Solutions Manual

Fourth Edition

SIGNALS & SYSTEMS

Continuous and Discrete

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PREFACE

This manual contains solutions to all end-of-chapter problems and all computer exercises contained in the Fourth Edition of *Signals and Systems: Continuous and Discrete*. The manual is divided into two separate parts. Part I contains solutions to the end-of-chapter problems and Part II contains solutions to the computer exercises.

All computer exercises are developed using MATLAB as are all end-of-chapter problems specifying the use of MATLAB. In several parts of this manual, especially in Chapters 8 and 9, MathCAD is used for a little variety in a few problems.

Thanks go to Carol Baker for her expert typing skills and for her help in assembling the final product.

While we have done our best to insure that the problem solutions contained herein are correct, it is inevitable that manuals such as this are never perfect. We apologize in advance for any frustration caused by such errors.

R.E.Z.
W.H.T.
D.R.F.

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PART I

SOLUTIONS TO END-OF-CHAPTER PROBLEMS

CHAPTER 1

Problem 1-1

(a) Write the acceleration as

$$a(t) = \begin{cases} \alpha t, & t \le t_0 \\ 0, & t > t_0 \end{cases}$$

Thus the velocity and position are, respectively, given by

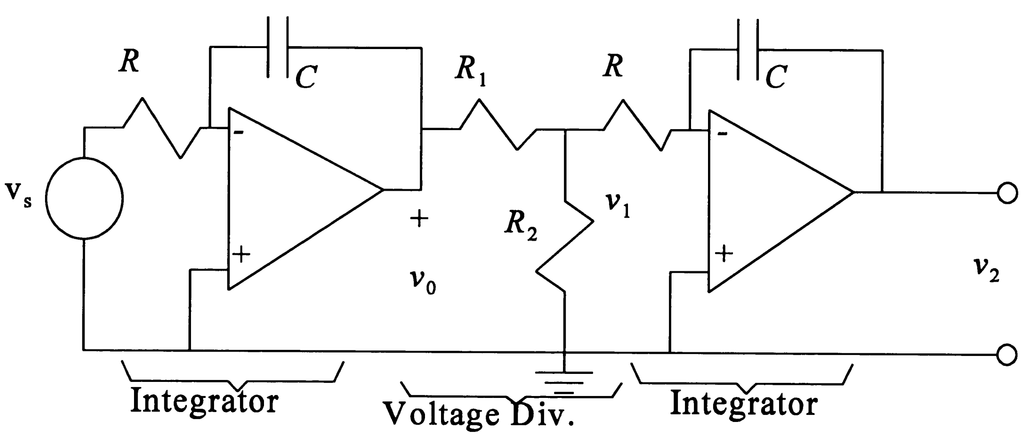
$$v(t) = \int_0^t a(\lambda)d\lambda = \begin{cases} \alpha t^2/2, & t \le t_0 \\ \alpha t_0^2/2, & t > t_0 \end{cases}$$

and

$$x(t) = \int_0^t v(\lambda)d\lambda = \begin{cases} \alpha t^3/6, & t \le t_0 \\ \alpha t_0^3/6 + \alpha t_0^2(t - t_0)/2, & t > t_0 \end{cases}$$

For $t_0 = 72$ s and $\alpha = 5/9$ m/s^2, we have $x(t) = (5/54)t^3$, $t \le 72$ s. At $t = t_0 = 72$ s (burnout), we have $x(t_0) = 35.56$ km.

(b) See the figure below for the integrator.



$$v_2(t) = -\frac{1}{RC}\int_0^t v_1(\lambda)d\lambda$$

Assume that $R_2 << R$. The input impedance to the op-amp integrator is therefore much larger than the output impedance of the previous stage, and

$$v_1(t) = \frac{R_2}{R_1 + R_2} v_0(t)$$

From Example 1-2,

$$v_0(t) = -\frac{1}{RC}\left(\frac{\beta t^2}{2}\right) = -\frac{\beta t^2}{2RC}$$

Therefore,

$$v_2(t) = -\frac{1}{RC}\int_0^t \frac{R_2}{R_1 + R_2}\left(-\frac{\beta}{2RC}\right)\lambda^2 d\lambda$$

Integrating and setting $t = t_0$, we obtain

$$v_2(t_0) = \frac{R_2}{R_1 + R_2}\left(\frac{\beta t_0^2}{2RC}\right)\left(\frac{t_0}{3RC}\right) = 10 \text{ V}$$

The second factor on the right is 10 V because of the maximum output limitation on the first integrator. Thus, we require that

$$\frac{R_2}{R_1 + R_2}\left(\frac{t_0}{3RC}\right) = 1$$

For example, from Example 1-2 we have $RC = 0.36$ s. With $t_0 = 72$ s and $R_1 = 10$ k ohms, we get $R_2 = 152$ ohms.

Problem 1-2

(a) Let $n = 0, 1, 2, 3, ..., N$. Then

$$\begin{aligned} v(T) &= v(0) + Ta(T) \quad (a) \\ v(2T) &= v(T) + Ta(2T) \quad (b) \\ &\cdots \\ v(NT) &= v[(N-1)T] + Ta(NT) \quad (c) \end{aligned}$$

Substitute (a) into (b) and so on until (c is reached. This gives

$$v(NT) = v(0) + T\sum_{n=1}^{N} a(nT)$$

(b) Let $n = 0, 1, 2, 3, ..., N$. Then

$$\begin{aligned} v(T) &= v(0) + (T/2)[a(0) + a(T)] \quad (a) \\ v(2T) &= v(T) + (T/2)[a(T) + a(2T)] \quad (b) \\ &\cdots \\ v(NT) &= v[(N-1)T] + (T/2)\{a[(N-1)T] + a(NT)\} \quad (c) \end{aligned}$$

Substitute (a) into (b) and so on until (c) is reached. The result is as given in the problem statement.

Problem 1-3

(a) A maximum departure of the weight from equilibrium of 1 cm requires a spring constant of

$$K = \frac{Ma_{\max}}{x_{\max}} = \frac{(0.002)(20)}{0.01} = 4 \text{ kg/s}^2$$

(b) For a minimum increment of 0.5 mm = 0.0005 m, we have

$$\Delta a_{\min} = \frac{K\Delta x_{\min}}{M} = \frac{4(0.0005)}{0.002} = 1 \text{ m/s}^2$$

(c) The velocity is given by

$$v_r(t) = \int_0^t a(\lambda)d\lambda = \int_0^t 20\,d\lambda = \begin{cases} 20t, & 0 \le 50 \text{ s} \\ 1000, & t > 50 \text{ s} \end{cases}$$

Problem 1-4

K is the same as in Example 1-1 because M, x_{max}, and a_{max} are the same. Also, Δa_{min} is the same. The velocity profile is

$$v_r(t) = \begin{cases} \int_0^t 20\,d\lambda = 20t, \; 0 \le t < 10 \\ 200, \; 10 \le t < 20 \\ 200 + \int_{20}^t 20\,d\lambda = 200 + 20(t - 20), \; 20 \le t < 30 \\ 400, \; t > 30 \end{cases}$$

Problem 1-5

From (1-15) and using the $x(t)$ given in the problem, we have

$$\begin{aligned} s(t) &= \cos(\omega_0 t) + \alpha\beta\cos[\omega_0(t - 2\tau)] \\ &= [1 + \alpha\beta\cos(2\omega_0\tau)]\cos(\omega_0 t) + \alpha\beta\sin(2\omega_0\tau)\sin(\omega_0 t) \\ &= A(\tau)\cos[\omega_0 t - \theta(\tau)] \\ &= A(\tau)\cos\theta(\tau)\cos(\omega_0 t) + A(\tau)\sin\theta(\tau)\sin(\omega_0 t) \end{aligned}$$

Set coefficients of like sin/cos terms equal on each side of the identity to obtain

$$\begin{aligned} A(\tau)\cos\theta(\tau) &= 1 + \alpha\beta\cos(2\omega_0\tau) \\ A(\tau)\sin\theta(\tau) &= \alpha\beta\sin(2\omega_0\tau) \end{aligned}$$

Square and add to obtain

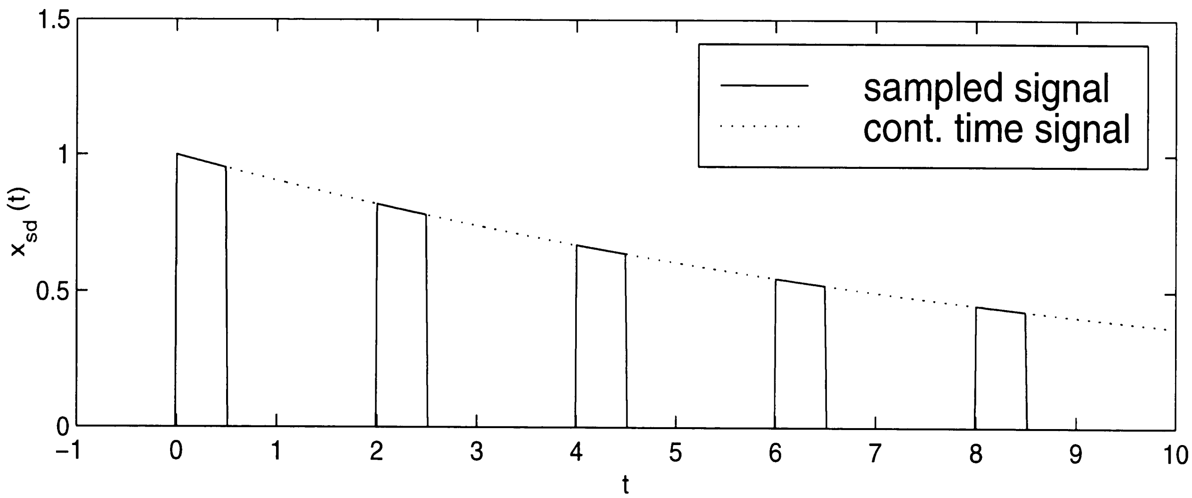
$$A(\tau) = \sqrt{1 + 2\alpha\beta\cos(2\omega_0\tau) + (\alpha\beta)^2}$$

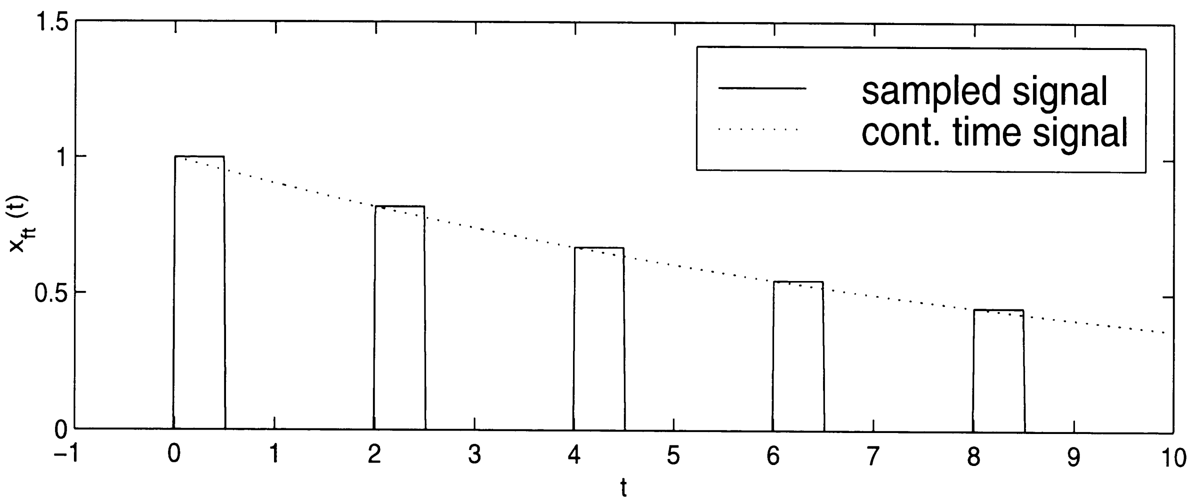
Divide the second equation by the first to obtain

$$\frac{\sin\theta(\tau)}{\cos\theta(\tau)} = \tan\theta(\tau) = \frac{\alpha\beta\sin(2\omega_0\tau)}{1 + \alpha\beta\cos(2\omega_0\tau)}$$

Problem 1-6

Sketches of the analog and sampled signals for both cases are shown below[(a) top and (b) bottom]:





Problem 1-7

(a) The impulse-sampled signal is

$$x_{\text{imp. samp}}(t) = \cos(2\pi t) \sum_{n=-\infty}^{\infty} \delta(t - 0.1n)$$

$$= \sum_{n=-\infty}^{\infty} \cos(2\pi t)\,\delta(t - 0.1n)$$

$$= \sum_{n=-\infty}^{\infty} \cos(0.2\pi n)\,\delta(t - 0.1n)$$

where property (1-59) for the unit impulse has been used to get the last result.
(b) The unit-pulse train sampled signal is

$$x_{\text{unit pulse samp}}(t) = \cos(2\pi t) \sum_{n=-\infty}^{\infty} \delta[t - 0.1n]$$

$$= \sum_{n=-\infty}^{\infty} \cos(2\pi t)\,\delta[t - 0.1n]$$

$$= \sum_{n=-\infty}^{\infty} \cos(0.2\pi n)\,\delta[t - 0.1n]$$

where the fact that the unit pulse is 1 for its argument 0 and 0 otherwise has been used.

Problem 1-8

(a) The signal can be developed in terms of equations as follows:

$$\Pi(0.1t) = \begin{cases} 1, & |0.1t| \le 1/2 \\ 0, & \text{otherwise} \end{cases}$$

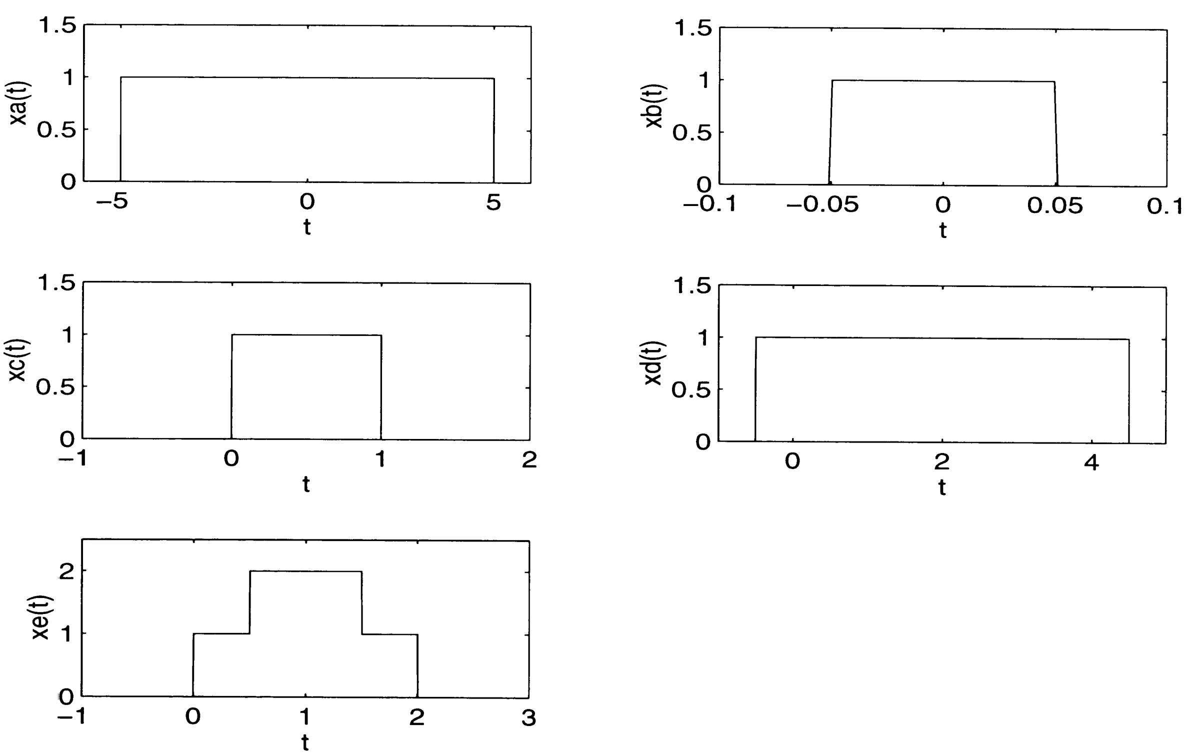
$$= \begin{cases} 1, & |t| \le 10/2 = 5 \\ 0, & \text{otherwise} \end{cases}$$

This is a rectangular pulse of amplitude 1 between -5 and 5 and 0 otherwise. A sketch will be given at the end of the problem solution.
(b) Following a procedure similar to that of (a) one finds that this is a rectangular pulse of amplitude 1 between -0.05 and 0.05 and 0 otherwise. A sketch will be given at the end of the problem solution.
(c) This is a rectangular pulse of amplitude 1 between 0 and 1 and 0 otherwise. A sketch will be given at the end of the problem solution.
(d) This is a rectangular pulse of amplitude 1 between 0.5 and 4.5 and 0 otherwise. A sketch will be given at the end of the problem solution.

(e) The first term of this signal is a rectangular pulse of amplitude 1 between 0 and 2 and 0 otherwise. The second term is a rectangular pulse of amplitude 1 between 0.5 and 1.5 and 0 otherwise. Where both pulses are nonzero, the total amplitude is 2; where only one pulse is nonzero the amplitude is 1. A sketch is provided below.

The MATLAB program below uses the special function given in Section 1-6 (page 32) of the text to provide the plots.

```
%       Sketches for Problem 1-8
%
t = -6:0.0015:6;
xa = pls_fn(0.1*t);
xb = pls_fn(10*t);
xc = pls_fn(t - 0.5);
xd = pls_fn((t - 2)/5);
xe = pls_fn((t - 1)/2) + pls_fn(t - 1);
subplot(3,2,1),plot(t, xa,'-w'), axis([-6 6 0 1.5]),xlabel('t'),ylabel('xa(t)')
subplot(3,2,2),plot(t, xb,'-w'), axis([-.1 .1 0 1.5]),xlabel('t'),ylabel('xb(t)')
subplot(3,2,3),plot(t, xc,'-w'), axis([-1 2 0 1.5]),xlabel('t'),ylabel('xc(t)')
subplot(3,2,4),plot(t, xd,'-w'), axis([-1 5 0 1.5]),xlabel('t'),ylabel('xd(t)')
subplot(3,2,5),plot(t, xe,'-w'), axis([-1 3 0 2.5]),xlabel('t'),ylabel('xe(t)')
```



Problem 1-9

(a) $2\pi f_0 = 50\pi$, so $T_0 = 1/f_0 = 1/25 = 0.04$ s. (b) $2\pi f_0 = 60\pi$, so $T_0 = 1/f_0 = 1/30 = 0.0333$ s. (c) $2\pi f_0 = 70\pi$, so $T_0 = 1/f_0 = 1/35 = 0.0286$ s. (d) We have $50\pi = 2\pi m f_0$ and $60\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 5 Hz with $m = 5$ and $n = 6$. (e) We have $50\pi = 2\pi m f_0$ and $70\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 5 Hz with $m = 5$ and $n = 7$.

Problem 1-10

(a) $|A| = 4.2426$; angle(A) = 0.7854 radians; $B = 5.0 + j\,8.6603$, so Re(B) = 5 and Im(B) = 8.6603. (b) $A+B = 8.0 + j11.6603$. (c) $A - B = -2.0 - j5.6603$. (d) $A*B = -10.9808 + j40.9808$. (e) $A/B = 0.4098 - j0.1098$.

Problem 1-11

(a) $2\pi f_0 = 10\pi$, so $T_0 = 1/f_0 = 1/5 = 0.2$ s. (b) $2\pi f_0 = 17\pi$, so $T_0 = 1/f_0 = 1/8.5 = 0.1176$ s. (c) $2\pi f_0 = 19\pi$, so $T_0 = 1/f_0 = 1/9.5 = 0.1053$ s. (d) We have $10\pi = 2\pi m f_0$ and $17\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 0.5 Hz with $m = 10$ and $n = 17$. (e) We have $10\pi = 2\pi m f_0$ and $19\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 0.5 Hz with $m = 10$ and $n = 19$. (f) We have $17\pi = 2\pi m f_0$ and $19\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 0.5 Hz with $m = 17$ and $n = 19$.

