Probability Distributions for Two Random Variable

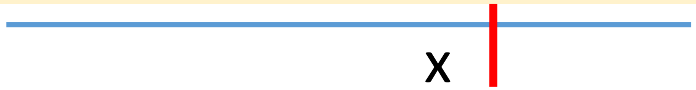
In certain experiments, we may be interested in observing several quantities as they occur.

Examples are:

- The magnitude and phase angle of the noise affecting the transmitted signal in a communication system.
- The altitude and speed of a moving target.
- The temperature, pressure, and wind speed at some location
- The input X to a system and the output Y.
- The length and width of a manufactured part.
- The diameter and thickness of a cylindrical disk model.

CDF of one variable X:

$$F_X(x) = P(X \le x)$$



In the case of two quantities X and Y, the outcome of the experiment is a point in the x-y plane.

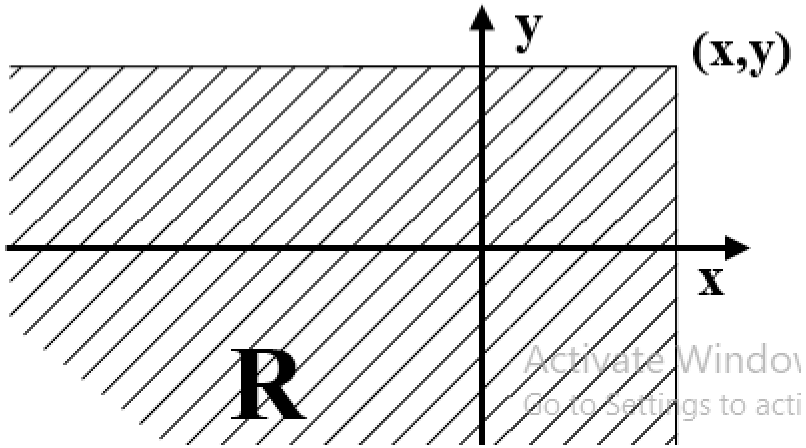
The joint cumulative distribution function of two r.v X and Y is defined as:

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

Note that $F_{X,Y}(x,y)$ is the intersection of the two events

Event A = $\{X \le x\}$

Event B = $\{Y \le y\}$



Discrete Two Dimensional Distribution:

Two random variables X and Y are called discrete if the sample space of the experiment can assume only countably finite or at most countably infinite pairs of values (x_1,y_1), (x_2,y_2), ……

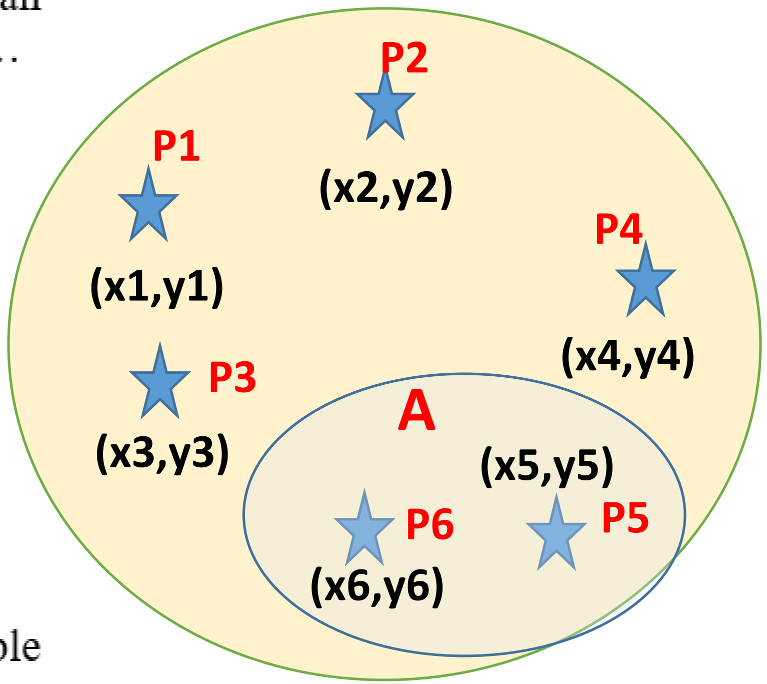
The joint probability mass function of (X) and (Y) is: $P(x,y) = P(X=x, Y=y)$

Such that: $P(X=x, Y=y) \geq 0$

$$\sum_{all\ x} \sum_{all\ y} P(X=x, Y=y) = 1$$

$$F_{X,Y}(x,y) = \sum_{u \leq x} \sum_{v \leq y} P(X=x, Y=y)$$

Same conditions as those for the single random variable.

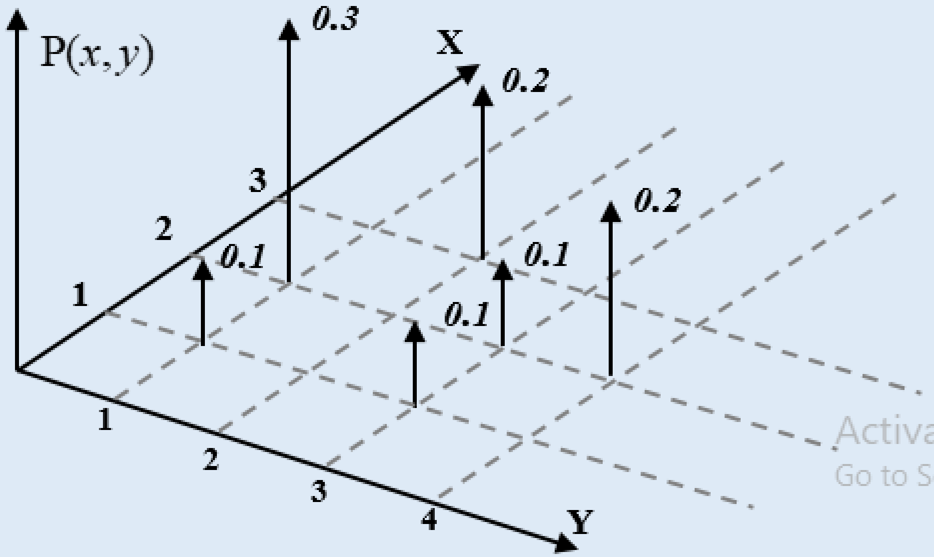


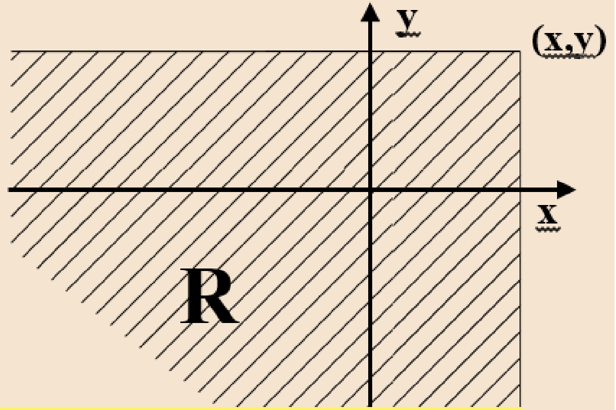
If A is an event defined on S, then $P(A) = \sum_{x} \sum_{y} P(X=x, Y=y);\ x, y \in A$

The joint probability mass function in the two-dimensional space is often represented in a table as shown in the figure.

$P(X=1, Y=1) = 0.1)$

X \ Y	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0





$$F_{X,Y}(x,y) = \sum_{u \leq x} \sum_{v \leq y} P(X=x, Y=y)$$

Relations Related to Two Random Variables

Marginal Distributions of a Discrete Distribution

$$P(X = x) = \sum_{all\ y} P(X = x, Y = y)$$

This is the probability that (X) may assume a value (x), while (Y) may assume any value, which we ignore. Likewise,

$$P(Y = y) = \sum_{all\ x} P(X = x, Y = y)$$

X \ Y	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

Independence of Discrete Random Variable

Two discrete random variables X and Y are independent if:

$$P(X = x, Y = y) = P(X = x)P(Y = y);\ \text{for all x and y}$$

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(X)\, f_X(x)\, dx$$

Expected Value of a Function of Two Random Variables X and Y

The expected value of a function g(x,y) of two random variables (X) and (Y) is

$$E\{g(x, y)\} = \sum_{all\ x} \sum_{all\ y} g(x, y)\, P(X = x, Y = y)$$

Probability Mass Function of a Function of Two Random Variables X and Y

When (X) and (Y) are discrete random variables, we may obtain the probability mass function $P\{Z = z\}$ of some function $Z = g(X,Y)$ by summing all probabilities for which $g(X,Y)$ equals the value of (z). That is,

$$P(Z = z) = \sum_{g(x,y)=z} \sum P(X = x , Y = y)$$

Will be investigated in a later lecture when we study the transformation of random variables

Correlation Coefficient

The term association is also often used

The correlation coefficient between two random variables (X) and (Y) is a measure of similarity between X and Y and is defined as

$$\rho_{XY} = \frac{E\{(X - \mu_X)(Y - \mu_Y)\}}{\sigma_X \ \sigma_Y} = \frac{E(XY) - \mu_X \mu_Y}{\sigma_X \ \sigma_Y}$$

ρ_{XY} is bounded between $-1 \leq \rho_{XY} \leq 1$

More on the correlation coefficient will given in future lectures

When $\rho_{XY} = 0$, (X) and (Y) are said to be uncorrelated.

