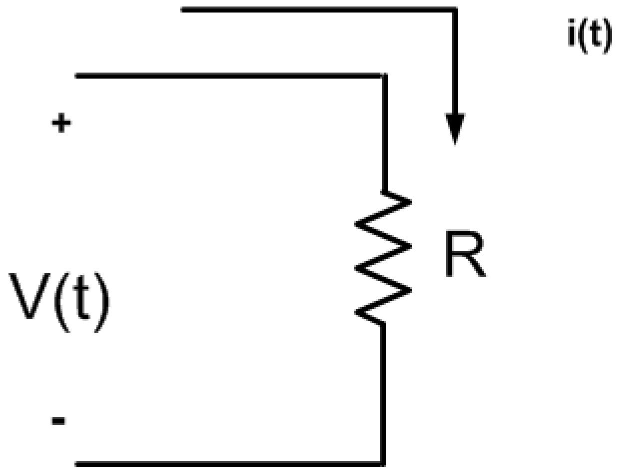
$$\cos(\omega t \mp 90) = \pm\sin\omega t$$

$$V(t) = 4\cos(30t + 140)\,V.$$

$$\therefore \vec{V} = 4\angle 140°\,V.$$

Phasor relationships for circuit elements:

1) Resistor :



$$V(t) = Ri(t)$$

$$Vm\ e^{j(\omega t+\theta v)} = RIm\ e^{j(\omega t+\emptyset_i)}$$

$$Vm\ e^{j\theta v} = RIme^{j\emptyset i}$$

$$Vm\ \angle\theta v = RIm\angle\emptyset i$$

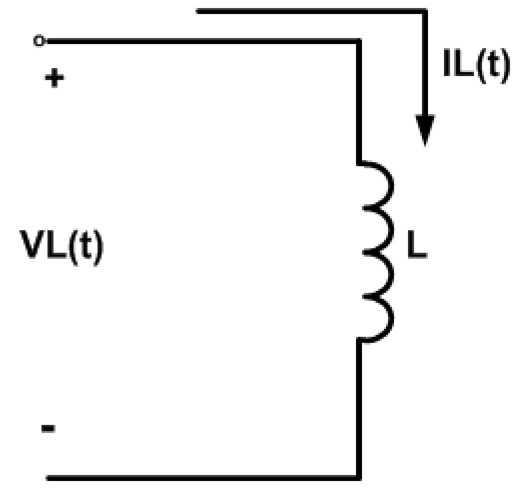
$$\vec{V} = R\vec{I}$$

$$Vm = RIm$$

$$\theta v = \emptyset i$$

■ ***voltage and current of a resistor are in phase***

2) **Inductor:**



$$V(t) = L\frac{di(t)}{dt}$$

$$Vm\ e^{j(\omega t+\theta v)} = L\frac{d}{dt}\left(Im\ e^{j(\omega t+\emptyset i)}\right)$$

$$Vm\ e^{j(\omega t+\theta v)} = j\omega LIm\ e^{j(\omega t+\emptyset i)}$$

$$Vme^{j\theta v} = j\omega L\ Im\ e^{j\emptyset i}$$

$$Vm\angle\theta v = j\omega L\ Im\ \angle\emptyset i$$

$$\vec{V} = j\omega L\ \vec{I}$$

$$Vm\angle\theta v = \omega L\ \angle 90°\ .\ Im\angle\emptyset i$$

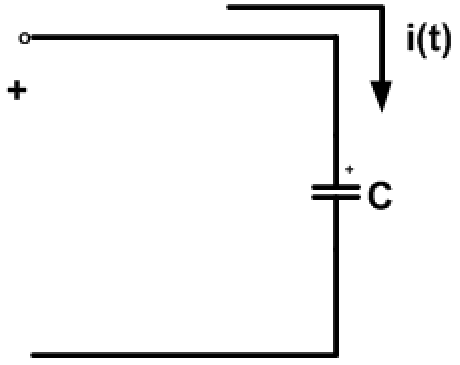
$$Vm\angle\theta v = \omega L\ Im\ \angle\emptyset i + 90°$$

$$\therefore Vm = \omega LIm$$

$$\theta v = \emptyset i + 90°$$

▪***The voltage leads the current by*** 90°.

3. Capacitor



$$i(t) = C\frac{dV(t)}{dt}$$

$$Im\ e^{j(\omega t+\emptyset i)} = C\frac{d}{dt}\left(Vm\ e^{j(\omega t+\theta v)}\right)$$

$$Im\ e^{j(\omega t+\emptyset i)} = j\omega CVm\ e^{j(\omega t+\theta v)}$$

$$Im\ e^{j\emptyset i} = j\omega CVm\ e^{j\theta v}$$

$$Im\angle\emptyset i = j\omega C\ Vm\angle\theta v$$

$$\vec{I} = j\omega C\ \vec{V}$$

$$Im\angle\emptyset i = \omega C\angle 90°\,.Vm\angle\theta v$$

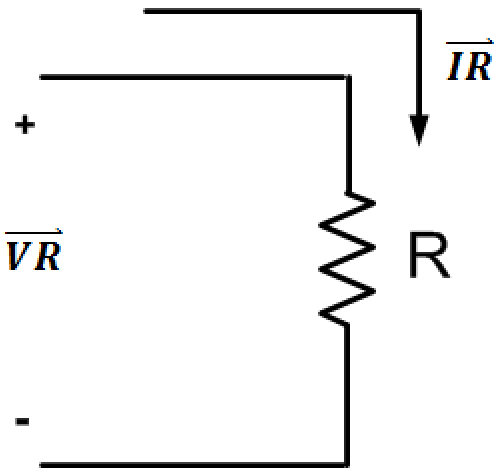
$$Im\angle\emptyset i = \omega C\ Vm\angle\theta v + 90°$$

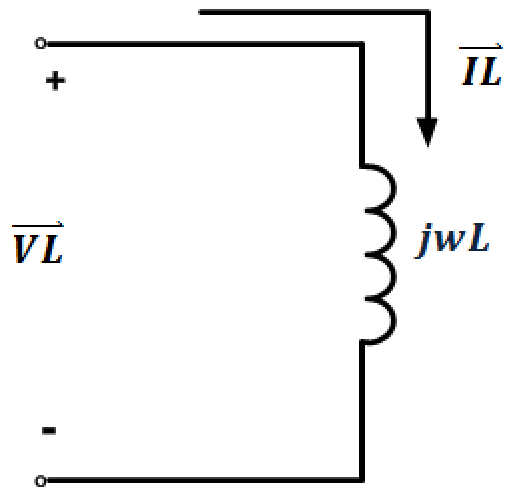
$$Im = \omega C\ Vm$$

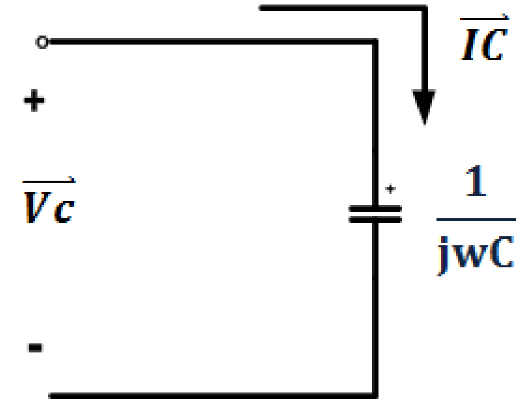
$$\emptyset i = \theta v + 90°$$

■ ***The current leads the voltage by*** 90°

Phasor Relationships for circuit elements







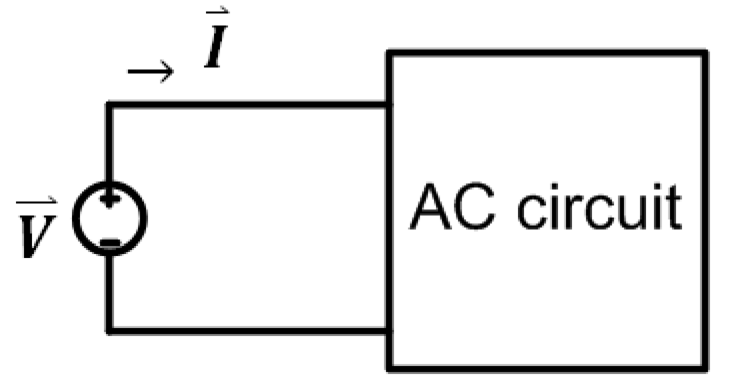
$$\vec{V_R} = R * \vec{I_R}$$

$$\vec{V_L} = j\omega L * \vec{I_L}$$

$$\vec{V_C} = \frac{1}{j\omega C}\,\vec{I_C}$$

$$\vec{V} = Z(j\omega)\,\vec{I}$$

Impedance and Admittance:



$$\boldsymbol{Z}(\boldsymbol{j\omega}) = \frac{\vec{\boldsymbol{V}}}{\vec{\boldsymbol{I}}}$$; Impedance in Ω

Or

$$\vec{\boldsymbol{V}} = \boldsymbol{Z}(\boldsymbol{j\omega}).\overrightarrow{\boldsymbol{I}}$$

$$\mathrm{Y}(\boldsymbol{j\omega}) = \frac{\vec{\boldsymbol{I}}}{\vec{\boldsymbol{V}}}$$; Admittance in Siemens

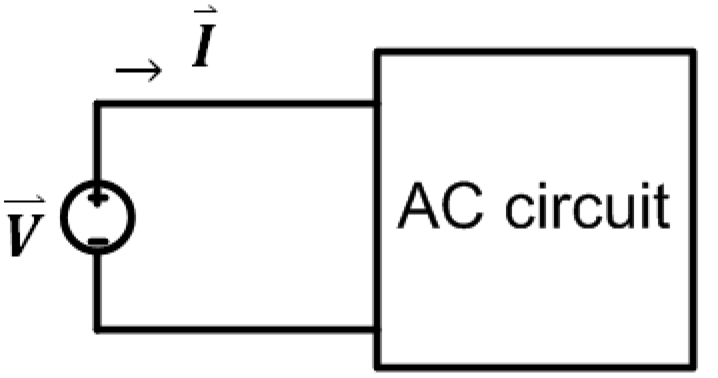
Or

$$\vec{\boldsymbol{I}} = \boldsymbol{Y}(\boldsymbol{j\omega}).\overrightarrow{\boldsymbol{V}}$$

$$\boldsymbol{Z}(\boldsymbol{j\omega}) = \frac{\mathbf{1}}{\boldsymbol{Y}(\boldsymbol{j\omega})}$$

Element	Impedance	Admittance
R	$Z(j\omega) = R$	$Y(j\omega) = \frac{1}{R}$
C	$Z(j\omega) = \frac{1}{j\omega C}$	$Y(j\omega) = j\omega C$
L	$Z(j\omega) = j\omega L$	$Y(j\omega) = \frac{1}{j\omega L}$

Impedance : Z(jω)



$$Z(j\omega) = \frac{\vec{V}}{\vec{I}} = \frac{Vm\angle\theta v}{Im\angle\theta i}$$

$$Z(j\omega) = \frac{Vm}{Im}\angle\theta v - \theta i$$

$$Z(j\omega) = |Z|\angle\theta z$$

The unit of impedance is ohm
Impedance is not a phasor but a complex number that can be written in polar or Cartesian forms.

$$\vec{Z} = R + jX$$

$$R \equiv Real\ part$$

$$X \equiv Reactive\ part.$$

$$Z = |Z|\angle\theta z$$

Or

$$Z = R + jX$$

$$|Z| = \sqrt{R^2 + X^2} \quad magnitude$$

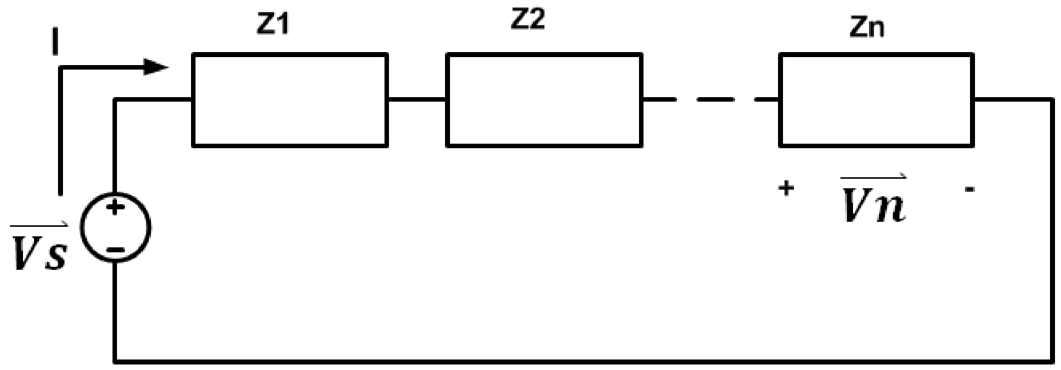
$$\theta z = \tan^{-1}\frac{X}{R} \quad phase$$

Where,

$$X = |Z|\sin\theta z$$

$$R = |Z|\cos\theta z$$

Application of KVL for Phasors



KVL :

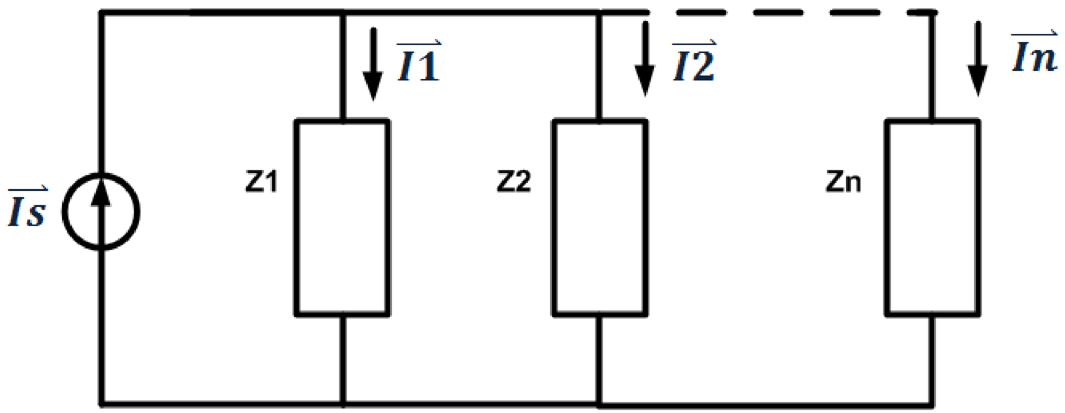
$$Vs(t) = V1(t) + V2(t) + \cdots Vn(t)$$

$$\overrightarrow{Vs} = \overrightarrow{V1} + \overrightarrow{V2} + \cdots \overrightarrow{Vn}$$

$$Zeq = Z1 + Z2 + Z3 + \cdots Zn$$

$$\overrightarrow{Vn} = \frac{Zn}{Z1 + Z2 + Z3 + \cdots Zn} \cdot \overrightarrow{Vs}$$