Problem 1-12

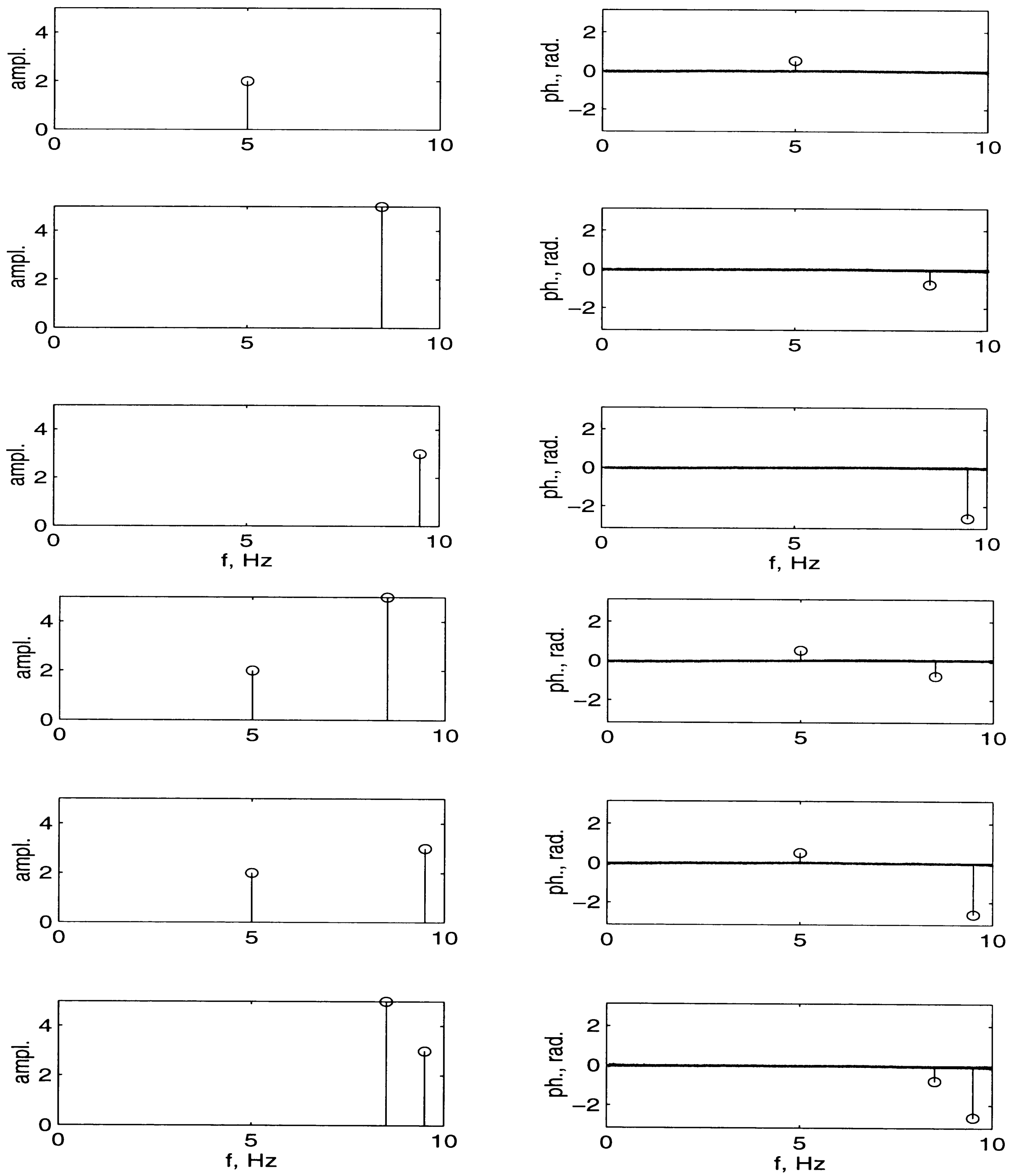
(a) Written as the real part of rotating phasors:

$$x_a(t) = \mathrm{Re}[2e^{j(10\pi t + \pi/6)}];\quad x_b(t) = \mathrm{Re}[5e^{j(17\pi t - \pi/4)}]$$
$$x_c(t) = \mathrm{Re}[3e^{j(10\pi t - \pi/3 - \pi/2)}] = \mathrm{Re}[3e^{j(10\pi t - 5\pi/6)}]$$
$$x_d(t) = \mathrm{Re}[2e^{j(10\pi t + \pi/6)} + 5e^{j(17\pi t - \pi/4)}];\quad x_e(t) = \mathrm{Re}[2e^{j(10\pi t + \pi/6)} + 3e^{j(10\pi t - 5\pi/6)}]$$
$$x_f(t) = \mathrm{Re}[5e^{j(17\pi t - \pi/4)} + 3e^{j(10\pi t - 5\pi/6)}]$$

(b) In terms of counter rotating phasors, the signals are:

$$x_a(t) = [e^{j(10\pi t + \pi/6)} + e^{-j(10\pi t + \pi/6)}];\quad x_b(t) = [2.5e^{j(17\pi t - \pi/4)} + 2.5e^{j(17\pi t - \pi/4)}]$$
$$x_c(t) = [1.5e^{j(10\pi t - 5\pi/6)} + 1.5e^{-j(10\pi t - 5\pi/6)}]$$
$$x_d(t) = [e^{j(10\pi t + \pi/6)} + e^{-j(10\pi t + \pi/6)} + 2.5e^{j(17\pi t - \pi/4)} + 2.5e^{-j(17\pi t - \pi/4)}]$$
$$x_e(t) = [e^{j(10\pi t + \pi/6)} + e^{-j(10\pi t + \pi/6)} + 1.5e^{j(10\pi t - 5\pi/6)} + 1.5e^{-j(10\pi t - 5\pi/6)}]$$
$$x_f(t) = [2.5e^{j(17\pi t - \pi/4)} + 2.5e^{-j(17\pi t - \pi/4)} + 1.5e^{j(10\pi t - 5\pi/6)} + 1.5e^{-j(10\pi t - 5\pi/6)}]$$

(c) Single-sided spectra are plotted below. Double-sided amplitude spectra are obtained by halving the lines and taking mirror image; phase spectra are obtained by taking antisymmetric mirror image.



Problem 1-13

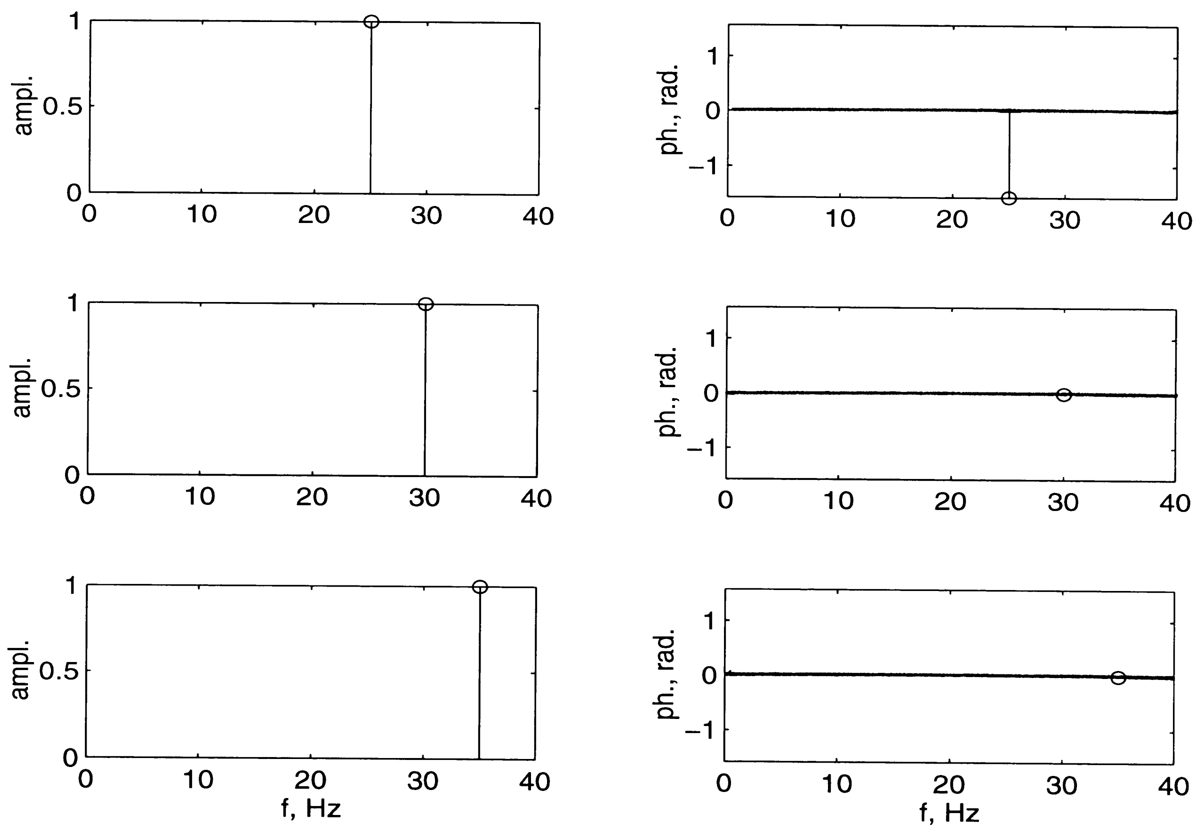
(a) Written as the real part of rotating phasors:

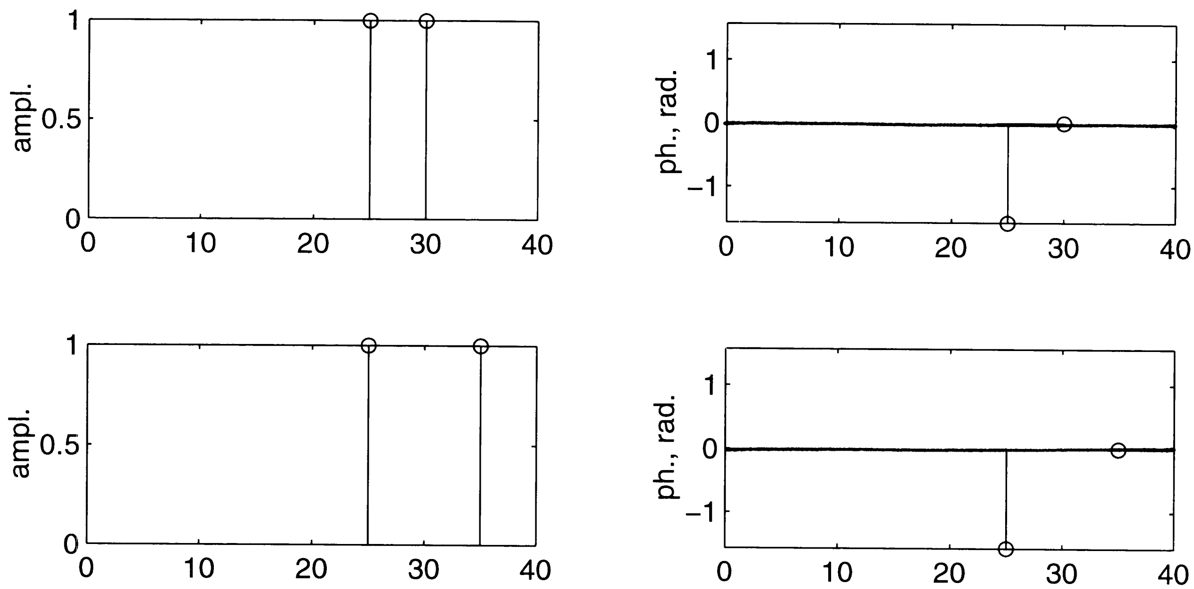
$$x_a(t) = \text{Re}[e^{j(50\pi t - \pi/2)}];\ x_b(t) = \text{Re}[e^{j60\pi t}];\ x_c(t) = \text{Re}[e^{j70\pi t}]$$
$$x_d(t) = \text{Re}[e^{j(50\pi t - \pi/2)} + e^{j60\pi t}];\ x_e(t) = \text{Re}[e^{j(50\pi t - \pi/2)} + e^{j70\pi t}]$$

(b) In terms of counter rotating phasors, the signals are:

$$x_a(t) = \text{Re}[0.5e^{j(50\pi t - \pi/2)} + 0.5e^{-j(50\pi t - \pi/2)}];\ x_b(t) = \text{Re}[0.5e^{j60\pi t} + 0.5e^{-j60\pi t}]$$
$$x_c(t) = \text{Re}[0.5e^{j70\pi t} + 0.5e^{-j70\pi t}]$$
$$x_d(t) = \text{Re}[0.5e^{j(50\pi t - \pi/2)} + 0.5e^{-j(50\pi t - \pi/2)} + 0.5e^{j60\pi t} + 0.5e^{-j60\pi t}]$$
$$x_e(t) = \text{Re}[0.5e^{j(50\pi t - \pi/2)} + 0.5e^{-j(50\pi t - \pi/2)} + 0.5e^{j70\pi t} + \ + 0.5e^{-j70\pi t}]$$

(c) The single-sided amplitude and phase spectra are shown below. See Prob. 1-12c for comments on obtaining double-sided spectra from single-sided spectra.

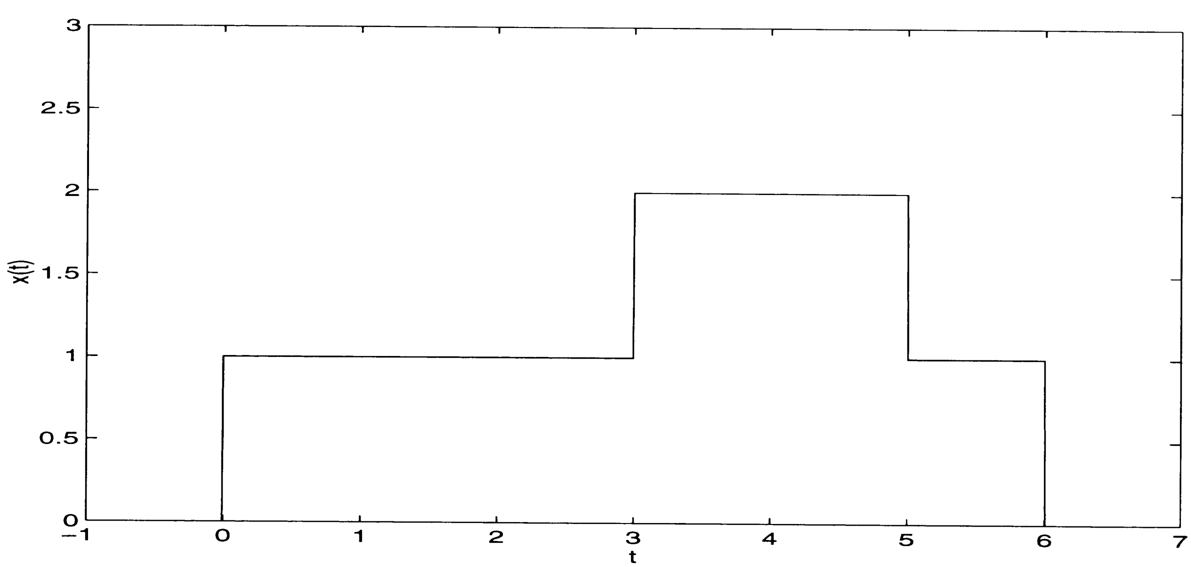




Problem 1-14

(a) A sketch is given below:

From the figure, it is evident that $x(t) = u(t) + u(t-3) - u(t-5) - u(t-6)$.



(b) The derivative of $x(t)$ is $dx(t)/dt = \delta(t) + \delta(t-3) - \delta(t-5) - \delta(t-6)$

Problem 1-15

Note that

$$\sin(\omega_0 t + \theta) = \frac{1}{2j} e^{j\theta} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\theta} e^{-j\omega_0 t}$$
$$= \frac{1}{2} e^{j(\theta - \pi/2)} e^{j\omega_0 t} + \frac{1}{2} e^{-j(\theta - \pi/2)} e^{-j\omega_0 t}$$

Thus we conclude the following:

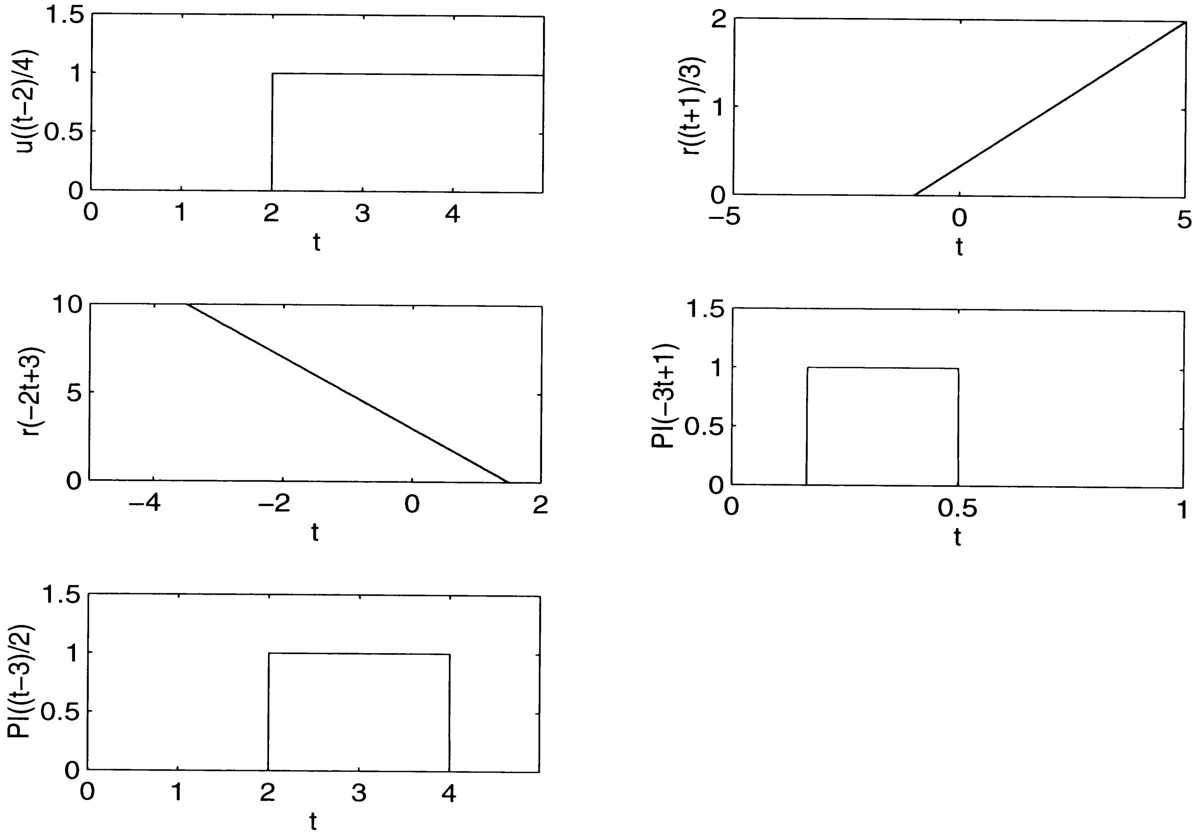
(1) The amplitude spectrum does not change;

(2) The phase spectrum has a $-\pi/2$ radian phase shift with respect to the cosine-convention phase spectrum. This destroys the odd symmetry present in the phase spectrum using the real part convention.