When $\rho_{XY} = +1$, (X) and (Y) will have maximum positive correlation

When $\rho_{XY} = -1$, (X) and (Y) will have maximum negative correlation.

The magnitude of the correlation coefficient indicates the strength of the association. For example, a correlation of $\rho = 0.9$ suggests a strong, positive association between two variables, whereas a correlation of $\rho = -0.2$ suggest a weak, negative association . A correlation close to zero suggests no linear association between two variables.

EXAMPLE: The joint probability mass function of two random variables X and Y is given in the table below:

X \ Y	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

1- Find the marginal pmf $P(X=x)$ and $P(Y=y)$
2- Find $P(X \geq 2)$
3- Find the mean and variance for both X and Y

SOLUTION: Marginal pmf's

$$P(X=x)=\sum_{all\ y} P(X=x, Y=y),$$

$$P(Y=y)=\sum_{all\ x} P(X=x, Y=y)$$

$P(X=1)=0.2$ $\quad$ $P(Y=1)=0.4$

$P(X=2)=0.6$ $\quad$ $P(Y=2)=0.2$

$P(X=3)=0.2$ $\quad$ $P(Y=3)=0.2$

$P(Y=4)=0.2$

$$\boldsymbol{P(X=1)=P(X=1,Y=1)+P(X=1,Y=2)+P(X=1,Y=3)+P(X=1,Y=4)=0.2}$$

$$\boldsymbol{P(Y=1)=P(X=1,Y=1)+P(X=2,Y=1)+P(X=3,Y=1)=0.4}$$

$P(X \geq 2)=P(X=2)+P(X=3)=0.6+0.2=0.8$

The mean value and variance of X

$$\mu_X = E(X)=1(0.2)+2(0.6)+3(0.2)=2; \quad E(X^2)=1(0.2)+4(0.6)+9(0.2)=4.4$$

$$\sigma_X^2 = E(X^2) - E^2(X) = 4.4-4=0.4$$

The mean and variance of Y:

$$\mu_Y = E(Y)=1(0.4)+2(0.2)+3(0.2)+4(0.2)=2.2, \quad E(Y^2)=1(0.4)+4(0.2)+9(0.2)+16(0.2)=6.2$$

$$\sigma_Y^2 = E(Y^2) - E^2(Y) = 6.2-(2.2)(2.2)=1.36$$

EXAMPLE: The joint probability mass function of two random variables X and Y is given in the table below:

1- Are (X) and (Y) independent?
2- Find $E\{XY\}$
3- Find P(X > Y) and P(X = Y)
a. Find the correlation coefficient between X and Y.

X \ Y	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

$P(X=1)=0.2$

$P(Y=1)=0.4$

SOLUTION:

Check for independence: Consider all pairs of (x, y) for the condition starting with the point (1, 1).

$P(X=x,Y=y)\overset{?}{=}P(X=x)P(Y=y)\Rightarrow P(X=1,Y=1)\overset{?}{=}P(X=1)P(Y=1)\Rightarrow 0.1\neq(0.2)(0.4)$. Since the condition is violated at one point, we do not have to check all points. As such, X and Y are dependent.

$$E\{g(x,y)\}=\sum_{all\ x}\sum_{all\ y} g(x\ ,y)\ P(X=x,Y=y)$$

E (XY) = (1)(1)(0.1) + (1)(3)(0.1) + (2)(1)((0.3) + (2)(3)(0.1) + (2)(4)(0.2) +((3)(2)(0.2) = 4.4

Probability of Events: $P(A)=\sum_{x}\sum_{y} P(X=x,Y=y);\ x,y\in\ A$

Find P(X > Y): The set {X > Y} consists of all pairs that form the event A, where

A = {(2,1), (3,2), (3,1)}. Hence, P(A) = 0.3 + 0.2 +0 = 0.5

Find P(X = Y): The Set{X=Y} consists of the pairs {(1,1), (2,2), (3,3)} ⇒ P(X=Y) = 0.1 + 0 + 0 = 0.1

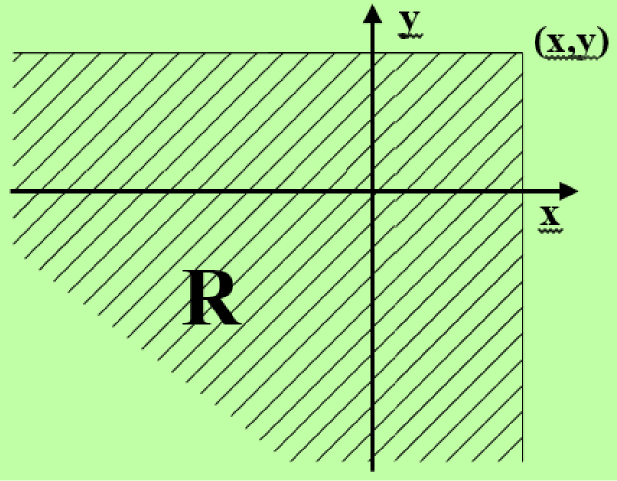
Correlation Coefficient: The correlation coefficient is defined as:

$$\rho_{XY}=\frac{E\{(X-\mu_X)(Y-\mu_Y)\}}{\sigma_X\ \sigma_Y}=\frac{E(XY)-\mu_X\mu_Y}{\sigma_X\ \sigma_Y}=\frac{4.4-(2)(2.2)}{\sqrt{0.4}\sqrt{1.36}}=0$$

Note that in this example, X and Y are uncorrelated, yet they are dependent

Continuous Two Dimensional Distributions

Two random variables X and Y are called continuous if the sample space of the experiment is uncountable and infinite. The joint cumulative distribution function $F_{X,Y}(x,y)$ of a continuous distribution is given as:



$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv$$

where, $f_{X,Y}(x,y)$ is the joint probability density function, with the following properties:

Properties of the joint pdf

Extension of the 1-D properties

1- $f_{X,Y}(x,y) \geq 0$

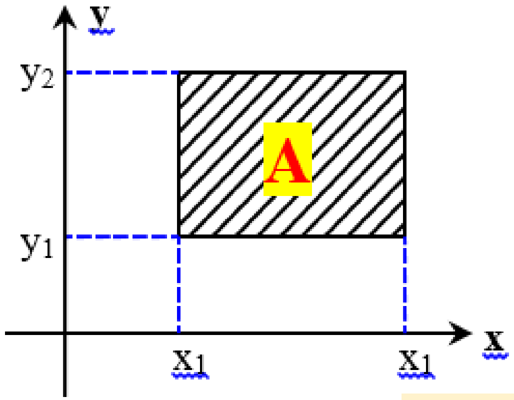
2- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

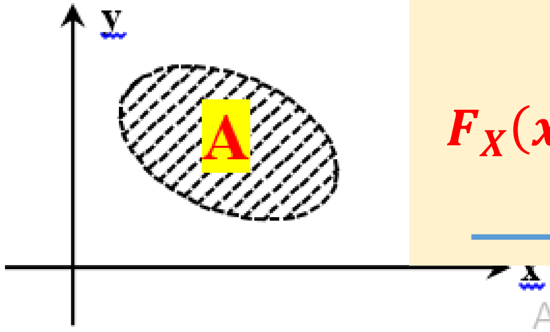
3- $P(x_1 < X \leq x_2\ ,\ y_1 < Y \leq y_2)$

$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx\,dy;\ \text{A is a rectangular region}$$

And in general

$$P(A) = P(x, y \in A) = \iint_{x,y\in A} f_{X,Y}(x,y)\,dx\,dy$$

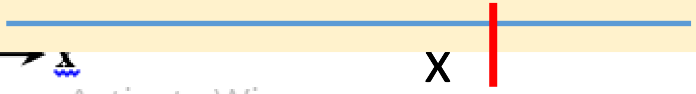




$$P(a \leq X \leq b) = \int_a^b f_X(u)\,du$$

CDF of one variable X:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)\,du$$



Marginal Distributions of a Continuous Distribution

For a continuous distribution, we have

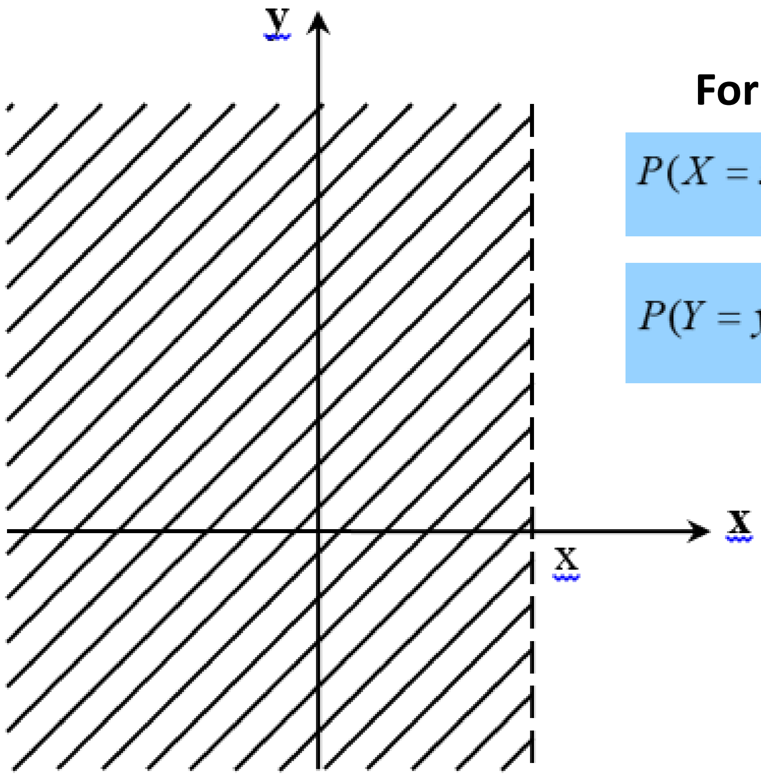
$$F_X(x) = P(X \le x) = \int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv\right) du$$

