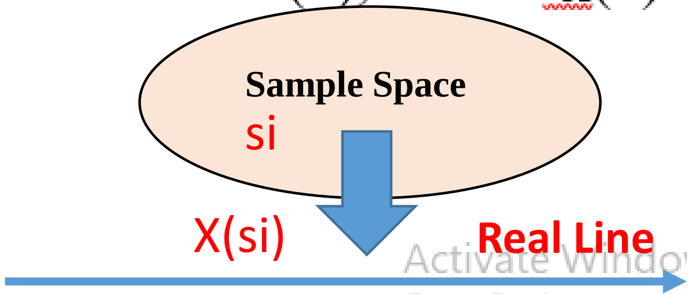
Single Random Variables and Probability Distributions: Basic Concepts

- **Informal Definition**: A random variable is a variable whose values depend on the outcomes of a random experiment.
- **Formal Definition**: A random variable is a real-valued function whose domain is the sample space defined on a probability space. It maps outcomes from the sample space along with their probabilities on the real line.
- The random variable is given an uppercase letter X, Y, Z, … while the values assumed by this random variable are given lowercase letters x, y, z, …
- Associated with each discrete r.v (X) is a Probability Mass Function $P(X = x)$. This density function **is the sum** of all probabilities associated with the outcomes in the sample space that get mapped into (x) by the mapping function (random variable X).
- Associated with each continuous r.v (X) is a Probability Density Function (pdf) $f_X(x)$. This $f_X(x)$ **is not** the probability that the random variable (X) takes on the value (x), rather $f_X(x)$ is a continuous curve having the property that:

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$



Examples on Discrete Random Variables

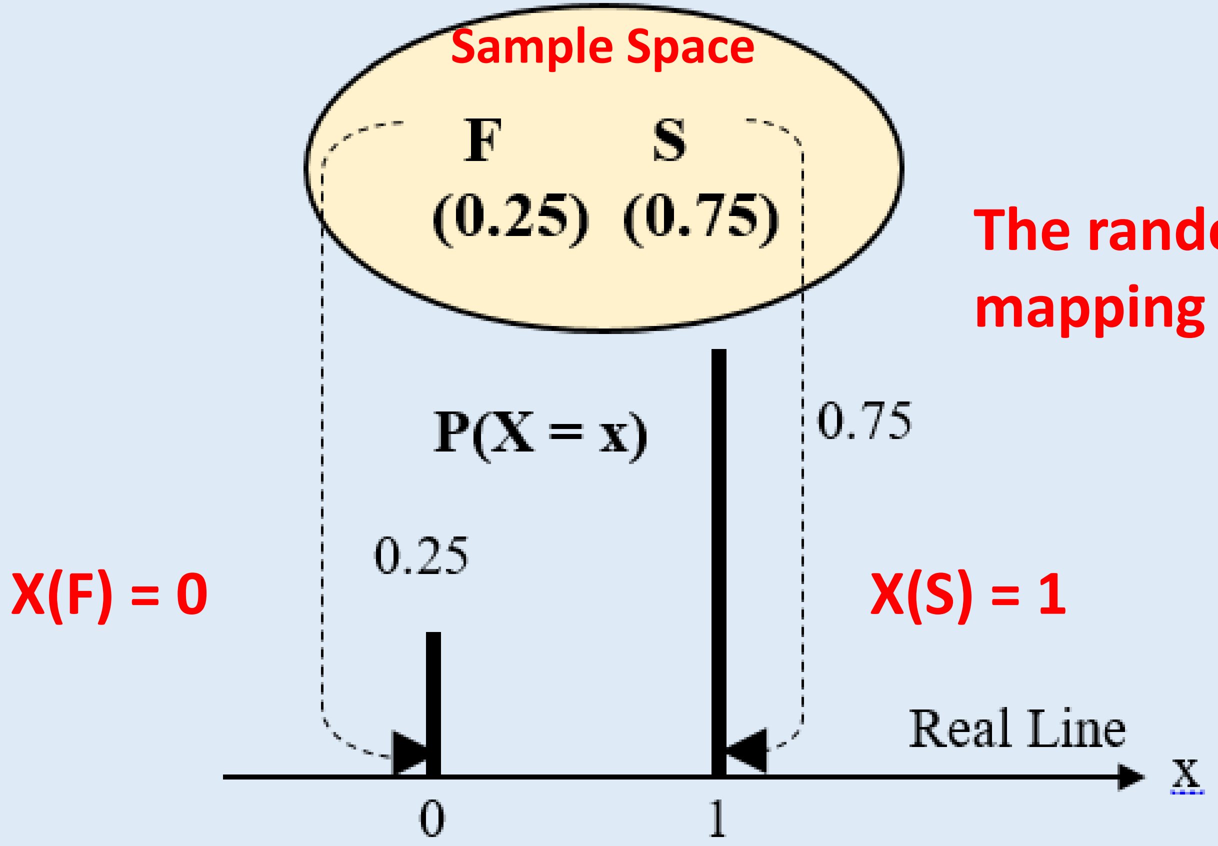
- **Example:** The sample space for an experiment is S={-1, 0, 1, 5}. List all possible values of the following random variables:
- $X = (s-1)^2$
- $Y = 1 + s + s^2$
- **Solution**: We note that **the random variable is a real-valued function of the elements of the sample space**.
- X assumes the values X = {4, 1, 0, 16}; (one to one mapping)
- Y assumes the values Y = {1, 3, 31}; (note that both -1 and 0 get mapped into 1)

Examples on Discrete Random Variables

EXAMPLE: A chance experiment has two possible outcomes, a success with probability 0.75 and a failure with probability 0.25. Define the random variable X (mapping function) as:

X = 1 if outcome is a success
X = 0 if outcome is a failure

SOLUTION:



The random Variable is a mapping function X(s)

Probability Mass Function

Examples on Discrete Random Variables

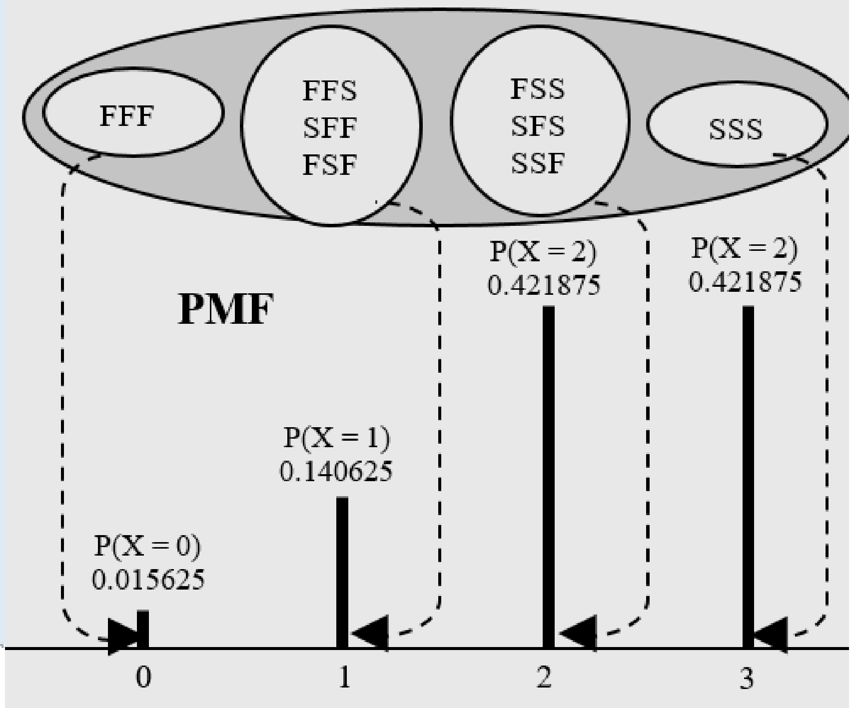
EXAMPLE: A chance experiment has two possible outcomes, a success with probability 0.75 and a failure with probability 0.25. The experiment is independently repeated 3 times in a row.

a. Find the sample space.
b. Define a random variable (X) as: X = number of successes in the three trials.
c. Find the probability mass function $P(X = x)$.

1	2	3
(S,F)	(S,F)	(S,F)
2*2*2=8 outcomes Example: P(FFF)=0.25*0.25*0.25		

SOLUTION: in the table below we show the possible outcomes and the mapping process.

Sample Outcome	$P(s_i)$	x	$P(X = x)$
F F F	$(0.25)^3$	0	$(0.25)^3 = 0.015625$
F F S	$(0.75)\,(0.25)^2$	1	$3 \times (0.75)\,(0.25)^2 = 0.140625$
S F F	$(0.75)\,(0.25)^2$		
F S F	$(0.75)\,(0.25)^2$		
S S F	$(0.75)^2\,(0.25)$	2	$3 \times (0.75)^2\,(0.25) = 0.421875$
S F S	$(0.75)^2\,(0.25)$		
F S S	$(0.75)^2\,(0.25)$		
S S S	$(0.75)^3$	3	$(0.75)^3 = 0.421875$



Examples on Discrete Random Variables

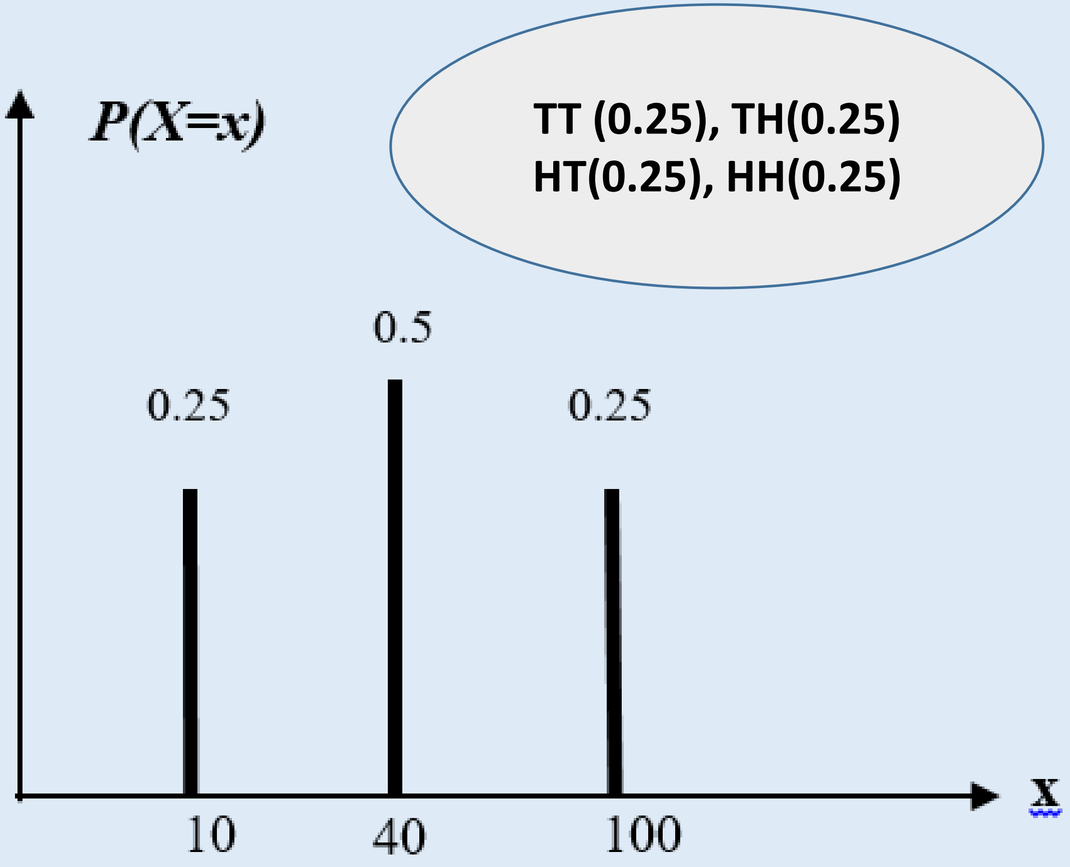
EXAMPLE: A chance experiment consists of flipping a fair coin twice. The outcome of the coin is independent from trial to trial. The profit, X, is a random variable, that is related to the experiment outcome as follows:

$X = 10$, if no heads appear
$X = 40$, if one head appears
$X = 100$, if two heads appear

Find the probability mass function of X

SOLUTION

$$P(X=10) = P(TT) = P(T)P(T) = (0.5)(0.5) = 0.25\,;$$
$$P(X=40) = P(HT) + P(TH) = 2(0.5)(0.5) = 0.5$$
$$P(X=100) = P(HH) = (0.5)(0.5) = 0.25$$



Probability Mass Function

EXAMPLE: Suppose that 5 persons including you and your friend line up at random. Let (X) be the number of people standing between you and your friend. Find the probability mass function for the random variable (X).

