Birzei University
Faculty of Engineering and Technology
Department of Electrical and Computer Engineering
Engineering Probability and Statistics ENEE 2307

Final Exam

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First Semester 2017-2018

Date: Wednesday January 24, 2017 Time: 120 minutes

Student's Name:	**Student's Number:**
Section (Instructor):	**Problems 1, 2, 3**

Opening Remarks:

- This is a 75-minute exam. Calculators are allowed, however, mobile phones, books, notes, formula sheets, and other aids are not allowed.
- You are required to show all your work and provide the necessary explanations everywhere to get full credit.

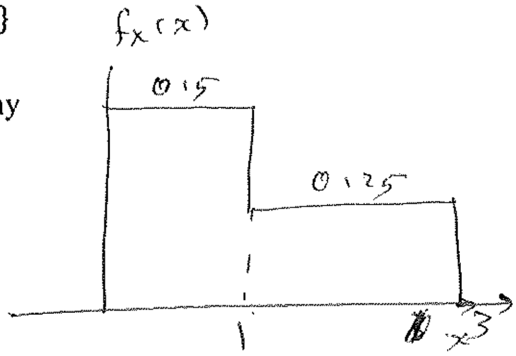
Problem 1: 18 Points

The probability density function of a continuous random variable X is given as:

$$f_X(x) = \begin{cases} 0.5, & 0 \le x < 1 \\ 0.25, & 1 < x \le 3 \\ 0, & otherwise \end{cases}$$

Events A and B are defined as: $A = \{0 \le x < 0.5\}$, $B = \{0 \le x < 2.5\}$

a. Find $P(A)$ and $P(B)$
b. Are events A and B disjoint (mutually exclusive)? Explain why
c. Are events A and B independent? Explain why
d. Find $P(A|B)$
e. If $Y = 3X + 1$, Find the mean (or expected) value of Y.

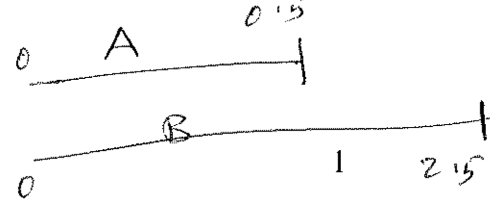


$P(A) = \int_0^{0.5} f_X(x)dx = (0.5)(0.5) = 0.25$

$P(B) = \int_0^{2.5} f_X(x)dx = 0.5 + (1.5)(0.25) = 0.875$

b. since $A \cap B \neq \phi \Rightarrow$ A & B are not disjoint

c. Need to find $P(A \cap B) = P(X \le 0.5 \cap X \le 2.5)$
$= P(X \le 0.5)$
$= 0.25$



$P(A \cap B) \stackrel{?}{=} P(A)P(B)$

$(0.25) \stackrel{?}{=} (0.25)(0.875)$

Answer NO $\Rightarrow$ A and B are not independent

d. $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{0.25}{0.875} = 0.2857$

e. $y = 3x + 1$

$\mu_y = 3\mu_x + 1$

$\mu_x = \int_0^1 0.5\, x\, dx + \int_1^3 0.125\, x\, dx$

$= 0.25 + 0.125\,(9 - 1) = 1.25$

$\mu_y = 3 \times 1.25 + 1$

$\mu_y = 4.75$

Problem 2: 16 Points

Let X be an exponential random variable with the pdf

$$f_X(x) = \begin{cases} 2e^{-2x}, & x \geq 0, \\ 0, & x \leq 0 \end{cases}$$

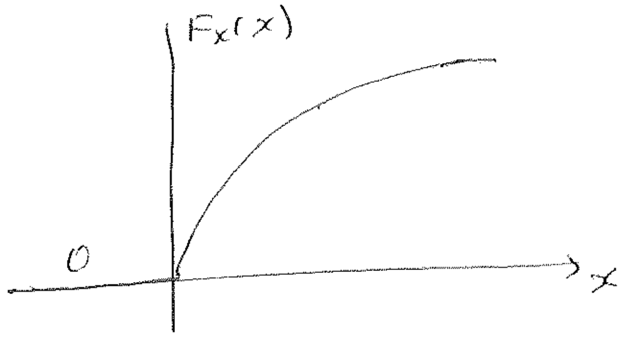
a. Find and sketch the cumulative distribution function of X defined as
$F_X(x) = P(X \leq x), 0 \leq x < \infty$