But, $$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)du$$

and, $$f_X(x) = \frac{d}{dx}F_X(x)$$

➔ $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$; Marginal pdf of X

➔ $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$; Marginal pdf of Y



Region for which:
{X ≤ x}

For the discrete case

$$P(X=x) = \sum_{all\ y} P(X = x, Y=y)$$

$$P(Y=y) = \sum_{all\ x} P(X = x, Y=y)$$

Independence of Continuous Random Variable

Theorem: Two random variables (X) and (Y) are independent if:

$$F_{X,Y}(x,y) = F_Y(x)F_Y(y);\ \text{for all x and y}\ \text{or, equivalently,}\ f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Proof: Two events A and B are independent when P(A ∩ B) = P(A) P(B)

Let: A: event {X ≤ x}
B: event {Y ≤ y}

Hence, P{X ≤ x , Y ≤ y} = P(X ≤ x) P(Y ≤ y) $\Rightarrow F_{X,Y}(x,y) = F_Y(x)F_Y(y)$

$$f_{X,Y}(x,y) = \frac{\partial^2 F}{\partial x \partial x} \Rightarrow F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)dvdu$$

$$f_X(x) = \frac{dF_X(x)}{dx};\Rightarrow F_X(x) = \int_{-\infty}^{x} f_X(u)du$$

$$f_Y(y) = \frac{dF_Y(Y)}{dy};\Rightarrow F_Y(y) = \int_{-\infty}^{y} f_Y(v)dv$$

Expected Value of a Function of Two Random Variables X and Y

The expected value of a function g(x,y) of two random variables (X) and (Y) is

$$E\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x\ ,y) f_{X,Y}(x,y)\,dx\,dy$$

Conditional Density Functions

Let X and Y be discrete random variables. The conditional probability mass function of Y given X = x, is given by

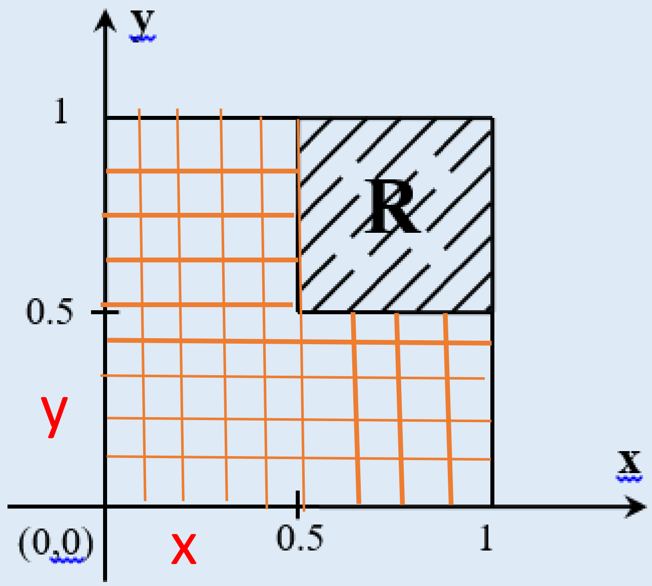
$$P_{Y/X}(y) = P(Y = y\ /\ X = x) = \frac{P(X = x\ ,Y = y)}{P(X = x)}$$

If (X) and (Y) are continuous random variables, the conditional pdf of Y given X = x is given by

$$f_{Y/X}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

EXAMPLE: The joint pdf of two random variables X and Y is given by

$$f_{X,Y}(x,y)=\begin{cases} k\,x\,y\ , & 0\le x\le 1,\ 0\le y\le 1, \\ 0\ , & otherwise \end{cases}$$



a- Find k so that $f_{X,Y}(x,y)$ is a proper pdf.

b- Find P(X ≥ 0.5 , Y ≥ 0.5)

c- Find the marginal pdf's $f_X(x)$ and $f_Y(y)$

d- Are (X) and (Y) independent?

SOLUTION:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx=1 \;\Rightarrow\; k\int_0^1 x\left(\int_0^1 y\,dy\right)dx=1$$

$$k\int_0^1 x\left(\frac{y^2}{2}\Big|_0^1\right)dx=\frac{k}{2}\int_0^1 x\,dx = \frac{k}{2}\frac{x^2}{2}\Big|_0^1=\frac{k}{4} \;\Rightarrow\; \frac{k}{4}=1 \;\Rightarrow\; k=4$$

a- $$P(X\ge 0.5, Y\ge 0.5)=\int_{0.5}^{1}\int_{0.5}^{1} 4\,x\,y\,dx\,dy=4\left(\frac{x^2}{2}\Big|_{0.5}^{1}\right)\left(\frac{y^2}{2}\Big|_{0.5}^{1}\right)= 0.75\times 0.75 = 0.5625$$

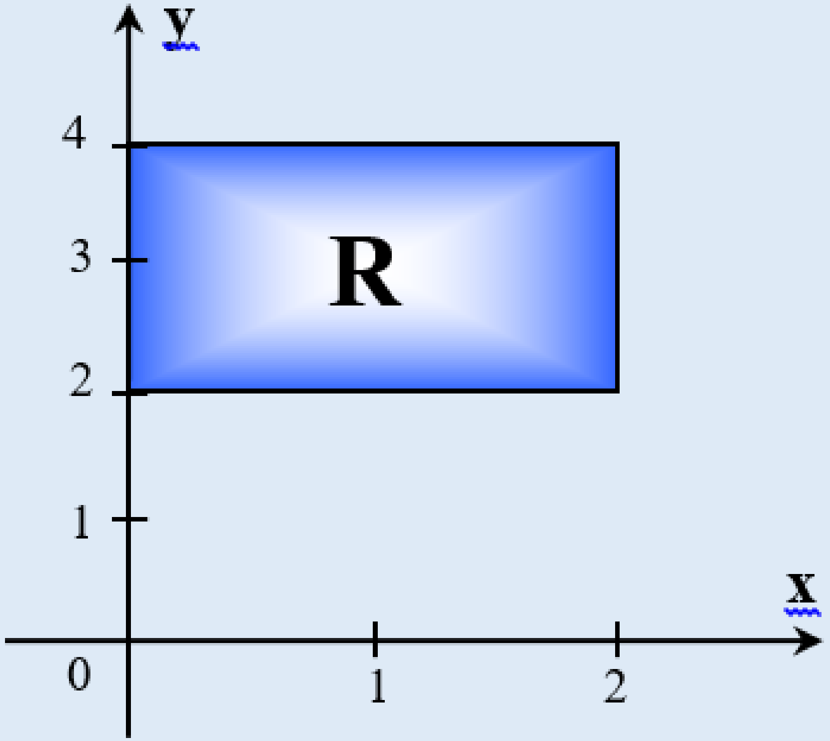
b- $$f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy \Rightarrow f_X(x) = \int_0^1 4\,x\,y\,dy \Rightarrow 4\,x\,\frac{y^2}{2}\Big|_0^1 = 2\,x;\ 0\le x\le 1$$

$$f_Y(y) = \int_0^1 4\,x\,y\,dx \Rightarrow 4\,y\,\frac{x^2}{2}\Big|_0^1 = 2\,y;\ 0\le y\le 1$$

Since $f_{XY}(x,y)=f_X(x)f_Y(y) \Rightarrow 4xy=(2x)(2y)\Rightarrow$ X and Y are independent

EXAMPLE: The joint pdf of two random variables X and Y is given

$$f_{X,Y}(x,y) = \frac{1}{8}(6 - x - y) \quad ; \quad 0 \le x \le 2 \,,\; 2 \le y \le 4$$



1- Find $f_X(x)$ and $f_Y(y)$.
2- Find the conditional pdf $f_{Y/X}(y)$.
3- Find $P(2 \le Y \le 3)$
4- Find $P(2 \le Y \le 3 / x = 1)$
5- Find $P(2 \le Y \le 3 / 0 \le x \le 1)$
6- Are X and Y independent?