Problem 1-16

A MATLAB script is provided below to show all the plots:

```
%       Plots for Problem 1-16
%
t = -5:.001:5;
xa = stp_fn((t-2)/4);
xb = rmp_fn((t+1)/3);
xc = rmp_fn(-2*t+3);
xd = pls_fn(-3*t+1);
xe = pls_fn((t-3)/2);
subplot(3,2,1),plot(t,xa,'-w'),xlabel('t'),ylabel('u((t-2)/4)'),...
        axis([0 5 0 1.5])
subplot(3,2,2),plot(t,xb,'-w'),xlabel('t'),ylabel('r((t+1)/3)')
subplot(3,2,3),plot(t,xc,'-w'),xlabel('t'),ylabel('r(-2t+3)'),...
        axis([-5 2 0 10])
subplot(3,2,4),plot(t,xd,'-w'),xlabel('t'),ylabel('PI(-3t+1)'),...
        axis([0 1 0 1.5])
subplot(3,2,5),plot(t,xe,'-w'),xlabel('t'),ylabel('PI((t-3)/2)'),...
        axis([0 5 0 1.5])
```



Problem 1-17

From (1-37)

$$u_{-3}(t) = \begin{cases} t^2/2, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, from (1-35a) with $i = -3$,

$$u_{-4}(t) = \int_{-\infty}^{t} \frac{1}{2}\lambda^2 u(\lambda)d\lambda = \begin{cases} t^3/(2\times 3), & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

In general,

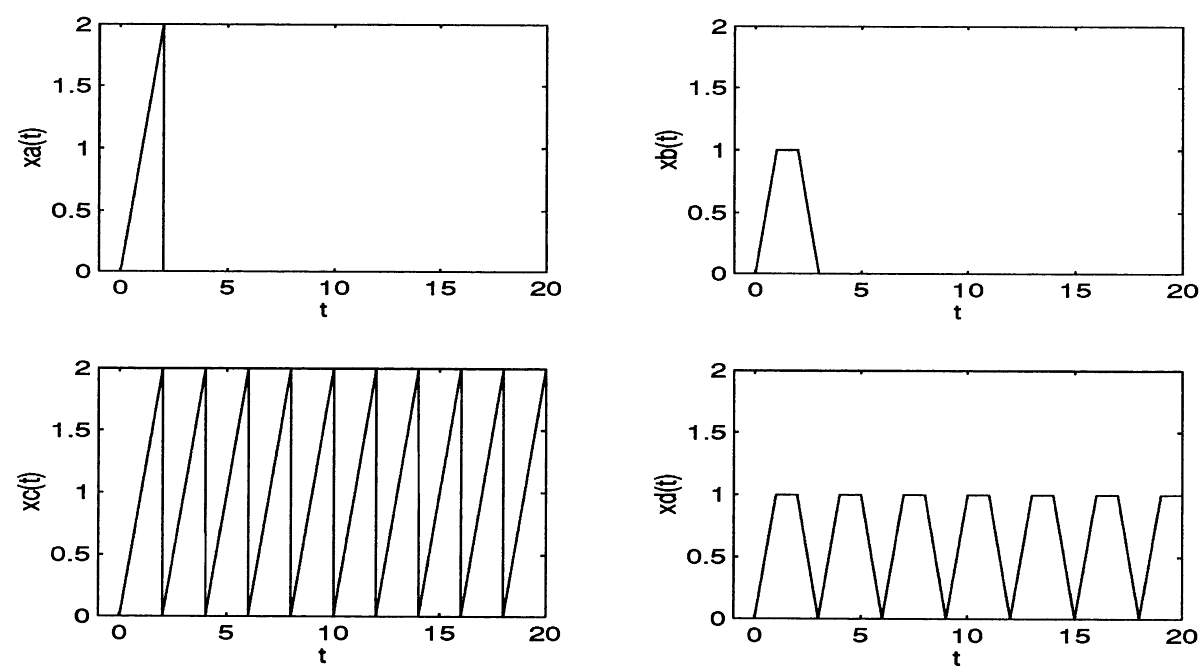
$$u_{-n}(t) = \begin{cases} t^{n-1}/(n-1)!, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 1-18

A MATLAB script is given below for making the plots. Note that use was made of functions to plot the repetitive signals in (c) and (d).

```
%      Plots for Problem 1-18
%
clf
t = -1:.005:20;
xa = xa_fn(t);
xb = xb_fn(t);
xc = xa;
xd = xb;
for n = 1:10
       xc = xc + xa_fn(t - 2*n);
       xd = xd + xb_fn(t - 3*n);
end
subplot(2,2,1),plot(t,xa),xlabel('t'),ylabel('xa(t)'),...
       axis([-1 20 0 2])
subplot(2,2,2),plot(t,xb),xlabel('t'),ylabel('xb(t)'),...
       axis([-1 20 0 2])
subplot(2,2,3),plot(t,xc),xlabel('t'),ylabel('xc(t)'),...
       axis([-1 20 0 2])
subplot(2,2,4),plot(t,xd),xlabel('t'),ylabel('xd(t)'),...
       axis([-1 20 0 2])
```

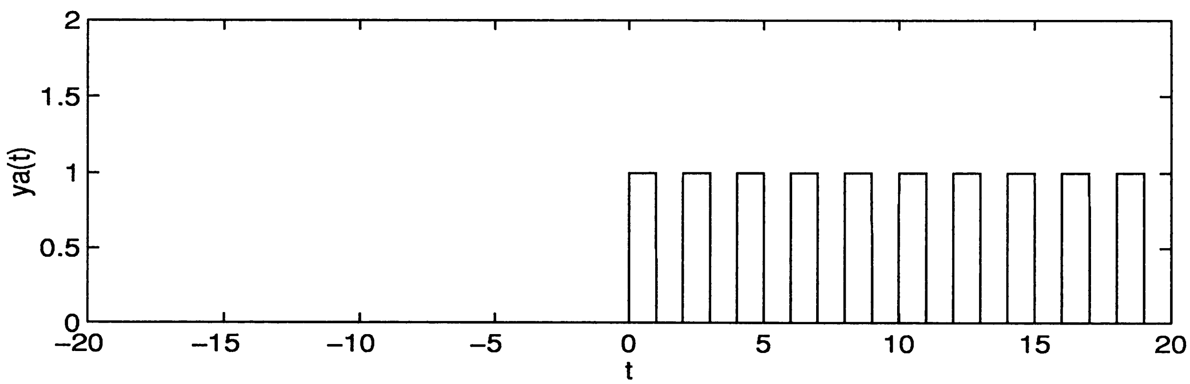
```
%      Function to compute xa for problem 1-18
%

function x = xa_fn(t)
x = rmp_fn(t).*stp_fn(2 - t);
%      Function to compute xb for problem 1-18
%

function x = xb_fn(t)
x = rmp_fn(t)-rmp_fn(t-1)-rmp_fn(t-2)+rmp_fn(t-3);
```

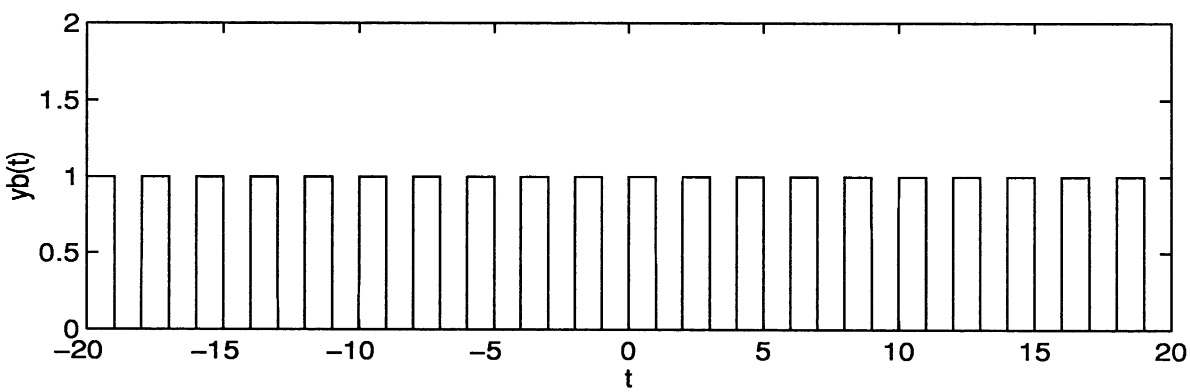


Problem 1-19

(a) Note that for $n = 0$ the summand can be written as $\Pi(t- 1/2)$. The MATLAB script below provides the plots for parts (a) and (c). The signal in part (a) is not periodic because it starts at $t = 0$. The signal of part (c) is periodic because it starts at $t = -\infty$.

```
%       Plots for Problem 1-19
%
clg
t = -20:.005:20;
ya = pls_fn(t - 0.5);
yb = pls_fn(t - 0.5);
for n = 1:10
        ya = ya + pls_fn(t-.5-2*n);
        yb = yb + pls_fn(t-.5-2*n)+ pls_fn(t-.5+2*n);
end
subplot(2,1,1),plot(t, ya, '-w'),xlabel('t'),ylabel('ya(t)'),...
        axis([-20 20 0 2])
subplot(2,1,2),plot(t, yb, '-w'),xlabel('t'),ylabel('ya(t)'),...
        axis([-20 20 0 2])
```





Problem 1-20

Representations for the signals are given below (others may be possible):

$$x_a(t) = \sum_{n=0}^{\infty} r(t - 3n)u(2 - t - 3n)$$

$$x_b(t) = \sum_{n=0}^{\infty} u(t - 4n)u(2 - t - 4n)$$

$$x_c(t) = \sum_{n=0}^{\infty} 2\delta(t - 2.5n)$$

$$x_d(t) = \sum_{n=0}^{\infty} \frac{2}{3}u(t - 3n)r(3 - t - 3n)$$

Problem 1-21

One possible representation for each (these follow from the results of Prob. 1-17) is:

$$x_1(t) = 2u_{-3}(t - 1)u(2 - t) + u(t - 2)u(4 - t) + 2u_{-3}(5 - t)u(t - 4)$$

$$x_2(t) = \frac{3}{2}u_{-4}(t)u(2 - t) + \frac{4}{9}u_{-3}(5 - t)u(t - 2)$$

Problem 1-22

One possible representation for each is

$$x_1(t) = u(t) + r(t - 1) - 2r(t - 2) + r(t - 3) - u(t - 4)$$

$$x_2(t) = u(t) - 2u(t - 1) + 2u(t - 2) - u(t - 3)$$

$$x_3(t) = r(t) - r(t - 1) - r(t - 3) + r(t - 4)$$

$$x_4(t) = r(t) - 2u(t - 1) - r(t - 2)$$

Problem 1-23

(a) First note that the integral of the function is 1, no matter what the value for ϵ:

$$I = \int_{-\infty}^{\infty} \delta_\epsilon(t)dt = \int_{0}^{\infty} \frac{1}{\epsilon}e^{-t/\epsilon}dt = -e^{-t/\epsilon}\Big|_0^\infty = 1$$

Second, note that the pulse becomes infinitely narrow and infinitely high as $\epsilon \to \infty$.

(b) Use the integral

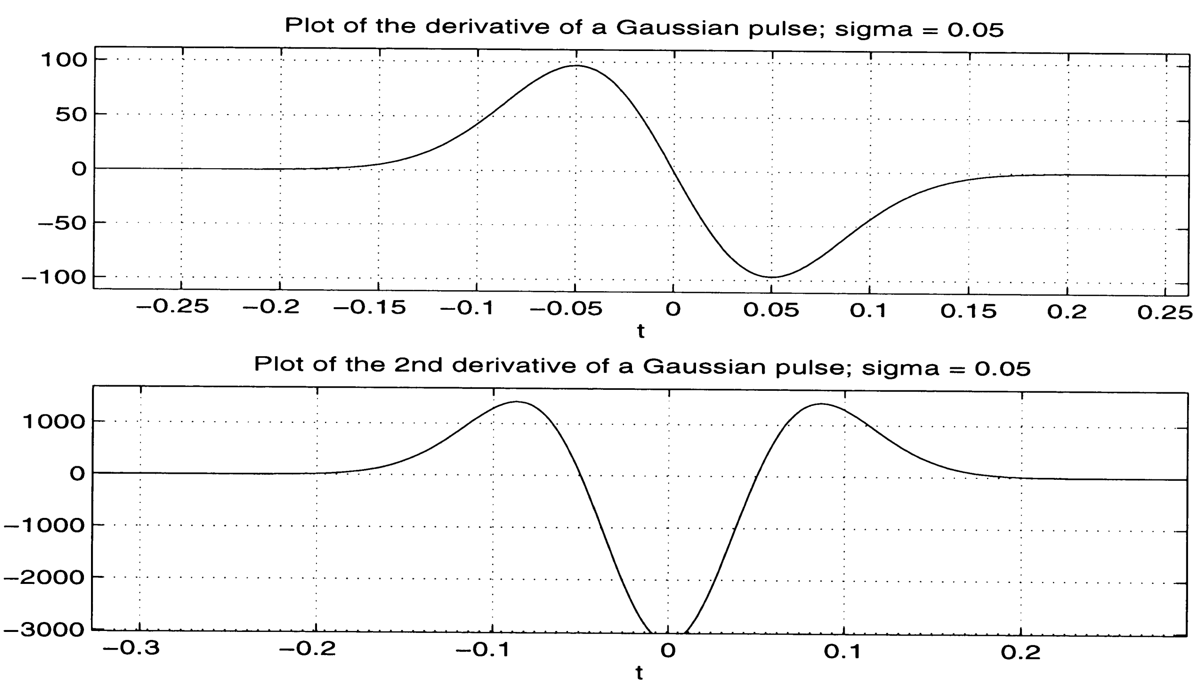
$$\int_{-\infty}^{\infty} e^{-\alpha t^2} dt = \sqrt{\frac{\pi}{\alpha}}$$

to show that the area under the given function is 1. Then note that as $\sigma \to 0$, the function becomes infinitely narrow and infinitely high. Thus the properties of a delta function are satisfied.

Problem 1-24

A MATLAB script using symbolic operations is given below for plotting the desired functions:

```
%       Plots for Problem 1-24
%
format short
sigma = 0.05;
y = 'exp(-t^2/(2*0.05^2))/sqrt(2*pi*0.05^2)';
y_prime = diff(y)
y_dbl_prime = diff(y_prime)
subplot(2,1,1),ezplot(y_prime),...
        title(['Plot of the derivative of a Gaussian pulse; sigma = ',num2str(sigma)])
subplot(2,1,2),ezplot(y_dbl_prime),...
        title(['Plot of the 2nd derivative of a Gaussian pulse; sigma = ',num2str(sigma)])
```



Problem 1-25

Using the stated rules, the first derivative is

$$\frac{dh(t)}{dt} = e^{-\alpha t}\frac{du(t)}{dt} - \alpha e^{-\alpha t}u(t) = 1\times\delta(t) - \alpha e^{-\alpha t}u(t)$$

The second derivative is

$$\frac{dh^2(t)}{dt^2} = \frac{d\delta(t)}{dt} + \alpha^2 e^{-\alpha t}u(t) - \alpha e^{-\alpha t}\frac{du(t)}{dt} = \frac{d\delta(t)}{dt} + \alpha^2 e^{-\alpha t}u(t) - \alpha\,\delta(t)$$

Problem 1-26

(a) The integral is zero because the delta function is outside the range of integration.
(b) The integral evaluates as follows:

$$\int_0^5 \cos(2\pi t)\,\delta(t-2)\,dt = \cos(4\pi) = 1$$

(c) This integral can be evaluated as

$$\int_0^5 \cos(2\pi t)\,\delta(t-0.5)\,dt = \cos(\pi) = -1$$

(d) The value of this integral is 0:

$$\int_{-\infty}^{\infty} (t-2)^2\,\delta(t-2)\,dt = (2-2)^2 = 0$$

(e) This integral evaluates to

$$\int_{-\infty}^{\infty} t^2\,\delta(t-2)\,dt = 2^2 = 4$$

Problem 1-27

(a) Using (1-66), this integral becomes

$$\int_{-\infty}^{\infty} e^{3t}\ddot{\delta}(t-2)\,dt = (-1)^2 \frac{d^2}{dt^2} e^{3t}\bigg|_{t=2} = 9e^6$$

(b) Again applying (1-66), we have

$$\int_{0}^{10} \cos(2\pi t)\,\dddot{\delta}(t-0.5)\,dt = (-1)^3 \frac{d^3}{dt^3}\cos(2\pi t)\bigg|_{t=0.5} = -(2\pi)^3 \sin(2\pi t)\big|_{t=0.5} = 0$$

(c) Using (1-66) we get

$$\int_{-\infty}^{\infty} \left[e^{-3t} + \cos(2\pi t)\right]\dot{\delta}(t)\,dt = (-1)\frac{d}{dt}\left[e^{-3t} + \cos(2\pi t)\right]_{t=0} = -\left[-3e^{-3t} - 2\pi\sin(2\pi t)\right]_{t=0} = 3$$

Problem 1-28

Match coefficients of like derivatives of $\delta(t)$ on either side of the given equations:
(a) In this case, we obtain

$$10 = 3 + C_3 \text{ or } C_3 = 7;\ C_1 = 5;\ 2 + C_2 = 6 \text{ or } C_2 = 4$$

(b) The resulting equations are

$$3 + C_1 = 0 \text{ or } C_1 = -3;\ C_2 = 0;\ C_3 = 0;\ C_4 = 0;\ C_5 = 0$$

Problem 1-29

(a)
- (1) This plots to a triangle 2 units high, centered on $t = 0$, and going from $t = -2$ to 2.
- (2) This is a rectangle of unit height starting at $t = 0$ and ending at $t = 10$.
- (3) This is a step of height 2 starting at $t = 0$ with an impulse of unit area at $t = 2$ superimposed.
- (4) This an impulse of area 2 at $t = 2$.

(b) One possible representation is

$$x(t) = r(t+4) - r(t+2) + u(t) - 3r(t-4) + 3r(t-5)$$

Problem 1-30

A possible representation is

$$x(t) = 2u(t) - u(t-2) + u(t-4) - r(t-6) + r(t-8)$$

Problem 1-31

(a) Using the sifting property of the delta function, we get

$$\int_{-\infty}^{\infty} t^3 \delta(t-3)\,dt = t^3\big|_{t=3} = 27$$

(b) Using (1-66), we get

$$\int_{-\infty}^{\infty} [3t + \cos(2\pi t)]\,\dot{\delta}(t-5)\,dt = (-1)\frac{d}{dt}[3t + \cos(2\pi t)]_{t=5} = -[3 - 2\pi\sin(2\pi t)]_{t=5} = -3$$

(c) From (1-66) we have

$$\int_{-\infty}^{\infty} (1+t^2)\,\dot{\delta}(t-1.5)\,dt = (-1)\frac{d}{dt}(1+t^2)\big|_{t=1.5} = -(2t)\big|_{t=1.5} = -3$$

Problem 1-32

(a) A possible representation is

$$x_a(t) = A[2u(t) - 2u(t-T) + u(t-2T) - u(t-3T)]$$

(b) One representation of this signal is

$$x_b(t) = r(t) - 2r(t-1) + 2r(t-3) - r(t-4)]$$

(c) One way of writing this signal is

$$x_c(t) = r(t-1) - 2r(t-2) + r(t-3) + 0.5[u(t-1.5) - u(t-2.5)]$$

Problem 1-33

(a) This is a decaying exponential starting at $t = 0$. Its energy is

$$E = \int_0^\infty e^{-20t}\,dt = \left.\frac{e^{-20t}}{-20}\right|_0^\infty = \frac{1}{20}\ \text{J}$$

(b) This is a rectangular pulse starting at $t = 0$ and ending at $t = 15$. Its energy is

$$E = \int_{-\infty}^{\infty} [u(t) - u(t-15)]^2\,dt = \int_0^{15} 1^2\,dt = 15\ \text{J}$$

(c) This is a cosine burst starting at $t = 0$ and ending at $t = 2$. It contains 10 cycles. Its energy is calculated as

$$E = \int_{-\infty}^{\infty} \cos^2(10\pi t)\,[u(t)u(2-t)]^2\,dt = \int_0^2 \cos^2(10\pi t)\,dt = \int_0^2 \left[\frac{1}{2} + \frac{1}{2}\cos(10\pi t)\right]dt = 1\ \text{J}$$

(d) This is a triangle going from $t = 0$ and ending at $t = 2$ of unit height. For $0 \le t \le 1$ its equation is just t. The integral of t^2 from 0 to 1 can be doubled to yield the total energy with the result

$$E = 2\int_0^1 t^2\,dt = 2\left.\frac{t^3}{3}\right|_0^t = \frac{2}{3}\ \text{J}$$

Problem 1-34

(a) Note that $x_1(t)$ is symmetrical about $t = 2$. Therefore

$$E_1 = 2\left[\int_0^2 1^2 dt + \int_1^2 t^2 dt\right] = 2\left[t\,\Big|_0^2 + \frac{t^3}{3}\Big|_1^2\right] = \frac{20}{3}\ \text{J}$$

(b) Note that $x_2^2(t) = 1$ for t between 0 and 3, and is 0 otherwise. Therefore

$$E_2 = \int_0^3 1^2 dt = 3\ \text{J}$$

(c) Note that $x_3(t)$ is symmetrical about $t = 2$ which allows the energy to be calculated as

$$E_3 = 2\left[\int_0^1 t^2 dt + \int_1^2 1^2 dt\right] = 2\left[\frac{t^3}{3}\Big|_0^1 + t\,\Big|_1^2\right] = \frac{8}{3}\ \text{J}$$

(d) Note that $x_4^2(t)$ is symmetrical about $t = 1$ which allows the energy of $x_4(t)$ to be calculated as

$$E_4 = 2\int_0^1 t^2 dt = 2\frac{t^3}{3}\Big|_0^1 = \frac{2}{3}\ \text{J}$$

Problem 1-35

Only (a) and (b) are energy signals. For (a)

$$E_a = \int_0^2 t^2 dt = \frac{t^3}{3}\Big|_0^2 = \frac{8}{3}\ \text{J}$$

For (b), we note that it is symmetric about $t = 1.5$. It is a ramp from 0 to 1 and constant from 1 to 1.5, which yields

$$E_b = 2\left[\int_0^1 t^2 dt + \int_1^{1.5} 1^2 dt\right] = 2\left[\frac{t^3}{3}\Big|_0^1 + t\,\Big|_1^{1.5}\right] = 2[1/3 + 1.5 - 1] = 5/3\ \text{J}$$

The other functions are semi-infinite in extent, so their squares will integrate to infinity.

