جامعة بيرزيت
BIRZEIT UNIVERSITY

Department of Electrical and Computer Engineering
First Semester, 2018/2019
Probability and Statistics - ENEE2307
Final Exam, January 6, 2019
Time Allowed: 120 Minutes.

د. جبران

Part I

Name: ....

ID: ....

Section: ....

Question #	SOC	Max Grade	Achieved
1		18	
2		18	
3		16	
Total		52	

Opening Remarks:

- This is a 120-minutes exam. Calculators are allowed. Books, notes, formula sheets, and other aids are not allowed.
- You are required to show all your work and provide the necessary explanations everywhere to get full credit.

Question#1 [18 Points]

A. A shipment of 8 motors contains 6 defective and 2 non-defective motors. Three motors are chosen at random (without replacement) and classified. Find the probability that at least two of the chosen motors are defective.

X: defective

defective / non-defective

$$\boxed{6 \,/\, 2} \quad N=8$$

$$\boxed{x \,/\, n-x} \quad n=3$$

$$P(X \geq 2) = ? \quad , \; p(X=x) = \frac{\binom{6}{x}\binom{2}{3-x}}{\binom{8}{3}} \quad x: 0,1,2,3$$

$$p(X \geq 2) = p(X=2) + p(X=3)$$

$$p(X=2) = \frac{\binom{6}{2}\binom{2}{1}}{\binom{8}{3}} = \frac{\frac{6!}{4! \times 2!} \times \frac{2!}{1! \times 1!}}{\frac{8!}{5! \times 3!}} = \frac{\frac{6 \times 5}{2} \times 2}{\frac{8 \times 7 \times 6}{3 \times 2}} = \frac{15}{28}$$

$$p(X=3) = \frac{\binom{6}{3}\binom{2}{0}}{\binom{8}{3}} = \frac{6 \times 5 \times 4}{3 \times 2} \times \frac{3 \times 2}{8 \times 7 \times 6} = \frac{5}{14} \rightarrow p(X \geq 2) = \frac{15}{28} + \frac{10}{28} = \frac{25}{28}$$

B. The number of cars that arrive at a certain intersection follows the Poisson distribution with a rate of 2 cars/min. What is the probability that at least one car arrives in a one- minute period.

$$\lambda = 2 \frac{\text{cars}}{\text{min}} , \quad p(X=x) = e^{-\lambda T} \frac{(\lambda T)^x}{x!} , \quad x = 0, 1, 2, \ldots$$

$$p(x \geq 1)\Big|_{T = 1\,min} = ? = 1 - p(X=0) = 1 - e^{-2 \times 1} \frac{(2 \times 1)^{x=0}}{x!}$$

$$= 1 - e^{-2}$$

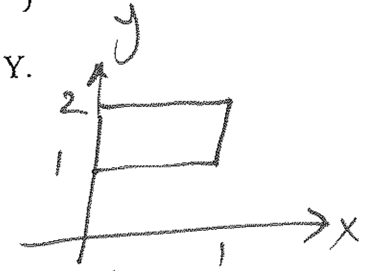
$$= 0.8646$$

Question#2 [18 Points]

The joint probability density function of two random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} KX(1-Y) & (0 \le X < 1), (1 \le Y < 2) \\ 0 & o.w \end{cases}$$

A. Find K such that $f_{X,Y}(x,y)$ is a valid joint probability density function of X and Y.



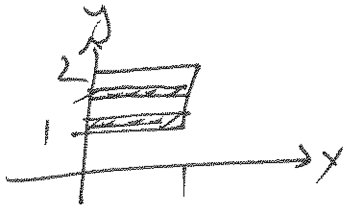
$$\int_{-\infty}^{\infty}\int f_{x,y}(x,y)\,dy\,dx = 1 = \int_0^1\int_1^2 Kx(1-y)\,dy\,dx$$

$$= \int_0^1 Kx\left(y-\frac{y^2}{2}\right)\Big|_1^2 dx = K\int_0^1 x\left[\left(2-\cancel{2}\right)-\left(1-\tfrac{1}{2}\right)\right]dx = \int_0^1 Kx - \frac{1}{2}x\,dx$$

$$= \frac{-K}{2}\frac{x^2}{2}\Big|_0^1 = \frac{-K}{4}(1-0) = \frac{-K}{4} = 1 \longrightarrow \boxed{K=-4}$$

$$f_{x,y}(x,y) = \begin{cases} -4x(1-y) & 0 \le x < 1,\ 1 \le y < 2 \\ 0 & o.w \end{cases}$$