Solution: Continued

1- $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_2^4 \frac{1}{8}(6-x-y)\,dy = \frac{1}{8}(6-2x); 0 \le x \le 2$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^2 \frac{1}{8}(6-x-y)\,dx = \frac{1}{4}(5-y) \quad ; \quad 2 \le y \le 4$

2- $f_{Y/X}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{\frac{1}{8}(6-x-y)}{\frac{1}{8}(6-2x)} = \dfrac{(6-x-y)}{(6-2x)} \quad ; \quad 0 < x \le 2,\ 2 \le y \le 4$

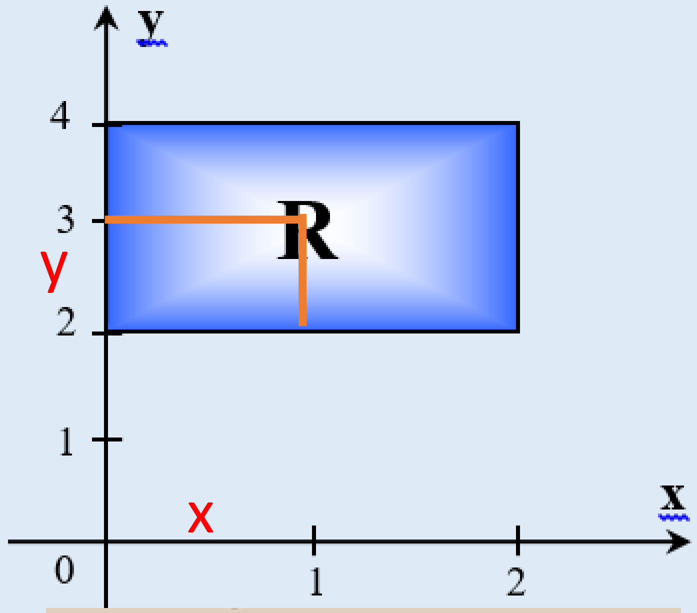
$\dfrac{(5-y)}{4}$ **For x = 1**

3- $P(2 \le Y \le 3) = \int_2^3 f_Y(y)\,dy = \int_2^3 \frac{1}{4}(5\text{-}y)\,dy = \frac{5}{8}$

4- For x=1, $f_{Y/X}(y/x=1) = \dfrac{(5-y)}{4} \Rightarrow P(2 \le Y \le 3/x=1) = \int_2^3 \dfrac{5-y}{4}\,dy = \dfrac{5}{8}$.

5- $P(2 \le \underset{A}{Y} \le 3 / 0 \le \underset{B}{X} \le 1) = \dfrac{P(2 \le Y \le 3 \cap 0 \le X \le 1)}{P(0 \le X \le 1)} = \dfrac{\int_{x=0}^{1}\int_{y=2}^{3} [(6-x-y)/8]\,dydx}{\int_0^1 [(6-2x)/8]\,dx} = \dfrac{3/8}{5/8} = \dfrac{3}{5}$

6- Since $f_{XY}(x,y) \ne f_X(x) f_Y(y) \Rightarrow \frac{1}{8}(6-x-y) \ne \frac{1}{8}(6-2x)\frac{1}{4}(5-y) \Rightarrow X$ and Y are dependent



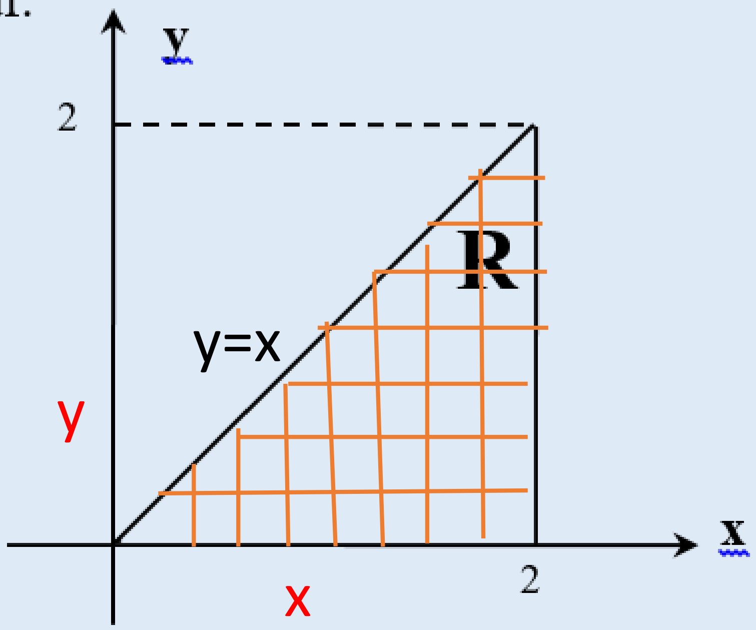
1- Find $f_X(x)$ and $f_Y(y)$.
2- Find the conditional pdf $f_{Y/X}(y)$.
3- Find P(2≤Y≤3)
4- Find $P(2 \le Y \le 3/x=1)$
5- Find $P(2 \le Y \le 3/0 \le x \le 1)$
6- Are X and Y independent?

$\frac{1}{8}(6-2x); 0 \le x \le 2$

$\frac{1}{4}(5-y) \quad ; \quad 2 \le y \le 4$

EXAMPLE: Two random variables (X) and (Y) have the joint pdf:

$$f_{XY}(x,y)=\begin{cases}\frac{5}{16}\,x^2\,y\,, & 0\le y\le x\le 2\\ 0\,, & otherwise\end{cases}$$



a- Verify that $f_{X,Y}(x,y)$ is a valid pfd.

b- Find the marginal density functions of X and Y.

c- Are X and Y statistically independent?

d- Find P{X≤1} , P{Y≤0.5} , P{XY≤1}

SOLUTION:

a- $$\iint_R f_{X,Y}(x,y)\,dx\,dy=\int_0^2\int_0^x \frac{5}{16}x^2y\,dy\,dx$$

$$\frac{5}{16}\int_0^2 x^2\left(\int_0^x y\,dy\right)dx=\frac{5}{16}\int_0^2 x^2\left(\frac{y^2}{2}\bigg|_0^x\right)dx=\frac{5}{16}\int_0^2\frac{x^4}{2}\,dx=\frac{5}{16}\times\frac{1}{2}\times\frac{x^5}{5}\bigg|_0^2=\frac{5}{16}\times\frac{1}{2}\times\frac{32}{5}=1$$

b- $$f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\Rightarrow f_X(x)=\int_0^x\frac{5}{16}x^2\,y\,dy\Rightarrow\frac{5}{16}x^2\,\frac{y^2}{2}\bigg|_0^x=\frac{5}{32}x^4,\,0\le x\le 2$$

$$f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\Rightarrow f_Y(y)=\int_y^2\frac{5}{16}x^2\,y\,dx\Rightarrow\frac{5}{16}\,y\,\frac{x^3}{3}\bigg|_y^2=\frac{5}{48}\,y(8-y^3);\,0\le y\le 2$$

c- Since $f_{XY}(x,y) \neq f_X(x) f_Y(y) \Rightarrow \frac{5}{16} x^2 y \neq \left(\frac{5}{32} x^4\right)\left(\frac{5}{48} y(8 - y^3)\right) \Rightarrow X$ and Y are dependent .

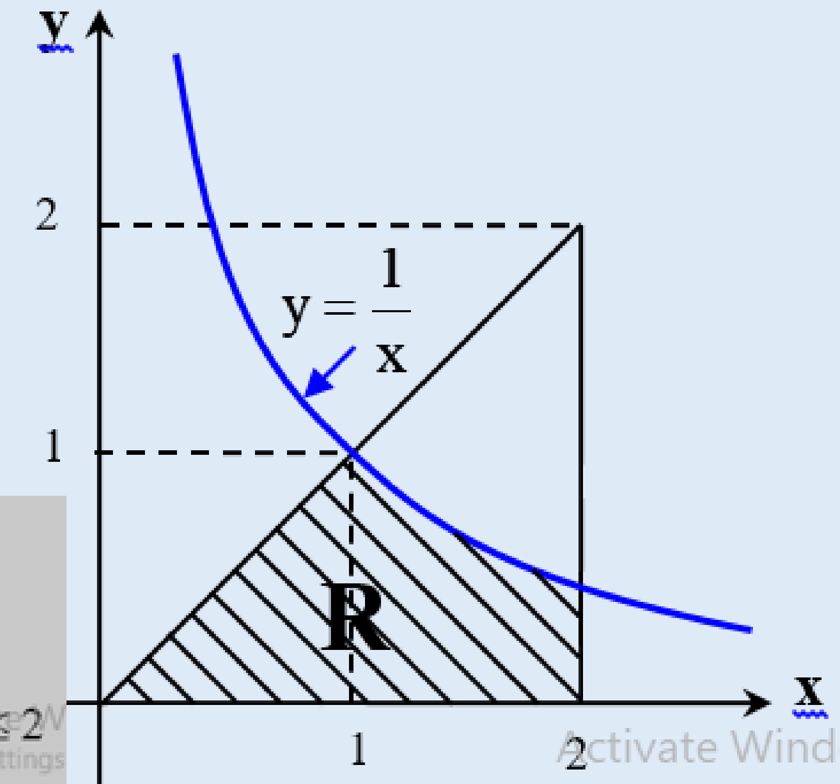
d- $P(X \leq 1) = \int_0^1 f_X(x)\, dx = \int_0^1 \frac{5}{32} x^4\, dx = \frac{5}{32} \frac{x^5}{5}\Big|_0^1 = \frac{1}{32} = 0.03125$

$$P(Y \leq 0.5) = \int_0^{0.5} f_Y(y)\, dy = \int_0^{0.5} \frac{5}{48} y(8 - y^3)\, dy = \frac{5}{48}\left(4y^2 - \frac{y^4}{4}\right)\Big|_0^{0.5} = \frac{105}{1024} = 0.1025$$

$$P(XY \leq 1) = P(Y \leq \frac{1}{X}) = \iint_R f_{XY}(x,y)\, dy\, dx$$

$$= \int_0^1 \int_0^x \frac{5}{16} x^2 y\, dy\, dx + \int_1^2 \int_0^{\frac{1}{x}} \frac{5}{16} x^2 y\, dy\, dx$$