Problem 1-36

The average powers of (a) - (c) are ½ W; for (d) and (e), the powers are 1 W. These are obtained by squaring the amplitudes of the separate frequency components, dividing by 2 to get power, and adding. This is permissible since the sinusoids have frequencies that are integer multiples of a fundamental frequency.

Problem 1-37

(a) $P = 2^2/2 = 2$ W; (b) $P = 5^2/2 = 12.5$ W; (c) $P = 3^2/2 = 4.5$ W; (d) $P = 2^2/2 + 5^2/2 = 14.5$ W; (e) $P = 2^2/2 + 3^2/2 = 6.5$ W; (f) $P = 5^2/2 + 3^2/2 = 17$ W.

Problem 1-38

(a) Power:

$$P_a = \lim_{T\to\infty} \frac{1}{2T}\left[\int_0^1 1^2\,dt + \int_1^2 6^2\,dt + \int_2^T 4^2\,dt\right] = 0 + 0 + \lim_{T\to\infty}\frac{16(T-2)}{2T} = 8 \text{ W}$$

(b) Energy:

$$E_b = \int_0^1 1^2\,dt + \int_1^2 6^2\,dt = 37 \text{ J}$$

(c) Energy:

$$E_c = \int_0^\infty e^{-10t}\,dt = -\frac{e^{-10t}}{10}\bigg|_0^\infty = \frac{1}{10} \text{ J}$$

(d) Power:

$$P_d = \lim_{T\to\infty}\frac{1}{2T}\int_0^T \left[e^{-5t}+1\right]^2 dt = \lim_{T\to\infty}\frac{1}{2T}\int_0^T\left[e^{-10t} + 2e^{-5t} + 1\right]dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\left[-\frac{e^{-10t}}{10} - \frac{2e^{-5t}}{5} + t\right]_0^T = \frac{1}{2} \text{ W}$$

(e) Power: similarly to (d), it can be shown that P_e = ½ W. (f) Neither: it can be shown that both the power and energy are infinite. (g) Power: P_g = ½ W. (h) Neither: $E_h = \infty$ and $P_h = 0$.

Problem 1-39

(a) Yes. The frequencies of its separate components are commensurable: $f_1 = 3\times1$ Hz and $f_2 = 5\times1$ Hz. Therefore, the fundamental frequency is 1 Hz and the period is 1 s.
(b) Its amplitude spectrum consists of a line of height 2 at 3 Hz and a line of height 4 at 5 Hz. Its phase spectrum consists of a line of height $-\pi/3$ at 3 Hz and a line of height $-\pi/2$ at 5 Hz.
(c) Written as the sum of counter-rotating phasors, the signal is

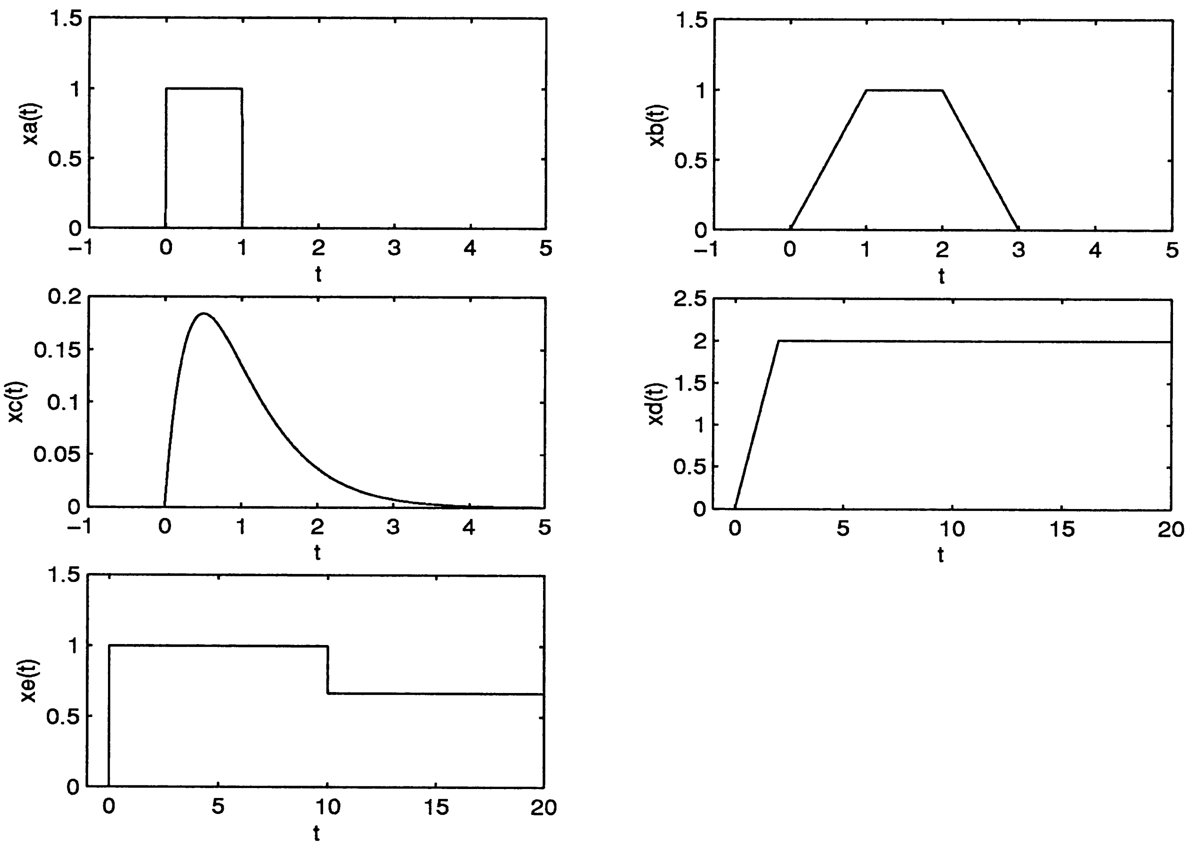
$$x(t) = e^{-j\pi/3} e^{j6\pi t} + e^{j\pi/3} e^{-j6\pi t} + 2 e^{-j\pi/2} e^{j10\pi t} + 2 e^{j\pi/2} e^{-j10\pi t}$$

(d) See (b): for the amplitude spectrum, halve the lines and take the mirror image about $f = 0$; for the phase spectrum, take the antisymmetric image about $f = 0$.
(e) It is clear that it is a power signal because it is the sum of sinusoids whose frequencies are harmonics of a fundamental frequency. The total power is $2^2/2 + 4^2/2 = 10$ W.

Problem 1-40

(a) through (c) are energy signals; (d) and (e) are power signals. By applying the definitions of energy and power, (1-75) and (1-76), respectively, the energies are $E_a = 1$ J, $E_b = 5/3$ J, $E_c = 1/32$ J, $P_d = 2$ W, and $P_e = 2/9$ W. The MATLAB script given below plots these signals:

```
%       Plots for Problem 1-40
%
t = -1:.005:20;
xa = stp_fn(t) - stp_fn(t-1);
xb = rmp_fn(t) - rmp_fn(t-1) - rmp_fn(t-2) + rmp_fn(t-3);
xc = t.*exp(-2*t).*stp_fn(t);
xd = rmp_fn(t) - rmp_fn(t-2);
xe = stp_fn(t) - (1/3)*stp_fn(t-10);
subplot(3,2,1),plot(t,xa,'-w'),xlabel('t'),ylabel('xa(t)'),...
        axis([-1 5 0 1.5])
subplot(3,2,2),plot(t,xb),xlabel('t'),ylabel('xb(t)'),...
        axis([-1 5 0 1.5])
subplot(3,2,3),plot(t,xc),xlabel('t'),ylabel('xc(t)'),...
        axis([-1 5 0 .2])
subplot(3,2,4),plot(t,xd),xlabel('t'),ylabel('xd(t)'),...
        axis([-1 20 0 2.5])
subplot(3,2,5),plot(t,xe),xlabel('t'),ylabel('xe(t)'),...
        axis([-1 20 0 1.5])
```



Problem 1-41

(a) Only (1) is periodic; $f_1 = 2.5$ Hz $= 0.5m$ and $f_2 = 3$ Hz $= 0.5n$ where the integers m and n are 5 and 6, respectively. The fundamental frequency is 0.5 Hz and the period is 2 s.
(b) Signals (1) and (2) are power signals. Their powers are both 1 W.
(c) Only signal (3) is an energy signal; its energy is 1/20 J. Signal (4) is neither energy nor power.

Problem 1-42

By definition, the average power is

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

For a periodic signal $x(t) = x(t + T_0)$, and the integral can be broken into segments one period long plus the end pieces that are less than a period. Because of periodicity, these integrals are equal with the exception of the end pieces. Thus, we can write the integral as

$$\int_{-T}^{T} |x(t)|^2 dt = 2N \int_{t_0}^{t_0 + T_0} |x(t)|^2 dt + \epsilon_{-N} + \epsilon_N$$

where the latter two terms represent the integrals over the end intervals. Also, $2T = 2NT_0 + \Delta t_1 + \Delta t_2$. The latter two time segments are the lengths of the end intervals which are less than a period. Thus, the power expression becomes

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T \to \infty} \frac{1}{2T} \left[\frac{2N \int_{t_0}^{t_0 + T_0} |x(t)|^2 dt + \epsilon_{-N} + \epsilon_N}{2NT_0 + \Delta t_1 + \Delta t_2} \right] = \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} |x(t)|^2 dt$$

Problem 1-43

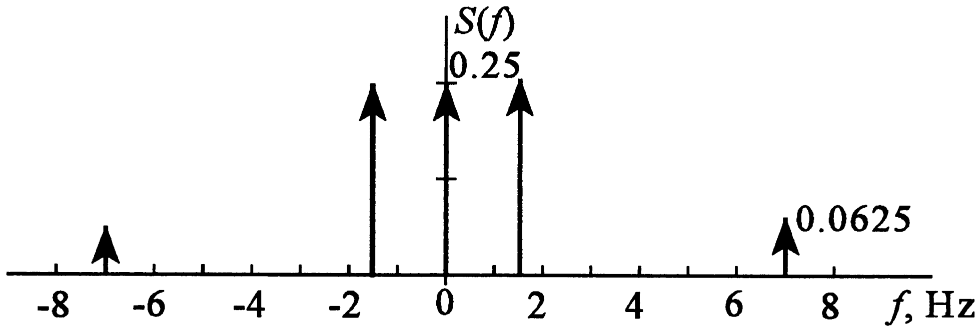
Use the trigonometric identity for $\sin^2(x) = ½ - ½ \cos(2x)$ to write the signal as

$$x(t) = \frac{1}{2} - \frac{1}{2}\cos(14\pi t - \pi/3) + \cos(3\pi t - \pi/3) = \frac{1}{2} + \frac{1}{2}\cos(14\pi t - \pi/3 + \pi) + \cos(3\pi t - \pi/3)$$
$$= \frac{1}{2} + \frac{1}{2}\cos(14\pi t + 2\pi/3) + \cos(3\pi t - \pi/3)$$

(a) Its single-sided amplitude spectrum consists of a line of height ½ at $f = 0$, a line of height 1 at $f = 1.5$ Hz, and a line of height ½ at $f = 7$ Hz. Its single-sided phase spectrum consists of no line at $f = 0$, a line of height $-\pi/3$ at $f = 1.5$ Hz, and a line of height $2\pi/3$ at $f = 7$ Hz.
(b) To get the double-sided amplitude spectrum, halve the lines in the single-sided spectrum and take its mirror image about $f = 0$. To get the double-sided phase spectrum, take the antisymmetric image of the single-sided spectrum about $f = 0$.

Problem 1-44

The signal has frequency components at 0, 1.5, and 7 Hz of amplitudes ½, 1, and ½, respectively. The power at dc is $(\frac{1}{2})^2 = 0.25$ W which is placed at the single frequency $f = 0$ Hz. The power at the other frequencies is split between the positive and corresponding negative frequency. Thus, at $f =$ 1.5 Hz we have $(1)^2/4 = 0.25$ W (one 2 in the denominator is from computing power in a sinusoid and the other 2 is from splitting it between positive and negative frequencies) and similarly at $f = -1.5$ Hz. At $f = 7$ Hz, we have a power of $(1/2)^2/4 = 0.0625$ W with a similar power at $f = -7$ Hz. All these are represented by impulses of the appropriate weights, so the plot is as shown below:



Problem 1-45

(a) Following the solution to Problem 1-44, we have spectral components at $f = 10$, 15, and 20 Hz of amplitudes 16, 6, and 4, respectively. The power in these components gets split between positive and negative frequencies. Thus, and $f = 10$ Hz we have a power of $(16)^2/4 = 64$ W with a corresponding power at $f = -10$ Hz. At $f = 15$ Hz we have a power of $(6)^2/4 = 9$ W with a corresponding power at $f = -15$ Hz. Finally, at $f = 20$ Hz we have a power of $(4)^2/4 = 4$ W with a corresponding power at $f = -20$ Hz. Mathematically, this can be expressed as

$$S_x(f) = 64[\delta(f-10) + \delta(f+10)] + 9[\delta(f-15) + \delta(f+15)] + 4[\delta(f-20) + \delta(f+20)]$$

(b) The power contained between 12 and 22 Hz is

$$P[12 \le |f| \le 22 \text{ Hz}] = \int_{-22}^{-12} S_x(f)\,df + \int_{12}^{22} S_x(f)\,df = 2\int_{12}^{22} S_x(f)\,df = 26 \text{ W}$$

CHAPTER 2

Problem 2-1

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Problem 2-2

(a) First order; (b) first order (differentiate once to get rid of the integral on y); (c) zero order; (d) first order; (e) second order.

Problem 2-3

(a), (b), (c), and (e) are fixed; (d) is not because of the time-varying coefficient, t^2.

Problem 2-4

Only (c) and (d) are nonlinear. Superposition will not hold in (e) because of the term +10. As an example to show linearity, consider (d):

$$\frac{dy_1(t)}{dt} + t^2 y_1(t) = \int_{-\infty}^{t} x_1(\lambda)\,d\lambda$$

$$\frac{dy_2(t)}{dt} + t^2 y_2(t) = \int_{-\infty}^{t} x_2(\lambda)\,d\lambda$$

Multiply the first equation by a constant, say a, and the second equation by another constant, say b; add to obtain:

$$\frac{d[ay_1(t) + by_2(t)]}{dt} + t^2[ay_1(t) + by_2(t)] = \int_{-\infty}^{t} [ax_1(\lambda) + bx_2(\lambda)]\,d\lambda$$

This is of the same form as the original equation.

Problem 2-5

Noncausal. Consider $t = 0.25$, which gives $y(0.25) = x(0.5)$; i.e., the output depends on a future value of the input.

Problem 2-6

(a) Nonlinear. The proof is similar to Example 2-4 in the text. (b) Noncausal because of the +2 in the argument of x. Consider $t = 0$; the output at time 0 depends on the value of the input at time 2, or a future value.

Problem 2-7

(a) Linear. Consider the responses to two arbitrary inputs:

$$y_1(t) = x_1(t^2)$$
$$y_2(t) = x_2(t^2)$$

Multiply first by a and the second by b and add to get

$$ay_1(t) + by_2(t) = ax_1(t^2) + bx_2(t^2)$$

That is, for the input $ax_1(t) + bx_2(t)$, we replace t by t^2 to get the new output which is the right-hand side of the above equation.

(b) Time varying. Consider the response to the delayed input:

$$y_a(t) = x(t^2 - \tau)$$

Now consider the delayed output due to the undelayed input:

$$y(t - \tau) = x[(t - \tau)^2]$$

Clearly the two are not the same.

(c) Noncausal. Consider $t = 2$ which gives $y(2) = x(4)$; i.e., the output depends on a future value of the input.

(d) Not zero memory. This follows from (c) where it was found that the output does not depend only on values of the input at the present time only.

Problem 2-8

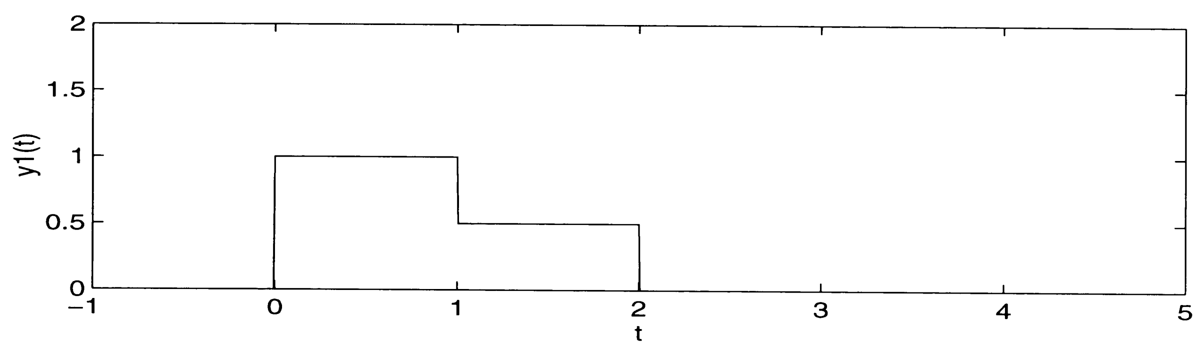
(a) Consider two inputs and the corresponding outputs:

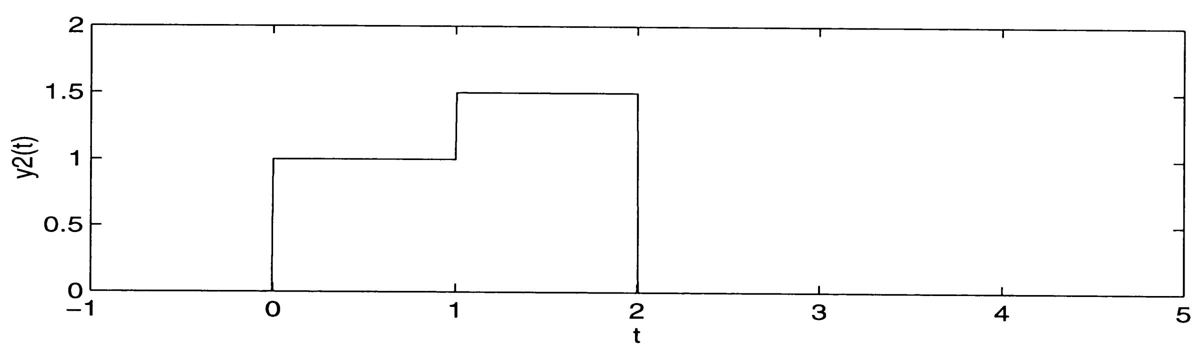
$$y_1(t) = x_1(t) + \alpha x_1(t - \tau_0)$$
$$y_2(t) = x_2(t) + \alpha x_2(t - \tau_0)$$

Multiply the top equation by a and the bottom by b (two constants); add to get

$$ay_1(t) + by_2(t) = ax_1(t) + bx_2(t) + \alpha[ax_2(t - \tau_0) + bx_1(t - \tau_0)]$$

This is of the same form as the original input/output relationship, so linarity is proved.
(b) The only way for the system to be zero memory is for τ_0 to be 0.
(c) It is causal only if $\tau_0 \geq 0$, for in that case the system doesn't respond before the input is applied.
(d) See the MATLAB plots below (α = 0.5 and 1.5 in that order):





Problem 2-9

(a) Consider two different inputs:

$$y_1(t) = \frac{1}{T_1 + T_2}\int_{t-T_1}^{t+T_2} x_1(\lambda)\,d\lambda$$

$$y_2(t) = \frac{1}{T_1 + T_2}\int_{t-T_1}^{t+T_2} x_2(\lambda)\,d\lambda$$

Multiply the first by a and the second by b (two arbitrary constants); add and rearrange to obtain

$$ay_1(t) + by_2(t) = \frac{1}{T_1 + T_2}\int_{t-T_1}^{t+T_2} [ax_1(\lambda) + bx_2(\lambda)]\,d\lambda$$

This is of the same form as the defining equation, so the system is linear.
(b) For causality, the output can't depend on future values of the input. This requires that $T_1 \geq 0$, $T_2 \leq 0$, and $T_1 > -T_2$.