B. Find the marginal pdf of Y; $f_Y(y)$.

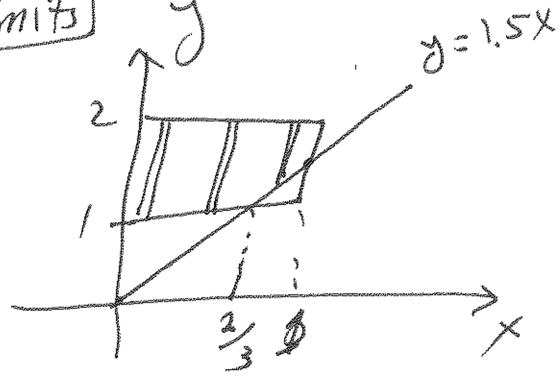


$$f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dx = \int_0^1 -4x(1-y)\,dx$$

$$= \frac{-4x^2}{2}(1-y)\Big|_0^1 = -2(1-y)$$

$$f_y(y) = \begin{cases} -2(1-y) & 1 \le y < 2 \\ 0 & o.w \end{cases} = \begin{cases} \frac{K}{2}(1-y) & 1 \le y \le 2 \\ 0 & o.w \end{cases}$$

most of the grade to be allocated for integration limits



C. Compute $P(1.5X \leq Y \leq 2)$.

$$P(1.5x \leq y \leq 2) = \int_0^{2/3}\int_1^2 -4x(1-y)\,dy\,dx + \int_{2/3}^{1}\int_{1.5x}^{2} -4x(1-y)\,dy\,dx$$

$$= \int_0^{2/3} -4x\left(y - \frac{y^2}{2}\right)\Big|_1^2 dx + \int_{2/3}^{1} -4x\left(y - \frac{y^2}{2}\right)\Big|_{1.5x}^{2} dx$$

$$= \int_0^{2/3} -4x\left[(2-2)-\left(1-\tfrac{1}{2}\right)\right] dx + \int_{2/3}^{1} -4x\left[(2-2)-\left(1.5x - \frac{(1.5)^2 x^2}{2}\right)\right] dx$$

$$= \int_0^{2/3} 2x\,dx + \int_{2/3}^{1} -4x\left(-1.5x + \frac{(1.5)^2}{2}x^2\right) dx$$

$$= \int_0^{2/3} 2x\,dx + \int_{2/3}^{1} \left(6x^2 - 4.5x^3\right) dx$$

$$= x^2\Big|_0^{2/3} + \left(\frac{6x^3}{3} - \frac{4.5x^4}{4}\right)\Big|_{2/3}^{1}$$

$$= \frac{4}{9} + \left(\frac{6}{3} - \frac{4.5}{4}\right) - \left(\frac{6}{3}\times\left(\frac{2}{3}\right)^3 - \frac{4.5}{4}\left(\frac{2}{3}\right)^4\right) = 0.949$$

Question#3 [16 Points]

X is a uniform random variable defined on the interval [0,4].

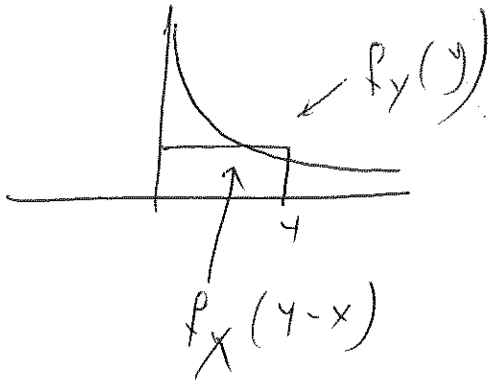
Y is another random variable with pdf given by $f_y(y) = \begin{cases} e^{-y} & 0 \le y \le \infty \\ 0 & otherwise \end{cases}$. A new random variable Z=X+Y is defined. Find the pdf of Z at Z=4?

$$f_z(z) = \int_{-\infty}^{\infty} f_x(x)\, f_y(z-x)\, dx = \int_{-\infty}^{\infty} f_y(y)\, f_x(z-y)\, dy$$

$$f_z(4) = \int_{-\infty}^{\infty} f_y(y)\, f_x(4-y)\, dy$$

$$= \int_{0}^{4} e^{-y} dy = -e^{-y} \Big|_0^4$$

$$= 1 - e^{-4} = 0.981$$



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Part II

Name: ....