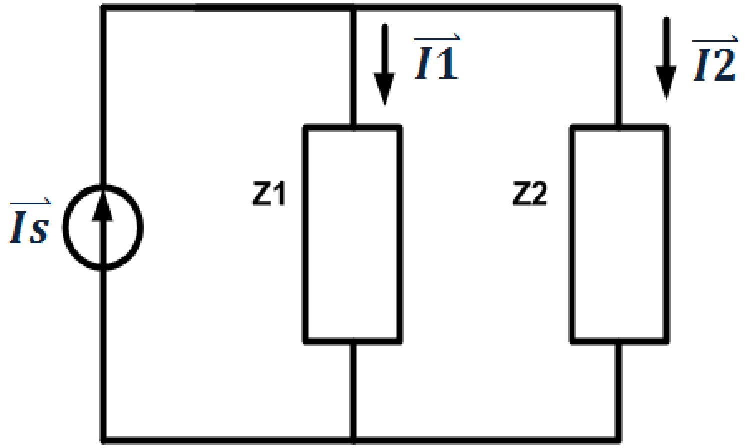
Application of KCL for phasors



KCL:

$$\overrightarrow{Is} = \overrightarrow{I1} + \overrightarrow{I2} + \cdots \overrightarrow{In}$$

Current Divider :



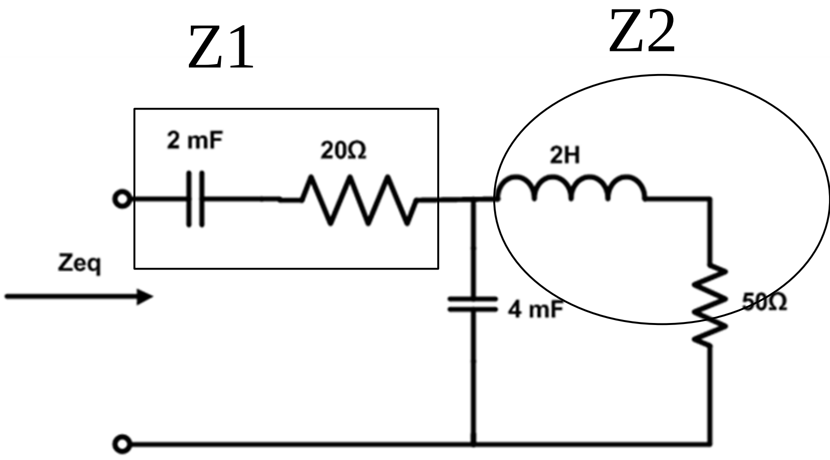
$$\overrightarrow{I1} = \frac{Z2}{Z1 + Z2} \cdot \overrightarrow{Is}$$

$$\overrightarrow{I2} = \frac{Z1}{Z1 + Z2} \cdot \overrightarrow{Is}$$

Find Zeq

$$Z_C(j\omega) = \frac{1}{j\omega C}$$

$$Z_L(j\omega) = j\omega L$$



$$\omega = 10\ rad/sec$$

$$Z1 = 20 + \frac{1}{j(10)(2)(10^{-3})} = 20 - j50$$

$$Z2 = 50 + j(10)(2) = 50 + j20$$

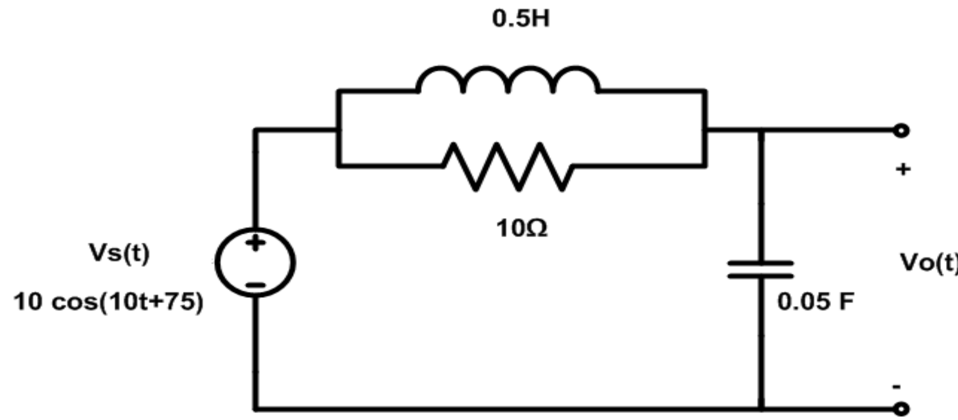
$$Z3 = (50 + j20)\ ||\ \left(\frac{1}{j(10)(4)(10^{-3})}\right)$$

$$Z3 = (50 + j20)\,||-j25$$

$$Z3 = \frac{(50 + j20)(-j25)}{50 + j20 - j25} = 12.38 - j23.76$$

$$Zeq = Z1 + Z3 = 32.38 - j73.76\ \Omega$$

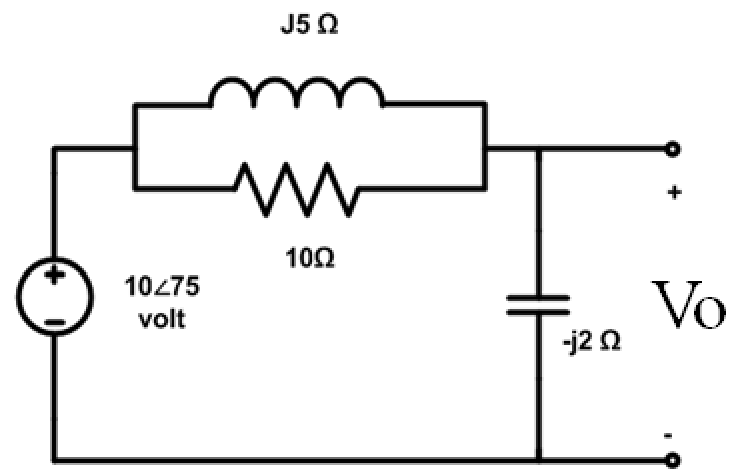
Calculate Vo(t)



$$Z_L(j\omega) = j\omega L = j5\ \Omega$$

$$Z_C(j\omega) = -j\frac{1}{\omega C} = -j2\ \Omega$$

$$\overrightarrow{Vs} = 10\angle 75°\ volt$$

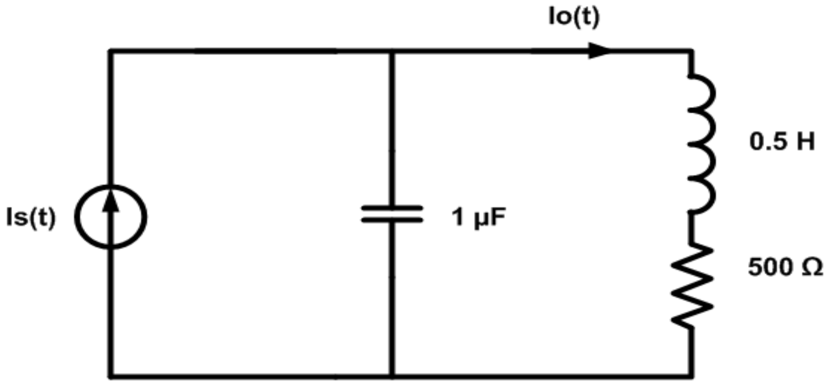


$$\overrightarrow{Vo} = \frac{-j2}{-j2 + (10||j5)} \cdot 10\angle 75°$$

$$\overrightarrow{V\mathrm{o}} = 7.071\angle -60^o\ volt$$

$$\therefore Vo(t) = 7.071\cos(10t - 60^o)\ volt$$

Calculate Io(t)



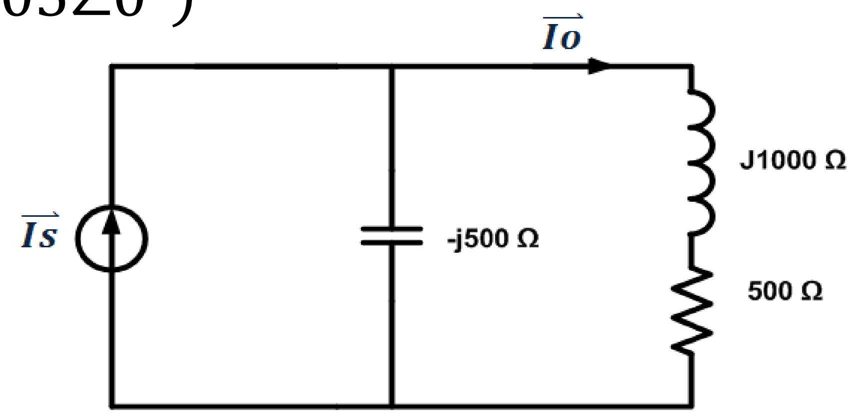
$$is(t) = 0.05\cos 2000t\ \ A$$

$$Z_C(j\omega) = -j\frac{1}{\omega C} = -j500\ \Omega$$

$$Z_L(j\omega) = j\omega L = j1000\ \Omega$$

$$\overrightarrow{Is} = 0.05\angle 0°\ A$$

$$\overrightarrow{Io} = \frac{-j500}{-j500 + 500 + j1000}\cdot(0.05\angle 0°)$$



$$\overrightarrow{Io} = 0.03535\angle -135°\ A$$

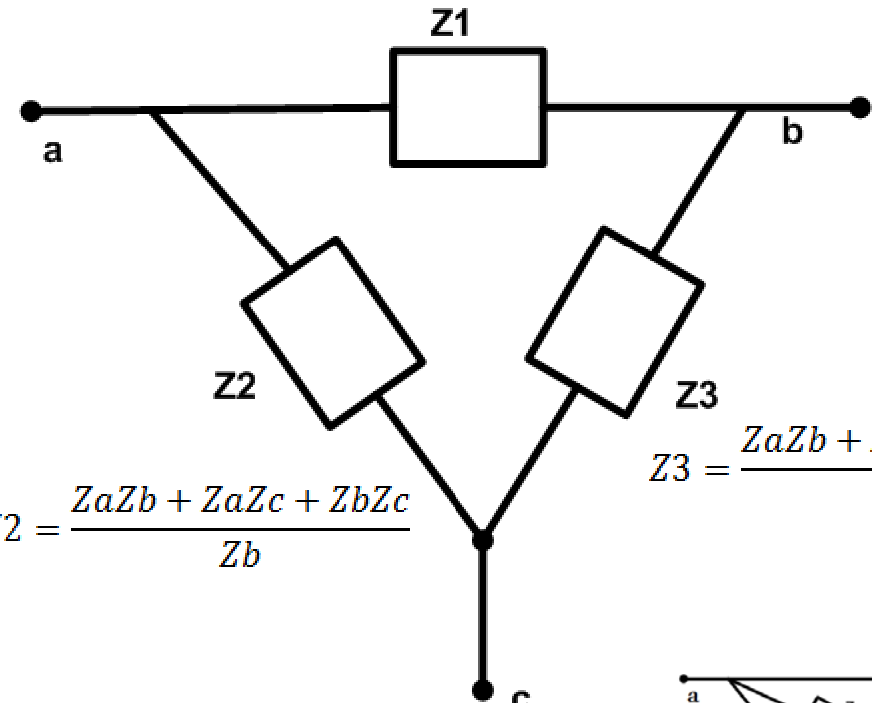
$$\therefore\ \ io(t) = 0.03535\cos(2000t - 135°)\ \ A$$

Delta – Wye & Wye to Delta transformation

$$Z1 = \frac{ZaZb + ZaZc + ZbZc}{Zc}$$

$$Z2 = \frac{ZaZb + ZaZc + ZbZc}{Zb}$$

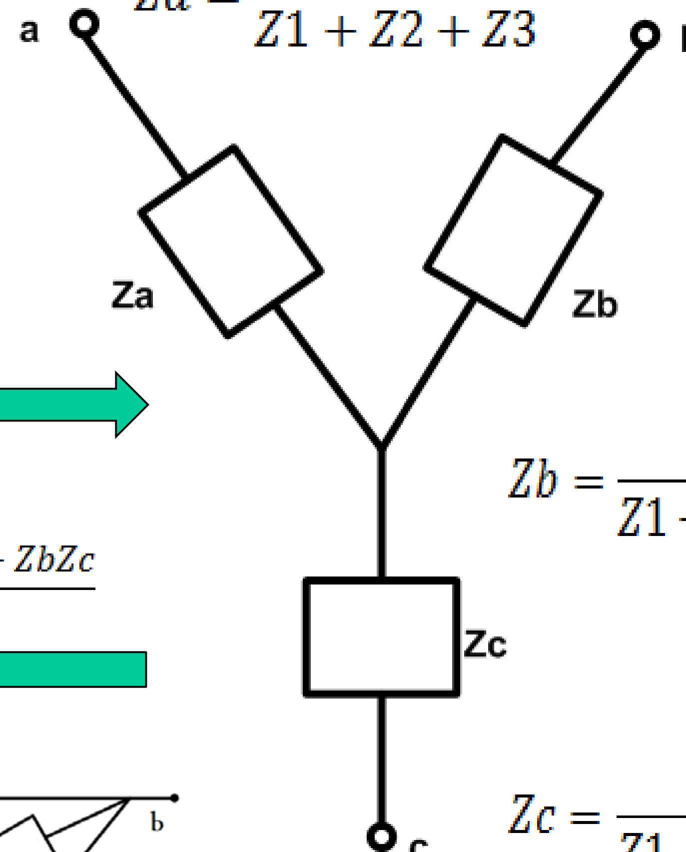
$$Z3 = \frac{ZaZb + ZaZc + ZbZc}{Za}$$

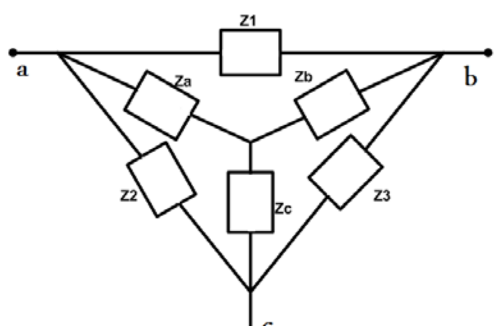


$$Za = \frac{Z1\,Z2}{Z1 + Z2 + Z3}$$

$$Zb = \frac{Z1\,Z3}{Z1 + Z2 + Z3}$$

$$Zc = \frac{Z3\,Z2}{Z1 + Z2 + Z3}$$

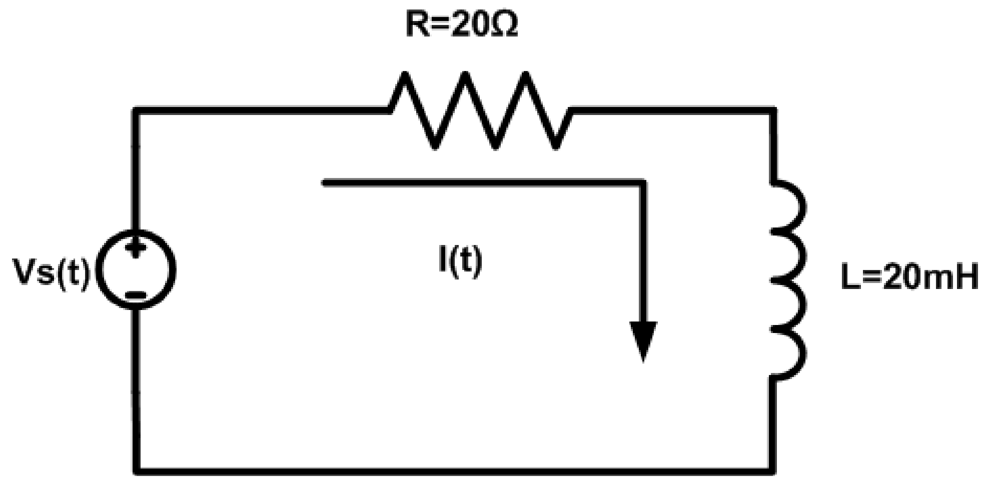




Series RL circuit

$$Vs(t) = 60\cos(200\pi\, t)\, volt$$

$$find\ i(t)$$



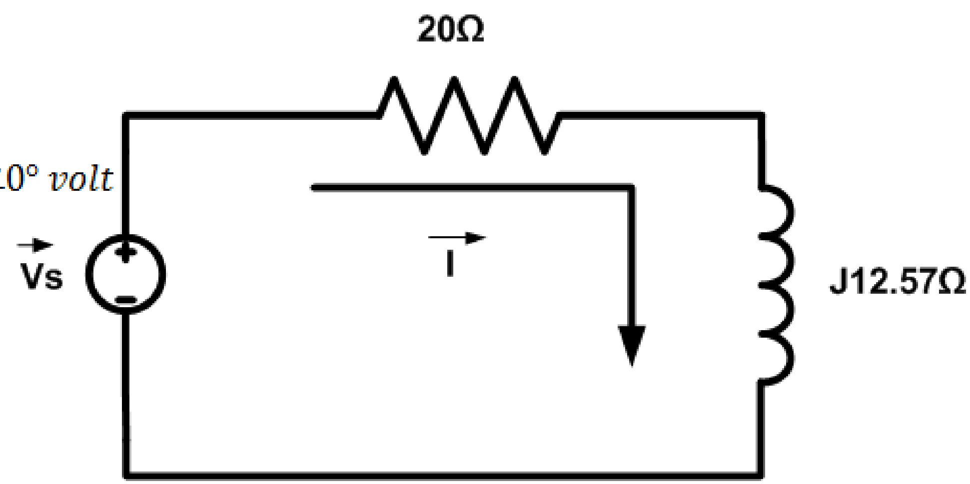
$$\vec{Vs} = \vec{VR} + \vec{VL}$$

$$60\angle 0^\circ = 20\vec{I} + j12.57\,\vec{I}$$

$$\vec{I} = \frac{60\angle 0^\circ}{20 + j12.57} = \frac{60\angle 0^\circ}{23.6\angle 32.1^\circ}$$