Problem 2-10

Using Kirchoff's voltage equation and Ohm's law, the appropriate equations are

$$x(t) = L\frac{di(t)}{dt} + y(t)$$

$$y(t) = Ri(t)$$

$$\frac{di(t)}{dt} = \frac{1}{R}\frac{dy(t)}{dt}$$

Substitute the last equation in the first and rearrange to obtain

$$\frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{R}{L}x(t)$$

(b) The proof is similar to those of Problems 2-8 and 2-9.
(c) Consider

$$\frac{dy(t-\tau)}{dt} = \frac{dy(t')}{dt'}\frac{dt'}{dt} \text{ where } t' = t - \tau$$

Thus

$$\frac{dy(t-\tau)}{dt} + \frac{R}{L}y(t-\tau) = \frac{R}{L}x(t-\tau)$$

which shows that the system is fixed.
(d) Note that the solution to the homogeneous equation is

$$y_H(t) = Ae^{-Rt/L},\ t > 0$$

Assume a complete solution of this form where A is time varying. Substitute into the differential equation of part (a) to obtain

$$A(t) = \int_0^t \frac{R}{L}x(\lambda)e^{R\lambda/L}d\lambda + A_0$$

Since the inductor current is assumed 0 at $t = 0$, this gives $A_0 = 0$, so the solution to the differential equation is

$$y(t) = \int_0^t \frac{R}{L}x(\lambda)\exp\left[-\frac{R}{L}(t-\lambda)\right]d\lambda$$

Problem 2-11

Property	a	b	c	d	e	f
Linear	X		X		X	
Causal	X	X	X	X		
Fixed	X	X				
Dynamic	X	X	X	X	X	X
Order	2	3	2	2	0	2

Problem 2-12

(a) Linear; first order; causal; time invariant. (b) Linear; first order; causal; time varying.
(c) Nonlinear; second order; causal; time invariant. (d) Nonlinear; zero order; causal; time invariant.

Problem 2-13

(a) Write the system equations for two arbitrary inputs:

$$y_1(t) = x_1(t)\cos(100\pi t)$$
$$y_2(t) = x_2(t)\cos(100\pi t)$$

Multiply the first equation by an arbitrary constant, a, and the second by another constant, b. Add to obtain

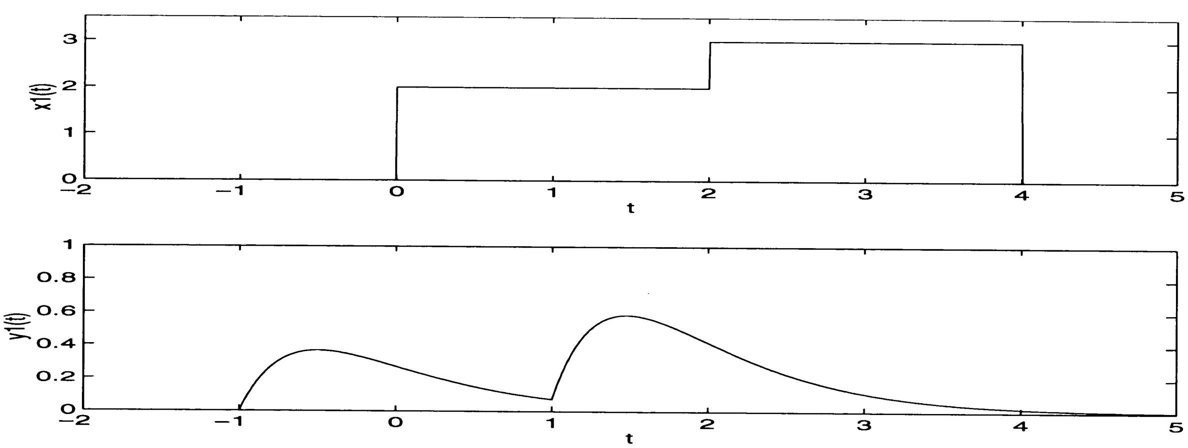
$$ay_1(t) + by_2(t) = [ax_1(t) + bx_2(t)]\cos(100\pi t)$$

This is of the same form as the system equation, so the system is linear. (b) The system is time-varying because of the $\cos(2\pi t)$ multiplying $x(t)$. (c) It is causal because the output does not depend on the future input. (d) It is instantaneous because the output depends only on the input at the present time.

Problem 2-14

(a) The system can't be causal because the output exists before the input is applied.
(b) Since the system is linear and fixed, superposition can be applied to get

$$y_{new}(t) = 2(t+1)\exp[2(t+1)]u(t+1) + 3(t-1)\exp[2(t-1)]u(t-1)$$



Problem 2-15

Given that

$$x_1(t) \rightarrow y_1(t) \text{ and } x_2(t) \rightarrow y_2(t)$$

where the arrow is read "produces the output", then

$$a_1 x_1(t) + a_2 x_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)$$

Now let

$$x_2(t) = b_1 x_{21}(t) + b_2 x_{22}(t)$$

By superposition, the response is

$$y_2(t) = b_1 y_{21}(t) + b_2 y_{22}(t)$$

where x_{21} produces the output y_{21}, etc. Thus

$$\begin{aligned} a_1 y_1 + a_2 y_2 &= a_1 y_1 + a_2[b_1 y_{21} + b_2 y_{22}] \\ &= a_1 y_1 + a_2' y_2' + a_2' y_3' \end{aligned}$$

where

$$a_2' = a_2 b_1;\ a_3' = a_2 b_2;\ y_2' = y_{21};\ y_3' = y_{22}$$

Thus, superposition can be extended to three or more inputs superimposed. Induction can be used to show it for the general case of N inputs.

Problem 2-16

(a) To show that

$$h(t)*[x_1(t) + x_2(t)] = h(t)*x_1(t) + h(t)*x_2(t)$$

The left-hand side can be written as

$$\begin{aligned}\text{LHS} &= \int_{-\infty}^{\infty} h(\lambda)[x_1(t-\lambda) + x_2(t-\lambda)]\,d\lambda \\ &= \int_{-\infty}^{\infty} h(\lambda)x_1(t-\lambda)\,d\lambda + \int_{-\infty}^{\infty} h(\lambda)x_2(t-\lambda)]\,d\lambda \\ &= h(t)*x_1(t) + h(t)*x_2(t)\end{aligned}$$

which proves the relationship.

(b) To show

$$h(t)*[x_1(t)*x_2(t)] = [h(t)*x_1(t)]*x_2(t)$$

The left-hand side can be written as

$$\text{LHS} = \int_{-\infty}^{\infty} h(\lambda)y(t-\lambda)\,d\lambda \text{ where } y(t) = \int_{-\infty}^{\infty} x_1(\eta)x_2(t-\eta)\,d\eta$$

Therefore

$$y(t-\lambda) = \int_{-\infty}^{\infty} x_1(\eta)x_2(t-\lambda-\eta)\,d\eta$$

and the left-hand side can be written as

$$\begin{aligned}\text{LHS} &= \int_{-\infty}^{\infty} h(\lambda)\left\{\int_{-\infty}^{\infty} x_1(\eta)x_2(t-\lambda-\eta)\,d\eta\right\}d\lambda = \int_{-\infty}^{\infty} h(\tau-\eta)\left\{\int_{-\infty}^{\infty} x_1(\eta)x_2(t-\tau)\,d\tau\right\}d\eta \\ &= \int_{-\infty}^{\infty} x_2(t-\tau)\left\{\int_{-\infty}^{\infty} x_1(\eta)h(\tau-\eta)\,d\eta\right\} = [h(t)*x_1(t)]*x_2(t)\end{aligned}$$

which proves the relationship.
(c) Simply note that a constant can be taken outside of an integral.
(d) Let

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau)d\tau$$

Look at the integrand components as functions of τ; $x(\tau)$ is 0 for $\tau < c$ and $\tau > d$, and $h(t - \tau)$ is 0 for $t - \tau < a$ and for $t - \tau > b$. There will be no overlap of the two signals in the integrand of a convolution if

$$t - a < c \text{ or } t < a + c$$
$$\text{and for } t - b > d \text{ or } t > b + d$$

That is, the result of the convolution will be zero for either of these two conditons.
(e) Integrate the convolution over all time, interchange the area and convolution integrals, change variables, and the result follows.

Problem 2-17

(a) There are three cases: (1) no overlap; (2) partial overlap; (3) full overlap. For case (1) the result is zero, and this holds for $t < -1$. For partial overlap, we have

$$y(t) = \int_{0}^{t+1} 2e^{-10\lambda}d\lambda = \frac{1}{5}\left[1 - e^{-10(t+1)}\right], \quad -1 \le t \le 1$$

For full overlap, the convolution integral is

$$y(t) = \int_{t-1}^{t+1} 2e^{-10\lambda}d\lambda = \frac{1}{5}\left[e^{-10(t-1)} - e^{-10(t+1)}\right], \quad t > 1$$

A sketch of the integrand for these cases will help in establishing the limits of integration.
(b) The convolution integral for this case is

$$y(t) = \int_{-\infty}^{\infty} \Pi\left[\frac{\lambda - 1}{2}\right]u(t - \lambda - 10)d\lambda = \int_{0}^{2} u(t - \lambda - 10)d\lambda = \begin{cases} \displaystyle\int_{0}^{t-10} d\lambda = t - 10, & 10 \le t < 12 \\ \displaystyle\int_{0}^{2} d\lambda = 2, & t \ge 12 \end{cases}$$

(c) The convolution integral for this case is

$$y(t) = \int_{-\infty}^{\infty} 2e^{-10\lambda} u(\lambda) u(t - \lambda - 2)\, d\lambda = 2\int_{0}^{\infty} e^{-10\lambda} u(t - \lambda - 2)\, d\lambda = \frac{1}{5}\left[1 - e^{-10(t-2)}\right] u(t - 2)$$

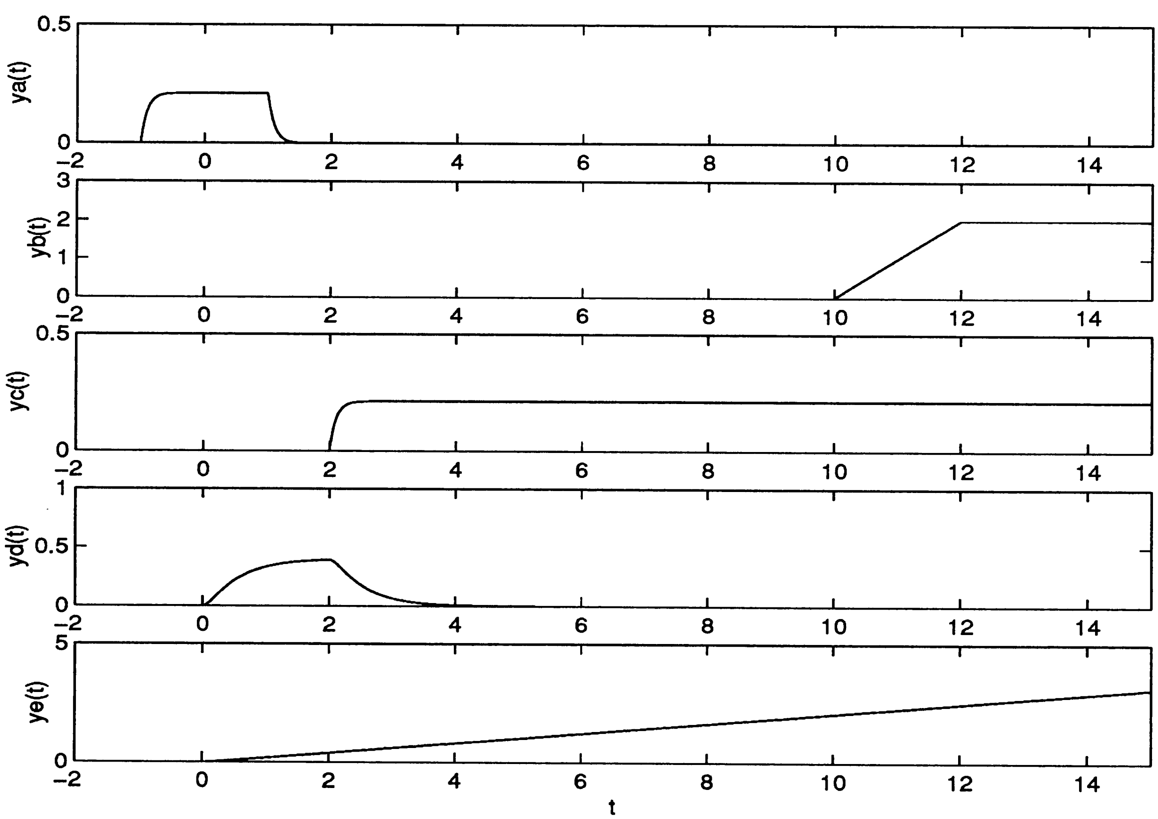
(d) The convolution integral for this case is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \left[e^{-2\lambda} - e^{-10\lambda}\right] u(\lambda) \left[u(t - \lambda) - u(t - \lambda - 2)\right] d\lambda \\ &= \left[0.5(1 - e^{-2t}) - 0.1(1 - e^{-10t})\right] u(t) - \left[0.5(1 - e^{-2(t-2)}) - 0.1(1 - e^{-10(t-2)})\right] u(t-2) \end{aligned}$$

which follows by noting that $h(t) = u(t) - u(t - 2)$.
(e) In this case, the convolution integral becomes

$$y(t) = \int_{-\infty}^{\infty} 2\lambda e^{-2(t-\lambda)} u(\lambda) u(t - \lambda)\, d\lambda = \left[t - 0.5(1 - e^{-2t})\right] u(t)$$

Sketches for all three cases are shown below.



Problem 2-18

The convolution is

$$y(t) = \Pi(t-0.5) * \sum_{n=-\infty}^{\infty} \delta(t-2n) = \sum_{n=-\infty}^{\infty} \Pi(t-0.5) * \delta(t-2n) = \sum_{n=-\infty}^{\infty} \Pi(t-0.5-2n)$$

This is a doubly infinite train of square pulses of unit width and height spaced by 2 units. The one at $t = 0$ starts at 0 and ends at 1.

Problem 2-19

(a) By KVL and Ohm's law:

$$\frac{L}{R}\frac{dh(t)}{dt} + h(t) = \delta(t)$$

where the forcing function being a delta function means that the response is the impulse response. For $t < 0$, the impulse response is zero because the input is 0 and the initial conditions are assumed 0. For $t > 0$, we solve the homogeneous differential equation to get the solution

$$h(t) = Ae^{-Rt/L}, \; t > 0$$

To find the initial condition required to fix A, integrate the differential equation (with impulse forcing function) through $t = 0$:

$$\int_{0-}^{0+} \frac{L}{R}\frac{dh(t)}{dt}dt + \int_{0-}^{0+} h(t)\,dt = \int_{0-}^{0+} \delta(t)\,dt = 1$$

From the form of $h(t)$, we see that it has a step discontinuity at $t = 0$ and therefore its derivative has an impulse at $t = 0$. Thus the second term on the left-hand side integrates to 0 (it only has a step discontinuity). The first term on the left-hand side integrates to $(L/R)[h(0+) - h(0-)]$. Thus, the above equation becomes $h(0+) = R/L = A$, and the impulse response becomes

$$h(t) = \frac{R}{L}e^{-Rt/L}u(t)$$

(b) Let the voltage across the resistor be $v_R(t)$ and the voltage across the inductor be $v_L(t)$. Thus

$$v_L(t) + v_R(t) = \delta(t)$$

But the voltage across the resistor cannot be proportional to an impulse because then the current around the loop would be proportional to an impulse and this means the inductor voltage (L times the derivative of the current) would be proportional to the derivative of an impulse. Since there is no derivative of an impulse on the right-hand side to balance it, this cannot be the case. Therefore, it must be true at time 0 that $v_L(t) = \delta(t)$ and the current possesses a step at time 0. In particular,

$$i(0+) = \frac{1}{L}\int_{0-}^{0+} \delta(t)\,dt = \frac{1}{L}$$

For $t > 0$, the current around the resistor-inductor loop must satisfy

$$L\frac{di(t)}{dt} + Ri(t) = 0 \text{ or } i(t) = Ae^{-Rt/L},\ t > 0$$

The constant A can be fixed by setting $i(0) = i(0+) = 1/L$. Thus, the same result is obtained for the impulse response as obtained in part (a).

Problem 2-20

Using voltage division with an impulse forcing function, the impulse response is

$$h(t) = \frac{1}{1+1+1}\delta(t) = \frac{1}{3}\delta(t)$$

Problem 2-21

(a) Use KCL at the junction of the three elements after expressing the currents in terms of input and output voltages. The currents through the inductor, top resistor, and output resistor are, respectively, given by

$$i_1(t) = \frac{1}{L}\int_{-\infty}^{t} [x(\lambda) - y(\lambda)]\,d\lambda;\ i_2(t) = \frac{x(t) - y(t)}{R_1};\ i_3(t) = \frac{y(t)}{R_2}$$

Using KCL, we obtain

$$\frac{1}{L}\int_{-\infty}^{t} [x(\lambda) - y(\lambda)]\,d\lambda + \frac{x(t) - y(t)}{R_1} = \frac{y(t)}{R_2}$$

When rearranged, this gives the result given in the statement of the problem.

(b) Let the input be a delta function. Then the output is the impulse response, and it obeys the differential equation

$$\frac{dh(t)}{dt} + a\,h(t) = b\frac{d\delta(t)}{dt} + a\,\delta(t)$$

where

$$a = \frac{R_1 R_2}{(R_1 + R_2)L};\; b = \frac{R_2}{R_1 + R_2}$$

Try a solution of the form

$$h(t) = Ae^{-at}u(t) + B\,\delta(t)$$

The delta function is necessary so that there is a derivative of a delta function on the left-hand side of the differential equation to match the one on the right-hand side. Substitute the assumed solution into the differential equation and match coefficients of like derivatives of delta functions to obtain

$$A = a(1 - b) = \frac{R_1^2 R_2}{(R_1 + R_2)^2 L};\; B = b = \frac{R_2}{R_1 + R_2}$$

Problem 2-22

Solution 1: Consider an impulse forcing function. KVL must be satisfied around the loop consisting of the impulse source, resistor, and inductor. The inductor must have an impulse of voltage across it since

$$\delta(t) = v_R(t) + v_L(t)$$

Otherwise, if an impulse appears across the resistor, the current is an impulse, and the voltage across the inductor, being proportional to the derivative of the current, would be the derivative of an impulse which isn't present on the forcing function side of the equation. Therefore, at $t = 0$,

$$v_L(t) = L\frac{di(t)}{dt} \text{ which gives } i(0+) = \frac{1}{L}\int_{0-}^{0+}\delta(t)dt = \frac{1}{L}$$

The system differential equation is

$$x(t) = R\,i(t) + L\frac{di(t)}{dt}$$

With $x(t) = \delta(t)$, we get for $t > 0$

$$i(t) = Ae^{-Rt/L}, \; t > 0$$

Using the initial condition for the current found above, we find that $A = 1/L$. Since the current is 0 for $t < 0$, the current around the loop for all time is

$$i(t) = \frac{1}{L} e^{-Rt/L} u(t)$$

The impulse response is the voltage across the inductor (with impulse input). This can be found as

$$h(t) = L\frac{d}{dt}\left[\frac{1}{L} e^{-Rt/L} u(t)\right] = -\frac{R}{L} e^{-Rt/L} + \delta(t)$$

Solution 2: By KVL

$$x(t) = R\,i(t) + y(t)$$

But

$$L\frac{di(t)}{dt} = y(t) \text{ or } i(t) = \frac{1}{L}\int_{-\infty}^{t} y(\lambda)\,d\lambda$$

Therefore

$$x(t) = \frac{R}{L}\int_{-\infty}^{t} y(\lambda)\,d\lambda + y(t) \text{ or } \frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{dx(t)}{dt}$$

With the input a unit impulse, we have

$$\frac{dh(t)}{dt} + \frac{R}{L}h(t) = \frac{d\delta(t)}{dt}$$

$h(t)$ must have a unit impulse term because there is a derivative of a unit impulse on the right-hand side. Taking the derivative of the impulse response, therefore, gives a term proportional to the derivative of a unit impulse to match the right-hand side. Thus we assume that

$$h(t) = A e^{-Rt/L} u(t) + B\,\delta(t)$$

Substitute into the differential equation and match coefficients of like derivatives of impulses on both sides to obtain $B = 1$ and $A = -R/L$.