=1/32 + 5/32 = 6/32



a- Verify that $f_{X,Y}(x,y)$ is a valid pfd.

b- Find the marginal density functions of X and Y.

c- Are X and Y statistically independent?

d- Find P{X≤1} , P{Y≤0.5} , P{XY≤1}

$\frac{5}{32} x^4, 0 \leq x \leq 2$

$\frac{5}{48} y(8 - y^3); 0 \leq y \leq 2$

Operations on Multiple Random Variables

Review: Basic operations on a single random variable

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\; dx$$

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(X)\, f_X(x)\; dx$$

$$E\{g_1(X) + g_2(X)\} = E\{g_1(X)\} + E\{g_2(X)\}$$

If $Y = aX + b$, then, $\mu_Y = a\,\mu_X + b$ and $\sigma_Y^2 = a^2\sigma_X^2$

Expected Value of a Function of Two Random Variables X and Y

The expected value of a function g(x,y) of two random variables (X) and (Y) is

$$E\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x\;,y) f_{X,Y}(x,y)\,dxdy$$

Since summation and integration are linear processes, we have:

$$E\{a_1 g_1(X,Y) + a_2 g_2(X,Y)\} = a_1 E\{g_1(X,Y)\} + a_2\{g_2(X,Y)\}$$

Application 1: Let $g(X,Y) = aX + bY$

Then, $E\{g(X,Y)\} = aE(X) + bE(Y) = a\,\mu_X + b\mu_Y$

Remark: This result applies in all cases whether X any Y are independent or not.

Application 2: Let X and Y be two independent random variables, then

$E(XY) = E(X)E(Y)$

$$E\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\,f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} x\,f_X(x)\,dx \int_{-\infty}^{\infty} y\,f_Y(y)\,dy = E(X)E(Y)$$

Note that for independent random variables X and Y $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

Remark: This result applies only when X any Y are independent.

Application 3: Let X and Y be two independent random variables, then

$$E\{g_1(X)g_2(Y)\} = E\{g_1(X)\}\,E\{g_2(Y)\}$$

$$E\{g_1(X)g_2(Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(X)\,g_2(Y)\,f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} g_1(X)\,f_X(x)\,dx \int_{-\infty}^{\infty} g_2(Y)\,f_Y(y)\,dy = E\{g_1(X)\}\,E\{g_2(Y)\}$$

Example: If X and Y are independent, find $E\{(X-\mu_X)(Y-\mu_Y)\}$

Solution: $E\{(X-\mu_X)(Y-\mu_Y)\} = E(X-\mu_X)E(Y-\mu_Y) = (\mu_X-\mu_X)((\mu_Y-\mu_Y) = 0$

Example: If X and Y are independent, find $E\{(X-\mu_X)^2(Y-\mu_Y)^2\}$

Solution: $E\{(X-\mu_X)^2(Y-\mu_Y)^2\} = E(X-\mu_X)^2 E(Y-\mu_Y)^2 = (\sigma_X^2)(\sigma_Y^2)$

Application 4: Let $Y = a_1X_1 + a_2X_2$, then

$$\sigma_Y^2 = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2 + 2\, a_1a_2\sigma_{X_1}\sigma_{X_2}\rho_{X_1X_2}$$

The correlation coefficient is a measure of association between the variables X1 and X2 and is bounded between $-1 \le \rho \le 1$ **.**

ρ >0 : positive association
ρ <0 : negative association
ρ =0 : no association

Proof:

$$\sigma_Y^2 = E\{(Y - \mu_Y)^2\} = E\{(a_1X_1 + a_2X_2 - a_1\mu_{X_1} - a_2\mu_{X_2})^2\}$$

$$= E\{[a_1(X_1 - \mu_{X_1}) + a_2(X_2 - \mu_{X_2})]^2\}$$

$$= E\{a_1^2(X_1 - \mu_{X_1})^2\} + E\{a_2^2(X_2 - \mu_{X_2})^2\} + 2\, a_1\, a_2\, E\{(X_1 - \mu_{X_1})(X_2 - \mu_{X_2})\}$$

$$\Rightarrow \sigma_Y^2 = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2 + 2\, a_1a_2\sigma_{X_1}\sigma_{X_2}\rho_{X_1X_2}$$

where, $\rho_{X_1X_2} = \dfrac{E\{(X_1 - \mu_{X_1})((X_2 - \mu_{X2}))\}}{\sigma_{X_1}\ \sigma_{X_2}}$ is the correlation coefficient

Application 5: If (X) and (Y) are *independent* random variables, then they are uncorrelated.

Proof:

$$E\{(X-\mu_X)(Y-\mu_Y)\} = E\{XY\} - \mu_Y E\{X\} - \mu_X E\{Y\} + \mu_X \mu_Y$$

$$= E\{XY\} - \mu_X \mu_Y$$

Since (X) and (Y) are *independent*, then $E\{XY\} = \mu_X \mu_Y$

Therefore, $\rho_{X,Y} = \dfrac{E\{(X - \mu_X)(Y - \mu_Y)\}}{\sigma_X \ \sigma_Y} = \dfrac{E\{XY\} - E\{X\}E\{Y\}}{\sigma_X \sigma_Y} = 0$

Remark: This result asserts that if X and Y are independent then they are uncorrelated ($\rho_{XY} = 0$). However, the converse is not necessarily true. That is, if $\rho_{XY} = 0$, then X and Y are not necessarily independent. The only exception is when X and Y are Gaussian. In this case, $\rho_{XY} = 0$ implied that X and Y are independent.

Application 6: Let $Y = a_1 X_1 + a_2 X_2$, and let X and Y be *independent* random variables, then

$$\sigma_Y^2 = a_1^2 \sigma_{X_1}^2 + a_2^2 \sigma_{X_2}^2$$

This result follows immediately from the above two results.

EXAMPLE: Two random variables X and Y are related by $Y = aX + b$, where X is a random variable with mean μ_X and variance σ_X^2

a. Find the mean and variance of Y
b. Find the correlation coefficient between X and Y.

SOLUTION:

$$Y = aX + b \Rightarrow \mu_Y = a\mu_X + b; \quad \sigma_Y^2 = a^2\sigma_X^2$$

$$\rho_{X,Y} = \frac{E\{(X - \mu_X)(Y - \mu_Y)\}}{\sigma_X \ \sigma_Y} = \frac{E\{(X - \mu_X)(aX + b - a\mu_X - b)\}}{\sigma_X \ (a\sigma_X)}$$

$$= \frac{E\{a(X - \mu_X)^2\}}{\sigma_X \ (a\sigma_X)} = \frac{a\sigma_X^2}{\sigma_X \ (a\sigma_X)} = 1$$

WE have maximum positive association between X and Y

EXAMPLE: Let X be a uniformly distributed random variable on the interval $0 \le X \le 10$ and zero elsewhere and let Y be another uniformly distributed random variable on $0 \le Y \le 20$ and zero elsewhere. Assuming that X and Y are independent, find

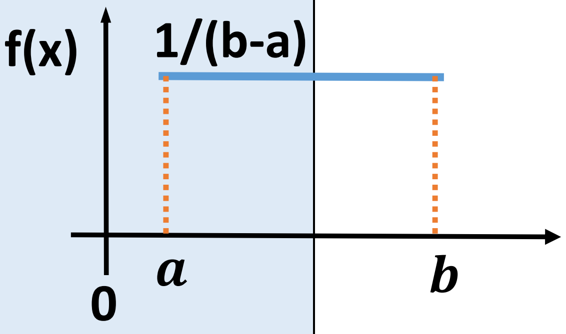
a. $P(X \le 4 \cap Y \le 8)$; b. $E(2X+Y)$; c. $\mathrm{Var}(2X+Y)$; d. $E(Z=XY)$ e. $Var(Z=XY)$;

SOLUTION: For the uniform random variable defined on an interval (a, b), we have

$$\mu = (a+b)/2; \quad \sigma^2 = (b-a)^2/12$$

$$\mu_X = (0+10)/2 = 5; \quad \sigma^2 = (10-0)^2/12 = 100/12$$

$$\mu_Y = (0+20)/2 = 10; \quad \sigma^2 = (20-0)^2/12 = 400/12$$



$$P(X \le 4 \cap Y \le 8) = P(X \le 4)P(Y \le 8); \text{ Due to independence}$$

a.