$$\therefore \vec{I} = 2.54\angle -32.1^\circ\, A$$

$$\vec{Vs} = 60\angle 0^\circ\ volt$$



$$\vec{VR} = 20\,\vec{I} = 50.8\angle -32.1\ volt$$

$$\vec{VL} = j\,12.57\,\vec{I} = 31.9\angle +57.9^\circ\ volt$$

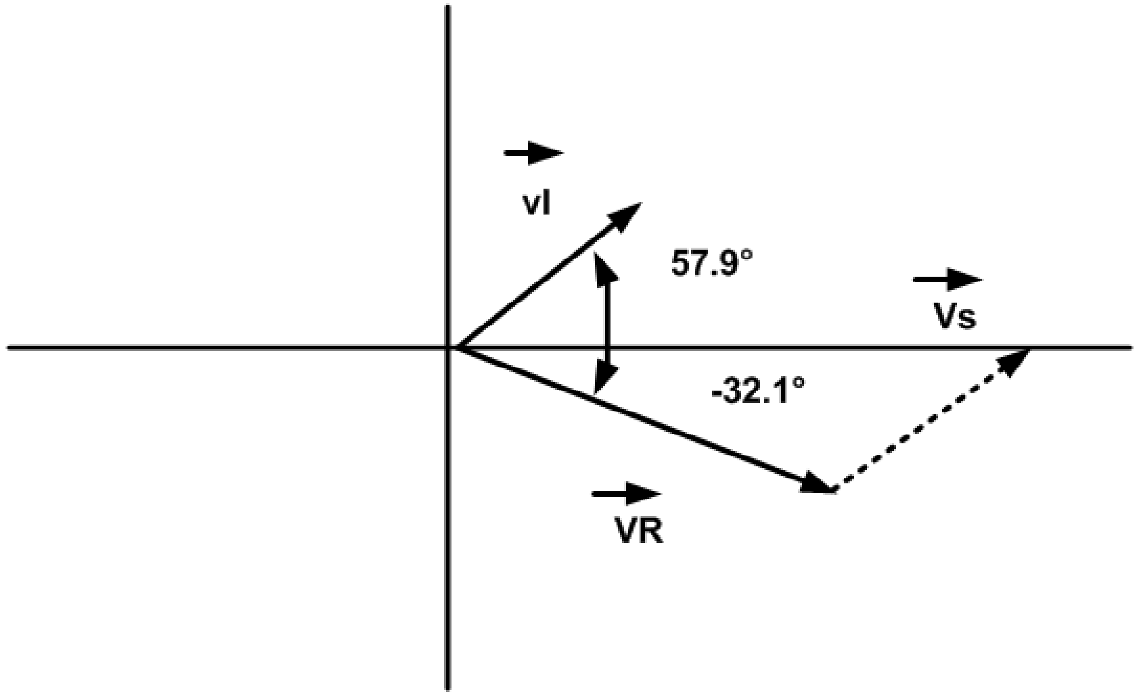
$$\vec{VL}\ \ leads\ \vec{VR}\ by\ 90^\circ$$

$$\vec{IL}\ lags\ \vec{Vs}\ by\ 32.1^\circ$$

$$Zeq = 20 + j12.57\,\Omega \qquad inductive$$

$$= 23.6\angle 32.1^\circ\,\Omega \qquad inductive$$

Phasor Diagram



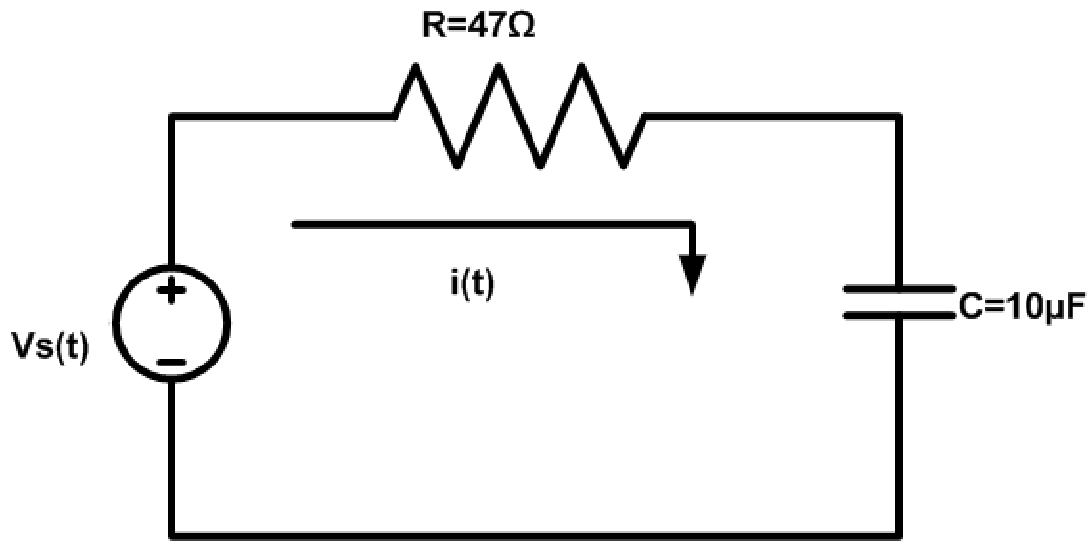
Series RC circuit

$$Vs(t) = 100\cos 600\pi t \ volt$$

$$ZR(jw) = 47\ \Omega$$

$$ZC(jw) = -j\frac{1}{wC} = -j\,53.1\ \Omega$$

$$\vec{Vs} = 100\angle 0^\circ \ volt$$



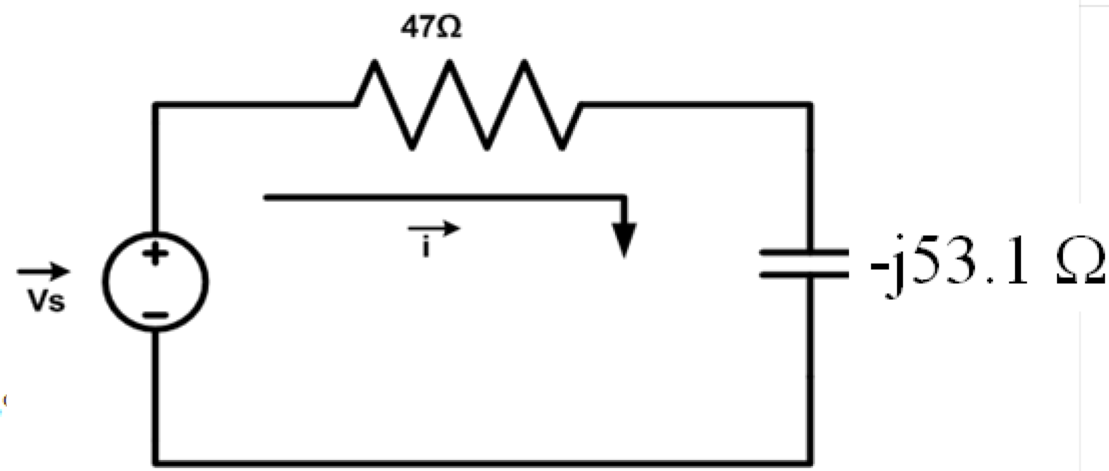
$KVL:$

$$\vec{Vs} = 47\ \vec{I} - j53.1\ \vec{I}$$

$$\vec{I} = \frac{\vec{Vs}}{47 - j53.1} = \frac{100\angle 0^\circ}{47 - j53.1}$$

$$\vec{I} = \frac{100\angle 0^\circ}{70.9\angle -48.5^\circ} \Rightarrow \vec{I} = 1.41\ \angle 48.5^\circ$$

$\vec{I}$ leads $\vec{Vs}$ by 48.5°



$\rightarrow$ *capacitive circuit*

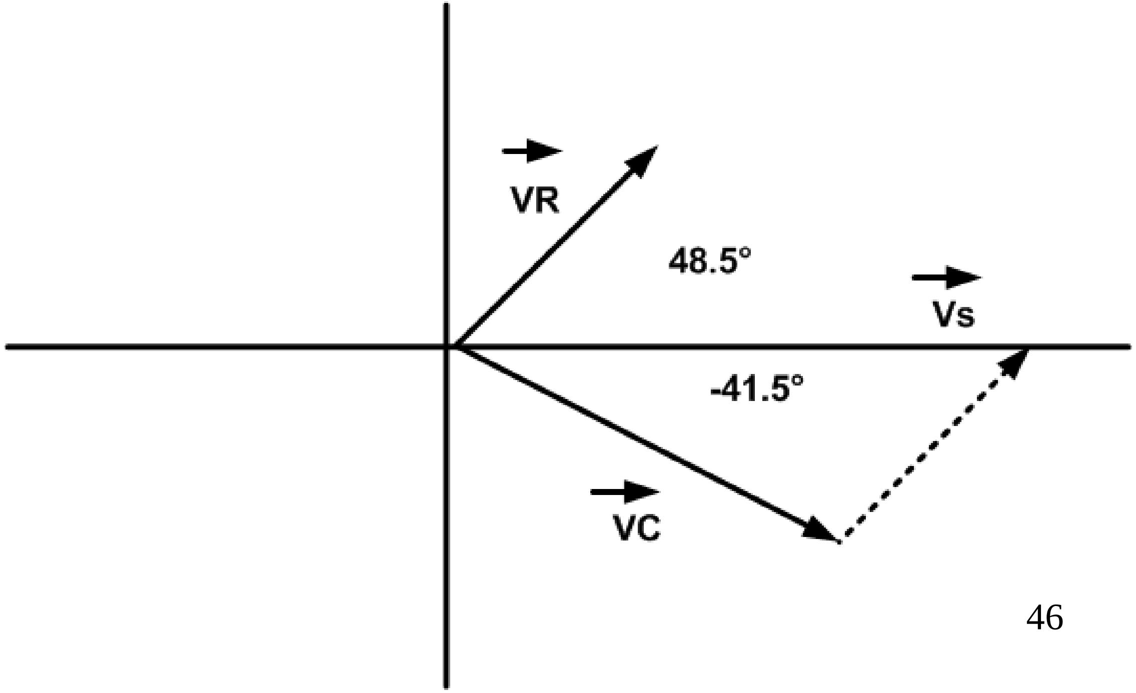
$Z(jw) = 47 - j53.1\ \Omega$ *capacitive*

$Z(jw) = 70.9\angle -48.5^{\circ}\ \Omega$ *capacitive*

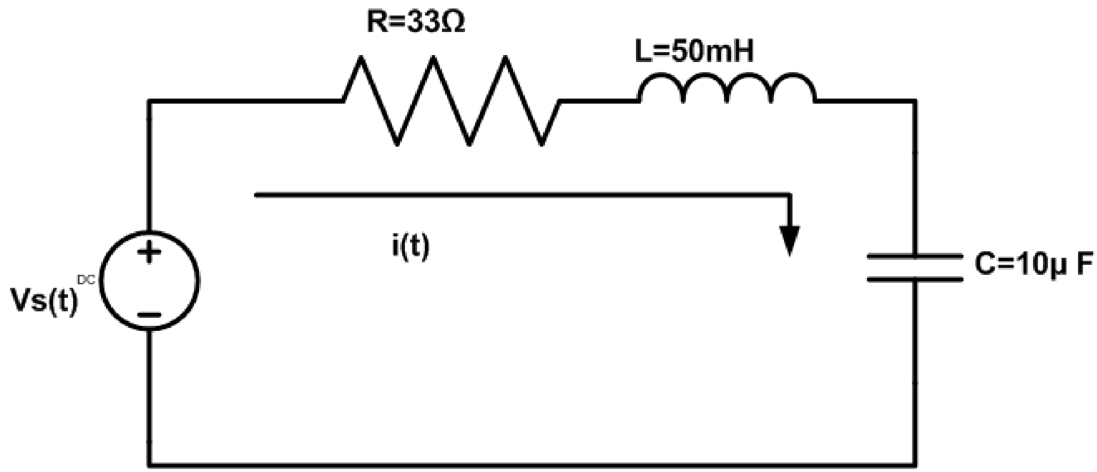
$\overrightarrow{VR} = 47\ \vec{I} = 66.3\ \angle 48.5^{\circ}\ volt$

$\overrightarrow{VC} = -J53.1\ \vec{I} = 74.9\angle -41.5^{\circ}$

$\overrightarrow{VC}$ *lags* $\vec{I}$ *by* 90°.