Problem 2-23

By KVL around the loop,

$$x(t) = R_1 i(t) + \frac{1}{C}\int_{-\infty}^{t} i(\lambda)\,d\lambda + R_2 i(t)$$

But $i(t) = y(t)/R_2$. Substitute this into the integro-differential equation and differentiate once to get

$$\frac{R_1 + R_2}{R_2}\frac{dy(t)}{dt} + \frac{y(t)}{R_2 C} = \frac{dx(t)}{dt}$$

One way to find the impulse response is to find the step response and differentiate it. The solution to the homogeneous equation is

$$a(t) = Ae^{-t/(R_1 + R_2)C},\ t > 0$$

With a step input, the right-hand side of the differential equation is an impulse. To get the required initial condition, we integrate the differential equation through $t = 0$:

$$\frac{R_1 + R_2}{R_2}\int_{0-}^{0+}\frac{da(t)}{dt}dt + \frac{1}{R_2 C}\int_{0-}^{0+} a(t)\,dt = \int_{0-}^{0+}\delta(t)\,dt = 1$$

To match the right-hand side, the integrand of the first term on the left-hand side must contain a unit impulse and, therefore, the second term on the left-hand side is proportional to a unit step. Hence the integral on the second term through $t = 0$ is 0 (a step discontinuity). The first term is a perfect differential. Thus, we obtain $a(0+) = R_2/(R_1 + R_2)$ as the required initial condition, and the step response is

$$a(t) = \frac{R_2}{R_1 + R_2}e^{-t/(R_1 + R_2)C}u(t) \text{ and } h(t) = \frac{da(t)}{dt} = \frac{R_2}{R_1 + R_2}\left[\delta(t) - \frac{1}{(R_1 + R_2)C}e^{-t/(R_1 + R_2)C}u(t)\right]$$

Problem 2-24

(a) From the analysis of this circuit in Example 1-2, we have

$$y(t) = -\frac{1}{RC}\int_{-\infty}^{t} x(\lambda)\,d\lambda$$

If the input is a unit impulse, we obtain

$$h(t) = -\frac{1}{RC}\int_{-\infty}^{t} \delta(\lambda)\,d\lambda = -\frac{1}{RC}u(t)$$

(b) See the first equation above, or evaluate the superposition integral with the impulse response:

$$y(t) = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)\,d\lambda = \int_{-\infty}^{\infty}\left[-\frac{1}{RC}u(t-\lambda)\right]x(\lambda)\,d\lambda = -\frac{1}{RC}\int_{-\infty}^{t} x(\lambda)\,d\lambda$$

Problem 2-25

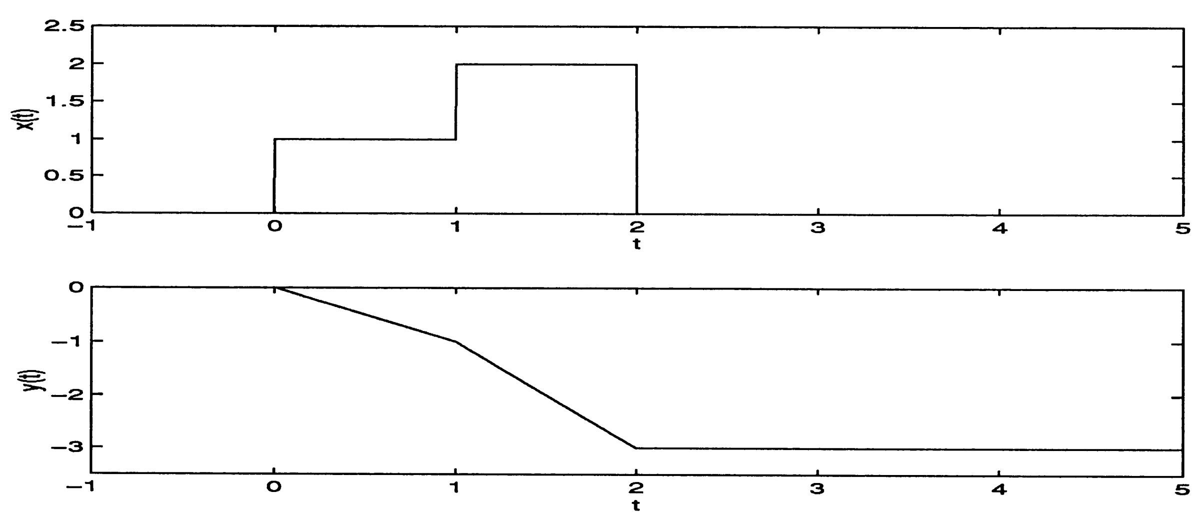
(a) To obtain the step response, integrate the impulse response, which yields

$$a(t) = -r(t)/RC$$

(b) Use superposition to obtain

$$y(t) = -[r(t) + r(t-1) - 2r(t-2)]/RC$$

The input and output are plotted below for $RC = 1$:

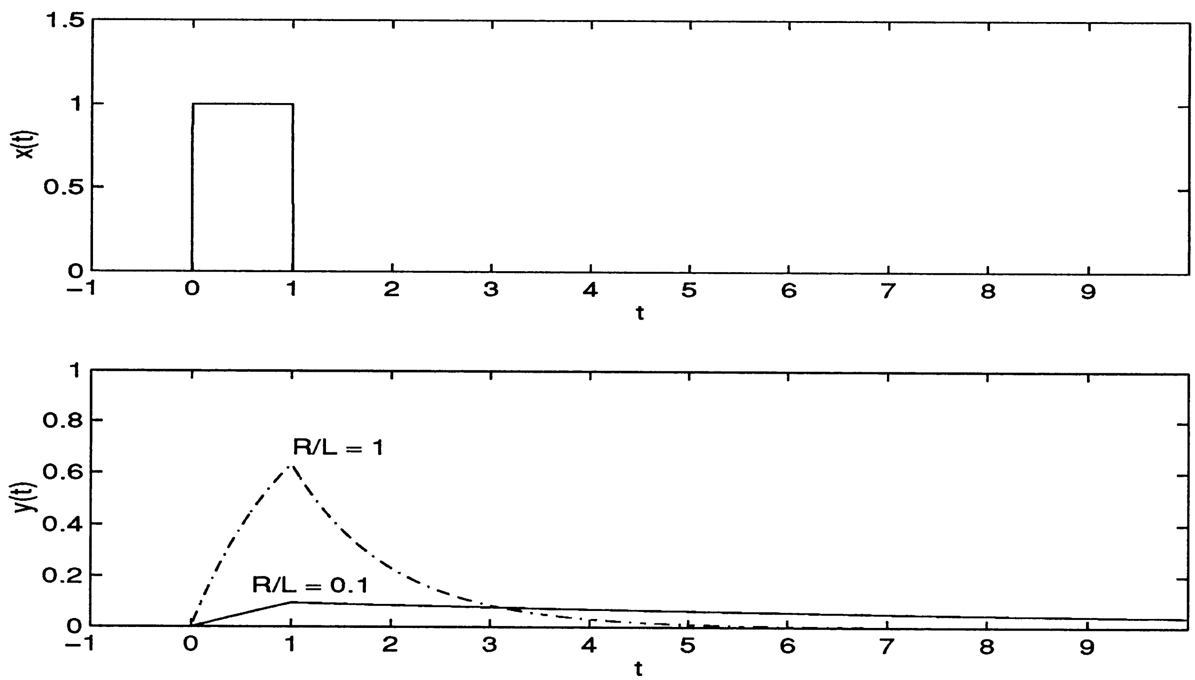


Problem 2-26

We use the impulse response found in Problem 2-19 to evaluate

$$y(t) = \int_{-\infty}^{\infty} \Pi(\lambda - 0.5)\frac{R}{L}e^{-R(t-\lambda)/L}u(t-\lambda)d\lambda = \begin{cases} 0, & t < 0 \\ 1 - e^{-Rt/L}, & 0 < t < 1 \\ \left[1 - e^{-R/L}\right]e^{-R(t-1)/L}, & t > 1 \end{cases}$$

An alternative solution is to write the input as the difference between two steps and use superpositon after finding the step response by integration of the impulse response. A sketch is provided below.



Problem 2-27

From Problem 2-22 and the given input, the superposition integral becomes

$$y(t) = \int_{-\infty}^{\infty} \Pi(\lambda - 0.5)\left[\delta(t-\lambda) - \frac{R}{L}e^{-R(t-\lambda)/L}u(t-\lambda)\right]d\lambda = \Pi(t-0.5) - \Pi(t-0.5) * \left[\frac{R}{L}e^{-Rt/L}u(t)\right]$$

The last term evaluates to the negative of the result given in the solution to Problem 2-26. A sketch is obtained by taking the plot for Problem 2-26, inverting it, and adding square pulse starting at t = 0 and ending at t = 1.

Problem 2-28

The derivative of the input is

$$\frac{dx(t)}{dt} = \delta(t) - \delta(t-1)$$

The step response is found by integrating the impulse response found in Problem 2-19:

$$a(t) = \int_{-\infty}^{t} \frac{R}{L} e^{-R\lambda/L} u(\lambda) d\lambda = [1 - \exp(-Rt/L)]\, u(t)$$

The output is

$$\begin{aligned} y(t) &= [\delta(t) - \delta(t-1)] * [1 - \exp(-Rt/L)]\, u(t) \\ &= [1 - \exp(-Rt/L)]\, u(t) - \{[1 - \exp[-R(t-1)/L]\}\, u(t-1) \end{aligned}$$

See the solution to Problem 2-26 for a plot.

Problem 2-29

(a) The step response is

$$a_s(t) = \int_{-\infty}^{t} h(\lambda)\, d\lambda = \exp(-Rt/L)\, u(t)$$

(b) The ramp response is

$$a_r(t) = \int_{-\infty}^{t} a_s(\lambda)\, d\lambda = \frac{L}{R}[1 - \exp(-Rt/L)]\, u(t)$$

Problem 2-30

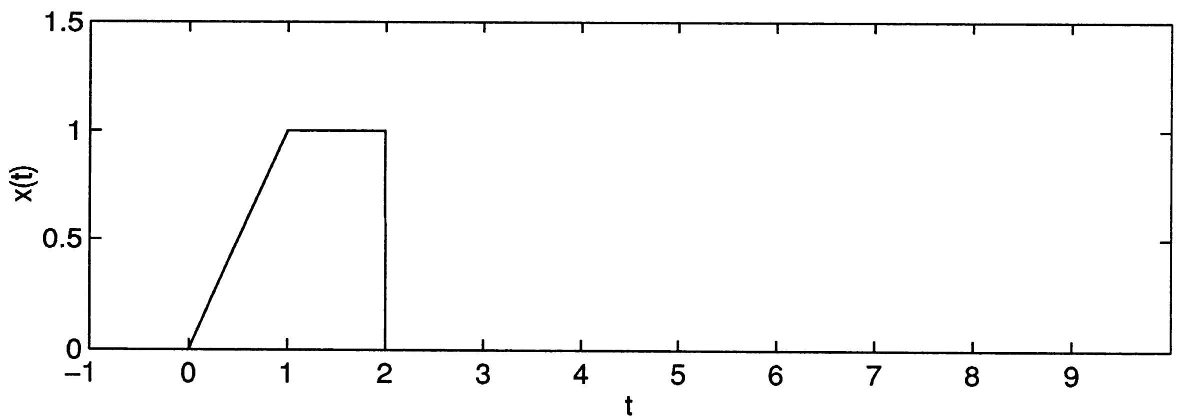
Write the input as

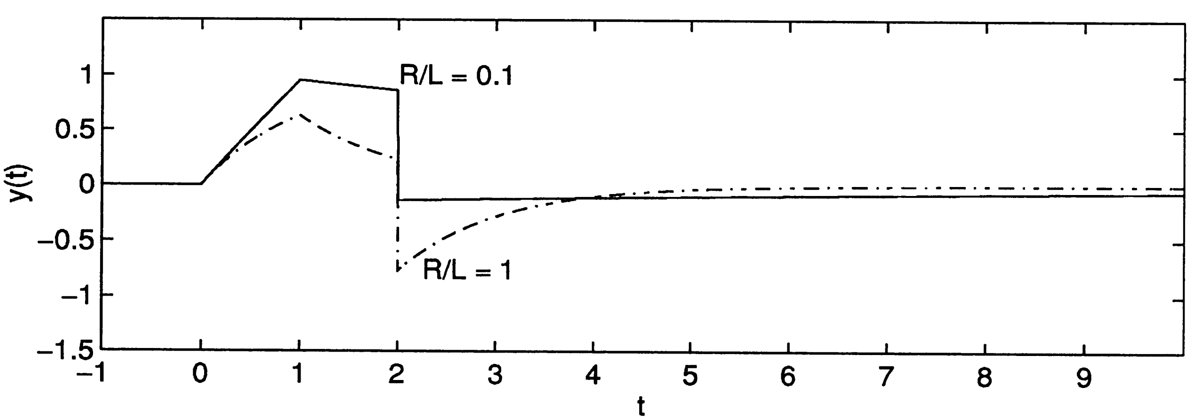
$$x(t) = r(t) - r(t-1) - u(t-2)$$

Thus, by superposition, the output is

$$y(t) = a_r(t) - a_r(t-1) - a_s(t-2)$$
$$= \frac{L}{R}[1 - \exp(-Rt/L)]u(t) - \frac{L}{R}\{1 - \exp[-R(t-1)/L)]u(t-1) - \exp[-R(t-2)/L]u(t-2)$$

Sketches of the input and output are given below:





Problem 2-31

(a) The impulse response is

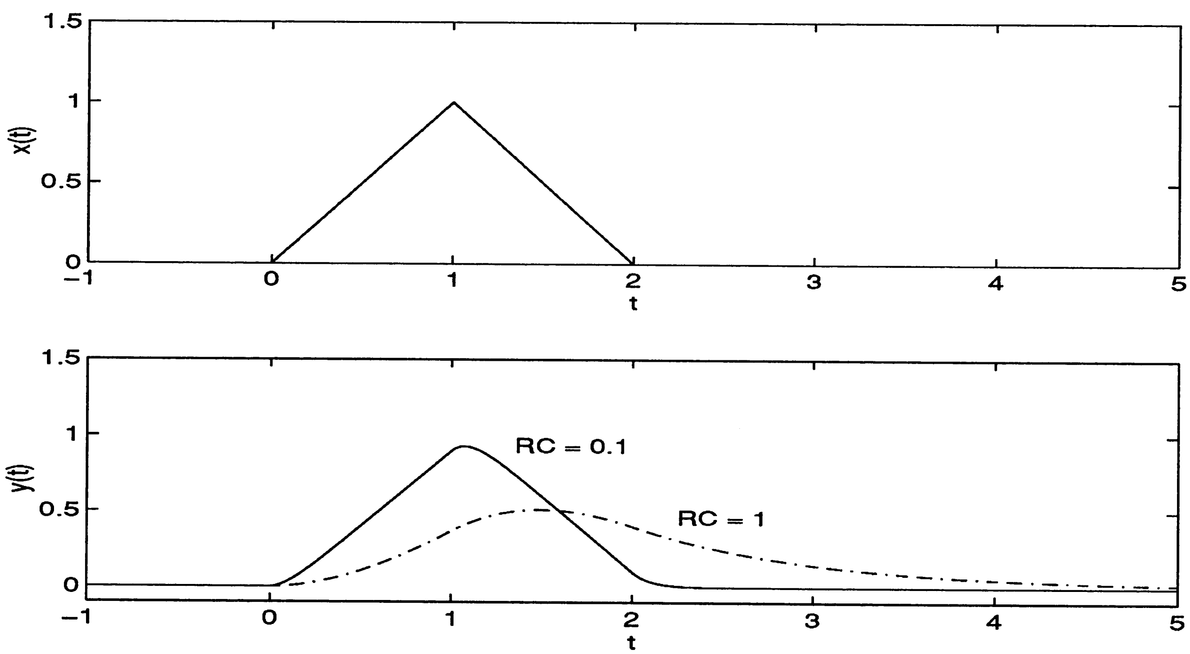
$$h(t) = \frac{1}{RC}\exp(-t/RC)\,u(t)$$

and, by integration, the step and ramp responses are

$$a_s(t) = [1 - \exp(-t/RC)]\,u(t) \text{ and } a_r(t) = r(t) - RC[1 - \exp(-t/RC)]\,u(t)$$

respectively. Using superposition, the output is

$$\begin{aligned} y(t) &= a_r(t) - 2a_r(t-1) + a_r(t-2) \\ &= r(t) - RC[1 - \exp(-t/RC)]\,u(t) \\ &\quad - 2\{r(t-1) - RC[1 - \exp(-(t-1)/RC)]\,u(t-1)\} \\ &\quad + r(t-2) - RC[1 - \exp(-(t-2)/RC)]\,u(t-2) \end{aligned}$$



(b) The response to $dx(t)/dt$ is the derivative of the response given in (a), which is

$$\begin{aligned} y(t) &= a_s(t) - 2a_s(t-1) + a_s(t-2) = [1 - \exp(-t/RC)]\,u(t) \\ &\quad - 2\{[1 - \exp(-(t-1)/RC)]\,u(t-1)\} + [1 - \exp(-(t-2)/RC)]\,u(t-2) \end{aligned}$$

Problem 2-32

(a) Use KCL at the output node to obtain

$$\frac{x(t) - y(t)}{R_1} = \frac{y(t)}{R_2} + C\frac{dy(t)}{dt}$$

When rearranged, this gives the answer given in the problem statement.
(b) The homogeneous equation for the impulse response is

$$R_1C\frac{dh(t)}{dt} + \left(1 + \frac{R_1}{R_2}\right)y(t) = 0$$

Assume a solution of the form

$$h(t) = Ae^{pt},\ t > 0$$

Substitute the assumed solution into the homogeneous differential equation to get the characteristic equation

$$R_1Cp + \left(1 + \frac{R_1}{R_2}\right) = 0 \text{ or } p = -\frac{R_1R_2C}{R_1 + R_2}$$

To get the required initial condition, integrate the differential equation, with impulse forcing function, through $t = 0$:

$$R_1C\int_{0-}^{0+}\frac{dh(t)}{dt}dt + \left(1 + \frac{R_1}{R_2}\right)\int_{0-}^{0+}h(t)dt = \int_{0-}^{0+}\delta(t) = 1$$

The second term on the left-hand side is discontinuous at $t = 0$, but contains no impulse function; the first term on the left-hand side must contain an impulse function to balance the impulse function on the right-hand side. The first integral has an integrand that is a perfect differential, so

$$R_1C[h(0+) - h(0-)] = 1 \text{ or } h(0+) = \frac{1}{R_1C}$$

Substituting for $h(0+) = A$ and p, we obtain the result for the impulse response given in the problem statement.