SOLUTION:

Number of different ways by which the 5 people can arrange themselves = 5!
This is the total number of points in the sample space.

Let A denotes your position in the line
B denotes the position of your friend

5	4	3	2	1	Server

The random variable (X) assumes four possible values 0, 1, 2, 3 as shown below:

$$\left.\begin{matrix} A\ B\ O\ O\ O \\ O\ A\ B\ O\ O \\ O\ O\ A\ B\ O \\ O\ O\ O\ A\ B \end{matrix}\right\} \Rightarrow (X = 0); \quad \left.\begin{matrix} A\ O\ B\ O\ O \\ O\ A\ O\ B\ O \\ O\ O\ A\ O\ B \end{matrix}\right\} \Rightarrow (X = 1)$$

$$\left.\begin{matrix} A\ O\ O\ B\ O \\ O\ A\ O\ O\ B \end{matrix}\right\} \Rightarrow (X = 2); \quad A\ O\ O\ O\ B\} \Rightarrow (X = 3)$$

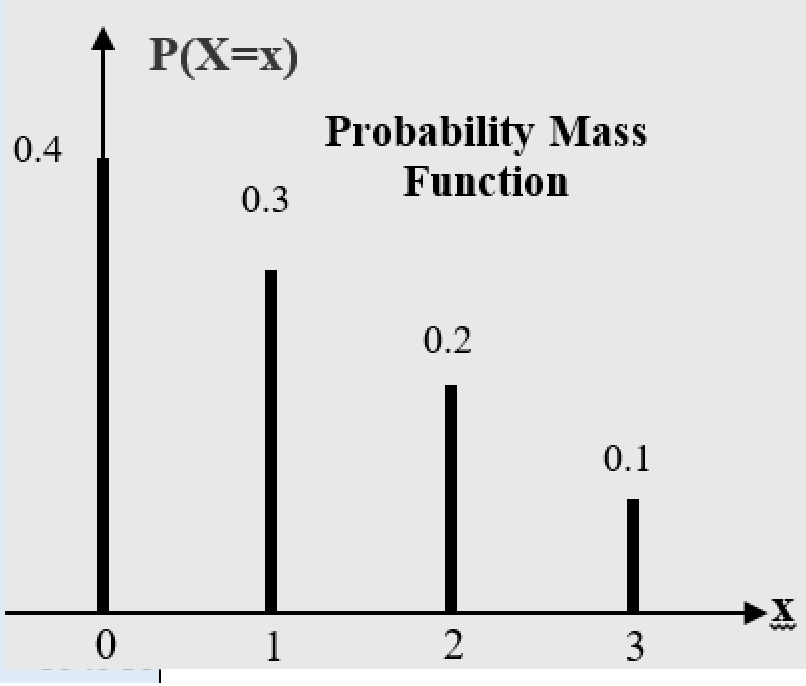
Any sequence similar to what is shown can be done in: $\underbrace{2!}_{\text{you and your friend}} \times \underbrace{3!}_{\text{the other people}}$

$$P(X=0) = \frac{4 \times 2! \times 3!}{5!} = 0.4$$

$$P(X=1) = \frac{3 \times 2! \times 3!}{5!} = 0.3$$

$$P(X=2) = \frac{2 \times 2! \times 3!}{5!} = 0.2$$

$$P(X=3) = \frac{1 \times 2! \times 3!}{5!} = 0.1$$



Discrete and Continuous Random Variables

The Cumulative Distribution Function

The cumulative distribution function of a r.v. X defined on a sample space (S) is given by:

$$F_X(x) = P(X \le x)$$

This is a general definition which applies to discrete as well continuous distributions, as we shall see next.

Properties of $F_X(x)$

1- $F_X(-\infty) = 0$

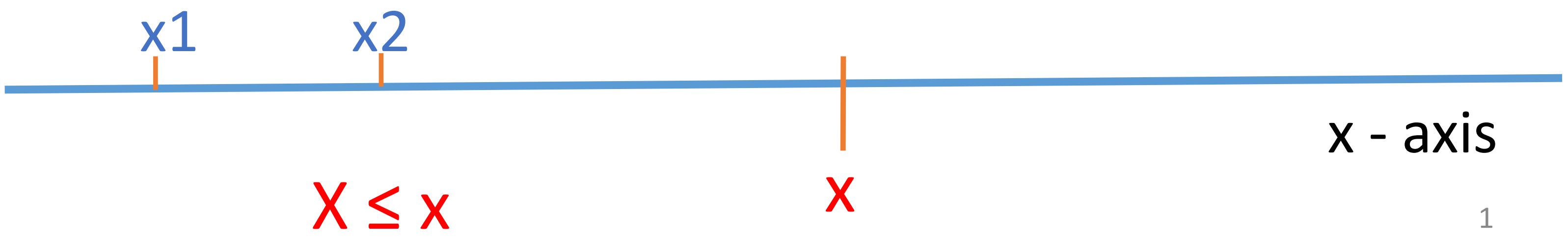
2- $F_X(\infty) = 1$

3- $0 \le F_X(x) \le 1$

4- $F_X(x_1) \le F_X(x_2)$ if $x_1 \le x_2$

5- $F_X(x^+) = F_X(x)$; function is continuous from the right

6- $P\{x_1 \le X \le x_2\} = F_X(x_2) - F_X(x_1)$

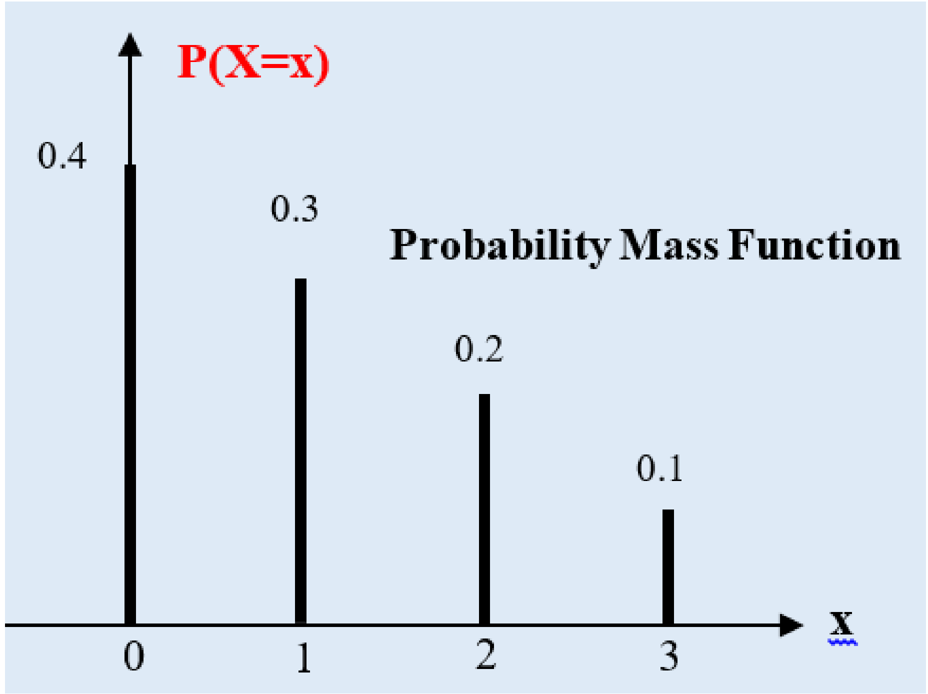


Discrete Random Variables and Distribution

Definition: A random variable and its distribution are of **discrete type** when the sample space of the random experiment is of countable nature and the corresponding cumulative distribution function $F_X(x)$ can be given as a summation of the form:

$$F_X(x) = \sum_{u=-\infty}^{x} P(X=u); \text{ Cumulative Distribution Function}$$

where $P(X=x)$ is the probability mass function (pmf).

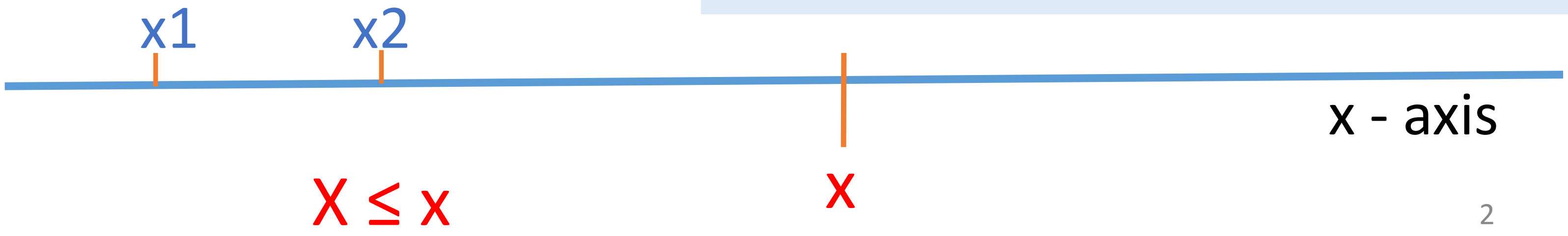


Properties of $P(X=x)$

1- $P(X=x) \geq 0;$ non-negative

2- $\sum_{u=-\infty}^{\infty} P(X=u) = 1$

3- $P(x_1 \leq X \leq x_2) = \sum_{u=x_1}^{x_2} P(X=u);$



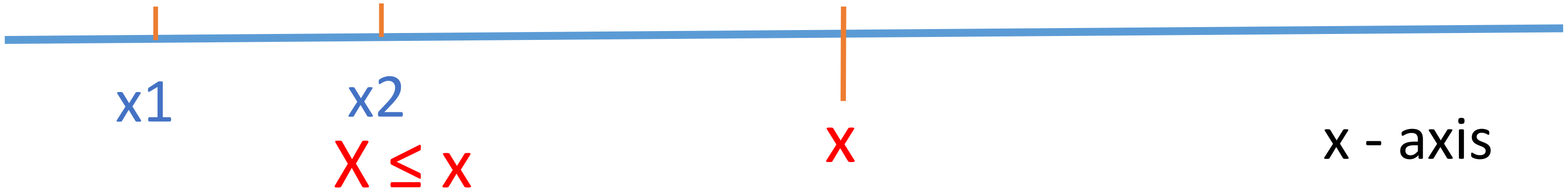
Continuous Random Variables and Distribution

Definition: A random variable and its distribution are of continuous type when the sample space of the random experiment is uncountable and the corresponding cumulative distribution function $F_X(x)$ can be given as an integral of the form:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

where $f_X(x)$ is the probability density function related to $F_X(x)$ by:

$$f_X(x) = \frac{d}{dx} F_X(x)$$



Properties of $f_X(x)$

1- $f_X(x) \geq 0$; nonnegative

2- $\int_{-\infty}^{\infty} f_X(u)\, du = 1$; Total area under the pdf is one.