Series RLC circuit



$$Vs(t) = 75\cos 400\pi t \;\; volt$$

$$KVL: \overrightarrow{Vs} = \overrightarrow{VR} + \overrightarrow{VL} + \overrightarrow{VC}$$

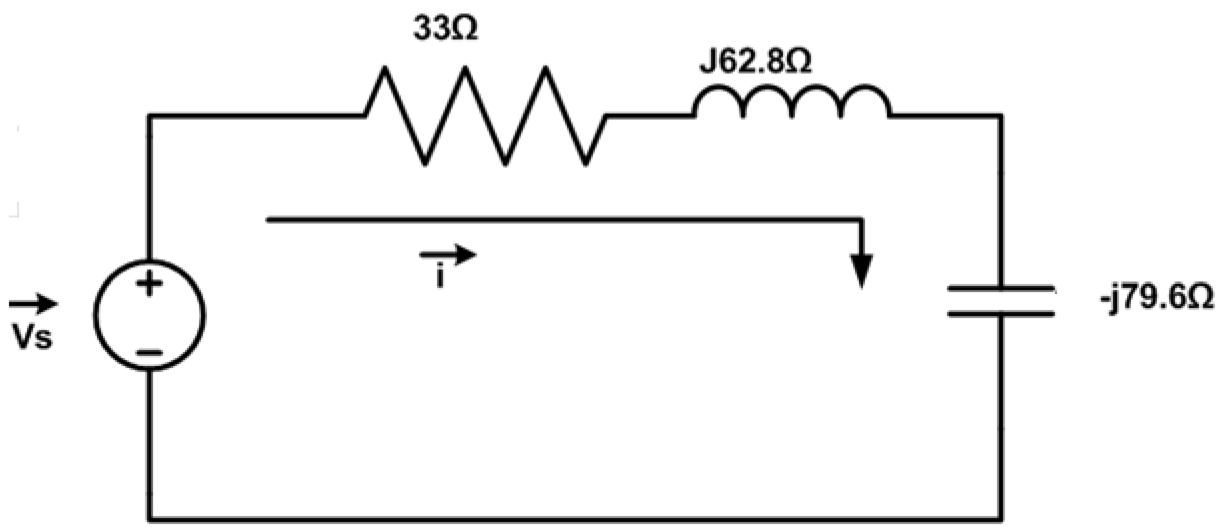
$$\overrightarrow{Vs} = 33\vec{I} + j62.8\vec{I} - j79.6\vec{I}$$

$$\vec{I} = \frac{\overrightarrow{Vs}}{33 - j16.8} = \frac{75\angle 0^\circ}{37\angle -27^\circ}$$

$$\vec{I} = 2.03\angle 27^\circ A$$

$\vec{I}$ *leads* $\overrightarrow{Vs}$ *by* 27°

$\therefore$ *capacitive circuit*



$\therefore$ *capacitive circuit*

$$Zeq = R + jwL - j\frac{1}{wC}$$

$$Zeq = 33 + j62.8 - j79.6$$

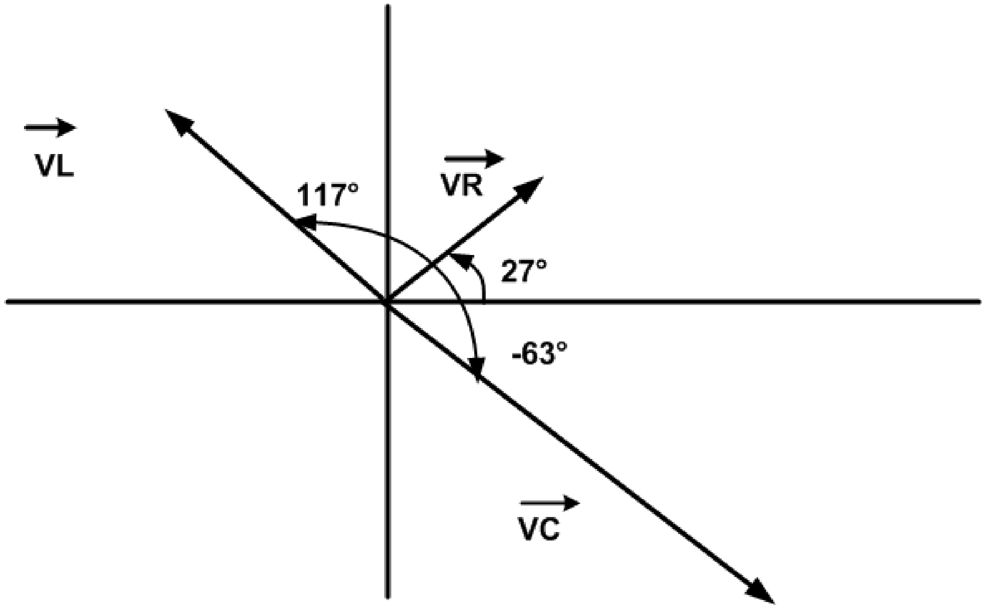
$$Zeq = 33 - j16.8\ \Omega \qquad capacitive$$

$$Zeq = 37\angle -27\ \Omega \qquad capacitive$$

$$\overrightarrow{VR} = R\vec{I} = 67\angle 27°\ volt$$

$$\overrightarrow{VL} = jwL\,\vec{I} = 127\angle 117°\ volt$$

$$\overrightarrow{VC} = -j\,\frac{1}{wC}\vec{I} = 162\angle -63\ \ volt$$



$$if\ jwL - j\frac{1}{wC} = 0$$

$$w = \frac{1}{\sqrt{LC}} \qquad Resonant\ frequency$$

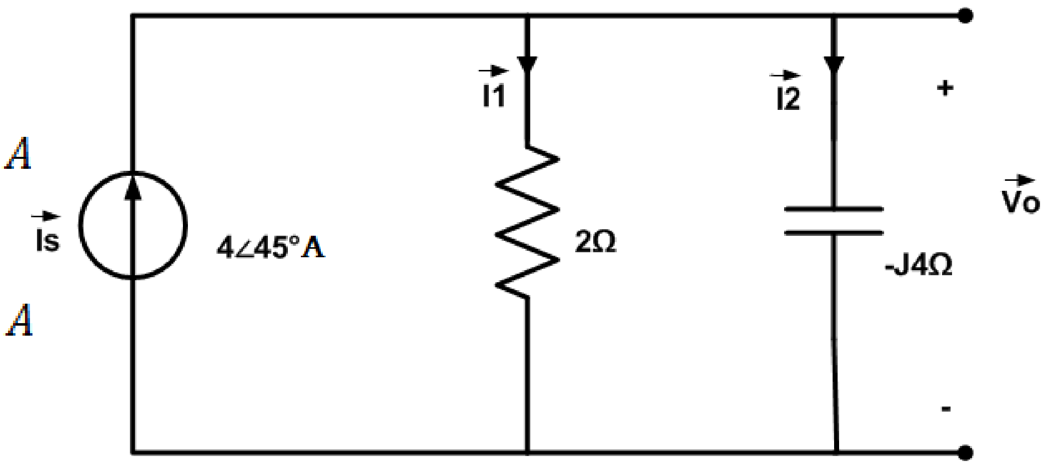
$$Zeq = R \qquad Resistive$$

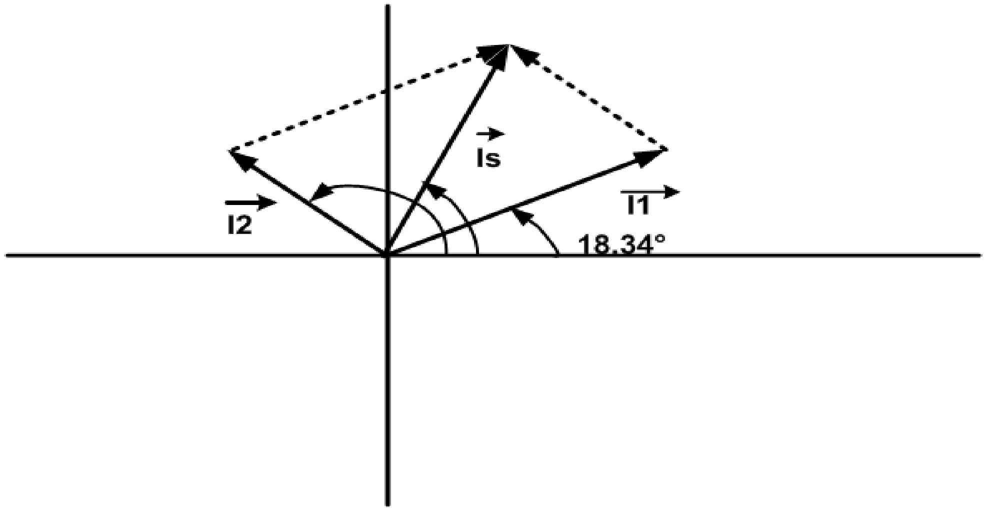
Find I1, I2, and Vo

$$\overrightarrow{I1} = \frac{-j4}{-j4+2} \cdot \overrightarrow{Is} = 3.578\ \angle 18.435°\ A$$

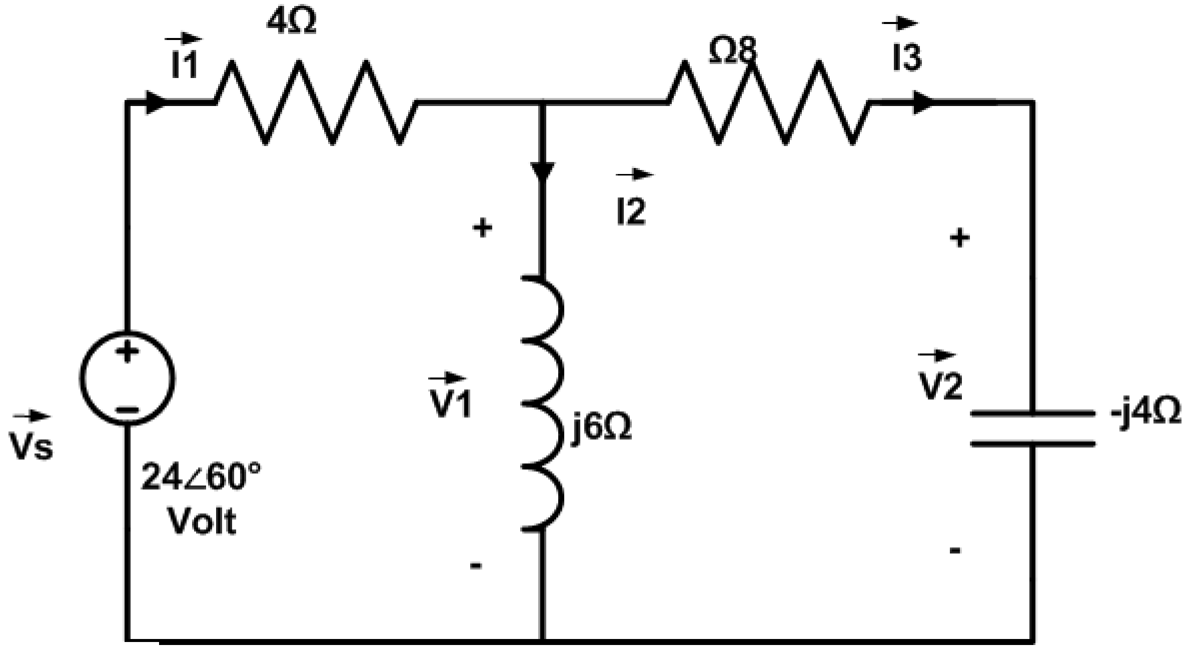
$$\overrightarrow{I2} = \frac{2}{-j4+2} \cdot \overrightarrow{Is} = 1.789\ \angle 108.435°\ A$$

$$\overrightarrow{Vo} = 2\ \overrightarrow{I1} = 7.156\ \angle 18.435°\ volt$$





Calculate all voltages and currents



$$Zeq = 4 + \left(j6\,||\,(8 - j4)\right)$$

$$Zeq = 9.604\angle 30.964°\ \Omega$$

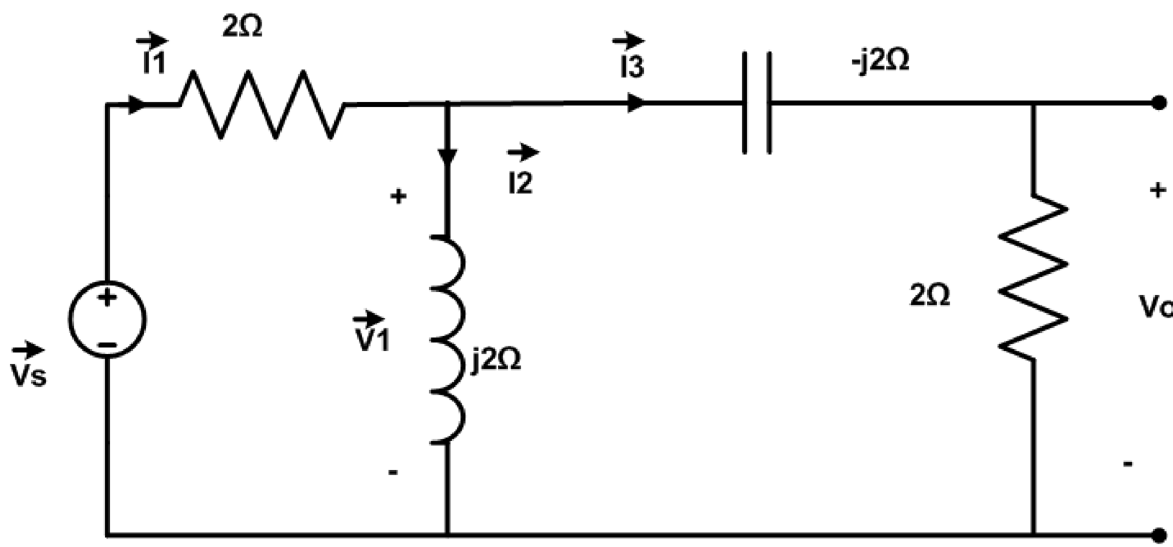
$$\overrightarrow{I1} = \frac{\overrightarrow{Vs}}{Zeq} = \frac{24\angle 60°}{9.604\angle 30.964°} = 2.498\angle 29.036°\ A$$

$$\overrightarrow{I3} = \frac{j6}{j6 + 8 - j4} \cdot \overrightarrow{I1} = 1.82\angle 105°\ A$$

$$\overrightarrow{I2} = \frac{8 - j4}{j6 + 8 - j4} \cdot \overrightarrow{I1} = 2.71\angle -11.58°\ A$$

$$\overrightarrow{V1} = j6\,\overrightarrow{I2} = 16.26\angle 78.42°\ volt$$

$$\overrightarrow{V2} = -j4\,\overrightarrow{I3} = 7.28\angle 15°\ volt$$



$$if\ \overrightarrow{Vo} = 8\angle 45^{\circ}\ volt\ ,\qquad find\ \overrightarrow{Vs}$$

$$\overrightarrow{I3} = \frac{\overrightarrow{Vo}}{2} = 4\angle 45^{\circ}\ A$$

$$\overrightarrow{V1} = (2 - j2)\overrightarrow{I3} = 11.314\ \angle 0^{\circ}$$

$$\overrightarrow{I2} = \frac{\overrightarrow{V1}}{j2} = 5.657\angle - 90^{\circ}\ A$$

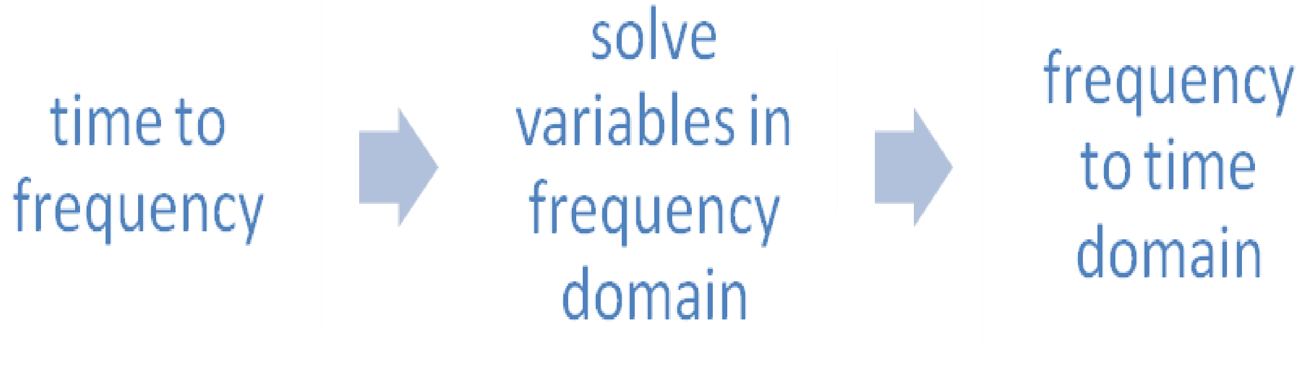
$$\overrightarrow{I1} = \overrightarrow{I2} + \overrightarrow{I3} = (2.828 - j2.829)A$$

$$\overrightarrow{Vs} = 2\overrightarrow{I1} + \overrightarrow{V1}$$

$$\overrightarrow{Vs} = 17.888\angle - 18.439^{\circ}\ volt$$

Steps to analyze Ac circuits

- Transform the circuit to the phasor or frequency domain.
- Solve the problem using circuit techniques (Nodal analysis , Mesh analysis , Superposition , … etc).
- Transform the resulting phasor to the time domain.