b. A random variable Y is defined by the transformation $Y = \sqrt{X+1}$, find and sketch the probability density function of Y.

a. $f_X(x) = 2e^{-2x} \quad x \geq 0$

$$F_X(x) = p(X \leq x) = \int_0^x 2e^{-2u}\,du = 2\left(\frac{-e^{-2y}}{2}\right)\Bigg|_0^x$$

$$= \begin{cases} 1 - e^{-2x} & x > 0 \\ 0 & x < 0 \end{cases}$$



b. $$f_Y(y) = \frac{f_X(x)}{|dy/dx|} = \frac{2e^{-2x}}{\frac{1}{2\sqrt{x+1}}}$$

$$\boxed{\frac{d}{dx}(x+1)^{1/2} = \frac{1}{2}(x+1)^{-1/2} = \frac{1}{2\sqrt{x+1}}}$$

$$= \sqrt{x+1}\; 4e^{-2x}$$

$$f_Y(y) = \begin{cases} 4y\, e^{-2(y^2-1)} & y \geq 1 \\ 0 & y < 1 \end{cases}$$

when $x = 0 \Rightarrow y = 1$

when $x \to \infty \Rightarrow y \to \infty$

Problem 3: 16 Points

The monthly profit made by the owner of a small shop is a Gaussian random variable X with mean \$3000 and standard deviation \$200. His monthly expenditure (مصاريف) Y, follows the Gaussian distribution with mean \$2000 and standard deviation \$150. Let Z be the random variable representing his monthly savings (توفير)

a. Find the mean and standard deviation of Z, assuming that X and Y are independent
b. Write down the probability density function of Z
c. Find $P(Z \geq \$1500)$

a. $\mu_X = 3000, \quad \sigma_X = 200$

$\mu_Y = 2000, \quad \mu_Y = 150$

$Z = X - Y$

$\mu_Z = \mu_X - \mu_Y = 3000 - 2000 = 1000$

$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 = (200)^2 + (150)^2 \Rightarrow \sigma_Z = 250$

b. $$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_Z^2}} e^{-(z-\mu_Z)^2/2\sigma_Z^2}$$

$$= \frac{1}{\sqrt{2\pi(250)^2}} e^{-(z-1000)^2/2(250)^2}$$

since both X and Y are Gaussian

c. $$P(Z \geq 1500) = 1 - P(Z \leq 1500) = 1 - \Phi\left(\frac{1500 - \mu_Z}{\sigma_Z}\right)$$