(c) Integrate the impulse response to obtain the given answer.
(d) Note that the input can be written as

$$x(t) = u(t) - u(t-1)$$

and use superposition to obtain the given answer.
(e) Duhamel's integral simply tells us to integrate the step response. This gives

$$y_r(t) = \int_0^t \frac{R_2}{R_1 + R_2}[1 - \exp(-\lambda/\tau)]d\lambda, \; t > 0$$

Integrating and putting in the limits gives the result in the problem statement.
(f) Using superposition,

$$y(t) = y_r(t) - 2y_r(t-1) + y_r(t-2)$$

Problem 2-33

(a) The frequency response function is given in terms of the impulse response by

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

From the impulse response given in Problem 2-29, this gives

$$H(j\omega) = \int_{-\infty}^{\infty}\left[\delta(t) - \frac{R}{L}\exp(-Rt/L)\,u(t)\right]\exp(-j\omega t)\,dt = \frac{j\omega}{R/L + j\omega}$$

(b) In terms of $f = \omega/2\pi$, the frequency response function is

$$H(f) = \frac{jf/f_3}{1 + jf/f_3} \text{ where } f_3 = \frac{R}{2\pi L}$$

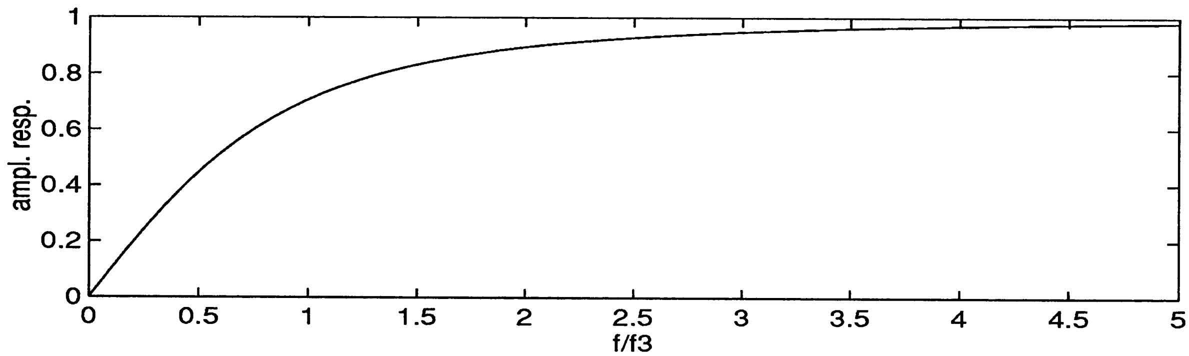
Taking the magnitude, the amplitude response function is

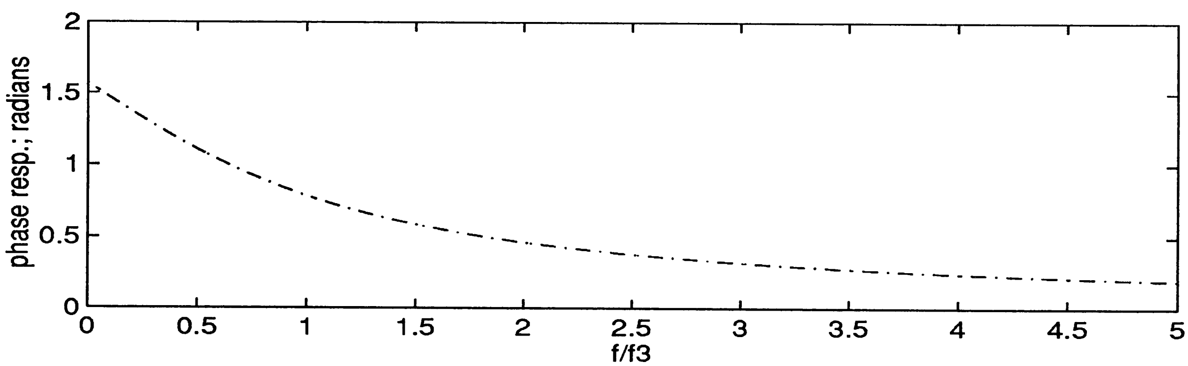
$$A(f) = \frac{|f|/f_3}{\sqrt{1 + (f/f_3)^2}}$$

(c) The phase response function is the argument of the frequency response function. It is given by

$$\theta(f) = \frac{\pi}{2} - \tan^{-1}(f/f_3)$$

Plots of the amplitude and phase response functions are given below:





Problem 2-34

(a) The frequency response is

$$H(j\omega) = \int_0^\infty \frac{1}{R_1C} e^{-(1/\tau + j\omega)t} dt = \frac{1}{R_1C} \frac{1}{\sqrt{1/\tau + j\omega}}, \ \tau = \frac{R_1R_2C}{R_1 + R_2}$$

(b) Let $f = \omega/(2\pi)$ and $f_3 = 1/(2\pi\tau)$. The amplitude response function is

$$A(f) = \frac{\tau}{R_1C} \frac{1}{1 + (f/f_3)^2}$$

and the phase response function is

$$\theta(f) = -\tan^{-1}(f/f_3)$$

(c) The steadystate response is

$$y(t) = A(1)\cos[2\pi t + \theta(1)] + A(2.5)\cos[5\pi t + \theta(2.5)]$$

Using the given parameter values, we obtain $\tau = 0.5$, $R_1C = 1$, and $f_3 = 1/\pi$. The amplitude and phase responses are

$$A(1) = \frac{0.5}{\sqrt{1+\pi^2}} = 0.1517;\ \theta(1) = -\tan^{-1}(\pi) = -72.35^\circ$$

$$A(2.5) = \frac{0.5}{\sqrt{1+(2.5\pi)^2}} = 0.0632;\ \theta(2.5) = -\tan^{-1}(2.5\pi) = -87.7^\circ$$

Thus, the steadystate output is

$$y(t) = 0.1517\cos[2\pi t - 72.35^\circ] + 0.0632\cos[5\pi t - 87.7^\circ]$$

Problem 2-35

From Problem 2-19 and the condition for BIBO stability, we obtain

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_0^{\infty} \frac{R}{L} e^{-Rt/L} dt = 1 < \infty$$

so the system is BIBO stable.

Problem 2-36

From Problem 2-22 and the condition for BIBO stability, we have

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left|\delta(t) - \frac{R}{L} e^{-Rt/L} u(t)\right| dt \le \int_{-\infty}^{\infty} \delta(t) dt + \frac{R}{L}\int_0^{\infty} e^{-Rt/L} dt = 1 + 1 = 2 < \infty$$

Therefore, the system is BIBO stable.

Problem 2-37

The condition for BIBO stability is

$$\int_{-\infty}^{\infty} \left| -\frac{1}{RC} u(t) \right| dt = \frac{1}{RC} r(t) \Big|_{-\infty}^{\infty} \to \infty$$

so the system is not BIBO stable.

Problem 2-38

The condition for BIBO stability is

$$\int_{-\infty}^{\infty} \left| \frac{1}{3} \delta(t) \right| dt = \frac{1}{3} < \infty$$

so the system is BIBO stable.

Problem 2-39

The impulse response, for time greater than 0, satisfies

$$\frac{d^2 h(t)}{dt^2} + \omega_0^2 h(t) = 0$$

Assume a solution of the form $A \exp(pt)$, substitute, and solve the resulting characteristic equation to get the roots

$$p_1 = j\omega_0 \text{ and } p_2 = -j\omega_0$$

Thus, for $t > 0$, the solution is

$$h(t) = A_1 e^{j\omega_0 t} + A_1 e^{-j\omega_0 t}$$

We can obtain the initial conditions by integrating the differential equation through 0. They are

$$\left. \frac{dh(t)}{dt} \right|_{t = 0+} = 1 \text{ and } h(0+) = 0$$

This gives the following impulse response:

$$h(t) = \frac{1}{\omega_0}\sin(\omega_0 t)\,u(t)$$

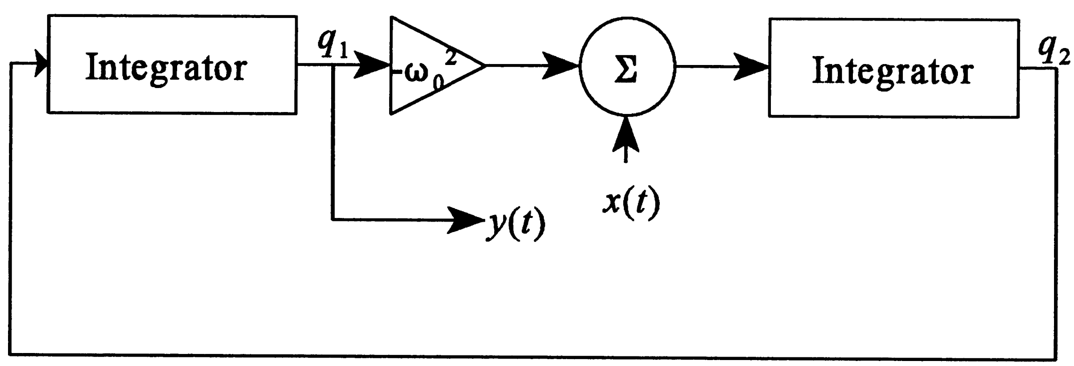
(b) Take the given equations and eliminate $q_2(t)$. Differentiate the first equation to get

$$\frac{d^2q_1(t)}{dt^2} = \frac{dq_2(t)}{dt}$$

Substitue into the second equation to get

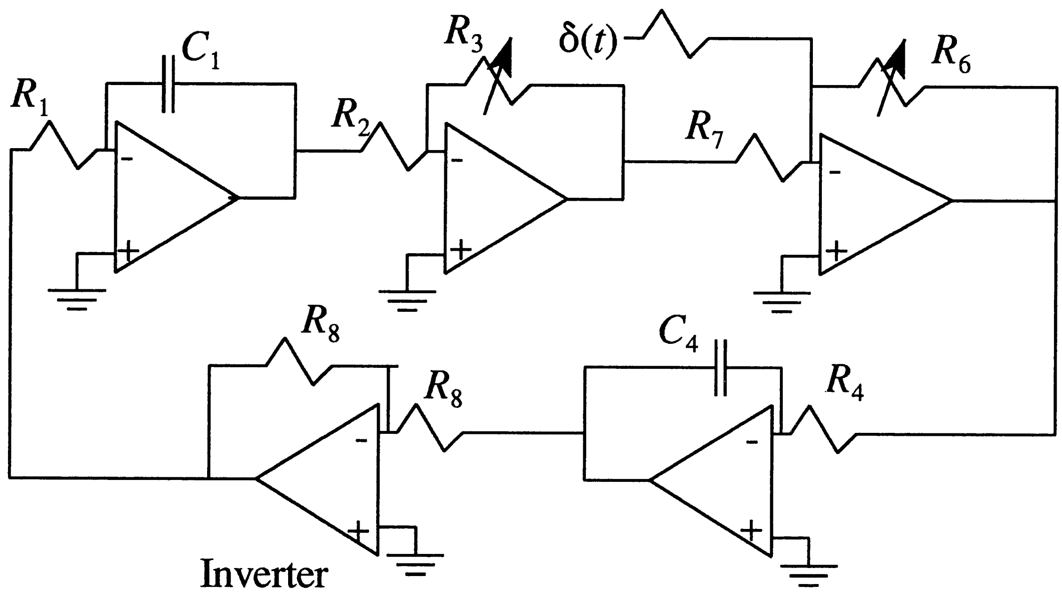
$$\frac{dq_2(t)}{dt} = -\omega_0^2 q_1(t) + x(t)$$

Let $q_1(t) = y(t)$ and the original equation results.
(c) A block diagram is given below:



(d) Possible values for the parameters given in the block diagram on the next page are

$$\frac{1}{R_1C_1} = \frac{1}{R_4C_4} = 1 \text{ and } 2\pi\times10^2 \le \frac{R_3}{R_2}, \frac{R_6}{R_7} \le 2\pi\times10^4$$



Problem 2-40

Differentiate the first equation to get

$$\frac{d^2q_1}{dt^2} = -\frac{1}{RC}\frac{dq_1}{dt} - \frac{dq_2}{dt} + \frac{1}{C}\frac{dx}{dt}$$

Substitute from the second equation to get

$$\frac{d^2q_1}{dt^2} = -\frac{1}{RC}\frac{dq_1}{dt} - \frac{1}{LC}q_1 + \frac{1}{C}\frac{dx}{dt}$$

Replace q_1 with v and rearrange to get (2-123).

Problem 2-41

(a) Consider the equation

$$(M_0 - kt)\frac{dh(t,\tau)}{dt} + (\alpha - k)h(t,\tau) = \delta(t-\tau)$$

where we will use initial conditions appropriate for the impulse response. To get the initial condition at $t = \tau$ we integrate the differential equation from $t = \tau^-$ to $t = \tau^+$. The first term on the left-hand side must be proportional to an impulse at $t = \tau$; the second term may have a jump discontinuity, but its integral through $t = \tau$ is 0. Thus, we have

$$(M_0 - k\tau)[h(\tau+,\tau) - h(\tau-,\tau)] + 0 = 1$$

By definition of the impulse response for a time-varying system, $h(\tau^-, \tau) = 0$. Thus the initial condition for the impulse response is

$$h(t = \tau+, \tau) = \frac{1}{M_0 - k\tau}$$

Now set the right-hand side of the differential equation for the impulse response to 0, separate variables, and integrate from $\lambda = \tau^+$ to $\lambda = t$ and on h from the above initial condition to $h(t, \tau)$:

$$\ln[(M_0 - k\tau)h(t,\tau)] = \ln\left[\frac{M_0 - kt}{M_0 - k\tau}\right]^{\frac{\alpha - k}{k}}, \; t > \tau > 0$$

Exponentiating both sides and dividing through by $M_0 - k\tau$, we obtain

$$h(t,\tau) = \frac{(M_0 - kt)^{\frac{\alpha-k}{k}}}{(M_0 - k\tau)^{\frac{\alpha}{k}}}, \; t > \tau > 0$$

(b) Since the step is applied at time zero, we evaluate the convolution integral from $\lambda = 0$ to t to get the step response. After substituting the given parameter values, the result is

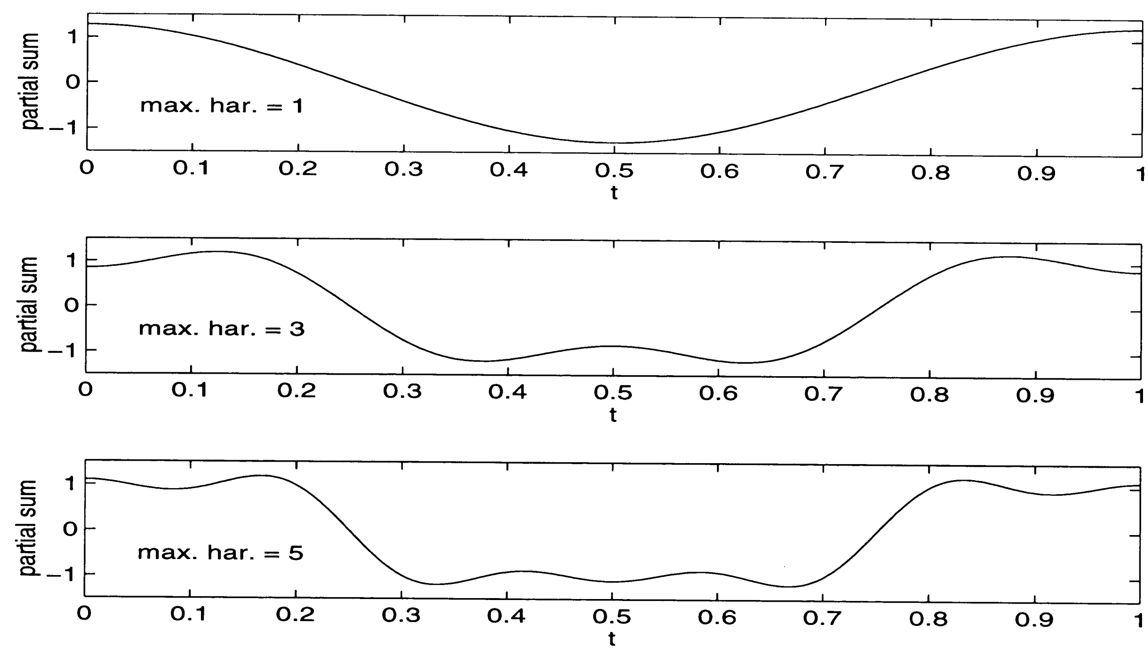
$$a_s(t,0) = \int_0^t h(t,\lambda)d\lambda = \int_0^t \frac{10-t}{10-\lambda}d\lambda = \left.\frac{10-t}{10-\lambda}\right|_0^t = \frac{t}{10}, \; t > 0$$

CHAPTER 3

Problem 3-1

A MATLAB program for computing the partial sums is given below with plots of the partial sums through harmonics 1, 3, and 5 following:

```
%       Program to give partial Fourier sums of an
%       even square wave of unit amplitude
%
n_max = input('Enter vector of highest harmonic values desired (odd) ');
N = length(n_max);
t = 0:.002:1;
omega_0 = 2*pi;
for k = 1:N
        n = [];
        nn = [];
        n = [1:2:n_max(k)]
        nn = [1:(n_max(k)+1)/2]
        a_n = (-1).^(nn+1)*4./(pi*n);          % Form vector of Fourier cosine-coefficients
        x = a_n*cos(omega_0*n'*t);  % Rows of cosine matrix are versus time; columns versus n,
                                    % so matrix multiply sums over n
        subplot(N,1,k),plot(t, x,'-w'), xlabel('t'), ylabel('partial sum'),...
        axis([0 1 -1.5 1.5]), text(.05,-.5, ['max. har. = ', num2str(n_max(k))])
end
```



Problem 3-2

To show that

$$I_2 = \int_0^{T_0} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0, & m \neq n \\ T_0/2, & m = n \end{cases}$$

Use the trigonometric identity

$$\cos(u)\cos(v) = \frac{1}{2}\cos(u+v) + \frac{1}{2}\cos(u-v)$$

Then I_2 becomes

$$I_2 = \frac{1}{2}\int_0^{T_0} \cos[(m-n)\omega_0 t]\,dt + \frac{1}{2}\int_0^{T_0} \cos[(m+n)\omega_0 t]\,dt$$

If $m \neq n$, $I_2 = 0$ because both integrals are over an integer number of cycles of cosine waveforms. If $m = n$, the second integral is still 0, but the first integral evaluates to T_0 (the argument of the cosine is 0 making the integrand 1). Thus the desired result follows.

For (3-13), use the identity $\sin(m\omega_0 t)\cos(n\omega_0 t) = ½ \sin[(m+n)\omega_0 t] + ½ \sin[(m-n)\omega_0 t]$. Now both integrals will be 0 no matter if $m \neq n$ or if $m = n$ because $\sin(0) = 0$.

In deriving (3-16) the steps are identical to those used in deriving (3-15) except that the identities (3-11) and (3-13) are used.

Problem 3-3

(a) Using a trigonometric identity (see Appendix F), this may be expanded as

$$x_1(t) = \frac{1}{2} + \frac{1}{2}\cos(2\omega_0 t)$$

The right-hand side is the trigonometric Fourier series for this signal.

(b) Using Euler's theorem, we have

$$x_2(t) = \cos(200\pi t) + j\sin(200\pi t)$$

which is the trogonometric Fourier series for this signal.

(c) Using appropriate trigonometric identities

$$x_3(t) = \sin(2\pi t)\left[\frac{1}{2} + \frac{1}{2}\cos(20\pi t)\right] = \frac{1}{2}\sin(2\pi t) + \frac{1}{4}\sin(18\pi t) + \frac{1}{4}\sin(22\pi t)$$

The right-hand side is the trigonometric Fourier series for this signal.
(d) Using appropriate trigonometric identities

$$\begin{aligned} x_4(t) &= \cos(20\pi t)\left[\frac{1}{2} + \frac{1}{2}\cos(40\pi t)\right]^2 \\ &= \frac{1}{4}\cos(20\pi t)\left[1 + 2\cos(20\pi t) + \cos^2(20\pi t)\right] \\ &= \frac{1}{4}\cos(20\pi t)\left[1 + 2\cos(20\pi t) + \frac{1}{2} + \frac{1}{2}\cos(40\pi t)\right] \\ &= \frac{5}{8}\cos(20\pi t) + \frac{5}{16}\cos(60\pi t) + \frac{1}{16}\cos(100\pi t) \end{aligned}$$

The right-hand side is the trigonometric Fourier series for this signal.

Problem 3-4

This an even square wave with zero average value. Because of the zero average value, $a_0 = 0$. Because of the eveness, $b_n = 0$ for all n. To find a_n, evaluate (3-15):

$$a_n = \frac{2}{T_0}\left[\int_{-T_0/2}^{-T_0/4} -A\cos(n\omega_0 t)\,dt + \int_{-T_0/4}^{T_0/4} A\cos(n\omega_0 t)\,dt + \int_{T_0/4}^{T_0/2} -A\cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2A}{T_0}\left[-\frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_{-T_0/2}^{-T_0/4} + \frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_{-T_0/4}^{T_0/4} - \frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_{T_0/4}^{T_0/2}\right] = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right), \text{ using } \omega_0 = \frac{2\pi}{T_0}$$

$$= \begin{cases} 0, & n \text{ even} \\ (-1)^{(n-1)/2}\dfrac{4A}{n\pi}, & n \text{ odd} \end{cases}$$

Problem 3-5

(a) Because the string shape is assumed even (origin centered at the peak location), we can use a cosine series. The dc level is 0; hence $a_0 = 0$. Since the string shape is even, only the a_m's are nonzero. They are given by

$$a_m = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(\lambda)\cos(m\omega_0\lambda)\,d\lambda = \frac{4}{T_0}\int_{0}^{T_0/2} x(\lambda)\cos(m\omega_0\lambda)\,d\lambda,\ \omega_0 = \frac{2\pi}{T_0}$$

where λ is a spatial variable. The equation describing the string between 0 and $T_0/2$ is

$$f(\lambda) = -4A\lambda/T_0 + A$$

so that the integral for the Fourier coefficients becomes

$$a_m = \frac{4}{T_0}\int_0^{T_0/4}\left(-\frac{4A}{T_0}\lambda + A\right)\cos(2\pi m\lambda/T_0)\,d\lambda$$

$$= -\frac{16A}{(2\pi m)^2}\int_0^{m\pi} u\cos(u)\,du + \frac{4A}{2\pi m}\int_0^{m\pi}\cos(u)\,du = \begin{cases} 0, & m \text{ even} \\ \dfrac{8A}{(m\pi)^2}, & m \text{ odd} \end{cases}$$

For $T_0 = 9$ inches and $A = 0.5$ inches as in Example 3-3, we obtain

$$a_m = \frac{4}{(m\pi)^2},\ m \text{ odd},\ \omega_0 = \frac{\pi}{18}$$

The Fourier series for the triangular string shape is

$$y(x) = \frac{4}{\pi^2}\left[\cos\left(\frac{\pi x}{18}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{18}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{18}\right) + \dots\right]$$

This differs from the result found in Example 3-3 in the following ways:

1. The new series has no dc term;
2. The fundamental frequency of the new series is half that of the old;
3. The amplitudes of the harmonics in the new seriew are half those of the old series.