Nodal Analysis

$find\ \vec{Io}\ using\ Nodal\ analysis$

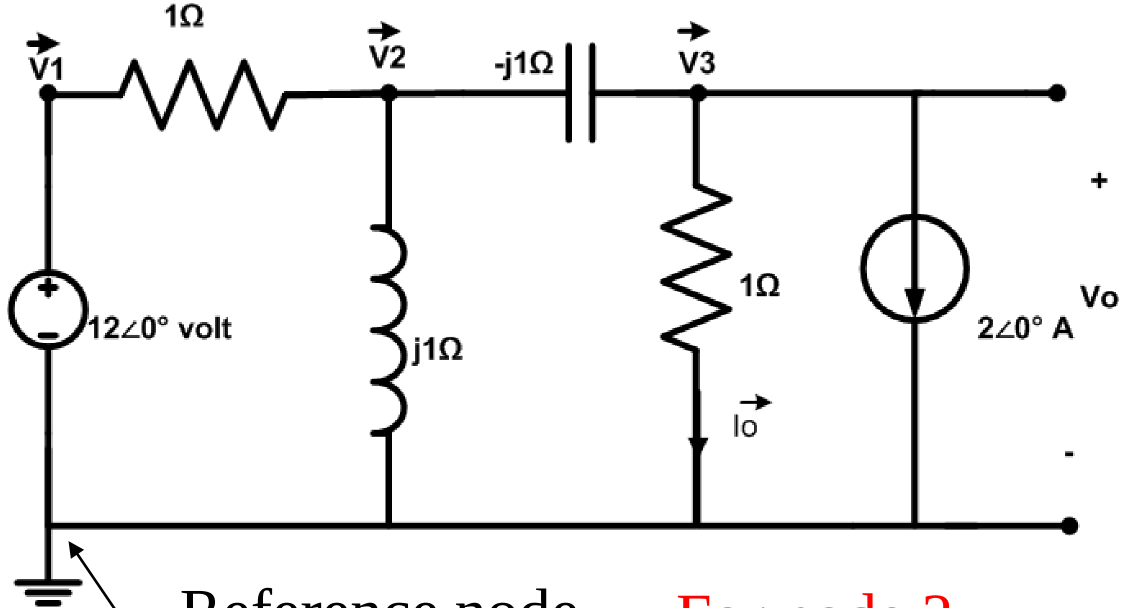
$$\vec{Io} = \frac{\vec{V3}}{1}$$

$\vec{V1} = 12\angle 0°$ $constrain\ equation$

$KCL\ at\ Node\ 2:$

$$\frac{\vec{V2}-\vec{V1}}{1} + \frac{\vec{V2}}{j1} + \frac{\vec{V2}-\vec{V3}}{-j1} = 0$$

$$-\vec{V1} + \vec{V2} - j\ \vec{V3} = 0$$



Reference node

For node 2

Self admittance = Y22 $= (1 + \frac{1}{j1} + \frac{1}{-j1}) = 1$

Mutual admittance Y23 = $-\frac{1}{-j1} = -j$

Mutual admittance Y21 = - 1

$KCL\ at\ node\ 3:$

$$-2\angle 0° = -\frac{1}{-j1}\vec{V2} + \left(\frac{1}{-j1} + 1\right)\vec{V3}$$

$$-2\angle 0° = -\frac{1}{-j1}\vec{V2} + (j1+1)\vec{V3}$$

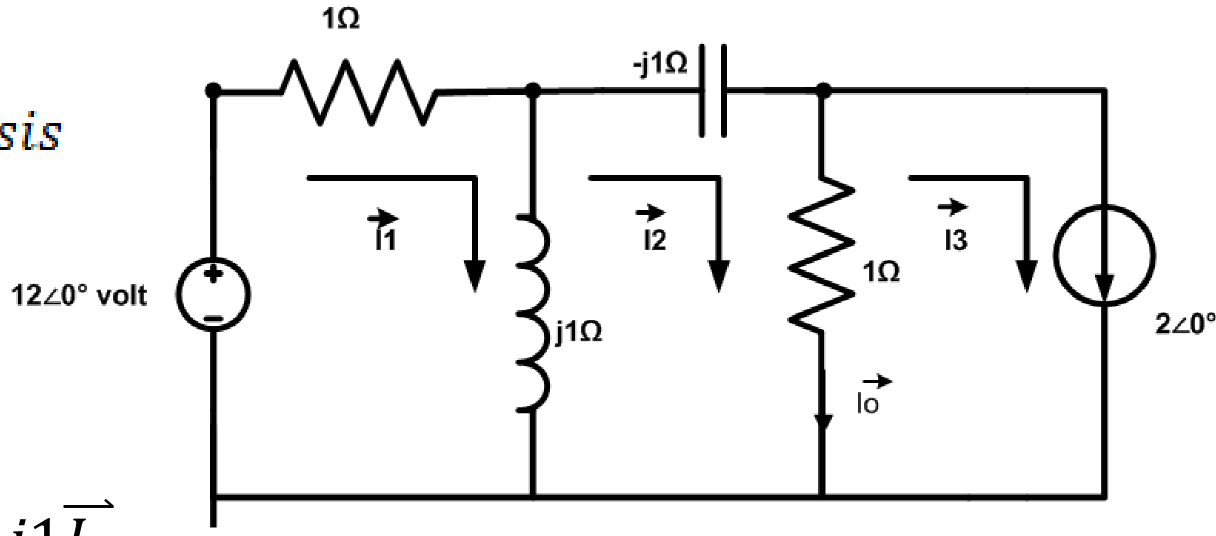
$solving\ for\ \overrightarrow{V3}$;

$$\overrightarrow{V3} = \left(\frac{8}{5} + j\,\frac{26}{5}\right) V$$

$$\therefore \overrightarrow{Io} = \frac{\overrightarrow{V3}}{1} = \left(\frac{8}{5} + j\,\frac{26}{5}\right) A$$

Mesh analysis :

find $\overrightarrow{Io}$ *using Mesh analysis*



$\overrightarrow{Io} = \overrightarrow{I2} - \overrightarrow{I3}$

$\overrightarrow{I3} = 2\angle 0^\circ\ A$ *Constrain equation*

KVL for mesh 1:

$$12 < 0^o = (1 + j1)\overrightarrow{I_1} - j1\overrightarrow{I_2}$$

KVL for mesh 2 :

$$0 = -j1\,\overrightarrow{I1} + (1 + j1 - j1)\overrightarrow{I2} - \overrightarrow{I3}$$

$$0 = -j1\,\overrightarrow{I1} + \overrightarrow{I2} - \overrightarrow{I3}$$

For mesh 1

Self impedance Z11 = + (1+j1)

Mutual impedance Z12 = - j1

Mutual impedance Z13 = - 0

solving for $\overrightarrow{I2}$ *and* $\overrightarrow{I3}$

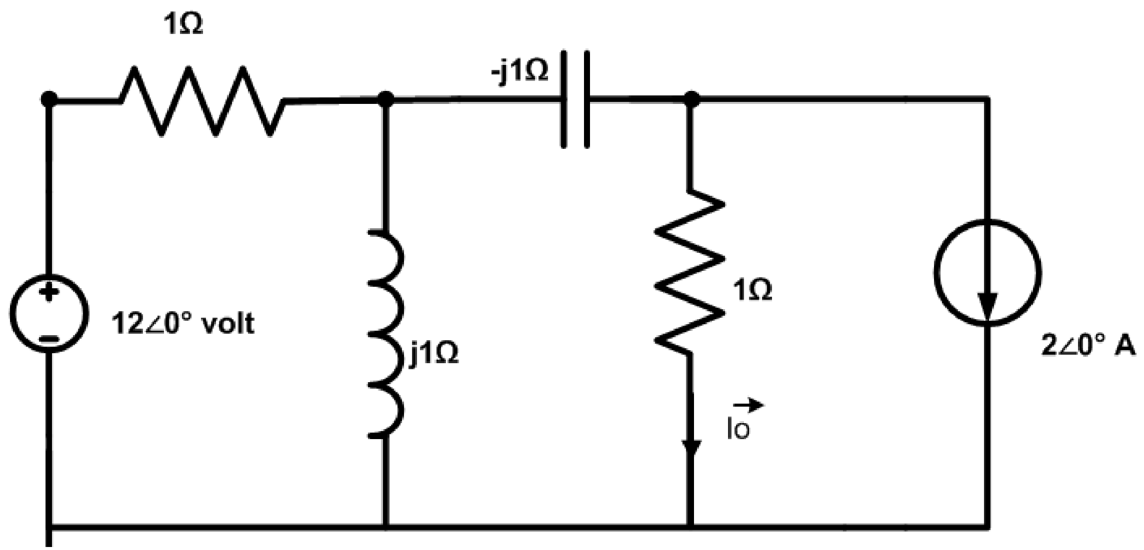
$$\overrightarrow{I2} = \left(\frac{18}{5} + j\frac{26}{5}\right) A$$

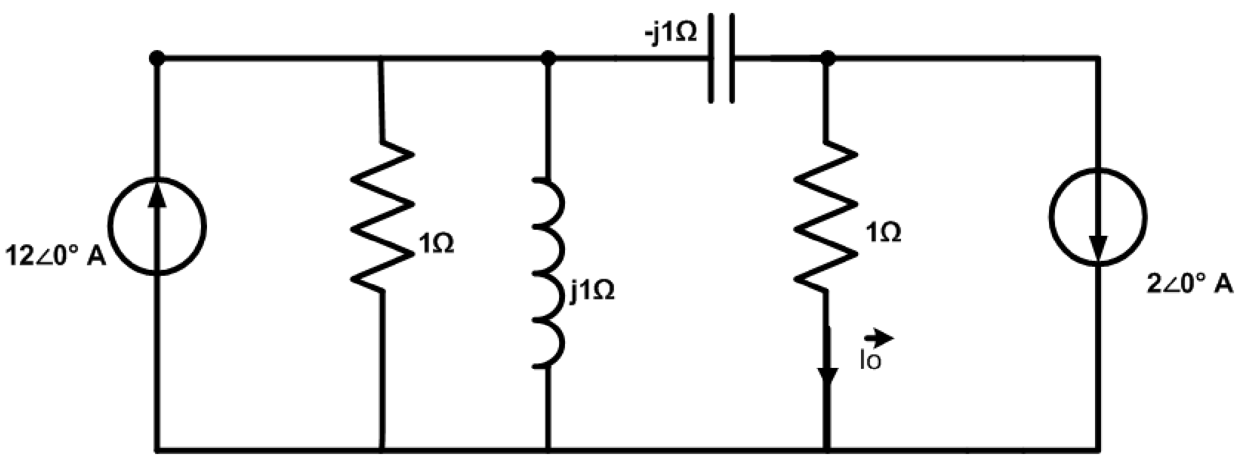
$$\overrightarrow{I3} = 2\angle 0^\circ\ A$$

$$\therefore\ \overrightarrow{Io} = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

Source Transformation

$find\ \overrightarrow{Io}\ using\ source\ transformation$

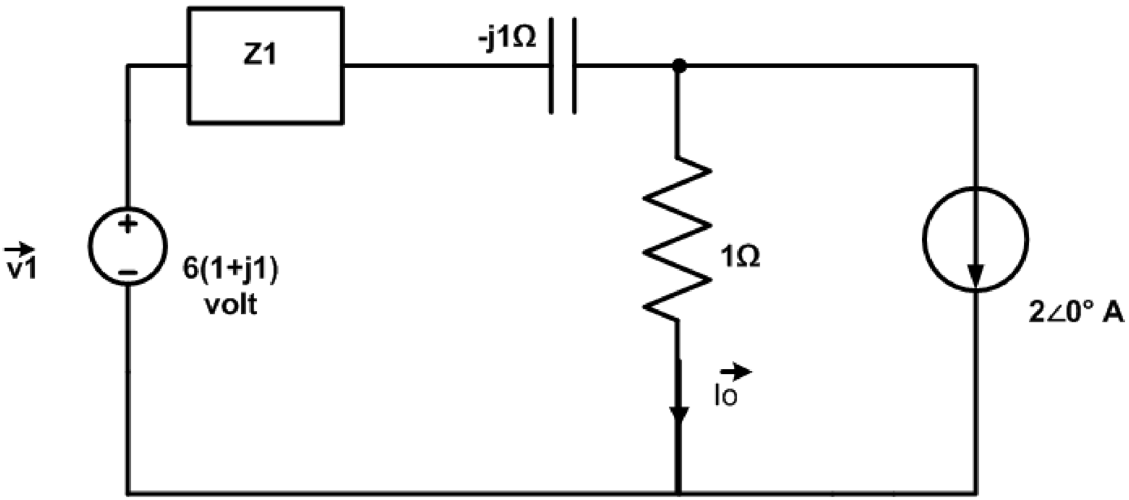


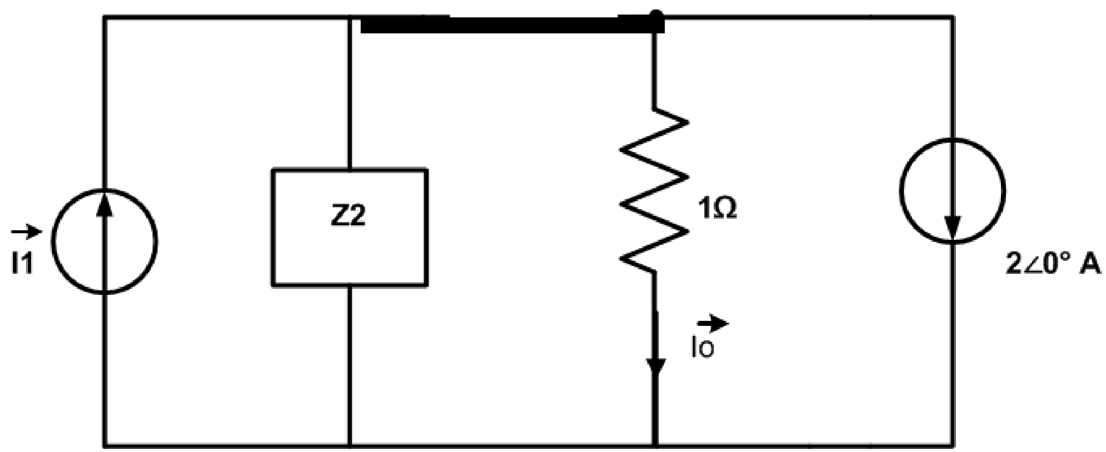


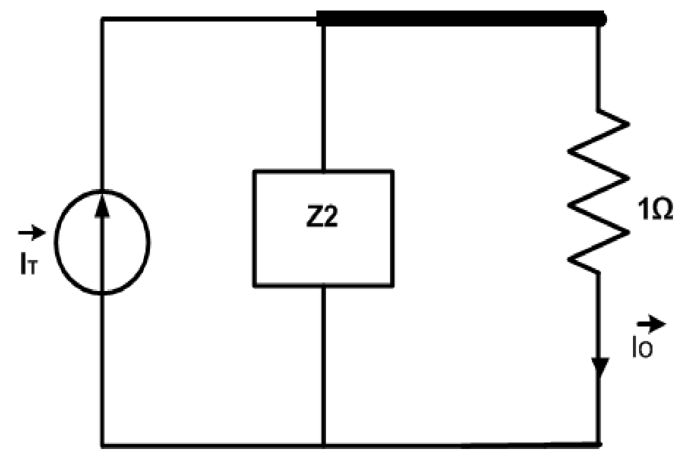
$$Z1 = 1\,||(j1) = \left(\frac{1}{2} + j\frac{1}{2}\right)\,\Omega$$