$$= 1 - \Phi\left(\frac{1500 - 1000}{250}\right)$$

$$= 1 - \Phi(2)$$

$$= 1 - 0.9772$$

$$= 0.0228$$

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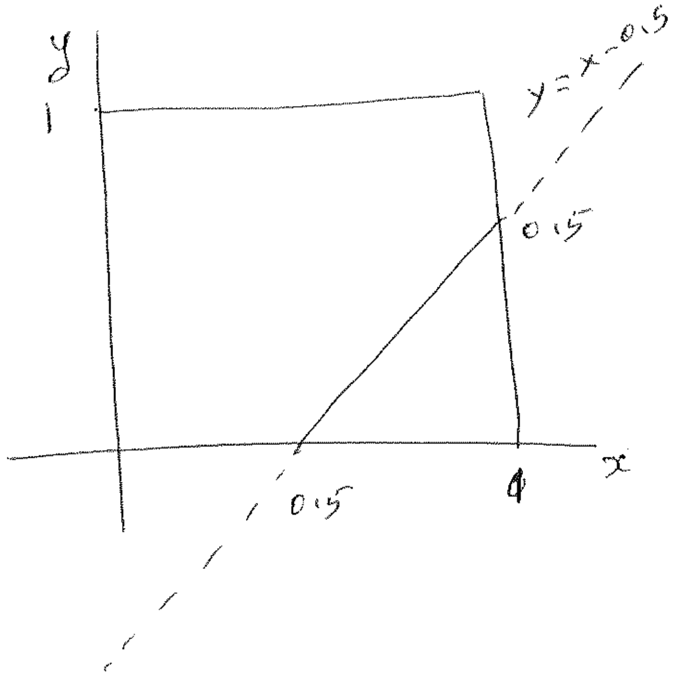
Problem 4: 17 Points

The joint probability density function of two random variables X and Y is:

$$f_{X,Y}(x,y)=\begin{cases}2-x-y, & 0\le x<1, \quad 0<y\le 1\\ 0, & otherwise\end{cases}$$

a. Find $P(Y-X\le -0.5)$
b. Find the marginal pdf's $f_X(x)$ and $f_Y(y)$
c. Are X and Y independent? Explain why



a. curve $y-x\le 0.5$

$y \le x-0.5$

$p(Y\le x-0.5)$

$$=\int_{x=0.5}^{1}\int_{0}^{x-0.5}(2-x-y)\,dy\,dx$$

$$=\int_{0.5}^{1}\left[2y-xy-\frac{y^2}{2}\right|_0^{x-0.5} dx = \int_{0.5}^{1}\left[2(x-0.5)-x(x-0.5)-\frac{(x-0.5)^2}{2}\right]dx$$

$= 0.125$

b. $f_X(x)=\int_0^1(2-x-y)\,dy = \left(2y-xy-\frac{y^2}{2}\right|_0^1 = (2-x-0.5)$

$$\Rightarrow\begin{cases}\frac{3}{2}-x & 0\le x\le 1\\ 0 & o.w\end{cases}$$

c. $f_Y(y)=\int_0^1(2-x-y)\,dx = \left(2x-\frac{x^2}{2}-xy\right)\Big|_0^1 = \left(2-\frac{1}{2}-y\right)\Rightarrow\begin{cases}\frac{3}{2}-y & 0<y\le 1\\ 0 & o.w\end{cases}$

since $f_{X,Y}(x,y) \neq f_X(x)\, f_Y(y)$

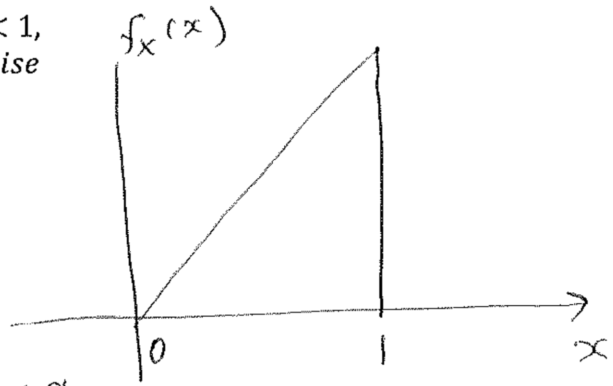
$$2 - x - y \neq \left(\frac{3}{2} - x\right)\left(\frac{3}{2} - y\right)$$