$$= \left[\int_0^4 \left(\frac{1}{10}\right)dx\right]\left[\int_0^8 \left(\frac{1}{20}\right)dy\right] = \left(\frac{4}{10}\right)\left(\frac{8}{20}\right)$$

b. $E(2X+Y) = 2E(X) + E(Y) = 10 + 10 = 20$

c. $\mathrm{Var}(2X+Y) = 4\sigma_X^2 + \sigma_Y^2 = \frac{400}{12} + \frac{400}{12} = \frac{800}{12}$; $\rho_{X,Y} = 0$ since X any Y are independent

$$\sigma_Y^2 = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2$$

d. $E(Z) = E(XY) = E(X)\mathrm{E}(Y) = (5)(10) = 50$

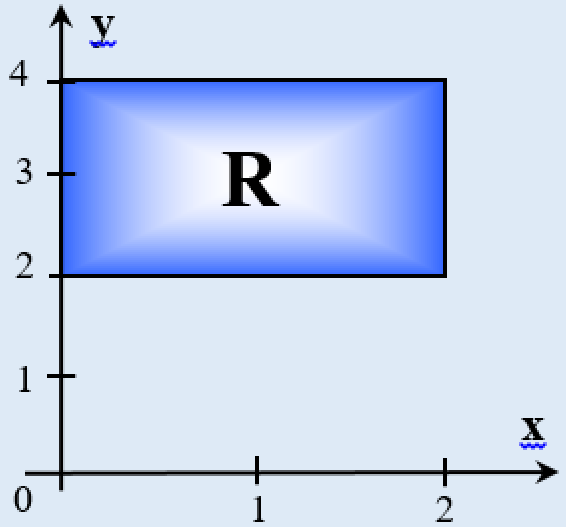
$$Var(Z) = E(Z^2) - (\mu_Z)^2 \Rightarrow E(X^2Y^2) - (\mu_Z)^2 = E(X^2)E(Y^2) - (\mu_Z)^2$$

e.

$$= \left\{\sigma_X^2 + (\mu_X)^2\right\}\left\{\sigma_Y^2 + (\mu_Y)^2\right\} - (\mu_X\mu_Y)^2 = \left\{\frac{100}{12} + 25\right\}\left\{\frac{400}{12} + 100\right\} - (50)^2$$

EXAMPLE: The joint pdf of two random variables X and Y is given

$$f_{X,Y}(x,y)=\frac{1}{8}(6-x-y)\quad ;\quad 0\le x\le 2\,,\ 2\le y\le 4$$



1- Find the correlation coefficient $\rho_{X,Y}$

2- Define Z = X + Y, find μ_Z and σ_Z^2

SOLUTION:

1- $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy=\int_2^4 \frac{1}{8}(6-x-y)\,dy=\frac{1}{8}(6-2x); 0\le x\le 2$

$f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^2 \frac{1}{8}(6-x-y)\,dx=\frac{1}{4}(5-y)\quad ;\quad 2\le y\le 4$

$\mu_X = \int_0^2 x f_X(x)\,dx=\int_0^2 x\frac{1}{8}(6-2x)\,dx=\frac{5}{6};\qquad \mu_Y = \int_2^4 y f_Y(y)\,dy=\int_2^4 y\frac{1}{4}(5-y)\,dy=\frac{17}{6}$

$\sigma_X^2 = \int_0^2 (x-\mu_X)^2 f_X(x)\,dx=\int_0^2 (x-\mu_X)^2\frac{1}{8}(6-2x)\,dx=\frac{11}{36};\ \sigma_Y^2 = \int_2^4 (Y-\mu_Y)^2 f_Y(y)\,dy=\int_2^4 (y-\mu_Y)^2\frac{1}{4}(5-y)\,dy=\frac{11}{36}$

$$Y = a_1X_1 + a_2X_2$$

$$\sigma_Y^2 = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2 + 2\,a_1a_2\sigma_{X_1}\sigma_{X_2}\rho_{X_1X_2}$$

$E\{XY\}=\int_0^2\int_2^4 x\,y\,f_{X,Y}(x,y)\,dy\,dx = \int_0^2\int_2^4 x\,y\,\frac{1}{8}(6-x-y)dy\,dx = \frac{7}{3}$

$$\rho_{X,Y} = \frac{E\{(X-\mu_X)(Y-\mu_Y)\}}{\sigma_X\ \sigma_Y}=\frac{E\{XY\}-E\{X\}E\{Y\}}{\sigma_X\sigma_Y}=-0.0909$$

2- $E\{Z\}=\mu_X+\mu_Y=3.6667;\qquad \sigma_Z^2 = \sigma_X^2+\sigma_Y^2+2\,\sigma_X\sigma_Y\rho_{X,Y}=0.5556$

Probability Density Function of a Sum of Two Independent Random Variables

- Let X and Y be two random variables with a joint pdf $f_{X,Y}(x,y)$ and let $Z = g(X,Y)$ be any continuous function of the variables X and Y.
- A function of a random variable is also a random variable. Therefore, Z is a random variable.
- The objective is to find $f_Z(z)$; the probability density function of Z.
- When (X) and (Y) are discrete random variables, we may obtain the probability mass function $P(Z=z)$ by summing all probabilities for which $Z = g(X,Y)$ equals the value of z considered, thus:

$$P(Z=z) = \sum_{g(x,y)=z} \sum P(X=x\ ,Y=y)\ .$$

- In the case of continuous random variables X and Y we first find $F_Z(z)$

$$F_Z(z) = \iint_{g(x,y)\le z} f_{X,Y}(x,y)\ dx\ dy$$

then, we find $$f_Z(z) = \frac{d\ F_z(z)}{dz}.$$

- In this section we consider the special case when $Z = X + Y$, where X and Y are independent.
- The general case will be treated in the next lecture.

Discrete Case: Example

EXAMPLE: The joint probability mass function of two random variables X and Y is given in the table below:

X \ Y	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

b. Define $Z = X + Y$. Find the pmf of Z.

c. Define $Z = |X - Y|$. Find the pmf of Z.

SOLUTION: PMF of Z = X + Y

The random variable Z assumes the values 2, 3, 4, 5, 6,

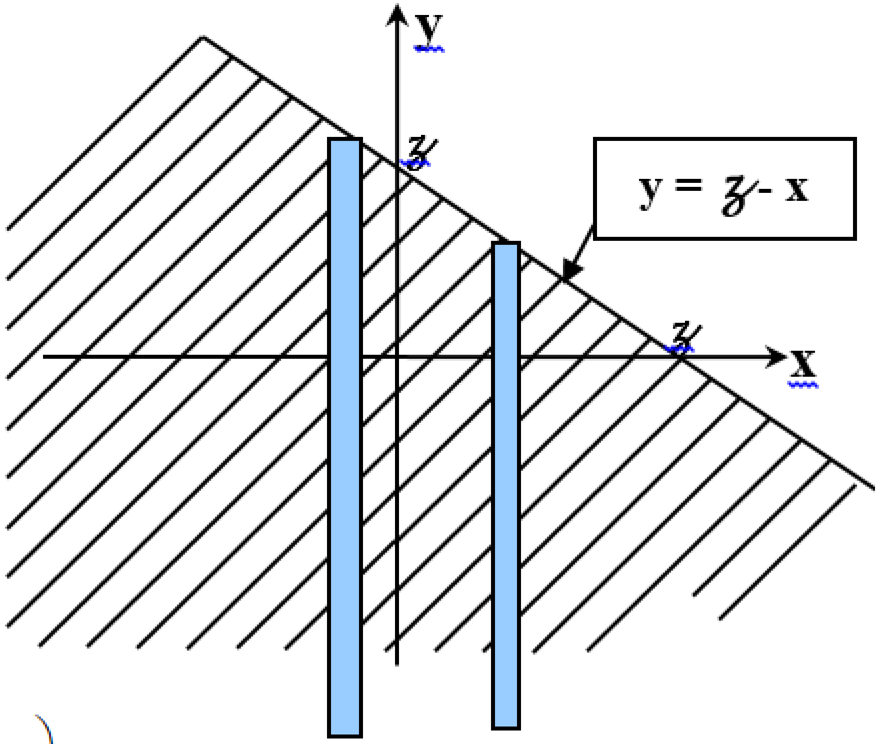
Z	2	3	4	5	6
P(Z=z)	P(X=1, Y=1) **0.1**	P(X=2, Y=1) **0.3**	P(X=1, Y=3) **0.1**	P(X=2, Y=3) + P(X=3, Y=2) 0.1+0.2 = **0.3**	P(X=2, Y=4) **0.2**

PMF of Z = |X –Y|: The random variable Z assumes the values 0, 1, 2.

Z	0	1	2
P(Z=z)	P(X=1, Y=1) **0.1**	P(X=2, Y=1)+ P(X=3, Y=2)+ P(X=2, Y=3)+ 0.3+0.2+0.1=**0.6**	P(X=1, Y=3)+ P(X=2, Y=4) 0.1+0.2 **0.3**

Theorem: Let Z = X + Y where X and Y are *independent* random variables, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$



Proof:

$$F_Z(z) = P(Z \le z) = P(X + Y \le z)$$

$$F_Z(z) = P(Y \le z - X) = \iint_R f_{X,Y}(x, y)\, dy dx$$

Since (X) and (Y) are *independent* random variables, then

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

$$F_z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy\, dx \Rightarrow F_z(z) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f_Y(y)\, dy \right) f_X(x)\, dx$$

$$F_z(z) = \int_{-\infty}^{\infty} f_X(x)\, F_Y(z - x)\, dx$$

But, $f_Z(z) = \dfrac{d\, F_z(z)}{dz}$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx \text{; the Convolution Integral.}$$

EXAMPLE: Let (X) and (Y) be two independent exponential random variables, such that:

$$f_X(x) = \begin{cases} \alpha\ e^{-\alpha x} & , \quad x \geq 0 \\ 0 & , \quad otherwise \end{cases} \; ; \; f_Y(y) = \begin{cases} \beta\ e^{-\beta y} & , \quad y \geq 0 \\ 0 & , \quad otherwise \end{cases}$$