$$Z1 = 1\,||\,(j1) = \left(\frac{1}{2} + j\frac{1}{2}\right)\ \Omega$$

$$V1 = 12\angle 0^\circ . Z1 = 6\ (1 + j1)\ volt$$







$$\overrightarrow{I_T} = \overrightarrow{I1} - 2\angle 0^\circ$$

$$\overrightarrow{I_T} = \left(\frac{10 + j4}{1 - j1}\right)\ A$$

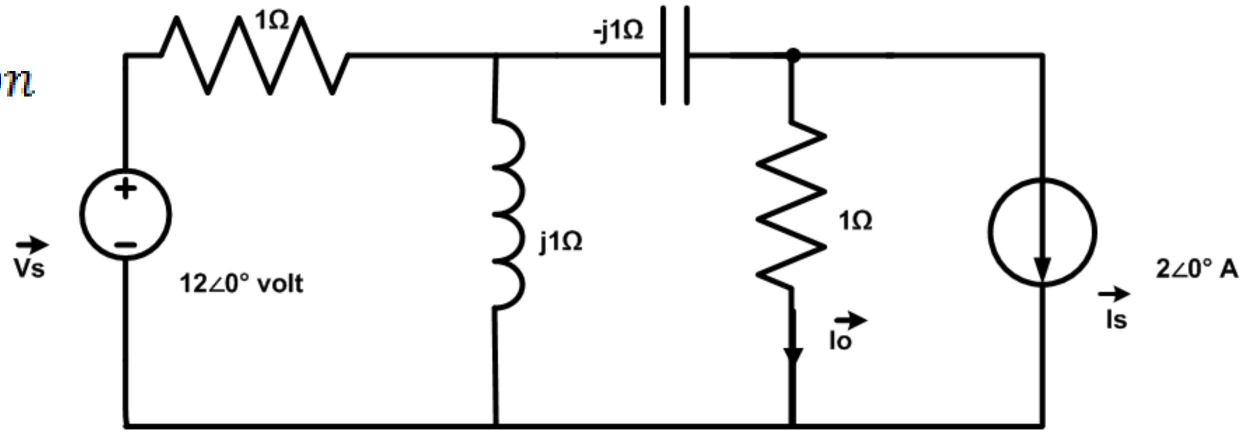
$$\overrightarrow{Io} = \frac{Z2}{Z2 + 1} \cdot \overrightarrow{I_T} = \left(\frac{8}{5} + j\frac{26}{5}\right)\ A$$

$$\overrightarrow{I1} = \frac{\overrightarrow{V1}}{Z2} = \frac{12\ (1 + j1)}{1 - j1}$$

$$Z2 = -j1 + Z1 = \left(\frac{1}{2} - j\frac{1}{2}\right)\ \Omega$$

Superposition

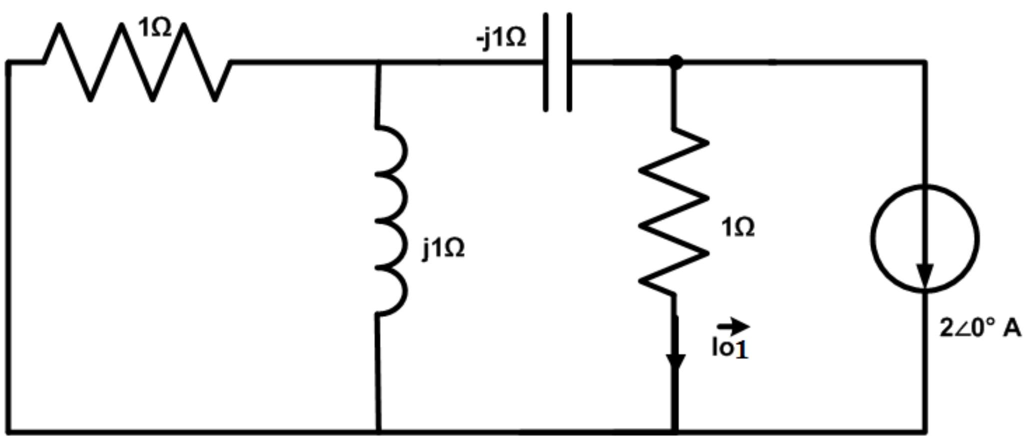
$find\ \overrightarrow{Io}\ using\ super\ position$



.

$\overrightarrow{Io} = \overrightarrow{Io1} + \overrightarrow{Io2}$

1) $let\ \overrightarrow{Vs}\ off\ , and\ \overrightarrow{Is}\ on\ .$



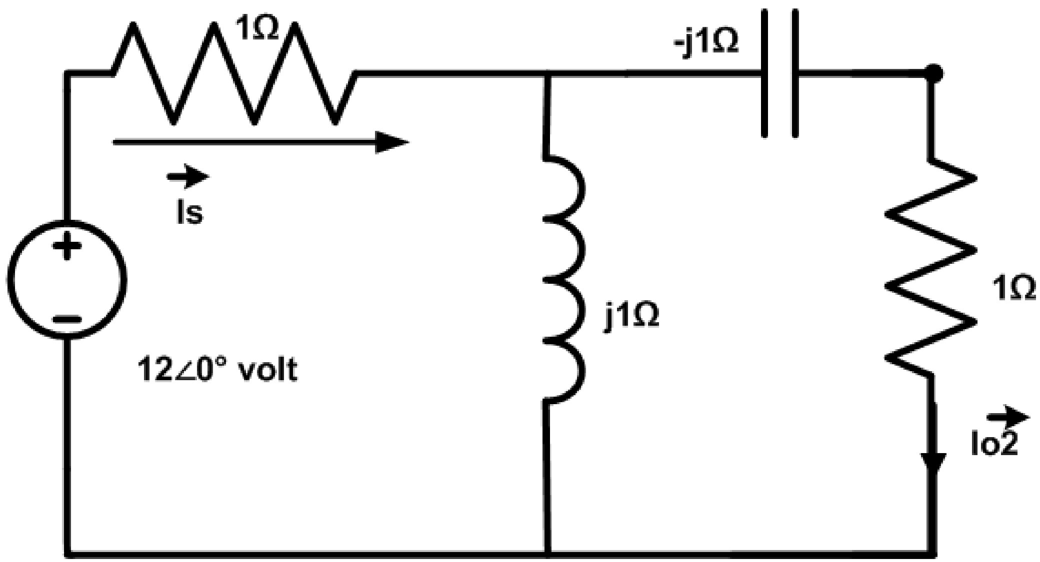
$$\overrightarrow{Io1} = -2\angle 0^\circ \cdot \frac{Z1}{Z1+1}$$

$$Z1 = -j1 + (1\,||\,j1) = -j1 + \frac{j1}{1+j1}$$

$$\overrightarrow{Io1} = \frac{-2}{2+j1}\ A.$$

2) $let\ \overrightarrow{Is}\ off\ , and\ \overrightarrow{Vs}\ on\ .$

$$\overrightarrow{Is} = \frac{12\angle 0^\circ}{Zeq}$$



$$Zeq = 1 + \left(j1 \,\|\, (1 - j1)\right) = 2 + j1\ \Omega$$

$$\therefore \overrightarrow{Is} = \frac{12\angle 0^\circ}{2 + j1}\ A$$

$$\overrightarrow{Io2} = \overrightarrow{Is} \cdot \frac{j1}{j1 + 1 - j1}$$

$$\overrightarrow{Io2} = \overrightarrow{Is} \cdot j1 = \frac{12}{1 - j2}\ A.$$

$$\therefore \overrightarrow{Io} = \overrightarrow{Io1} + \overrightarrow{Io2}$$

$$\overrightarrow{Io} = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

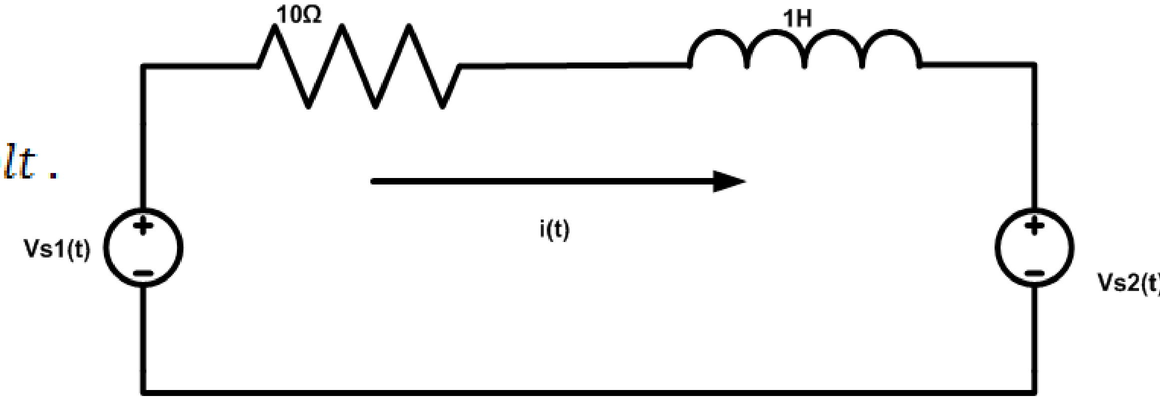
Power of super position :

$Vs1(t) = 100\cos 10t \ \ volt$

$Vs2(t) = 50\cos(20t - 10^\circ)\ volt\ .$

$note\ that\ w1 = 10\ {}^{rad}/_{sec}$

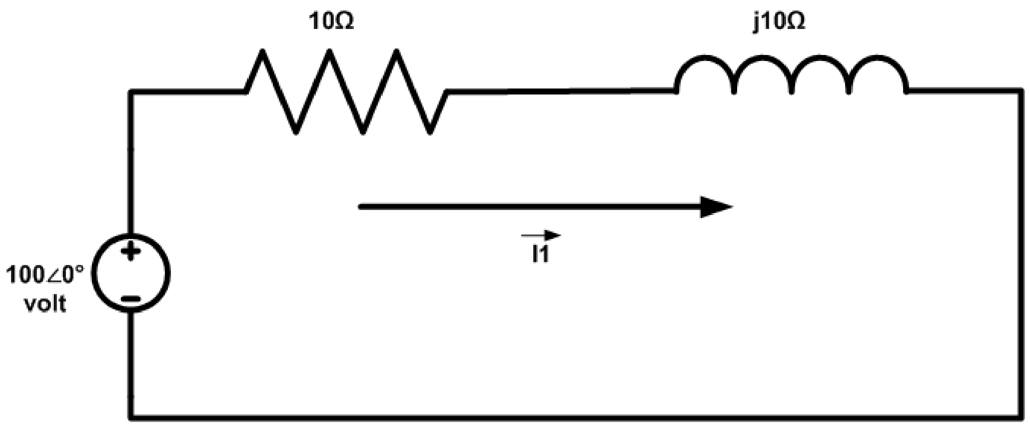
$and\ w2 = 20\ {}^{rad}/_{sec}$



$\therefore$ *super position is the only method of analysis .*

$i(t) = i1(t) + i2(t).$

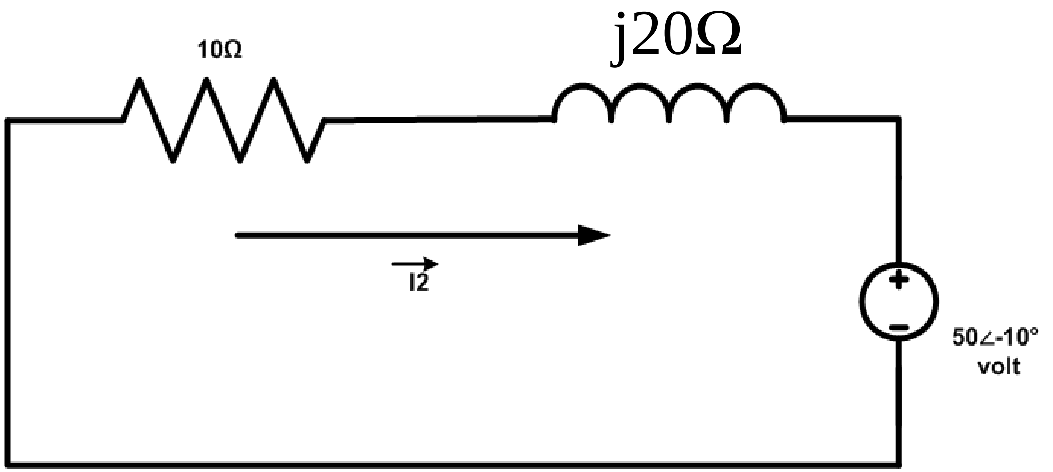
1) $let\ Vs2(t) off\ , and\ Vs1(t) on\ .$



$$\overrightarrow{I1} = \frac{100\angle 0^\circ}{10 + j10} = 7.07\angle - 45^\circ\ A$$

$\therefore i1(t) = 7.07\cos(10t - 45^\circ)\ A.$

2) $let\ Vs1(t) off, and\ Vs2(t) on.$



$$\vec{I2} = \frac{-50\angle-10^\circ}{10+j20} = \frac{50\angle170^\circ}{10+j20}$$

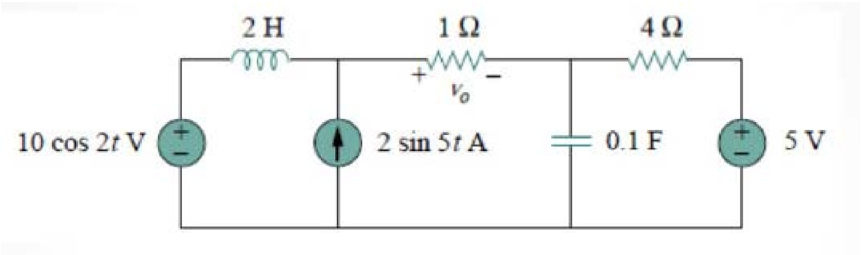
$$\vec{I2} = 2.24\angle106.57^\circ\ A$$

$$\therefore i2(t) = 2.24\cos(20t + 106.57^\circ)\ A.$$

$$\therefore i(t) = i1(t) + i2(t)$$

$$i(t) = 7.07\cos(10t - 45^\circ) + 2.24\cos(20t + 106.57^\circ)\ A$$

Superposition (dc+ac) sources



Find Vo(t) using the superposition theorem

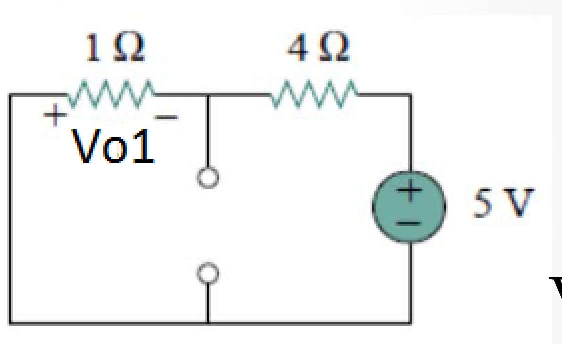
The steady state Vo(t)