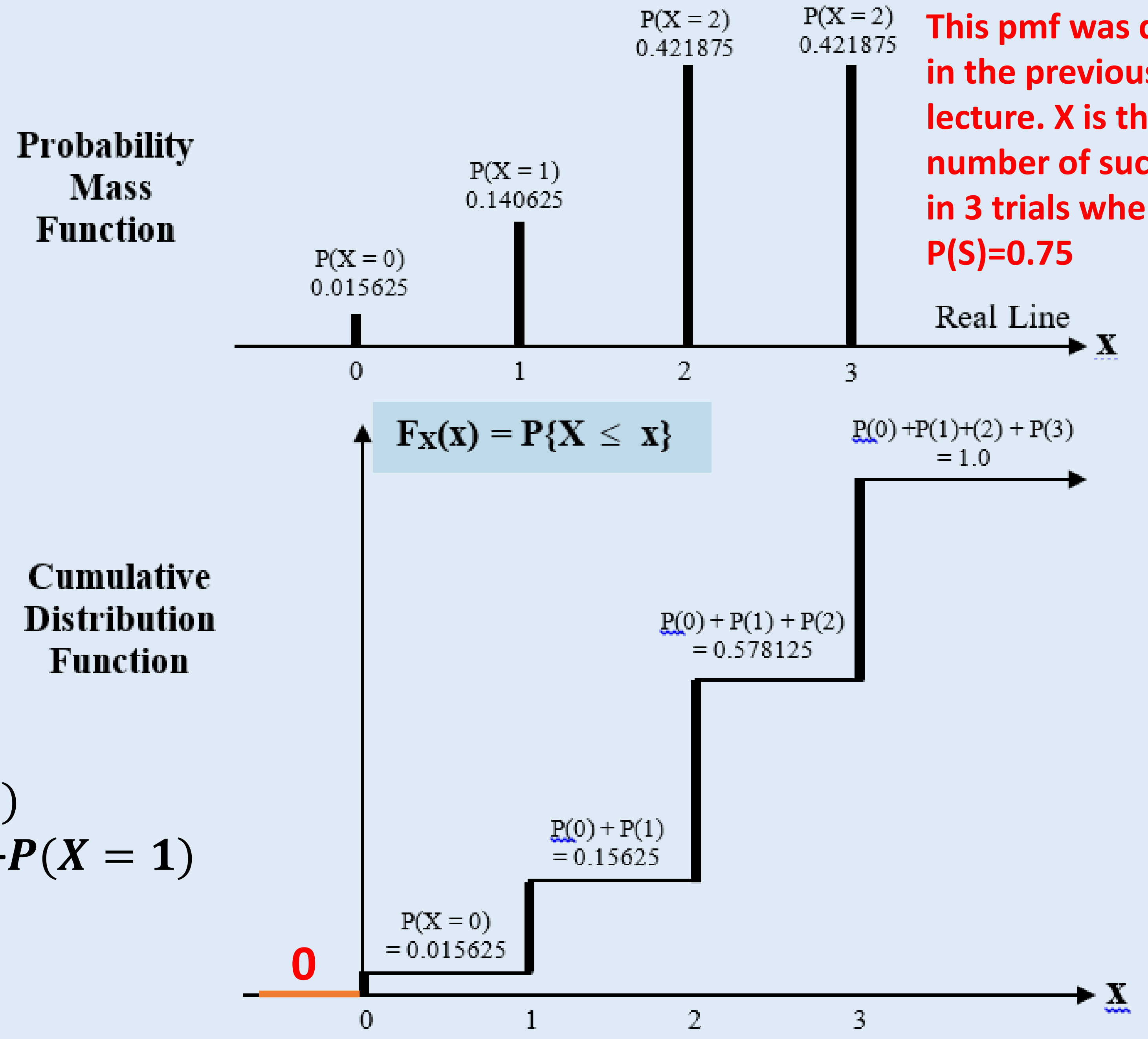
3- $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(u)\, du$; Probability is the area under $f_X(x)$ between x_1 and x_2.

Cumulative Distribution Function of a Discrete Random Variable

The pmf of a discrete random variable X is shown in the figure. Construct the cumulative distribution function defined as

$$F_X(x) = P(X \le x)$$

$$F_X(-0.5) = P(X \le -0.5) = 0$$
$$F_X(0^-) = P(X \le 0^-) = 0$$
$$F_X(0^+) = P(X \le 0^+) = P(X = 0)$$
$$F_X(0.5) = P(X \le 0.5) = P(X = 0)$$
$$F_X(1^+) = P(X \le 1^+) = P(X = 0) + P(X = 1)$$



This pmf was derived in the previous lecture. X is the number of successes in 3 trials where P(S)=0.75

EXAMPLE: Cumulative Distribution of a Continuous Random Variable

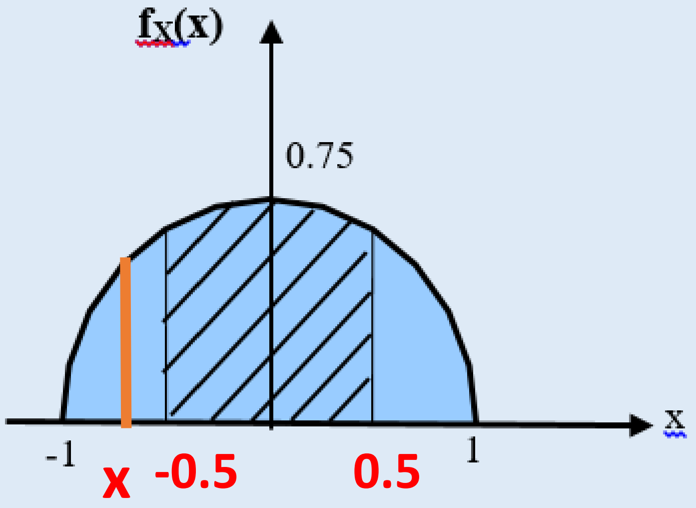
Let X be a random variable with the pdf: $f_X(x) = 0.75(1 - x^2)$, $-1 \le x \le 1$

1- Verify that fx(x) is indeed a valid pdf.

2- Find:

a- $F_X(x)$

b- $P(-0.5 \le X \le 0.5)$



SOLUTION:

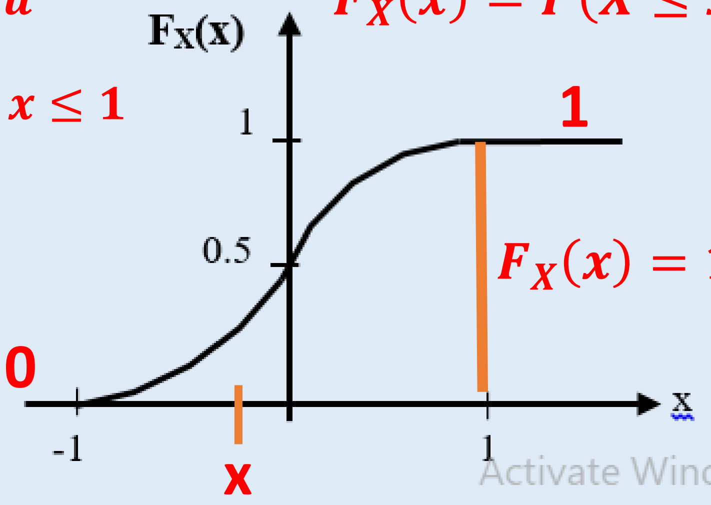
1- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \;\Rightarrow\; 2\int_{0}^{1} 0.75\,(1 - x^2)\, dx$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 2 \times (0.75\, u - 0.75 \frac{u^3}{3}\Big|_0^1 = 1.0$$

$$F_X(x) = \int_{-1}^{x} f_X(u) du$$

$$F_X(x) = P(X \le x)$$

2-a) $F_X(x) = \int_{-\infty}^{x} 0.75\,(1 - u^2)\, du = 0.5 + 0.75x - 0.25\,x^3$ $\quad -1 \le x \le 1$



$$F_X(x) = 1, x \ge 1$$

2-b) $P(-0.5 \le X \le 0.5) = \int_{-0.5}^{0.5} 0.75\,(1 - u^2)\, du$

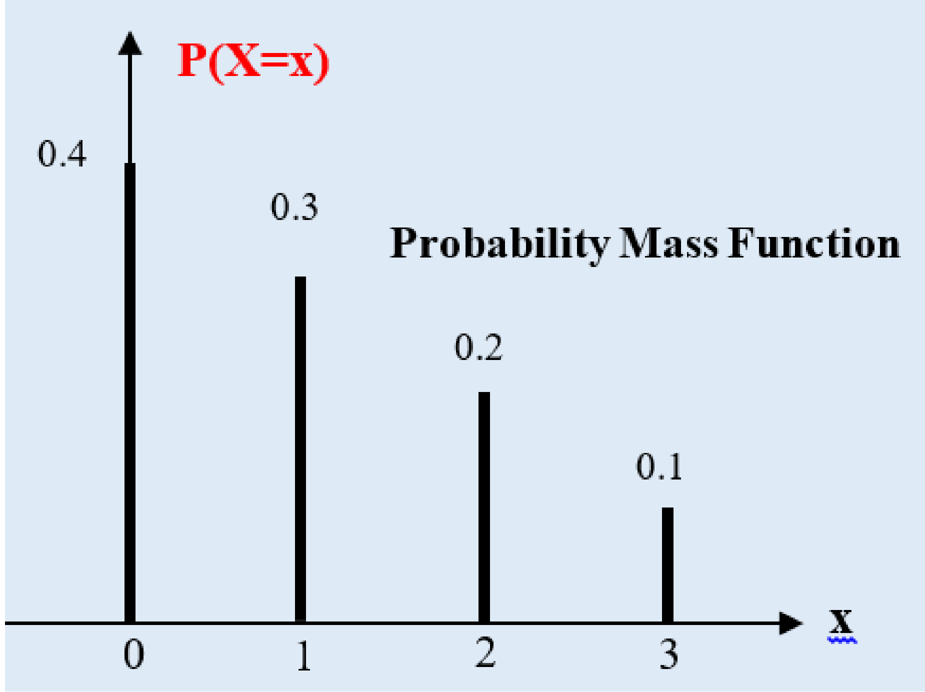
$$P(-0.5 \le X \le 0.5) == F_X(0.5) - F_X(0.5) = 0.6875$$

EXERCISE: Find x_0 such that $F_X(x_0) = \mathrm{P}(X \le x_0) = 0.95$

$$P(X \le x_0) = 0.95 = 0.5 + 0.75x_0 - 0.25\,x_0^3 \Rightarrow x_0 \cong 0.73$$

Examples on Discrete Random Variables

The pmf of a random variable X is as shown in the figure. For this distribution, we can compute a number of probabilities as:



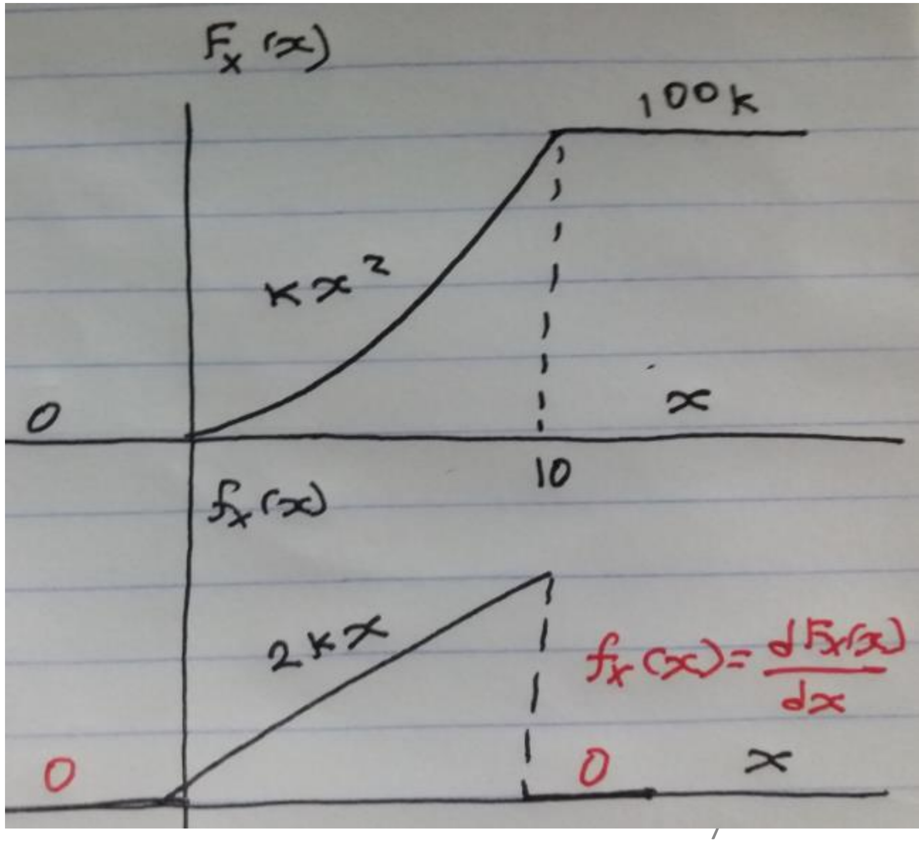
- $P(X \le 0.5) = P(X = 0) = 0.4$
- $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.9$
- $P(X < 2) = P(X = 0) + P(X = 1) = 0.7$
- $P(1 \le X \le 2) = P(X = 1) + P(X = 2) = 0.5$
- $P(1 \le X < 2) = P(X = 1) = 0.3$