Let Z = X + Y. Find $f_Z(z)$

$Z = X + Y$
$0 \leq z \leq \infty$

SOLUTION

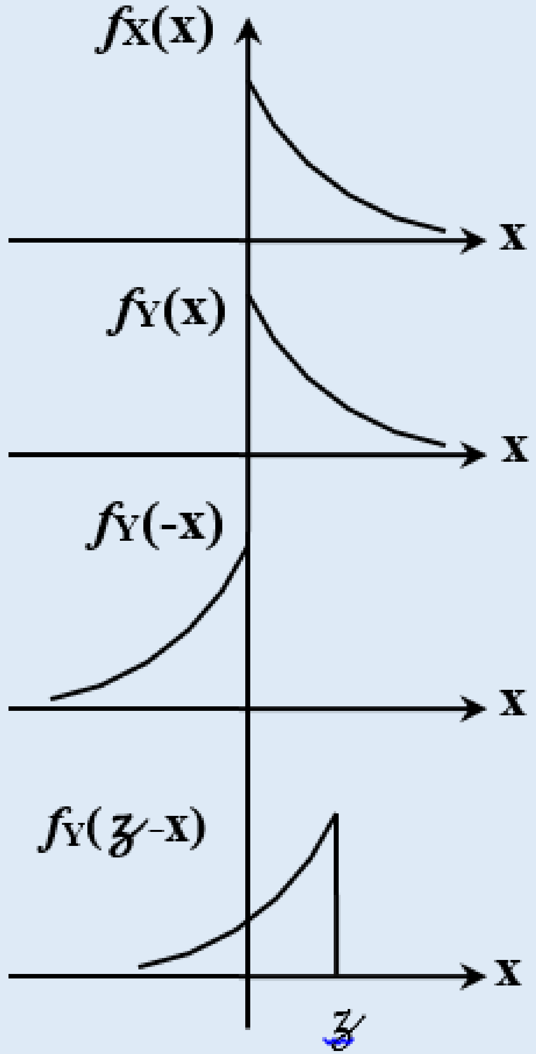
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$$

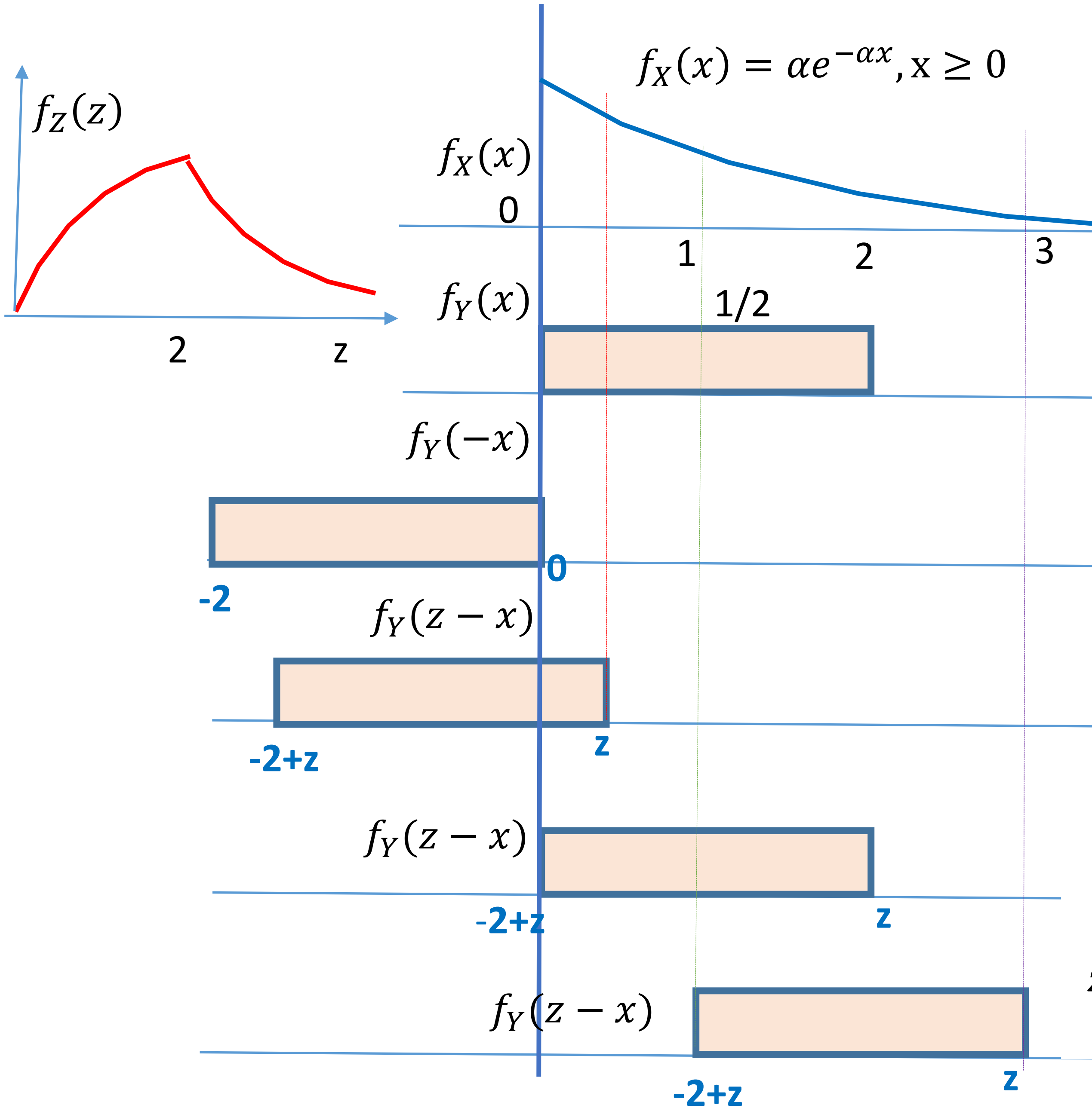
For z<0, $f_Z(z) = 0$

For $z \geq 0$, $f_Z(z) = \alpha\beta e^{-\beta z} \int_0^z e^{-(\alpha-\beta)x} dx$

$$f_Z(z) = \frac{\alpha\beta}{(\alpha-\beta)} c^{-\beta z}(1 - c^{-(\alpha-\beta)z})$$

$$f_Z(z) = \frac{\alpha\beta}{(\alpha-\beta)} (c^{-\beta z} - c^{-\alpha z}), \; z \geq 0$$





Example: Let X have an exponential distribution with parameter α and let Y be a uniform distribution over the interval (0,2). Define Z = X + Y, find the pdf of Z assuming X and Y are independent.

$$\mathbf{0 \leq z \leq \infty}$$

$$f_X(x) = \alpha e^{-\alpha x}, \mathrm{x} \geq 0$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$z \leq 0, f_Z(z) = 0$$

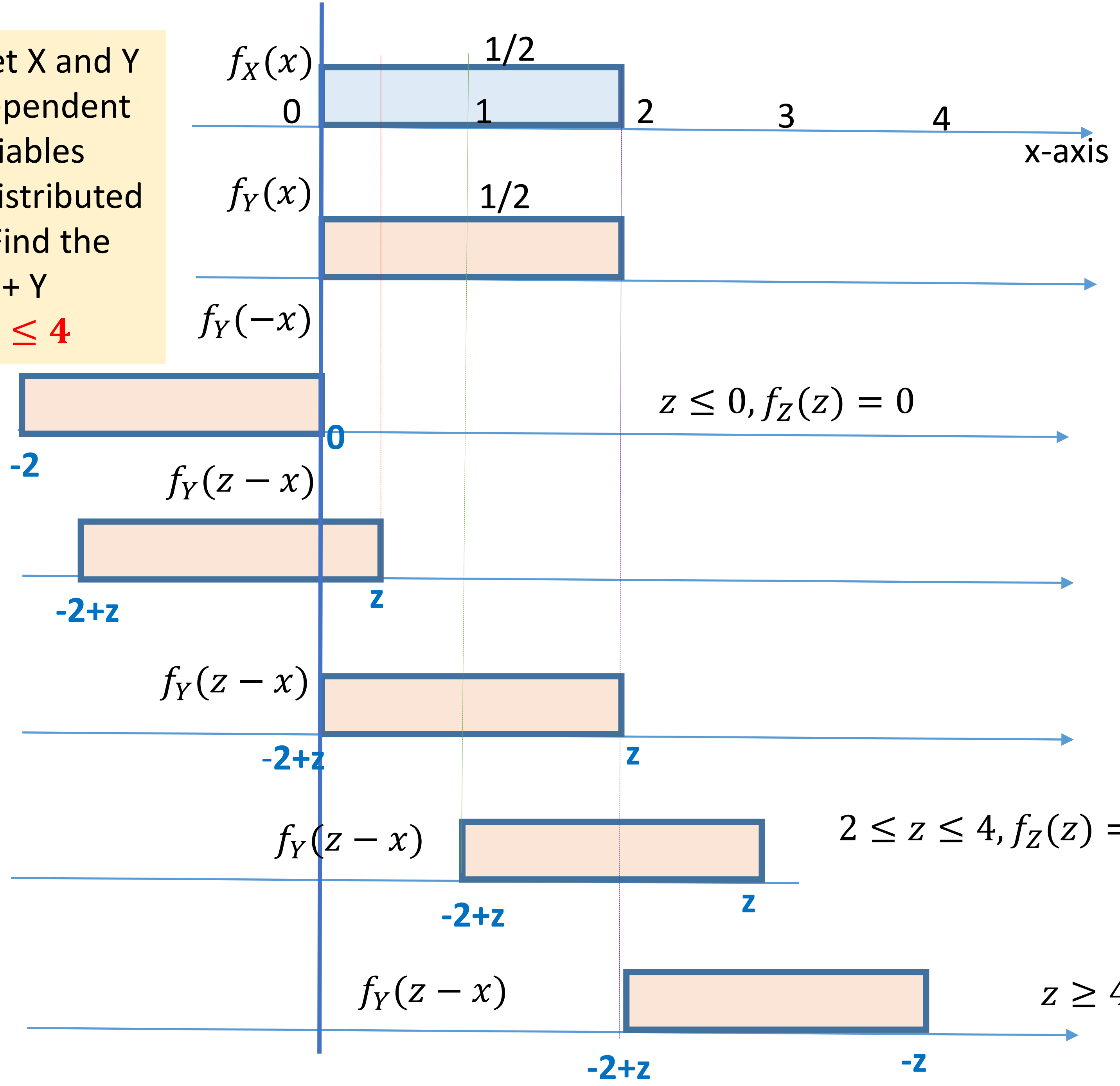
$$0 \leq z \leq 2, f_Z(z) = \int_0^z \alpha e^{-\alpha x} \times \frac{1}{2}\, dx = \frac{1}{2}(1 - e^{-\alpha z})$$