Solution:

- The circuit operates at three different frequencies

$$Vo(t) = Vo1(t) + Vo2(t) + Vo3(t)$$

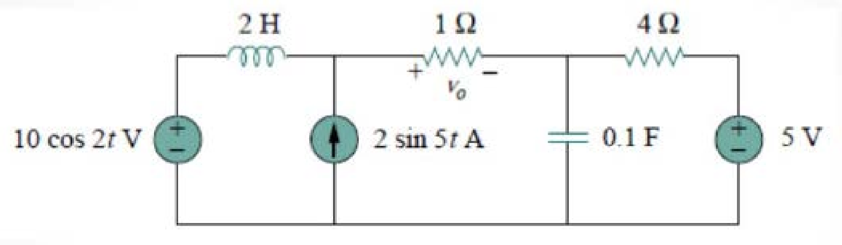
1. To find Vo1(t) (due to dc source)



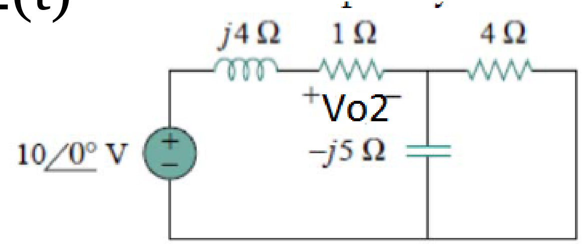
$$Vo1(t) = -1V$$

Inductor becomes short
Capacitor becomes open
The sinusoidal sources are off

Superposition (dc+ac) sources



2. To find Vo2(t)



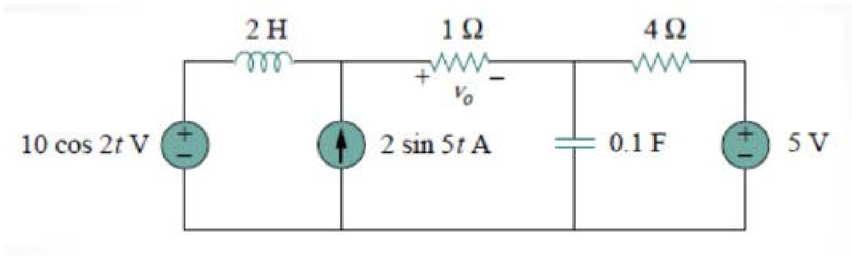
$10\cos 2t \rightarrow 10\angle 0^\circ, \quad \omega = 2\text{ rad/s}$

$2\text{ H} \rightarrow j\omega L = j4\ \Omega$

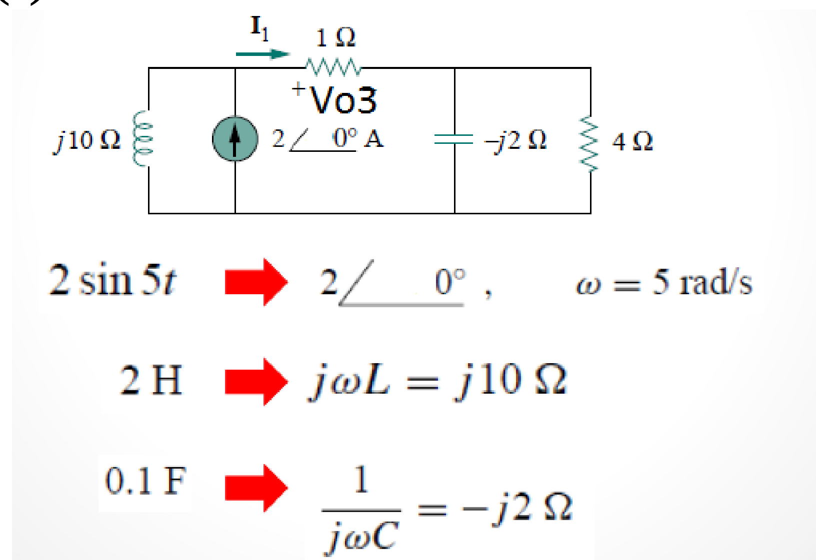
$0.1\text{ F} \rightarrow \dfrac{1}{j\omega C} = -j5\ \Omega$

Vo2(t) = 2.498 cos (2t – 30.79) V

Superposition (dc+ac) sources



3 . To find Vo3(t)

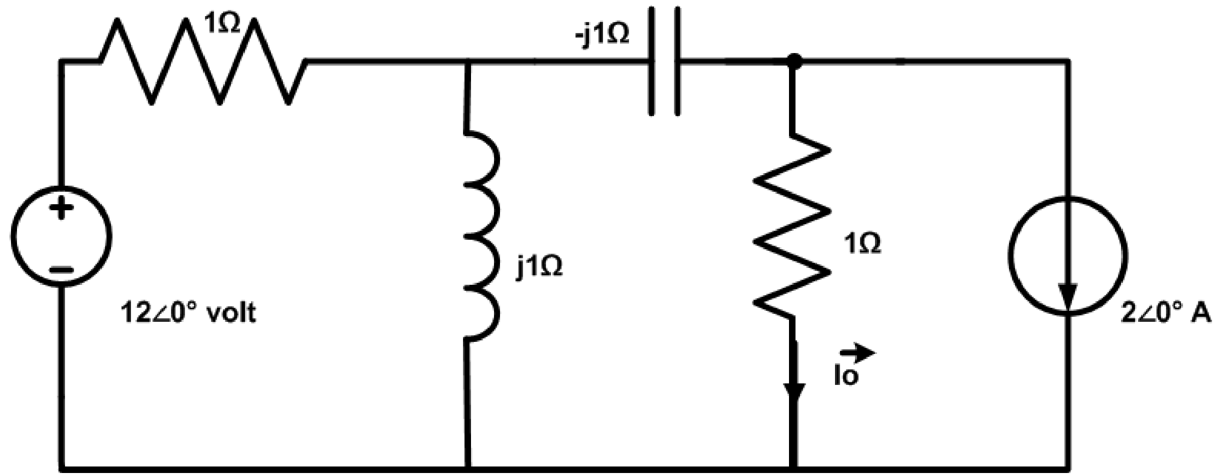


Vo3(t) = 2.33 sin(5t+10) V

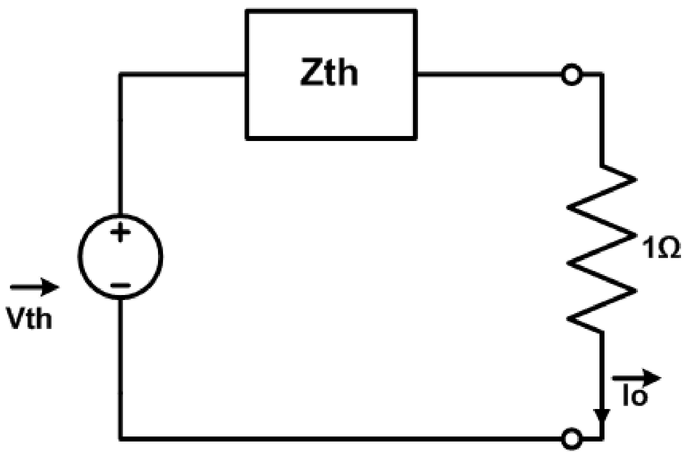
Vo(t) = -1+ 2.498 cos(2t-30.79)+2.33 sin(5t+10)

Thevenin's and Norton's theorems

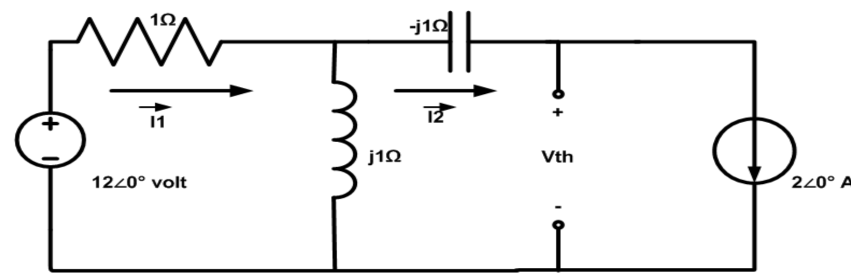
$find\ \overrightarrow{Io}\ using\ thevenin's\ theorem$



$$\overrightarrow{Io} = \frac{V_{TH}}{Z_{TH} + 1\ \Omega}$$



$to\ find\ \overrightarrow{V_{TH}}$



$$\overrightarrow{V_{TH}} = -(-j1)\overrightarrow{I2} + j1\left(\overrightarrow{I1} - \overrightarrow{I2}\right)$$

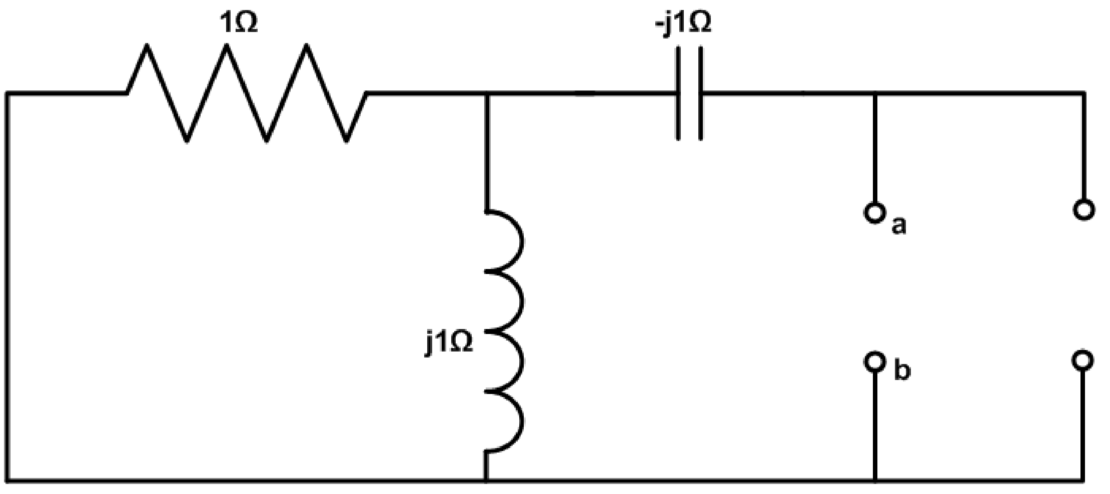
$$\overrightarrow{I2} = 2\angle 0^\circ \quad Cnstrain\ equation$$

$KVL\ \ for\ mesh\ 1:$

$$12\angle 0^\circ = (1 + 1j)\overrightarrow{I1} - j1\,\overrightarrow{I2}$$

$$\therefore \overrightarrow{I1} = \left(\frac{12 + j2}{1 + j1}\right) A \qquad \therefore \overrightarrow{V_{TH}} = \left(\frac{-2 + j12}{1 + j1}\right) volt$$

to find Z_{TH}, set all the independent source to zero.

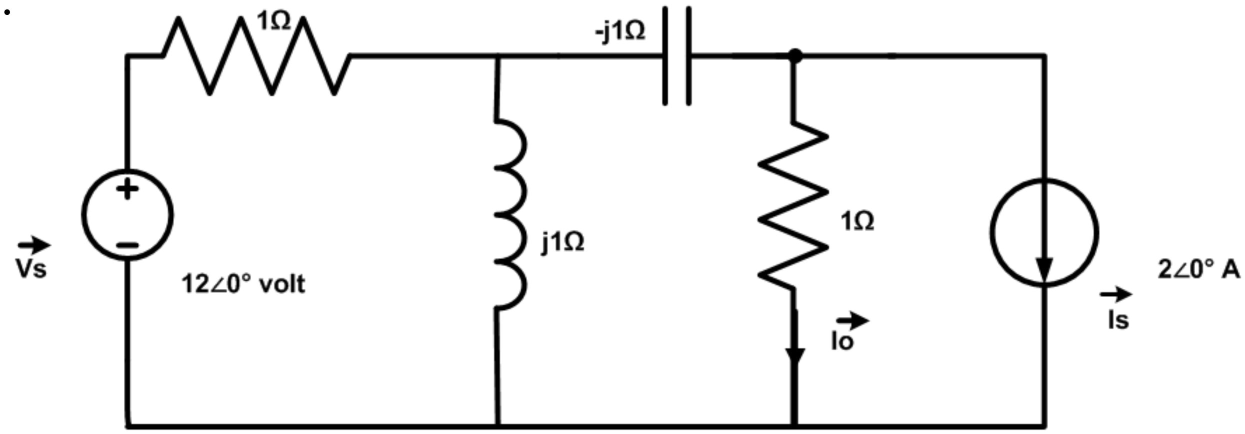


$$Z_{TH} = -j1 + (1||j1)$$

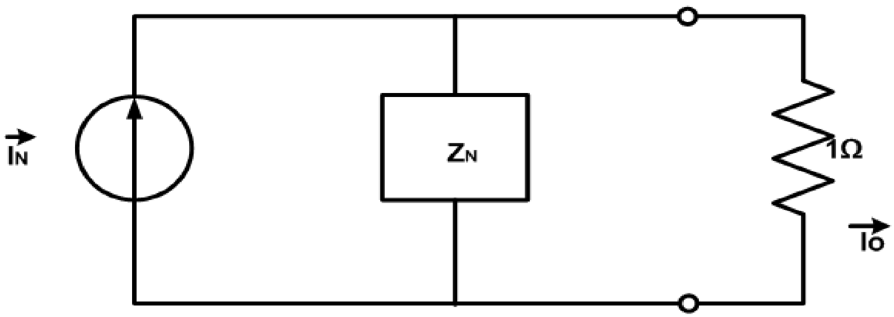
$$Z_{TH} = \left(\frac{1}{2} - j\frac{1}{2}\right)\ \Omega$$

$$\overrightarrow{Io} = \frac{\overrightarrow{V_{TH}}}{Z_{TH} + 1\ \Omega} \quad \Rightarrow \quad Io = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

Norton's Theorem :