Example on the Cumulative Distribution Function

- **Example:** Let X be a continuous random variable that has the following cumulative distribution function

$$F(x) = \begin{cases} 0 & x \le 0 \\ Kx^2 & 0 < x \le 10 \\ 100K & x > 10 \end{cases}$$

- Find K so that F(x) is a valid cumulative distribution function.
- Find P(X ≤ 5).
- Find the probability density function
- **Solution:** From the properties of the CDF, we should have
- $F_X(10) = 1 = 100K \Rightarrow K = 1/100$
- $F_X(5) = P(X \le 5) = \left(\frac{1}{100}\right)5^2 = \frac{1}{4}.$
- $\text{f}(x) = \frac{d}{dx}F(x) = \begin{cases} 0 & x \le 0 \\ \frac{2}{100}x & 0 < x \le 10 \\ 0 & x > 10 \end{cases}$



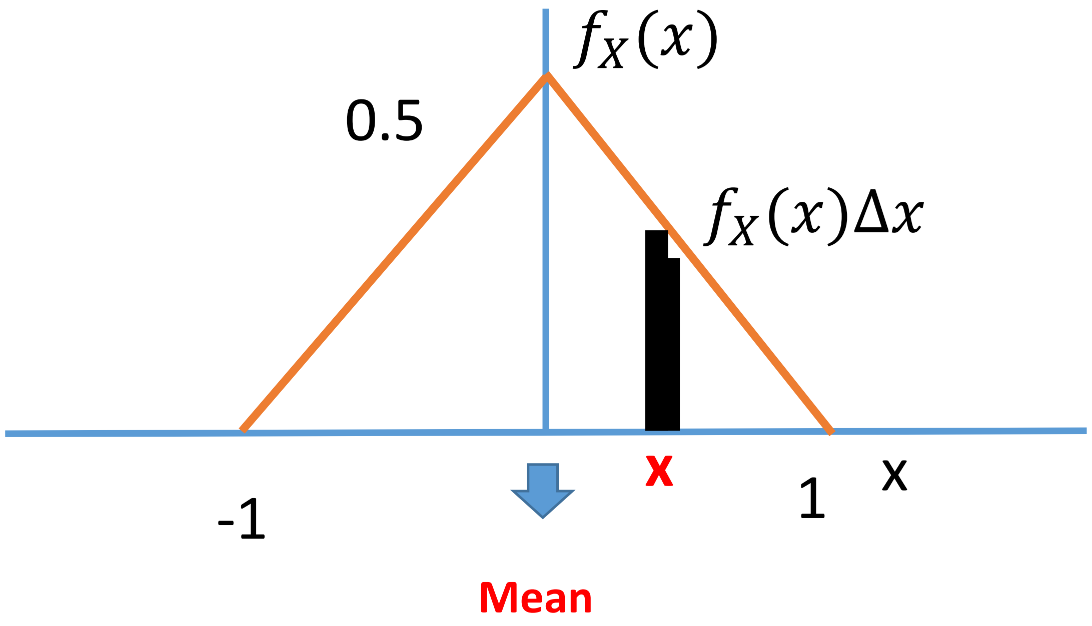
Mean and Variance of a Random Variable

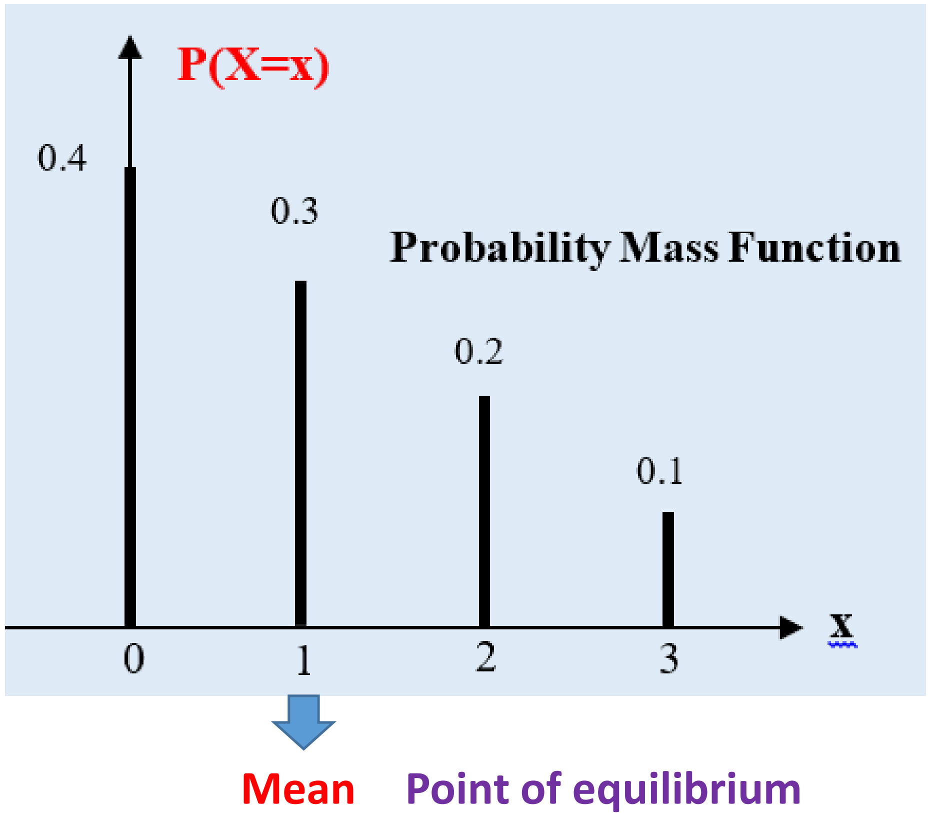
Definition: The mean value or expected value or average value of a random variable X is defined as:

$$\mu_X = E\{X\} = \sum x_i \, P(X = x_i) \qquad \text{if X is discrete}$$

$$\mu_X = E\{X\} = \int_{-\infty}^{\infty} x \, f_X(x) \, dx \qquad \text{if X is continuous}$$

The mean is analogous to the center of mass of a weight distribution





Mean and Variance of a Random Variable

Definition: The **variance** of a random variable X is defined as:

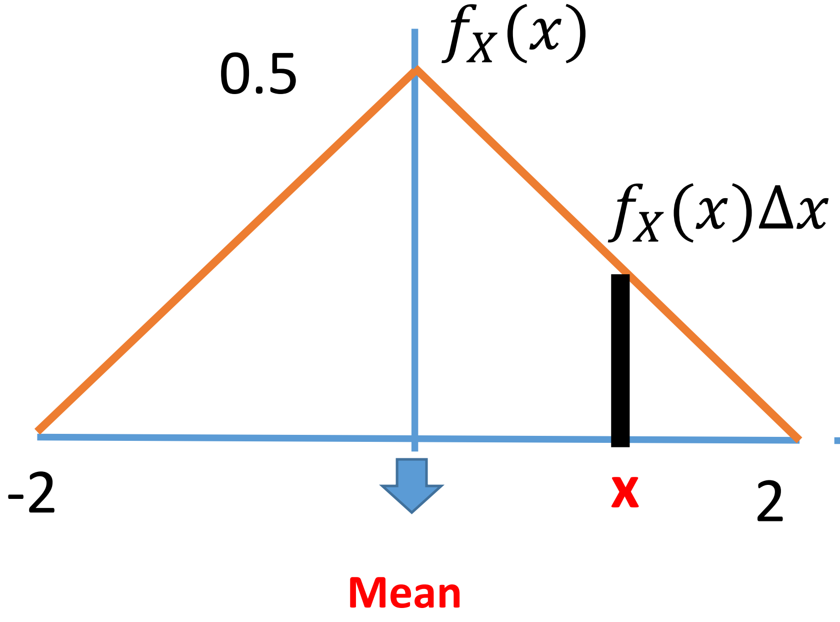
$$\sigma_X^2 = E\{(X-\mu_x)^2\} = \sum (X-\mu_x)^2 \; P(X=x_i) \quad \text{if X is discrete}$$

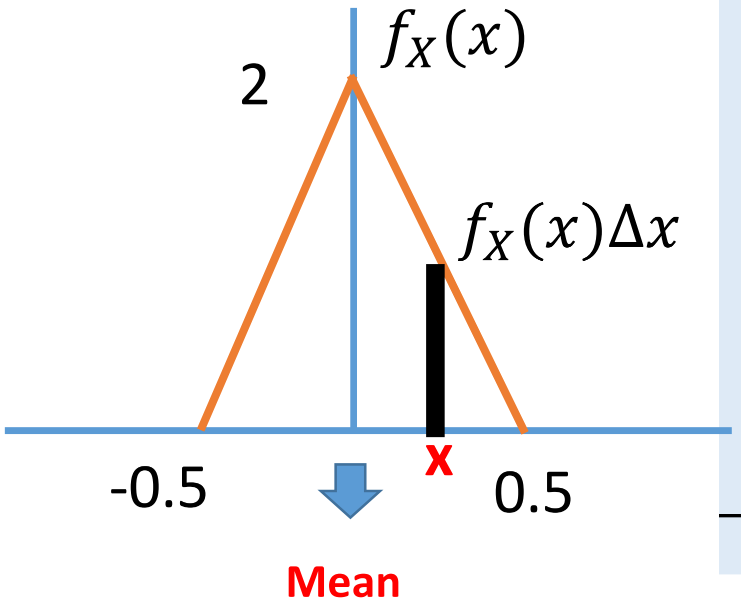
$$\sigma_X^2 = E\{(X-\mu_x)^2\} = \int_{-\infty}^{\infty} (x-\mu_x)^2 \; f_X(x)\,dx \quad \text{if X is continuous}$$

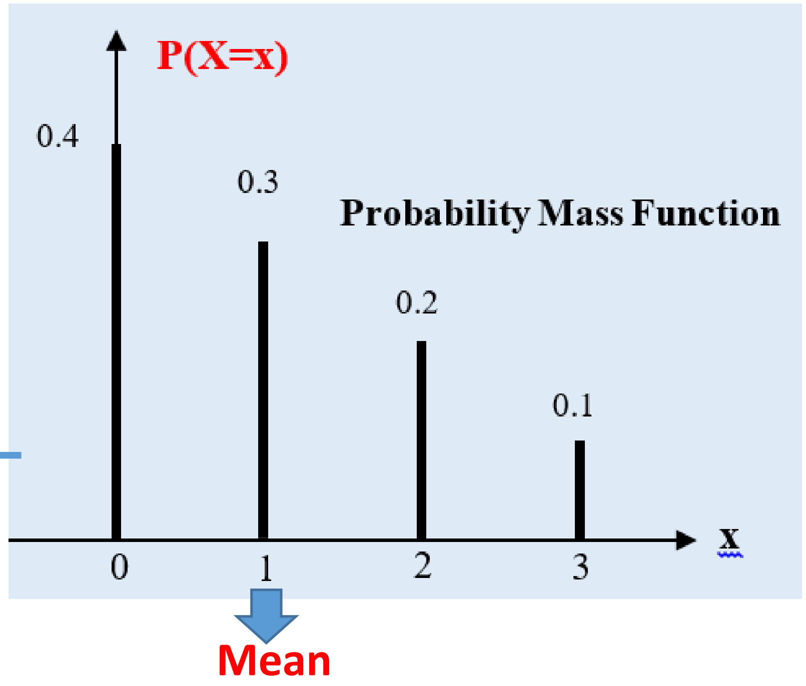
$$\sigma_X = \sqrt{\sigma_X^2} \quad \text{is the standard deviation}$$

The variance is analogous to the centralized moment of inertia

The variance is the measure of the spread of the distribution.