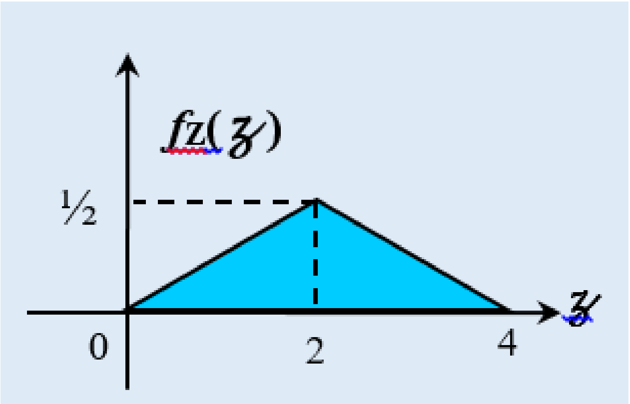
$$z = 2, f_Z(z) = \int_0^2 \alpha e^{-\alpha x} \times \frac{1}{2}\, dx = \frac{1}{2}(1 - e^{-2\alpha})$$

$$z \geq 2, f_Z(z) = \int_{-2+z}^{z} \alpha e^{-\alpha x} \times \frac{1}{2}\, dx = \frac{1}{2}(1 - e^{-2\alpha})e^{-\alpha(z-2)}$$

Example: Let X and Y be two independent random variables uniformly distributed over (0,2). Find the pdf of Z = X + Y

$0 \le z \le 4$





$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$z \le 0, f_Z(z) = 0$$

$$0 \le z \le 2, f_Z(z) = \int_0^z \frac{1}{2} \times \frac{1}{2}\ dx = \frac{z}{4}$$

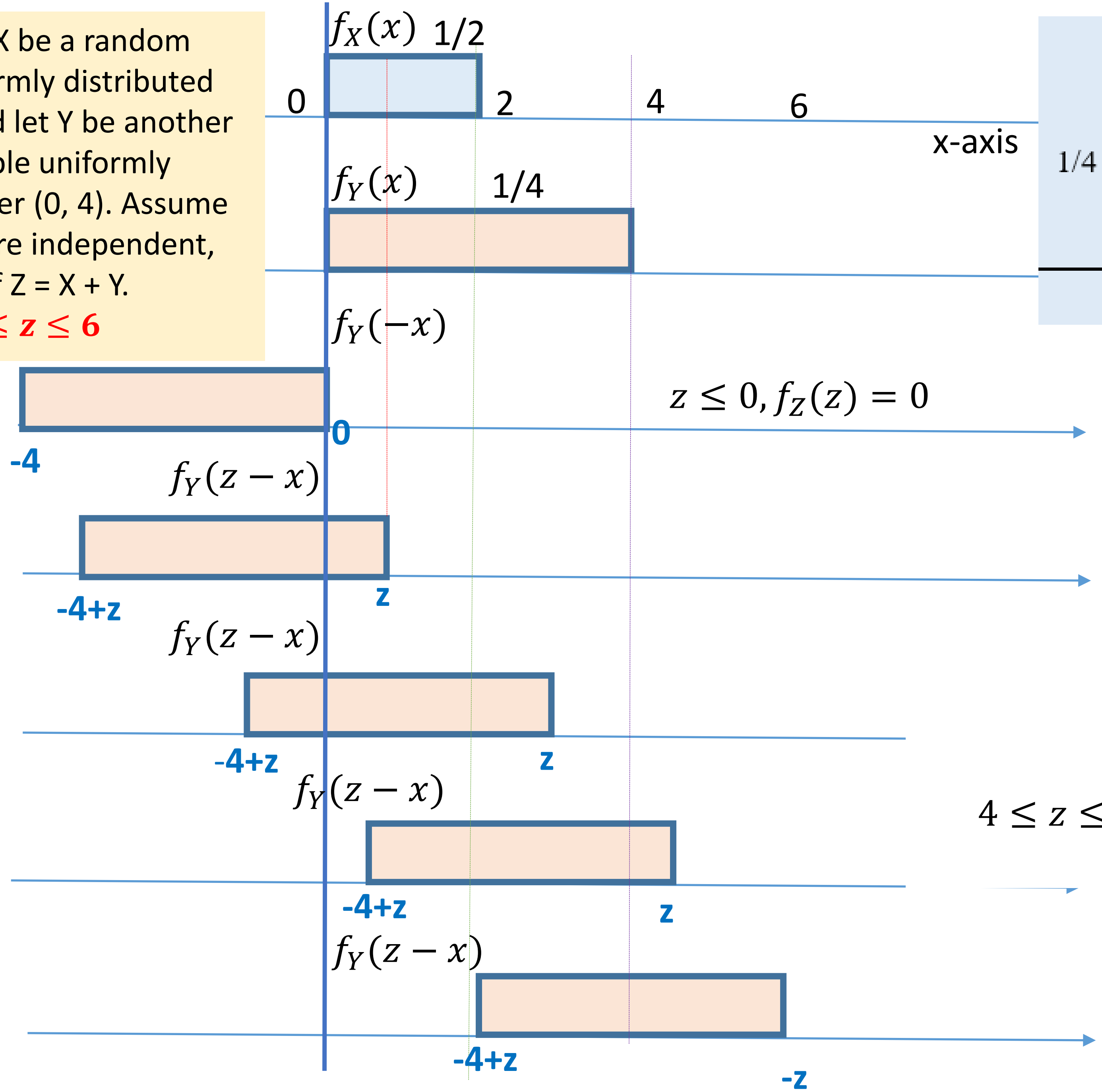
$$z = 2, f_Z(z) = \int_0^2 \frac{1}{2} \times \frac{1}{2}\ dx = \frac{1}{2}$$

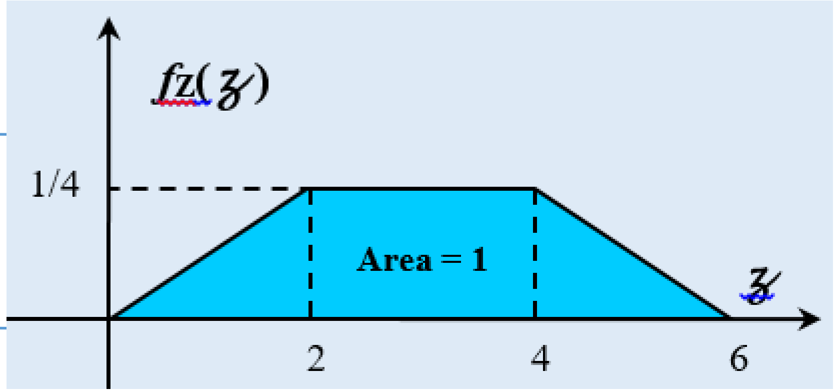
$$2 \le z \le 4, f_Z(z) = \int_{-2+z}^{2} \frac{1}{2} \times \frac{1}{2}\ dx = \frac{x}{4}\Big|_{-2+z}^{2} = \frac{1}{4}[4-z]$$

$$z \ge 4, f_Z(z) = 0$$

Example: Let X be a random variable uniformly distributed over (0, 2) and let Y be another random variable uniformly distributed over (0, 4). Assume that X and Y are independent, find the pdf of Z = X + Y.

$\mathbf{0 \leq z \leq 6}$





$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$z \leq 0, f_Z(z) = 0$$

$$0 \leq z \leq 2, f_Z(z) = \int_0^z \frac{1}{2} \times \frac{1}{4}\; dx = \frac{z}{8}$$

$$2 \leq z \leq 4, f_Z(z) = \int_0^2 \frac{1}{2} \times \frac{1}{4}\; dx = \frac{1}{4}$$

$$4 \leq z \leq 6, f_Z(z) = \int_{-4+z}^{2} \frac{1}{2} \times \frac{1}{4}\; dx = \frac{1}{8}[6-z]$$

$$z \geq 6, f_Z(z) = 0$$