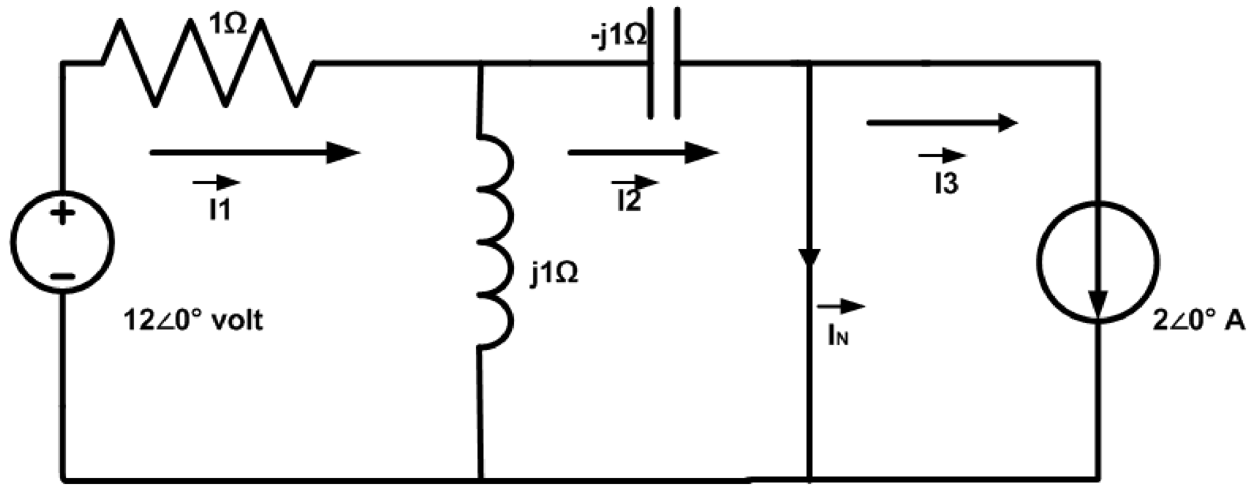


$find\ \overrightarrow{Io}\ using\ Norton's\ theorem$



$$\overrightarrow{Io} = \overrightarrow{I_N} \cdot \frac{Z_N}{Z_N + 1\Omega}$$

1) $to\ find\ I_N$



$$\overrightarrow{I_N} = \overrightarrow{I2} - \overrightarrow{I3}$$

$\overrightarrow{I3} = 2\angle 0°\ A\ costrain\ equation$

$KVL\ for\ mesh\ 1:$

$$12\angle 0° = (1 + j1)\overrightarrow{I1} - j1\ \overrightarrow{I2}$$

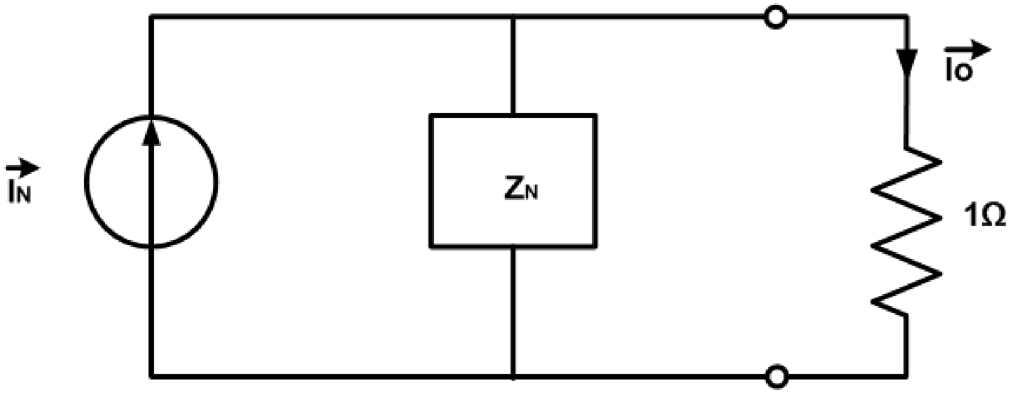
$KVL\ for\ mesh\ 2:$

$$0 = -j1\ \overrightarrow{I1} + (j1 - j1)\overrightarrow{I2} \Rightarrow 0 = -j1\ \overrightarrow{I1} \Rightarrow \therefore\ \overrightarrow{I1} = 0$$

$$\therefore \overrightarrow{I2} = 12\angle 90°\ A$$

$$\overrightarrow{I_N} = \overrightarrow{I2} - \overrightarrow{I3} = -2 + j12\ A$$

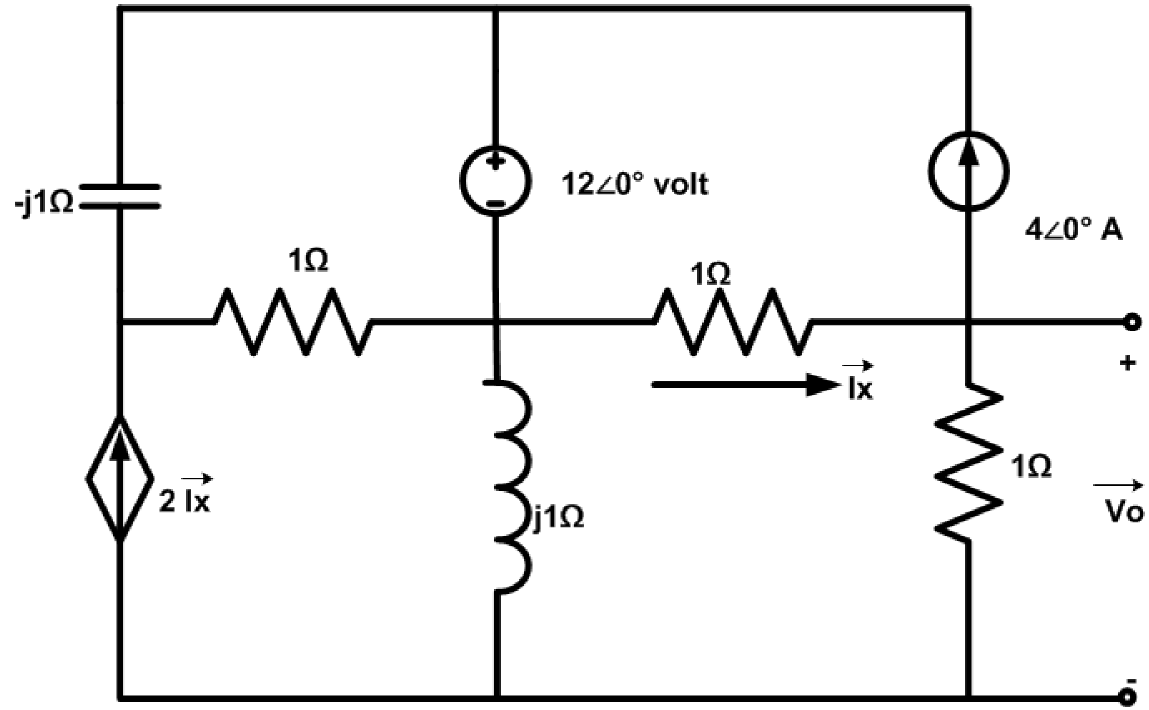
2) $Z_N = Z_{TH} = \left(\frac{1}{2} - j\frac{1}{2}\right)\ \Omega$

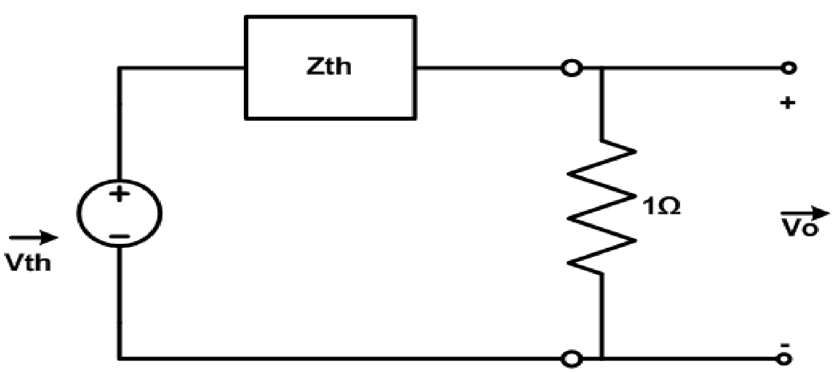


$$\overrightarrow{Io} = \overrightarrow{I_N} \cdot \frac{Z_N}{Z_N + 1\Omega}$$

$$\overrightarrow{Io} = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

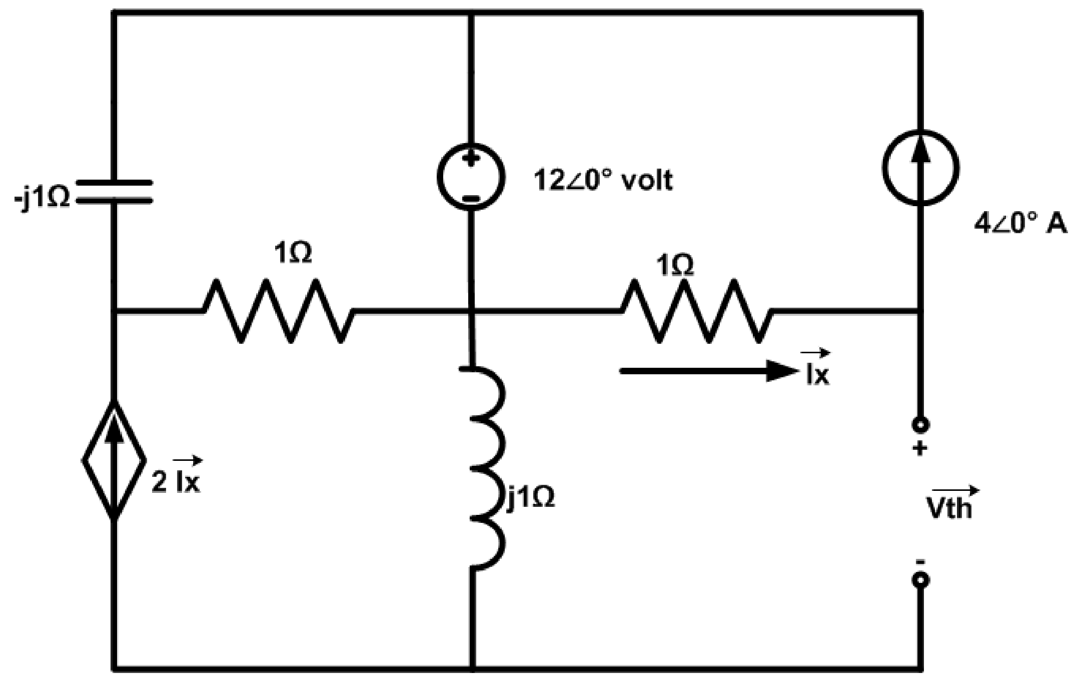
find $\overrightarrow{Vo}$ using Thevenin's theorem





$$\overrightarrow{Vo} = \frac{1\Omega}{1\Omega + Z_{TH}} \cdot \overrightarrow{V_{TH}}$$

1) $to\ find\ \overrightarrow{V_{TH}}$



$$V_{TH} = -1\Omega\,\overrightarrow{Ix} + j1\Omega\,(2\overrightarrow{Ix})$$

$$\overrightarrow{Ix} = 4\angle 0°\ A$$

$$\therefore\ \overrightarrow{V_{TH}} = (-4 + j8)volt.$$

2) $to\ find\ Z_{TH}$

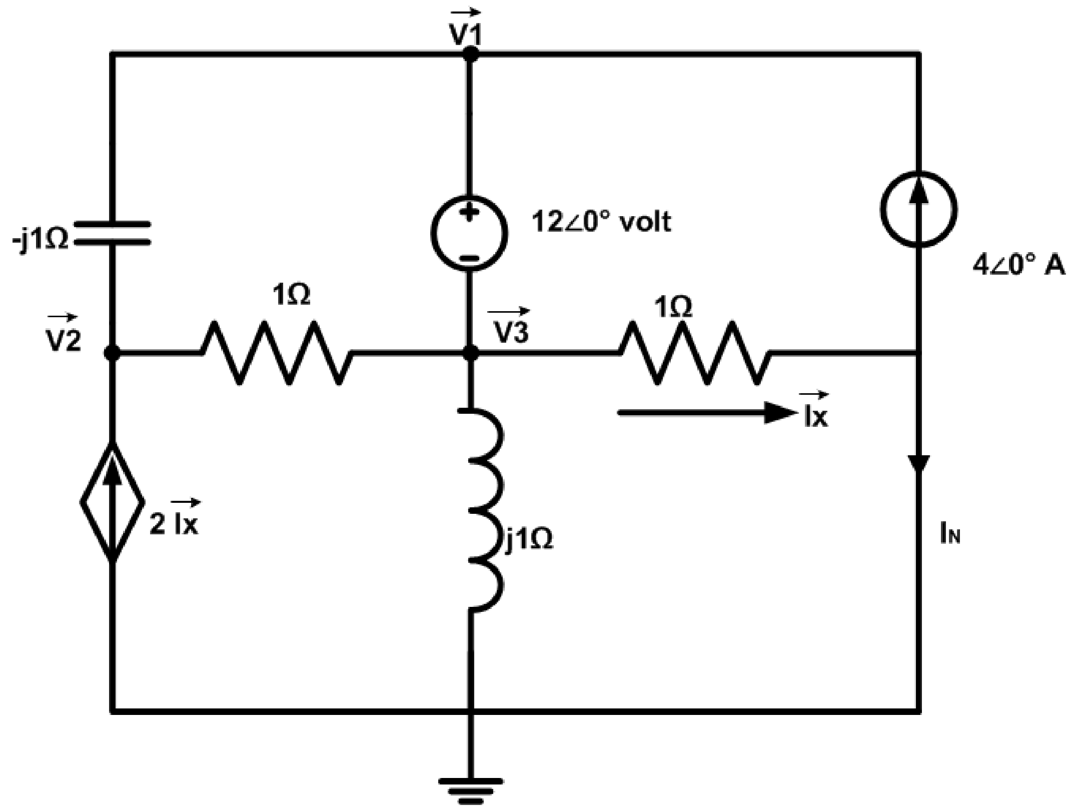
$$Z_{TH} = \frac{\overrightarrow{V_{TH}}}{\overrightarrow{I_N}}$$

$$Z_{TH} = \frac{\overrightarrow{V_T}}{\overrightarrow{I_T}}\bigg|\ all\ independent\ sources\ are\ set\ to\ zero.$$

to find $\overrightarrow{I_N}$

$$\overrightarrow{I_N} = \overrightarrow{Ix} - 4\angle 0^\circ$$

$$\overrightarrow{Ix} = \frac{\overrightarrow{V3}}{1\Omega} = \overrightarrow{V3}$$



Nodal analysis

$\overrightarrow{V1} - \overrightarrow{V3} = 12\angle 0^\circ$ *constrain equation*

KCL at node 2:

$$2\,\overrightarrow{Ix} = \left(1 + \frac{1}{-j1}\right)\overrightarrow{V2} + j1\,V1 - 1\,V3$$

KCL at super node (1,3)

$$4\angle 0^\circ = \left(\frac{1}{-j1}\right)V1 + \left(1 + 1 + \frac{1}{j1}\right)V3 - \left(1 + \frac{1}{-j1}\right)V2$$

Solving for $\overrightarrow{V3}$ ➡ $\overrightarrow{V3} = \left(\frac{j4}{1 - j1}\right)$

$$\therefore I_N = -\left(\frac{8 + j4}{1 + j1}\right) A$$

$$\therefore Z_{TH} = \frac{\overrightarrow{V_{TH}}}{\overrightarrow{I_N}}$$

$$Z_{TH} = (1 - j1)\Omega$$

$$\overrightarrow{Vo} = \frac{-4 + j8}{1 + 1 - j1} = 4\angle 143.13^\circ\ volt$$

$Z_{TH} = \frac{\overrightarrow{V_T}}{\overrightarrow{I_T}}\Big|$ *all independent sources are set to zero.*

$$\overrightarrow{V_T} = -1\,\overrightarrow{Ix} + j1\,\overrightarrow{Ix}$$

$$\overrightarrow{V_T} = (-1 + j1)\overrightarrow{Ix}$$

$$\overrightarrow{Ix} = -\overrightarrow{I_T}$$

$$\overrightarrow{V_T} = -(-1 + j1)\overrightarrow{I_T}$$

$$Z_{TH} = \frac{\overrightarrow{V_T}}{\overrightarrow{I_T}} = (1 - j1)\,\Omega$$