Mean and Variance of a Random Variable

Definition: For any random variable (X) and any continuous function Y = g (X), the expected value of g(X) is defined as:

$$E\{g(X)\} = \sum g(x_i)\, P(X = x_i);\ \text{x is discrete}$$

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx;\ \text{x is continuous}$$

Examples of $g(X)$ are: $g(X) = X$; $g(X)=X^2$; $g(X) = (X - \mu_X)^2$; $g(X) = (X - \mu_X)/\sigma_X$

Theorem: Let (X) be a random variable with mean μ_X, then:

$$\sigma_X^2 = E(X^2) - \mu_X^2$$

Proof: We assume that X is a continuous random variable (for a discrete r.v. we replace integration with summation and the result is the same)

$$\sigma_X^2 = E\{(X - \mu_X)^2\} = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\, dx = \int_{-\infty}^{\infty} (x^2 - 2x\mu_X + \mu_X^2) f_X(x)\, dx$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx - 2\mu_X \int_{-\infty}^{\infty} x\, f_X(x)\, dx + \mu_X^2 \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$\sigma_X^2 = E(X^2) - 2\mu_X\mu_X + \mu_X^2 \Rightarrow \sigma_X^2 = E(X^2) - \mu_X^2$$

Analogous to the parallel axis theorem: centralized moment of inertia equals the centralized plus the square of the center of mass

Theorem: Linear Transformation of a Random Variable

Let (X) be a random variable with mean μ_X and variance σ_X^2.

Define $Y = aX + b$; (a) and (b) are real constants, then:

$$\mu_Y = a\,\mu_x + b \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\sigma_Y^2 = a^2\,\sigma_X^2 \quad \ldots\ldots\ldots\ldots \quad (2)$$

Here, we only find the mean and variance of Y. In a later lecture, we will find also the pdf of Y.

Proof: We assume that X is a continuous random variable (for a discrete r.v. we replace integration with summation and the result is the same)

$$\mu_X = E\{X\} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

1- $$\mu_Y = E\{aX + b\} = \int_{-\infty}^{\infty} (ax + b)\, f_X(x)\, dx$$

$$= a\int_{-\infty}^{\infty} x\, f_X(x)\, dx + b\int_{-\infty}^{\infty} f_X(x)\, dx \;\rightarrow\; \mu_Y = a\,\mu_x + b$$

The variance is not influenced by the constant b. Only the mean is affected.

2- $$\sigma_Y^2 = E\{(Y - \mu_Y)^2\}$$

$$= E\{[(ax + b) - (a\mu_X + b)]^2\} = E\{[a(x - \mu_X)]^2\}$$

$$= a^2 E\{(x - \mu_X)^2\} \;\rightarrow\; \boldsymbol{\sigma_Y^2 = a^2\,\sigma_X^2}$$

Some Useful Properties of Expectation

$$E\{a\} = a \qquad \text{a is a constant}$$

$$E\{b\, g(X)\} = b\, E\{g(X)\} \qquad \text{b is a constant}$$

$$E\{ a g_1(X) + b g_2(X)\} = aE\{ g_1(X)\} + bE\{g_2(X)\}$$

Proof of the third result

$$E\{ a g_1(X) + b g_2(X)\} = \int_{-\infty}^{\infty} (a g_1(X) + b g_2(X)) f_X(x) dx$$

$$= \int_{-\infty}^{\infty} (a g_1(X)) f_X(x) dx + \int_{-\infty}^{\infty} (b g_2(X)) f_X(x) dx = aE\{ g_1(X)\} + bE\{g_2(X)\}$$

Examples

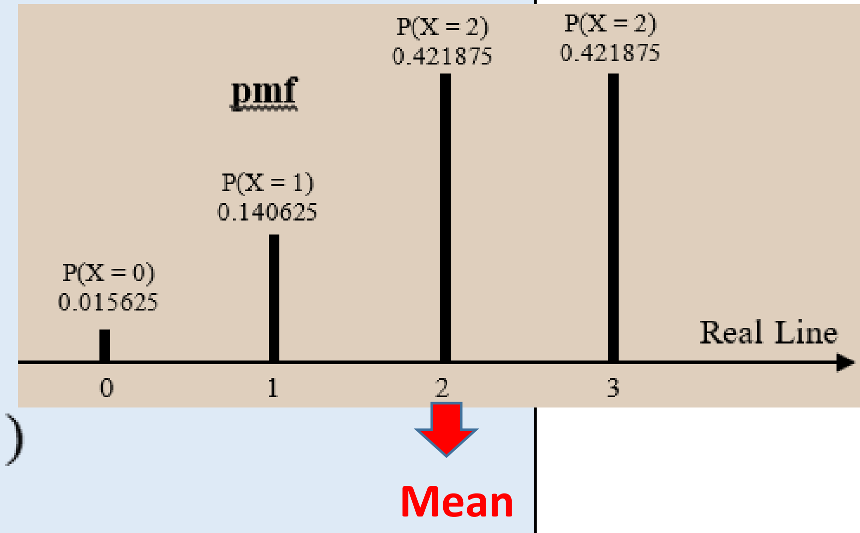
$$E\{ 2X + 3\} = 2E\{ X\} + 3$$

$$E\{ 2X^2 + 3X - 1\} = 2E\{ X^2\} + 3E\{ X\} + 1$$

$$E\{(X-1)^2 + e^x + 4\} = E\{ (X-1)^2\} + E\{e^x\} + 4$$

EXAMPLE: Mean and Variance of a Discrete Random Variable

Find the mean and the variance of the random variable with the pmf in the table below.



SOLUTION: Mean = $\mu_X = E\{X\} = \sum x_i \, P(X = x_i)$

$$\sum x_i \, P(X = x_i) = 2.25 = (3)(0.75) = np$$

$$Var(X) = \sigma_X^2 = E(X^2) - [E(X)]^2 \quad ; \quad E\{X^2\} = \sum x_i^2 \, P(X = x_i)$$

x	x^2	P(X = x)	x . P(X = x)	x^2 . P(X = x)
0	0	0.015625	0	0
1	1	0.140625	0.140625	0.140625
2	4	0.421875	0.843750	1.687500
3	9	0.421875	1.265625	3.796875
$\sum$			***2.25***	***5.625***

$$\sigma_X^2 = 5.625 - (2.25)^2 = 0.5625 = np(1-p)$$

EXAMPLE: Mean and Variance of the Uniform Distribution

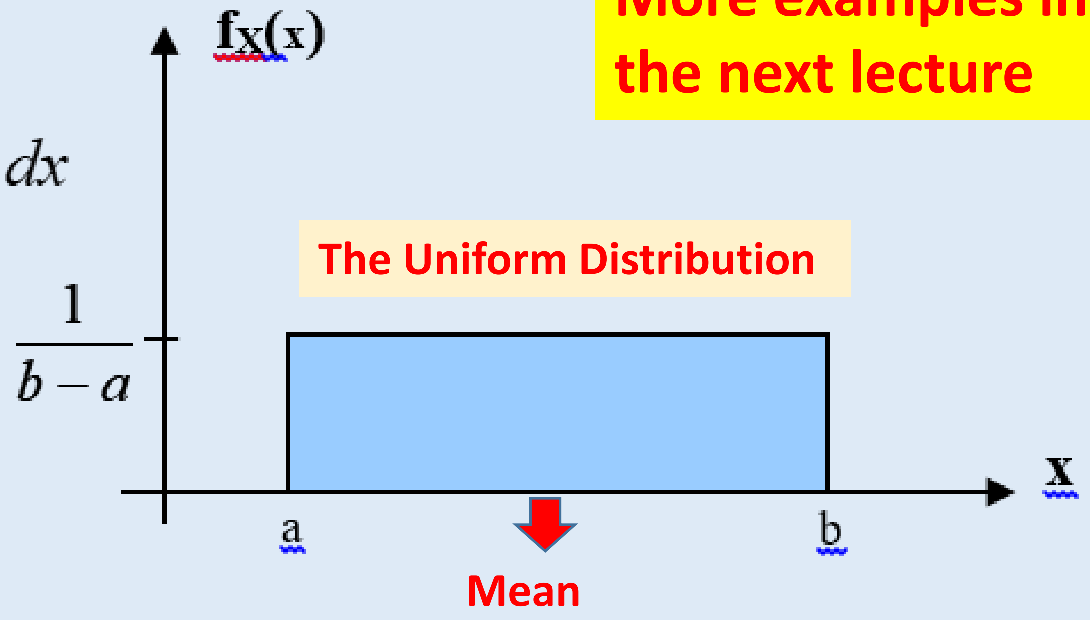
Find the mean and the variance of the uniform distribution shown in the figure.

SOLUTION:

More examples in the next lecture

$$Mean(X) = \mu_X = E\{X\} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$\mu_X = \int_a^b x \frac{1}{b - a} dx = \frac{a+b}{2}$$



$$Var(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$$

$$E\{X^2\} = \int_a^b x^2 \frac{1}{b - a} dx = \frac{b^3 - a^3}{3(b - a)} = \frac{a^2 + ab + b^2}{3}$$

$$\sigma_X^2 = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

Note that as (b – a) becomes larger, the variance also becomes larger but the mean value remains the same.

Median and Mode of a Continuous Distribution

Definition: Median of a Continuous Distribution

For a continuous random variable (X), the median of the distribution of (X) is defined to be a point (x_0) such that:

$$P(X < x_0) = P(X \geq x_0) \Rightarrow \int_{-\infty}^{x_0} f_X(x)dx = \int_{x_0}^{\infty} f_X(x)dx = \frac{1}{2}$$

Definition: Mode of a Continuous Distribution

If a random variable (X) has a pdf $f_X(x)$, then the value of X at which $f_X(x)$ is maximum is called the mode of the distribution.

MODE: Most probable value of X

Median and Mode of a Continuous Distribution

EXAMPLE: Median and Mode of a Continuous Distribution

Find the median and the mode for the random variable X with pdf: $f_X(x) = 2xe^{-x^2}, x \geq 0$

SOLUTION: The *median* is a point (x_0) such that

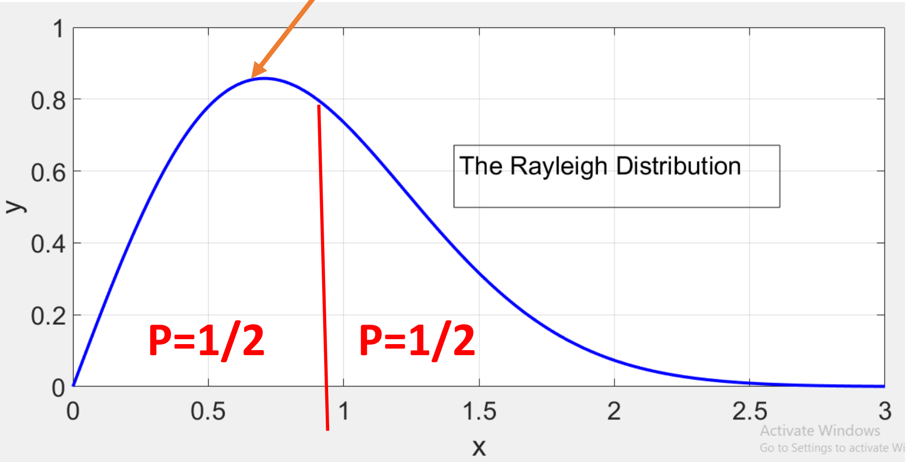
$$\int_0^{x_0} 2xe^{-x^2} dx = \int_{x_0}^{\infty} 2xe^{-x^2} dx = e^{-x_0{}^2} = 1/2$$

(x_0) is the solution to $e^{-x_0^2} = 0.5$ which results in $(x_0 = 0.832554)$

To find the *mode* we differentiate $f_X(x)$ with respect to x and set the derivative to zero

$$\frac{df(x)}{dx} = 2e^{-x^2} - 4x^2e^{-x^2} = 0,$$ the solution of which is $x_{mode} = 1/\sqrt{2}$.