ID: ....

Section: ....

Question #	SOC	Max Grade	Achieved
4		16	
5		16	
6		16	
Total		48	

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Question#4 [16 Points]

The data of an experiment is collected and found to be as shown in the table below. For these values find the best fitting curve equation $y = e^{ax}$ that describes the experiment results (y) versus the input (x).

x_i	-1	-1.2	-0.6	0.9	0.5	-0.1	-0.8	1	-0.2	1.3
y_i	2.7	3.3	1.8	0.4	0.6	1.1	2.2	0.3	1.2	0.2
$\ln(y_i)$	0.99	1.19	0.58	-0.916	-0.51	0.095	0.788	-1.2	0.18	-1.6

Hint: For the linear model $y = \alpha x + \beta$, we have

$$\begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\sum x_i^2 \alpha = \sum x_i y_i \longrightarrow \alpha = \frac{\sum x_i y_i}{\sum x_i^2} = \cancel{\frac{\sum x_i y_i}{7.24}} \; -7.79$$

$$y = e^{ax}$$

$$\ln(y) = ax \longrightarrow y_{new} = ax$$

$$a = \alpha = \frac{\sum x_i y_{i,new}}{\sum x_i^2} = \frac{-7.748}{7.24} = -1.07$$

-0.993
-1.4327
-0.35267
-0.82466
-0.25541
-0.00953
-0.63076
-1.20397
+0.0439
-2.0922

-7.748

Question#5 [16 Points]

Suppose that a random sample of size 100 is taken from an unknown distribution X with mean $\mu_x = 2$ and variance $\sigma_x^2 = 9$

A. Show that the sample average defined as $\hat{\mu}_x = \frac{1}{100}\sum_{i=1}^{100} x_i$ is an unbiased estimator for the mean value μ_x.

$$E\{\hat{\mu}_x\} = E\left\{\frac{1}{100}\sum_{i=1}^{100} x_i\right\} = \frac{1}{100}\sum_{i=1}^{100} E\{x_i\} = \frac{1}{100}\sum_{i=1}^{100} \mu_x = \frac{1}{100} \times 100\mu_x$$

$= \mu_x$ so $\hat{\mu}_x$ is unbiased estinator of μ_x

B. Find the probability that the sample mean $\hat{\mu}_x$ is less than 1.4

$$p(\hat{\mu}_x < 1.4) = ?$$

$$E\{\hat{\mu}_x\} = \mu_x = 2$$

$$var\{\hat{\mu}_x\} = \frac{\sigma_x^2}{n} = \frac{9}{100} = 0.09$$

$$p(\hat{\mu}_x < 1.4) = \phi\left(\frac{1.4-2}{\sqrt{0.09}}\right) = \phi\left(\frac{-0.6}{0.3}\right) = \phi(-2)$$

$$= 1-\phi(2) = 1 - 0.9772 = 0.0228$$

Question#6 [16 Points]

The following data were obtained from a Gaussian population, X, with an unknown mean μ and a known variance of 6.

8.81	15.91	20.39	8.63	13.80	21.73	27.23	7.32	14.83	20.26

A. Find a point estimate of the variance $\hat{\sigma}_x^2$.

$n=10$, $\hat{\sigma}_x^2 = \frac{1}{n-1}\sum_{i=1}^{10}(X_i - \hat{\mu}_x)^2$. $\hat{\mu}_x = \frac{1}{n}\sum X_i = \frac{1}{10} * 158.9 = 15.89$

$= \frac{1}{9}\{50.1264 + 0.0004 + 20.25 + 52.7076 + 4.3681 + 34.1056 + 128.5956 + 73.4449 + 1.1236 + 19.0969\} = \frac{383.8191}{9} = 42.6465$

B. Find a 95% confidence interval of the mean μ.

$1-\alpha = 0.95 \longrightarrow \alpha = 0.05 \longrightarrow \alpha/2 = 0.025 \longrightarrow z_{\alpha/2} = 1.96$

$p(\hat{\mu}_x - z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}} \le \mu_x \le \hat{\mu}_x + z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}}) = 0.95$

$p(15.89 - 1.96 * \sqrt{\frac{6}{10}} \le \mu_x \le 15.89 + 1.96 * \sqrt{\frac{6}{10}}) = 0.95$

$p(14.371 \le \mu_x \le 17.4082) = 0.95$

confidence interval.