$\Rightarrow X$ & Y are dependent

Problem 5: 17 Points

Suppose that a random sample $X_1, X_2, \ldots, X_{30}$ of size 30 is taken from a distribution with the following pdf:

$$f_X(x) = \begin{cases} 2x, & 0 \le x < 1, \\ 0, & otherwise \end{cases}$$



Define the sample average as: $\hat{\mu} = \frac{\sum_{i=1}^{30} X_i}{30}$

a. Find $P(X \le 0.7)$
b. Find the mean and variance of X
c. Find the mean and variance of $\hat{\mu}$
d. Use the central limit theorem to find $P(\hat{\mu} \le 0.7)$

a. $P(X \le 0.7) = \int_0^{0.7} 2x \, dx = x^2 \Big|_0^{0.7} = 0.49$

b. $E(X) = \int_0^1 x f_X(x) \, dx = \int_0^1 2x^2 \, dx = \frac{2x^3}{3} \Big|_0^1 = \frac{2}{3}$

$E(X^2) = \int_0^1 x^2 f_X(x) dx = \int_0^1 2x^3 \, dx = 2 \frac{x^4}{4} \Big|_0^1 = \frac{1}{2}$

$\sigma_X^2 = E(X^2) - E^2(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18} = 0.0555$

c. $E(\hat{\mu}) = \mu_X = 2/3$

$Var(\hat{\mu}) = \frac{\sigma_X^2}{30} = \frac{0.0555}{30} = 1.851 \times 10^{-3} \Rightarrow \sigma_{\hat{\mu}} = 0.043$

d. $P(\hat{\mu} \le 0.7) = \Phi\left(\frac{0.7 - 2/3}{\sigma_{\hat{\mu}}}\right) = \Phi\left(\frac{0.7 - 2/3}{0.043}\right)$

$= \Phi(0.7767)$

$\cong 0.7794$

Problem 6: 16 Points

The compressive strength, X, of concrete follows the Gaussian distribution with mean μ_X and variance σ_X^2. A civil engineer tests 12 specimens of concrete and obtains the following data (in psi)

2216	2237	2249	2204	2225	2301
2281	2283	2318	2255	2275	2295

a. Show that the sample mean $\hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n}$ is an unbiased estimator for the mean μ_X.

b. What is the unbiased estimators of σ_X^2 that he should use?

c. Based on the above data, find point estimates for μ_X and σ_X^2 of the compressive strength.

d. Construct a 95% confidence interval on the mean compressive strength μ_X

e. construct a 95% confidence interval on the variance σ_X^2 of the compression strength

a. $\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}$

$E[\hat{\mu}] = \frac{1}{n} \sum_{i=1}^{n} E(x_i) = \frac{1}{n} \cdot n \mu_X = \mu_X \Rightarrow$ unbiased estimator

b. $\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \hat{\mu})^2$ is an unbiased estimator of σ^2

c. $\hat{\mu} = \frac{\sum x_i}{12} = 2261.5$

$\hat{\sigma} = 36.19 \Rightarrow \hat{\sigma}^2 = \frac{1}{11} \sum (x_i - 2261.5)^2 = 1309.7$

d. $P\left(\hat{\mu} - t_{\alpha/2, 11} \frac{\hat{\sigma}}{\sqrt{n}} < \mu < \hat{\mu} + t_{\alpha/2, 11} \frac{\hat{\sigma}}{\sqrt{n}}\right)$

$P\left(2261.5 - 2.201 \frac{36.19}{\sqrt{12}} < \mu < 2261.5 + 2.201 \frac{36.19}{\sqrt{12}}\right)$

$P(2261.5 - 22.99 < \mu < 2261.5 + 22.99)$

$P(2238.51 < \mu < 2284.49)$

$$P\left(\frac{(n-1)\,\hat{a}^2}{\chi^2_{\alpha/2,\,11}} \le a^2 \le \frac{(n-1)\,\hat{a}^2}{\chi^2_{1-\alpha/2,\,11}}\right) = 0.95$$

$$\chi^2_{0.975,\,11}$$

$$P\left(\frac{11 \times 1309.7}{21.92} \le a^2 \le \frac{11 \times 1309.7}{3.816}\right) = 0.95$$

$$P\left(657.2 \le a^2 \le 3775.34\right) = 0.95$$

$$P\left(25.63 \le a \le 61.44\right) = 0.95$$