Mode



X0=0.832: Median

Median and Mode of a Discrete Distribution

Definition: Median of a Discrete Distribution

For a discrete random variable (X), the median of the distribution of (X) is defined to be a point (x_0) such that:

$$P(X \le x_0) > 0.5$$

$$P(X \ge x_0) > 0.5$$

Definition: Mode of a Discrete Distribution

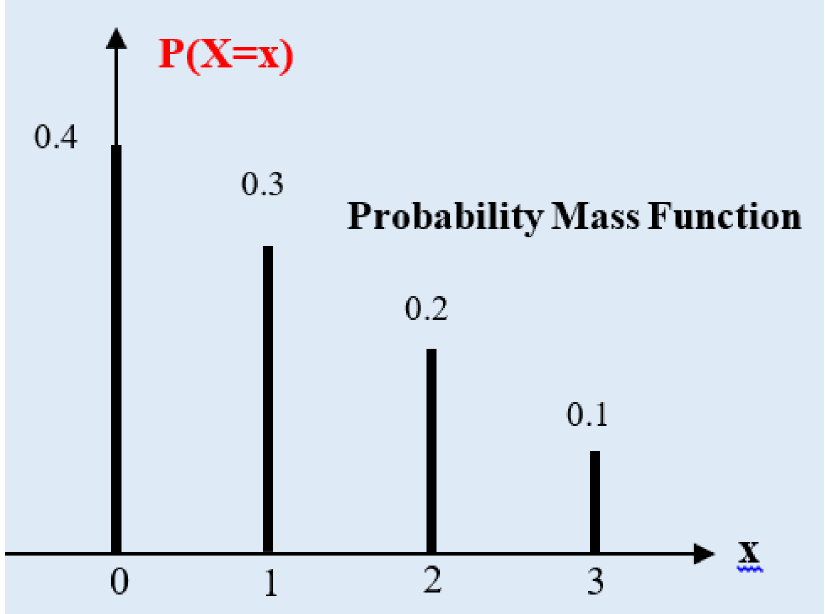
MODE: Most probable value of X

Given a discrete random variable X with a pmf $P(\mathrm{X} = \mathrm{x})$, its mode is the value $x_{\mathrm{mod}e}$ that is most likely to occur. Hence, the mode is equal to the value of $x_{\mathrm{mod}e}$ at which the probability mass function $P(\mathrm{X} = \mathrm{x})$ reaches its maximum.

Example: The mode of the distribution is the point **x=0** since it has the highest probability of occurrence.
As for the median, it is the point **x=1** since

$$P(X \le 1) = 0.4 + 0.3 = 0.7 > 0.5$$

$$P(X \ge 1) = 0.3 + 0.2 + 0.1 = 0.6 > 0.5$$



Review of Last Lecture and Additional Examples

From the Previous Lecture: Mean and variance

$$\mu_X = E\{X\} = \sum x_i\, P(X = x_i);\ \text{x is discrete}$$

$$\mu_X = E\{X\} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx;\ \text{x is continuous}$$

$$\sigma_X^2 = E\{(X - \mu_x)^2\} = \sum (X - \mu_x)^2\, P(X = x_i);\ \text{x is discrete}$$

$$\sigma_X^2 = E\{(X - \mu_x)^2\} = \int_{-\infty}^{\infty} (x - \mu_x)^2\, f_X(x)\, dx;\ \text{x is continuous}$$

Variance in terms of first and second moments

$$\sigma_X^2 = \mathrm{E}(X^2) - \mu_X^2$$

Linear Transformation of a random variable

Let (X) be a random variable with mean μ_X and variance σ_X^2.

Define $Y = aX + b$; (a) and (b) are real constants, then:

$$\mu_Y = a\,\mu_x + b \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\sigma_Y^2 = a^2\,\sigma_X^2 \quad \ldots\ldots\ldots\ldots \quad (2)$$

Expected value of a function of a random variable

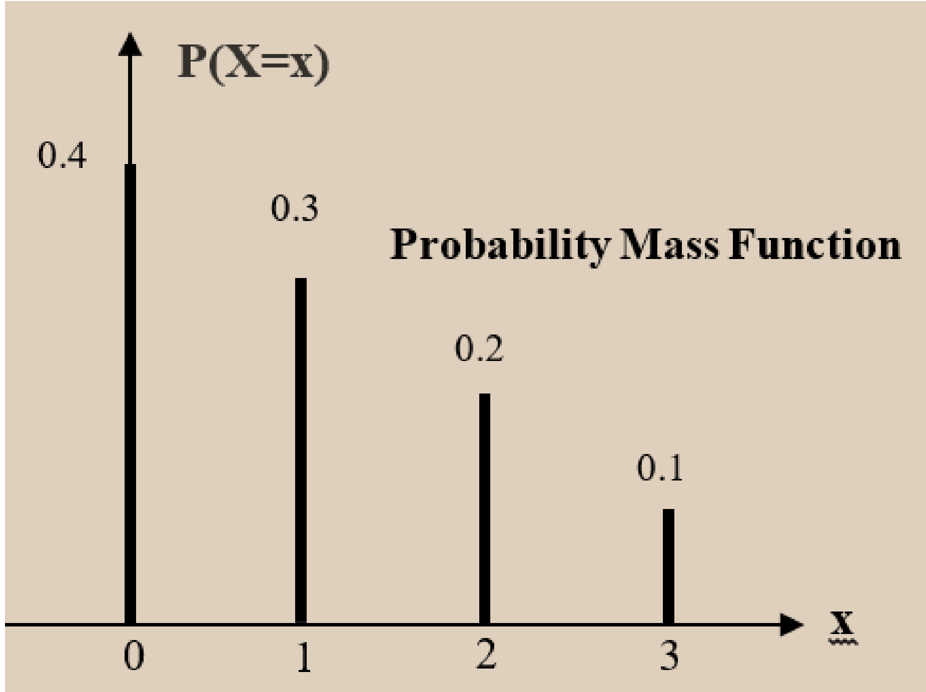
$$E\{g(X)\} = \sum g(x_i)\, P(X = x_i);\ \text{x is discrete}$$

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx;\ \text{x is continuous}$$

Review: Mean and Variance of a Discrete Random Variable

EXAMPLE: Let X be a discrete random variable with the following pmf:

$$P(X=0)=0.4\,,\qquad P(X=1)=0.3$$
$$P(X=2)=0.2\,,\qquad P(X=3)=0.1\,.$$



Find the mean and variance of X.

SOLUTION:

$$\mu_X = \boxed{E\{X\} = \sum x_i\, P(X = x_i)} = (0)(0.4)+(1)(0.3)+(2)(0.2)+(3)(0.1) = 1$$

$$\boxed{E\{X^2\} = \sum x_i^2\, P(X = x_i)} = 0(0.4)+1(0.3)+(2)^2(0.2)+(3)^2(0.1) = 2$$

$$Var(X) = \boxed{\sigma_X^2 = E(X^2) - [E(X)]^2} = 2-1 = 1$$

EXAMPLE: The Standard Random Variable

Let X be a r.v. with mean μ_X and variance σ_X^2. Define $\boxed{Z = \dfrac{X - \mu_X}{\sigma_X}}$

Find the mean and variance of Z.

SOLUTION:

Z can be written as: $\boxed{Z = \dfrac{X}{\sigma_X} - \dfrac{\mu_X}{\sigma_X}}$ which is of the form $Z = aX + b$

$$Mean(Z) = \mu_Z = E\{Z\} = \frac{1}{\sigma_X} E\{(X - \mu_X)\} = \frac{1}{\sigma_X}\{E(X) - E(\mu_X)\} = 0$$

$$Var(Z) = \sigma_Z^2 = \frac{1}{\sigma_X^2}\sigma_X^2 = 1$$

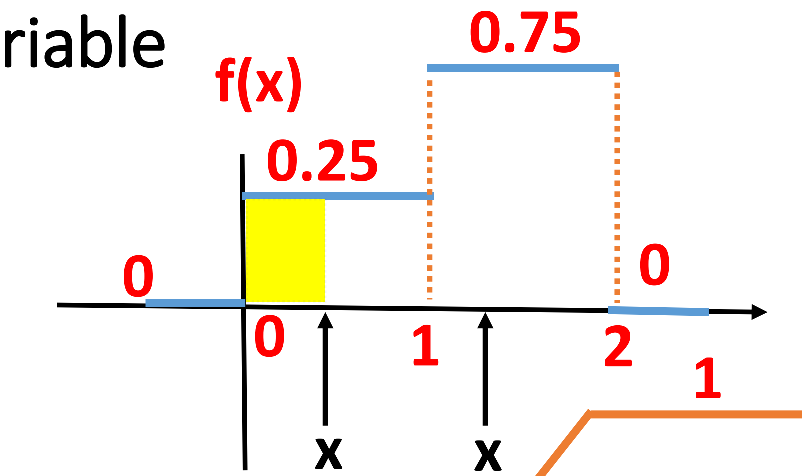
Let (X) be a random variable with mean μ_X and variance σ_X^2.
Define $Y = aX + b$; (a) and (b) are real constants, then:

$$\mu_Y = a\,\mu_x + b \quad \ldots\ldots\ldots\ldots \quad (1)$$

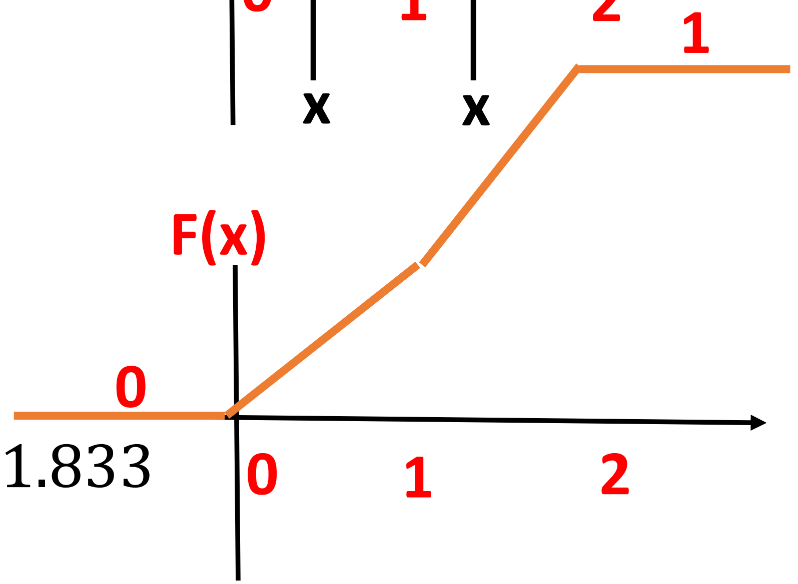
$$\sigma_Y^2 = a^2\,\sigma_X^2 \quad \ldots\ldots\ldots\ldots \quad (2)$$

Hence, the transformation above results in a standard random variable with mean 0 and variance = 1

Example on a Continuous Random Variable



- The pdf of a random variable X is: $f(x) = \begin{cases} 0.25, & 0 \le x \le 1 \\ 0.75, & 1 \le x \le 2 \\ 0 & otherwise \end{cases}$
    - Find the mean and variance
    - Find $\mathrm{P}(\mathrm{X} \le 1.5)$
    - Construct the cumulative distribution function



- $\mathrm{E(X)} = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 (0.25) x dx + \int_1^2 (0.75) x dx = 1.25$
- $\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^1 (0.25) x^2 dx + \int_1^2 (0.75) x^2 dx = 1.833$
- $\sigma^2 = \mathrm{E}(X^2) - \mu^2 = 1.833 - (1.25)^2 = 0.27$

- $F(x) = \begin{cases} \int_0^x (0.25) dx = 0.25x & ; 0 \le x \le 1 \\ \int_0^1 (0.25) dx + \int_1^x (0.75) dx = 0.25 + 0.75(x-1) & ; 1 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$

Example: Mean Value of a Continuous Random Variable

EXAMPLE: In the kinetic theory of gases, the distance, X, that a molecule travels between collisions is described by the exponential density function

$$f_X(x) = \frac{1}{\lambda}\, e^{\frac{-x}{\lambda}} \quad x > 0$$

Find the mean free path, defined as the average distance between collisions,

Solution: Mean Free Path is calculated as:

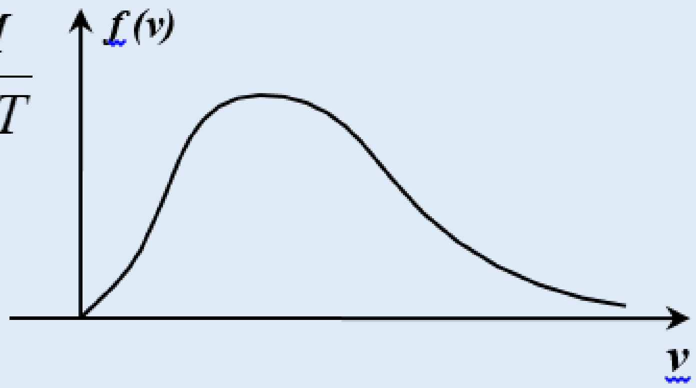
$$\boxed{\mu_X = E\{X\} = \int_0^{\infty} x\, f_X(x)\, dx}$$

; Use integration by parts to reach the final answer

$$= \int_0^{\infty} x \left(\frac{1}{\lambda}\right) e^{\frac{-x}{\lambda}}\, dx = \lambda$$

Maxwell's Distribution Law: The speed of gas molecules follows the distribution:

$$f(v) = 4\pi \left(\frac{k}{2\pi}\right)^{\frac{3}{2}} v^2 \, e^{\frac{-k\,v^2}{2}} \qquad v \geq 0; \quad k = \frac{M}{R\,T}$$



where v is the molecular speed
T is the gas temperature in Kelvin
R is the gas constant (8.31 J/mol.K)
M is the molecular mass of the gas

a- Find the average speed, $\bar{v}$
b- Find the root mean square speed v_{rms}
c- Find the most probable speed

SOLUTION:

a- $$\bar{v} = E(v) = \int_0^{\infty} v\, f(v)\, dv = \sqrt{\frac{8}{\pi\, k}}$$

b- $$E\{v^2\} = (v_{RMS})^2 = \int_0^{\infty} v^2 f(v)\; dv = \frac{3}{k} \Rightarrow v_{RMS} = \sqrt{\frac{3}{k}}$$

c- The most probable speed (mode of the distribution) is the speed at which $f(v)$ attains its maximum value. Therefore, we differentiate $f(v)$ with respect to (v), set the derivative to zero and solve for the maximum. The result is: $v_{\text{mode}} = \sqrt{\frac{2}{k}}$

The Binomial Distribution

Definition: The Binomial Experiment

Consider the random experiment consisting of n repeated trials such that:

a. The trials are independent.
b. Each trial results in only two possible outcomes, a success with probability p and a failure with probability (1-p)
c. The probability of a success, p, on each trial remains constant.

This experiment is called the binomial experiment.

1	**2**	**3**	**4**	**.**	**n**
(S,F)	**(S,F)**	**(S,F)**	**(S,F)**	**(S,F)**	**(S,F)**

The Binomial Distribution

Let X be the random variable representing the number of success in the n repeated trials. The random variable X is called the binomial distribution with parameters n and p and its probability mass function (pmf) is given as:

$$P(X=x)=\boxed{\binom{n}{x}}\boxed{p^{x}(1-p)^{n-x}},x=0,1,\ldots,n$$

$\binom{n}{x}$: Number of sequences with x successes, order not important
$p^x(1-p)^{n-x}$: probability of any one of the sequences

The mean and variance of X are
E(X) = np;   Var(x) = np(1-p)

Illustration of the Binomial Distribution

Let n in the above experiment be n = 4.

The possible outcomes along with their probabilities are given in the table below.

1	2	3	4
(S,F)	(S,F)	(S,F)	(S,F)

Sample Outcome	Probability	Value of X	Probability of X
FFFF	$(1-p)^4$	X = 0	$\binom{4}{0}(1-p)^4$
FFFS	$p(1-p)^3$	X=1	$4p(1-p)^3 = \binom{4}{1}p(1-p)^3$
FFSF	$p(1-p)^3$	X=1	
FSFF	$p(1-p)^3$	X=1	
SFFF	$p(1-p)^3$	X=1	
FFSS	$p^2(1-p)^2$	X=2	$6p^2(1-p)^2 = \binom{4}{2}p^2(1-p)^2$
FSSF	$p^2(1-p)^2$	X=2	
SSFF	$p^2(1-p)^2$	X=2	
SFFS	$p^2(1-p)^2$	X=2	
SFSF	$p^2(1-p)^2$	X=2	
FSFS	$p^2(1-p)^2$	X=2	
SSSF	$p^3(1-p)^1$	X=3	$4p^3(1-p)^1 = \binom{4}{3}p^3(1-p)^1$
SSFS	$p^3(1-p)^1$	X=3	
SFSS	$p^3(1-p)^1$	X=3	
FSSS	$p^3(1-p)^1$	X=3	
SSSS	p^4	X=4	$\binom{4}{4}p^4 = p^4$

Arrangements of elements of two different types k, n-k

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

0	0	1	1
0	1	1	0
1	1	0	0
1	0	0	1
1	0	1	0
0	1	0	1

$$\binom{4}{2} = \frac{4!}{2!\,(2)!} = 6$$

$$\binom{n}{x} p^x (1-p)^{n-x}$$

EXAMPLE: Suppose that the probability that any particle emitted by a radioactive material will penetrate a certain shield is 0.02. If 10 particles are emitted, find the probability that

a- Exactly one particle will penetrate the shield.
b- At least two particles will penetrate the shield.

SOLUTION: $\boxed{P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}}, n=10, p=0.02$

P=0.02

Radioactive Substance
10 particles are emitted

a- $P(X=1)=\binom{10}{1}(0.02)^1(1-0.02)^{10-1}$

b- $P(X\geq 2)=\sum_{x=2}^{10}\binom{10}{x}(0.02)^x(1-0.02)^{10-x}$

Note that

$$P(X=0)+P(X=1)+P(X\geq 2)=1$$

$$P(X\geq 2)=[1-P(X=0)-P(X=1)]$$

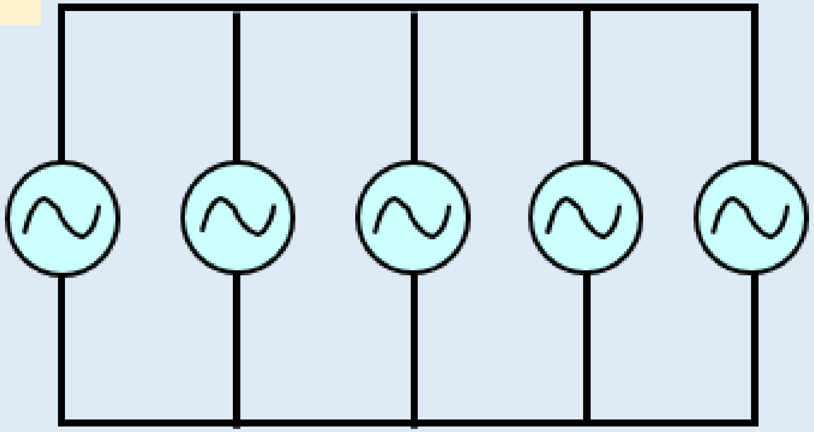
$$P(X\geq 2)=1-\left[\binom{10}{0}(0.02)^0(1-0.02)^{10-0}+\binom{10}{1}(0.02)^1(1-0.02)^{10-1}\right]$$

Usually, the shield is made of lead

EXAMPLE: Reliability of a Parallel System

Consider the parallel system shown in the figure. The system works if at least three of the five machines making up the system work. Find the reliability of the system assuming that the reliability of each unit is 0.9 over a given period.

SOLUTION: $P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$, n=5, p=0.9



Let X be the number of operating machines.

X: has a binomial distribution.

P(system works) = P(number of operating machines $\geq$ 3)

$$P(X\geq 3)=P(X=3)+P(X=4)+P(X=5)$$

$$P(system\ works)=P(X\geq 3)=\binom{5}{3}(p)^3(1-p)^2+\binom{5}{4}(p)^4(1-p)+\binom{5}{5}(p)^5$$

$$P(X\geq 3)=\binom{5}{3}(0.9)^3(1-0.9)^2+\binom{5}{4}(0.9)^4(1-0.9)+\binom{5}{5}(0.9)^5=0.9914$$

EXAMPLE: The process of manufacturing screws is checked every hour by inspecting 10 screws selected at random from the hour's production. If one or more screws are found defective, the production process is halted and carefully examined. Otherwise, the process continues. From experience, it is known that 1% of the screws produced are defective.

a. Find the probability that the production process continues at the end of an hour.
b. Find the probability that the production process is not halted for two consecutive hours.

SOLUTION: Let X be the number of defective items in the sample of 10 items.

a. P(system is not halted in one hour) = P(number of defective items is zero)

$$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$$, n=10, p=0.01

$$P(X=0)=\binom{10}{0}(p)^0(1-p)^{10-0}$$

$$P(X=0)=\binom{10}{0}(0.01)^0(0.99)^{10-0}=(0.99)^{10}=0.9043$$

First hour	Second hour
P(X=0)	P(X=0)

P(A∩B)=P(A)P(B); independent events

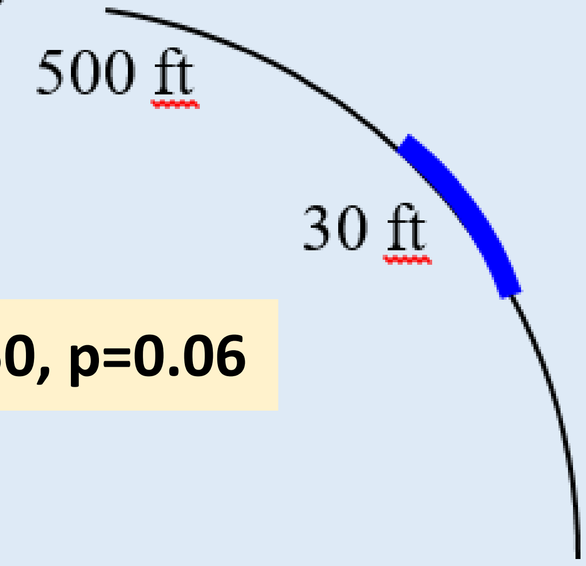
b. P(system is not halted for two hour) = P(X=0 in first hour)(PX=0) in second hour)

$$P(\textit{process continues for two hours})=(0.9043)(0.9043)$$

EXAMPLE: The captain of a navy gunboat orders a volley of 50 missiles to be fired at random along a 500-foot stretch of shoreline that he hopes to establish as a beachhead. Dug into the beach is a 30-foot long bunker serving as the enemy's first line of defense.

a. What is the probability that exactly three shells will hit the bunker?

b. Find the number of shells expected to hit the bunker.



SOLUTION: Let X: be the number of shells that hit the bunker

$$P(success) = \frac{30}{500} = 0.06$$

$$\boldsymbol{P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}}, \textbf{ n=50, p=0.06}$$

$$P(\text{x } successes\ in \text{ n trials}) = \binom{n}{x} (p)^x (1-p)^{n\text{ - x}}$$

For p = 0.06 and n = 50

$$P(3\ successes\ in\ 50\ shells) = \binom{50}{3} (0.06)^3 (1-0.06)^{25\text{ - }3} = \binom{50}{3} (0.06)^3 (0.94)^{22}$$

b. E(X) = n p = 50(0.06) = 3 shells.

Theorem: Mean Value of the Binomial Distribution

If (X) is a binomial r.v with parameters (n) and (p), then the expected value of X is

$$\boxed{\mu_X = E(X) = n\,p}$$

$$\mu_X = E\{X\} = \sum x_i\, P(X = x_i);\ \text{x is discrete}$$

Proof: $\mu_X = E(X) = \sum_{x=0}^{n} x \binom{n}{x} (p)^x (1-p)^{n-x} = \sum_{x=1}^{n} x \dfrac{n!}{x!(n-x)!} (p)^x (1-p)^{n-x}$

$$\mu_X = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} (p)^x (1-p)^{n-x}$$

Change of Variables: let $u = x - 1$. In terms of u, the summation above can be expressed as

$$\mu_X = \sum_{u=0}^{n-1} \frac{n(n-1)!}{u!(n-1-u)!} (p)^{u+1} (1-p)^{n-1-u}$$

$$\sum_{u=0}^{m} \binom{m}{u} p^u (1-p)^{m-u} = 1$$

Binomial with parameters m, p

Now, make the substitution $m = n - 1$ and take n and p out of the summation, we get

$$\mu_X = np \sum_{u=0}^{m} \frac{m!}{u!(m-u)!} (p)^u (1-p)^{m-u} = np \sum_{u=0}^{m} \binom{m}{u} (p)^u (1-p)^{m-u} = np$$

Note that the summation on the right hand side is one since this is the summation of all probabilities of a binomial distribution with parameters m and p.

Theorem: Variance of the Binomial Distribution

If (X) is a binomial r.v with parameters (n) and (p), then the variance of X is

$$\boxed{Var(x) = \sigma_X^2 = n\,p\,(1-p)}$$

Proof: We find it convenient to first evaluate the term $E(X(X-1))$ as follows:

$$E(X(X-1)) = \sum_{x=0}^{n} x(x-1)\binom{n}{x}(p)^x(1-p)^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1)\frac{n!}{x!(n-x)!}(p)^x(1-p)^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1)\frac{n!}{x(x-1)(x-2)!(n-x)!}(p)^x(1-p)^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!}(p)^x(1-p)^{n-x}$$

Change of Variables: As we have done before, let $u = x-2$ or $x = u+2$. The summation above becomes

$$E(X(X-1)) = \sum_{u=0}^{n-2} \frac{n(n-1)(n-2)!}{u!(n-2-u)!}(p)^{u+2}(1-p)^{n-2-u}$$

Theorem: Variance of the Binomial Distribution

$$E(X(X-1)) = \sum_{u=0}^{n-2} \frac{n(n-1)(n-2)!}{u!(n-2-u)!} (p)^{u+2}(1-p)^{n-2-u}$$

Next let $m = n-2$, and take out of the summation the terms n, (n-1) and p^2. The result is

$$E(X(X-1)) = n(n-1)p^2 \sum_{u=0}^{m} \frac{m!}{u!(m-u)!} (p)^{u}(1-p)^{m-u}$$

Again, the summation on the right hand side is one since it represents the sum of probabilities for a binomial distribution with parameters m and p. Therefore,

Binomial with parameters m, p

$$\sum_{u=0}^{m} \binom{m}{u} p^u (1-p)^{m-u} = 1$$

$$E(X(X-1)) = n(n-1)p^2$$

But $\quad E(X(X-1)) = E(X^2 - X) = E(X^2) - E(X)$

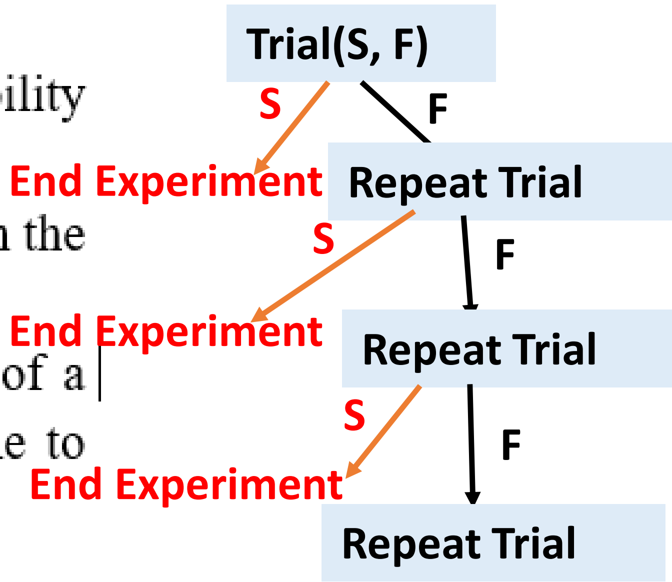
Or $\quad E(X^2) = E(X) + E(X(X-1))$

From which we conclude that: $\sigma_x{}^2 = E(X^2) - (\mu_X)^2 = np + n(n-1)p^2 - (np)^2$

This simplifies to $\quad \sigma_x{}^2 = np(1-p)$; $\quad$ This concludes the proof.

The Geometric Distribution

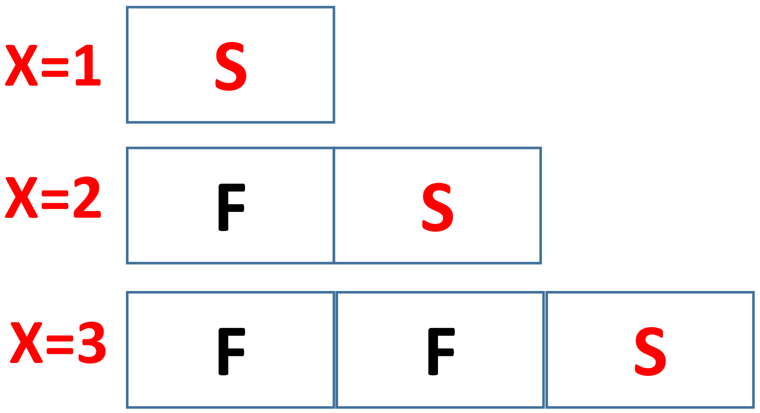
The Geometric Distribution



- Let the outcome of a trial be either a success with probability (p) or a failure with probability (1 – p). This is often called the Bernoulli trial.
- The trials are repeated independently until a success appears for the first time, at which the experiment ends.
- Let X be the number of times the experiment is performed to the first occurrence of a success. Then X is a discrete random variable with integer values ranging from one to infinity. The probability mass function of X is:

$$P(X=x) = P(\underbrace{F\ F\ F\ F\ \ldots F}_{x\,-\,1}\ S) = P(F)^{x-1}P(S)$$

$$= (1-p)^{(x-1)}p; \quad x = 1,\ 2,\ 3,\ \ldots$$



Details: Experiment Always Ends with a Success

X = 1 means a success was obtained on the first trial with prob. $P(X=1)=p$

X = 2 means a failure on first trial and a success on second: $P(X=2)=(1-p)p$

X = 3 means a failure on first two trials and a success on third: $P(X=3)=(1-p)^2 p$

The Geometric Distribution

Example: It is known that 5% of the items produced by a certain machine are defective. An inspector takes out one item at a time from the machine's production and examines it.

a. Find the probability that the first defective item is the fifth inspected item.

b. What is the probability that it takes the inspector less than 6 inspections to find a defective item.

Solution: p = 0.05 ; $P(X = x) = p(1-p)^{x-1}$

a. $P(X = 5) = (1-p)^{(5-1)}\, p = (0.95)^4(0.05)$

F	F	F	F	S

b. Here, we need to find

S		F	S		F	F	S		…….

$$P(X < 6) = P(X \leq 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$
$$P(X \leq 5) = (0.05) + (0.95)^1(0.05) + (0.95)^2(0.05) + (0.95)^3(0.05) + (0.95)^4(0.05)$$
$$P(X \leq 5) = 0.05[1 + (0.95)^1 + (0.95)^2 + (0.95)^3 + (0.95)^4]$$

Will prove this result in the next example: $P(X \leq k) = 1 - (1-p)^k$

$$P(X \leq 5) = 1 - (0.95)^5$$

The Geometric Distribution

Example: A production line has a 10 % defective rate. Find the probability that it takes 10 or more inspections to observe the first defective item.

Solution: p = 0.1, k = 10

F (1)	F (2)	…….	F (9)	S (10)

$P(X = x) = (1-p)^{(x-1)}\, p$

Need to find $P(X \geq k)$, for any p and k.

$$P(X \geq k) = P(X = k) + P(X = k+1) + P(X = k+2) + \cdots$$
$$P(X \geq k) = (1-p)^{k-1}(p) + (1-p)^{k}(p) + (1-p)^{k+1}(p) + \cdots$$
$$P(X \geq k) = p(1-p)^{k-1}[1 + (1-p)^1 + (1-p)^2 + (1-p)^3 + \cdots];$$

$$1 + u + u^2 + u^3 + \ldots = \frac{1}{1-u}$$

$$P(X \geq k) = p(1-p)^{k-1}\frac{1}{(1-(1-p))} = p(1-p)^{k-1}\frac{1}{p};$$ geometric series

$$\boldsymbol{P(X \geq k) = (1-p)^{k-1}}$$

In our example, p=0.1, k=10, hence $\boldsymbol{P(X \geq 10) = (0.9)^9 = 0.3874}$

From this we also conclude that: $\boldsymbol{P(X < k) = 1 - (1-p)^{k-1}}$

ALSO: $\boldsymbol{P(X \leq k) = 1 - (1-p)^{k}}$; **cumulative distribution function**

The Geometric Distribution

Example: A production line has a 20% defective rate. What is the minimum number of inspections, that would be necessary, so that the probability of observing a defective is more that 75%?

S(1)			
F(1)	S(2)		
F(1)	F(2)	S(3)	
F (1)	F (2)	…….	S (k)

Solution: p = 0.2

$P(X = x) = (1-p)^{(x-1)}\ p$

Need to find k so that $P(X \leq k) \geq 0.75$

From the previous example, we have

$\boldsymbol{P(X \geq k) = (1-p)^{k-1}}$

$\boldsymbol{P(X \leq k) = 1-(1-p)^{k}}$

What is the value of k such that

$$P(X \leq k) = 1-(1-0.2)^{k} \geq 0.75$$

This inequality is satisfied with k = 7 ($P(X \leq 7) = 0.79028$).

That is, we need at least 7 inspections to get the first defective item with a probability ≥ 0.75.

Theorem: Mean Value of the Geometric Distribution

The mean value of the geometric distribution with parameter p is given as:

$$\mu_X = E(X) = \frac{1}{p}$$

Proof: $\mu_X = E(X) = \sum_{x=1}^{\infty} x\, p(1-p)^{x-1} = p\left[1+2(1-p)+3(1-p)^2+4(1-p)^3+\ldots\right]$

$$= \frac{1}{(1-(1-p))^2} = \frac{1}{p^2}$$

Recall the geometric series

$$1+u+u^2+u^3+\ldots = \frac{1}{1-u}$$

The derivative of the series is

$$\frac{d}{du}\left(\frac{1}{1-u}\right) = \frac{1}{(1-u)^2} = 1+2u+3u^2+3u^2+\ldots$$

Making use of this result (with u = 1-p), the expected value of X becomes

$$\mu_X = p\frac{1}{(1-(1-p))^2} = \frac{1}{p}$$

Theorem: Variance of the Geometric Distribution

The variance of the geometric distribution with parameter p is given as $\sigma_X^2 = Var(X) = \dfrac{1-p}{p^2}$

Proof: To find the variance, we first find $E(X(X-1))$

$$E(X(X-1)) = \sum_{x=1}^{\infty} x(x-1)\, p(1-p)^{x-1} = p\{2(1)(1-p) + 3(2)(1-p)^2 + 4(3)(1-p)^3 + \ldots\}$$

$$E(X(X-1)) = p(1-p)\left[2(1) + 3(2)(1-p)^1 + 4(3)(1-p)^2 + \ldots\right]$$

$$= p(1-p)\frac{2}{p^3} = \frac{2(1-P)}{p^2}$$

Differentiating the geometric series twice with respect to u

$$\frac{d}{du}\left(\frac{1}{1-u}\right) = \frac{1}{(1-u)^2} = 1 + 2u + 3u^2 + 4u^3 + \ldots$$

$$\frac{d}{du}\left(\frac{1}{(1-u)^2}\right) = \frac{2}{(1-u)^3} = 2 + 3(2)u + 4(3)u^2 \ldots$$

Recall the geometric series

$$1 + u + u^2 + u^3 + \ldots = \frac{1}{1-u}$$

Making use of this result (with u = 1-p), we get $E(X(X-1)) = p(1-p)\dfrac{2}{p^3} = \dfrac{2(1-P)}{p^2}$

But, $E(X(X-1)) = E(X^2 - X) = E(X^2) - E(X) \Rightarrow E(X^2) = E(X) + E(X(X-1))$

From this we conclude that: $\sigma_x^2 = E(X^2) - (\mu_X)^2 = \dfrac{1}{p} + \dfrac{2(1-p)}{p^2} - \dfrac{1}{p^2} = \dfrac{(1-p)}{p^2}$

The Geometric Distribution

Example:

A production line has a 20% defective rate. What is the average number of inspections to obtain the first defective?

Solution: p = 0.2

$$P(X = x) = (1 - p)^{(x-1)}\, p$$

$$E(X) = \frac{1}{p} = \frac{1}{0.2} = 5$$

On the average, we need 5 inspections to get one defective item