(b) The error at the ends of the string ($x = \pm T_0/4$) is 0 because the triangular-wave approximation crosses the x-axis at these points.

Problem 3-6

(a) For the half-rectified sine wave

$$X_n = \frac{1}{T_0}\int_0^{T_0/2} A\sin(\omega_0 t)\, e^{-jn\omega_0 t}\,dt + \frac{1}{T_0}\int_0^{T_0/2}(0)\, e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0/2} A\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{j2T_0}\left[\frac{e^{j(1-n)\omega_0 t}}{j(1-n)\omega_0}\Bigg|_0^{T_0/2} + \frac{e^{j(1+n)\omega_0 t}}{j(1+n)\omega_0}\Bigg|_0^{T_0/2}\right] = \begin{cases} 0, & n \text{ odd} \\ \dfrac{A}{\pi(1-n^2)}, & n \text{ even and} \neq \pm 1 \\ \dfrac{A}{4j}, & n = \pm 1 \end{cases}$$

(b) For a full-rectified sine wave, the period is really $T_0/2$. Furthermore, it is an even signal, so

$$X_n = \frac{2A}{T_0}\int_0^{T_0/2} \sin(\omega_0 t)\cos(n\omega_0 t)\,dt$$

$$= \frac{A}{T_0}\int_0^{T_0/2} \{\sin[(1-n)\omega_0 t] + \sin[(1+n)\omega_0 t]\}\,dt$$

$$= \frac{A}{\omega_0 T_0}\left[-\frac{\cos[(1-n)\pi] - 1}{1-n} - \frac{\cos[(1+n)\pi] - 1}{1+n}\right], \quad n \neq \pm 1$$

$$= \begin{cases} 0, & n \text{ odd} \\ \dfrac{2A}{\pi(1-n^2)}, & n \text{ odd} \end{cases}$$

The results for $n = \pm 1$ have to be done by direct evaluation.

(c) For an even square wave

$$X_n = \frac{2A}{T_0}\left[\int_0^{T_0/4} \cos(n\omega_0 t)\,dt - \int_{T_0/2}^{T_0/2} \cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2A}{T_0}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_0^{T_0/4} - \frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_{T_0/4}^{T_0/2}\right] = \frac{2A\sin(n\pi/2)}{n\pi}$$

$$= \begin{cases} \dfrac{2A}{\pi|n|}, & n = \pm 1, \pm 5, \ldots \\ \dfrac{2A}{\pi|n|}, & n = \pm 3, \pm 7, \ldots \\ 0, & n \text{ even} \end{cases}$$

(d) For an even triangle signal,

$$X_n = \frac{2A}{T_0}\left[\int_0^{T_0/2}\left(1 - \frac{4t}{T_0}\right)\cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2A}{T_0}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_0^{T_0/2} - \frac{4}{T_0}\int_0^{T_0/2} t\cos(n\omega_0 t)\,dt\right]$$

The first term in the brackets is zero upon substitution of limits. The second term must be integrated by parts or looked up in a tables. We get

$$X_n = \frac{2A}{T_0}\left(-\frac{4}{T_0}\right)\left[\frac{t\sin(n\omega_0 t)}{n\omega_0}\Bigg|_0^{T_0/2} - \int_0^{T_0/2}\frac{\sin(n\omega_0 t)}{n\omega_0}\,dt\right]$$

The first term in the brackets again evaluates to 0. The remaining integral evaluates to

$$X_n = -\frac{4A}{n\pi T_0}\,\frac{\cos(n\omega_0 t)}{n\omega_0}\Bigg|_0^{T_0/2} = \begin{cases} \dfrac{4A}{(n\pi)^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Problem 3-7

Let

$$y(t) = \frac{dx(t)}{dt},\ y(t) = \sum_{n=-\infty}^{\infty} Y_n e^{jn\omega_0 t} \text{ and } x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

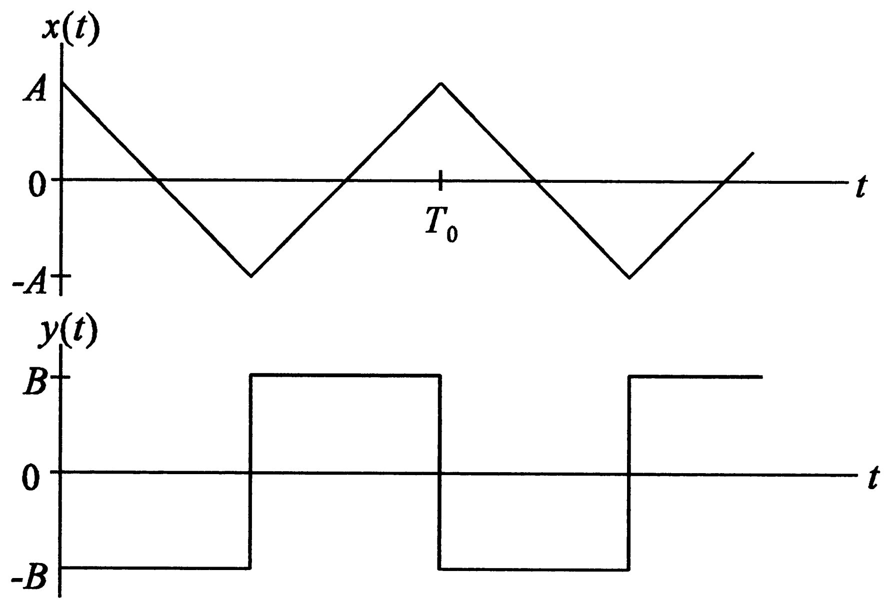
We calculate Y_n in terms of X_n as follows:

$$Y_n = \frac{1}{T_0}\int_{T_0} y(t)\, e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_{T_0} \frac{dx(t)}{dt}\, e^{-jn\omega_0 t}dt$$

Integrate the right-most integral by parts to obtain

$$Y_n = \frac{1}{T_0}\left[x(t)\,e^{-jn\omega_0 t}\Big|_0^{T_0} + jn\omega_0\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt\right] = \frac{1}{T_0}\left[x(T_0) - x(0) + jn\omega_0\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt\right] = jn\omega_0 X_n$$

Note that $\exp(jn\omega_0 T_0) = 1$ by using $\omega_0 = 2\pi/T_0$, and that $x(T_0) = x(0)$ by periodicity of $x(t)$.
(b) Differentiate a triangular wave to get a square wave as illustrated below:



The Fourier coeffients for the triangular wave are

$$X_n = \frac{4A}{(n\pi)^2}, \ n \text{ odd}$$

By graphical differentiation, we see that

$$B = \frac{2A}{T_0/2} = \frac{4A}{T_0}$$

From part (a) we have

$$Y_n = jn\omega_0 X_n = jn\omega_0\left(\frac{4A}{(n\pi)^2}\right) = j\frac{8A}{\pi n T_0}, \ n \text{ odd}$$

Using the value found for B in terms of A, we obtain the Fourier coeffients of a square wave knowing the Fourier coefficients of a triangular wave (or vice versa):

$$Y_n = j\frac{2B}{\pi n} = j\frac{8A}{n\pi T_0}, \ n \text{ odd } (B \text{ is the square wave amplitude})$$

Problem 3-8

(a) By trigonometric identities (see Appendix F), we may write $x(t)$ as

$$x(t) = [1 - \cos(5000\pi t)]\cos(20000\pi t) = \cos(20000\pi t) - 1/2\cos(15000\pi t) - 1/2\cos(25000\pi t)$$

Using the fact that the power in a sinusoid is ½ the square of its amplitude, and the fact that we can add powers of the separate harmonics of a harmonic sum of sinusoids, we find that

$$P_{\text{ave}, x(t)} = \frac{1}{2}(1)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{3}{4} \text{ W}$$

(b) From (a) it is seen that the signal $x(t)$ consists of components with frequencies 7500, 10000, and 12500 Hz. Only the first two are passed by the telephone system, so the output power is

$$P_{\text{ave}, y(t)} = \frac{1}{2}(1)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{5}{8} \text{ W; Ratio } = \frac{5}{6}$$

Problem 3-9

For a pulse of width τ, amplitude A, and period T_0, the complex exponential Fourier series coefficients are given by

$$X_n = \frac{A\tau}{T_0}\text{sinc}(nf_0\tau),\ f_0 = T_0^{-1}$$

By Parseval's theorem, the power contained in the frequency band $-\tau^{-1} \le f \le \tau^{-1}$ is

$$P[|nf_0| \le \tau^{-1}] = \sum_{n=-N}^{N} |X_n|^2$$

where $N = \lfloor 1/f_0\tau \rfloor$ where the notation means "the integer part less than". The total power in the signal is $A^2\tau/T_0$ so that the fraction of total power is

$$\frac{P[|nf_0| \le \tau^{-1}]}{P_{\text{total}}} = \frac{\tau}{T_0}\sum_{n=-N}^{N} \text{sinc}^2(nf_0\tau)$$

(a) For $T_0/\tau = 2$, the fraction of total power contained within the main lobe is

$$\begin{aligned}\frac{P[|nf_0| \le \tau^{-1}]}{P_{\text{total}}} &= \frac{1}{2}\sum_{n=-2}^{2} \text{sinc}^2(0.5n) \\ &= \frac{1}{2}[\text{sinc}^2(-1) + \text{sinc}^2(-1/2) + \text{sinc}^2(0) + \text{sinc}^2(1/2) + \text{sinc}^2(1)] \\ &= \frac{1}{2}[1 + 2(0.4053)] = 0.9053\end{aligned}$$

(b) For $T_0/\tau = 4$, the fraction of total power contained within the main lobe is

$$\begin{aligned}\frac{P[|nf_0| \le \tau^{-1}]}{P_{\text{total}}} &= \frac{1}{4}\sum_{n=-4}^{4} \text{sinc}^2(0.25n) \\ &= \frac{1}{4}[1 + 2\,\text{sinc}^2(0.25) + 2\,\text{sinc}^2(0.5) + 2\,\text{sinc}^2(0.75)] \\ &= 0.9030\end{aligned}$$

(c) For $T_0/\tau = 10$, the fraction of total power contained within the main lobe is

$$\frac{P[|nf_0| \le \tau^{-1}]}{P_{\text{total}}} = \frac{1}{10}\sum_{n=-10}^{10} \text{sinc}^2(0.1n)$$

$$= \frac{1}{10}[1 + 2\,\text{sinc}^2(0.1) + 2\,\text{sinc}^2(0.2) + 2\,\text{sinc}^2(0.3) + 2\,\text{sinc}^2(0.4) + 2\,\text{sinc}^2(0.5)$$

$$+ 2\,\text{sinc}^2(0.6) + 2\,\text{sinc}^2(0.7) + 2\,\text{sinc}^2(0.8) + 2\,\text{sinc}^2(0.9)] = 0.8067$$

Problem 3-10

The waveform is odd, so

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,[\cos(n\omega_0 t) - j\sin(n\omega_0 t)]\,dt$$

$$= -\frac{j}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(n\omega_0 t)\,dt = -\frac{2j}{T_0}\int_{-0}^{T_0/2} x(t)\sin(n\omega_0 t)\,dt$$

Substitute the equation for the waveform and use integral tables to evaluate (or integrate by parts):

$$X_n = -\frac{j4A}{T_0^2}\int_0^{T_0/2} t\sin(n\omega_0 t)\,dt = -\frac{j4A}{T_0^2}\left[\sin(n\omega_0 t)\Big|_0^{T_0/2} - (n\omega_0 t)\cos(n\omega_0 t)\Big|_0^{T_0/2}\right] = \frac{jA}{n\pi}(-1)^n$$

Thus, the exponential Fourier series is

$$x(t) = \frac{jA}{\pi}\left[\cdots + \frac{1}{3}e^{-j3\omega_0 t} - \frac{1}{2}e^{-j2\omega_0 t} + e^{-j\omega_0 t} - e^{j\omega_0 t} + \frac{1}{2}e^{j2\omega_0 t} - \frac{1}{3}e^{j3\omega_0 t} + \cdots\right]$$

$$= \frac{2A}{\pi}\left[\sin(\omega_0 t) - \frac{1}{2}\sin(2\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \cdots\right]$$

This is the same as (3-4) except for the constant out in front.

Problem 3-11

(a) Take the fundamental term in the series. It is

$$\frac{1}{1+j\pi} e^{j3\pi t/2} = X_1 e^{j2\pi f_0 t}$$

By matching exponents on both sides and solving for f_0, we find that f_0 = 3/4 Hz; T_0 = 1.333 s.
(b) The average value of the waveform is given by $X_0 = 1$.
(c) The third harmonic term is given by

$$X_{-3} e^{-j9\pi t/2} + X_3 e^{j9\pi t/2} = 2|X_3|\cos(9\pi t/2 + \underline{/X_3})$$

Thus, the amplitude of the 3rd harmonic term is $2/(1+9\pi^2)^{1/2}$.
(d) From (c), the phase is $-\tan^{-1}(3\pi)$ radians.
(e) Substitute the values for the amplitude and phase found in parts (c) and (d) into the expression found in part (c).

Problem 3-12

(a) The exponential Fourier series coefficients are given by

$$X_n = \frac{1}{2}\int_{-1}^{1} e^{-|t|} e^{-j2\pi n t/2} dt = \frac{1}{2}\int_{-1}^{1} e^{-|t|}[\cos(\pi n t) - j\sin(\pi n t)]\, dt = \int_{0}^{1} e^{-t}\cos(\pi n t)\, dt = \frac{1-(-1)^n e^{-1}}{1+(n\pi)^2}$$

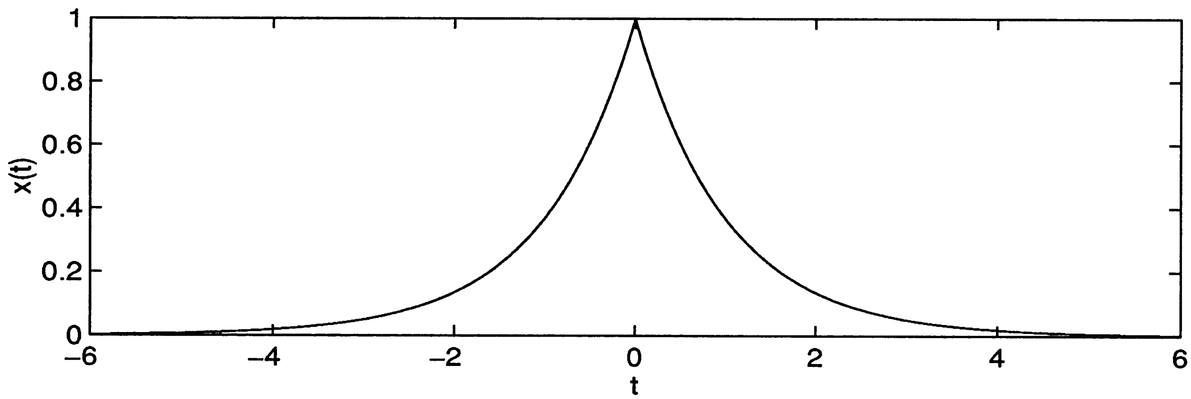
The second integral follows by the eveness of $\exp(-|t|)\cos(\pi n t)$ and oddness of $\exp(-|t|)\sin(\pi n t)$. Evaluation of the coefficients and substitution into the exponential Fourier series results in

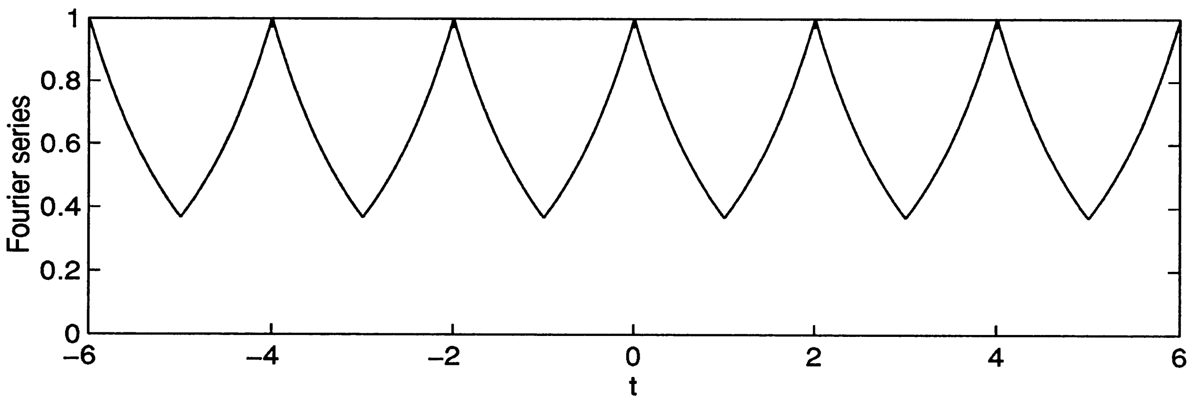
$$x(t) = \cdots + \frac{1-e^{-1}}{1+4\pi^2} e^{-j2\pi t} + \frac{1+e^{-1}}{1+\pi^2} e^{-j\pi t} + (1-e^{-1}) + \frac{1+e^{-1}}{1+\pi^2} e^{j\pi t} + \frac{1-e^{-1}}{1+4\pi^2} e^{j2\pi t} + \cdots$$

(b) The trigonometric Fourier series is

$$x(t) = (1-e^{-1}) + 2\frac{1+e^{-1}}{1+\pi^2}\cos(\pi t) + 2\frac{1-e^{-1}}{1+4\pi^2}\cos(2\pi t) + \cdots$$

A plot of the function and the sum of the Fourier series is given on the next page.





Problem 3-13

(a) Using Euler's theorem, the exponential Fourier series is

$$
\begin{aligned}
x_a(t) &= \left[\frac{e^{j20\pi t} + e^{-j20\pi t}}{2}\right]^2 \frac{e^{j10\pi t} - e^{-j10\pi t}}{2j} \\
&= \frac{1}{8j}\left[e^{j40\pi t} + 2 + e^{-j40\pi t}\right]\left[e^{j10\pi t} - e^{-j10\pi t}\right] \\
&= -\frac{1}{8j}e^{-j50\pi t} + \frac{1}{8j}e^{-j30\pi t} - \frac{1}{4j}e^{-j10\pi t} + \frac{1}{4j}e^{j10\pi t} - \frac{1}{8j}e^{j30\pi t} + \frac{1}{8j}e^{j50\pi t}
\end{aligned}
$$

(b) A series of steps similar to those used in part (a) results in

$$x_b(t) = \left[\frac{e^{j30\pi t} - e^{-j30\pi t}}{2j}\right]^3 + 2\frac{e^{j25\pi t} + e^{-j25\pi t}}{2}$$

$$= \frac{j}{8}\left[-e^{-j90\pi t} + 3e^{-j30\pi t} - 3e^{j30\pi t} + e^{j90\pi t}\right] + e^{j25\pi t} + e^{-j25\pi t}$$

$$= -\frac{j}{8}e^{-j90\pi t} + j\frac{3}{8}e^{-j30\pi t} + e^{-j25\pi t} + e^{j25\pi t} - j\frac{3}{8}e^{j30\pi t} + \frac{j}{8}e^{j90\pi t}$$

(c) A series of steps used to those above gives

$$x_c(t) = \left[\frac{e^{j40\pi t} - e^{-j40\pi t}}{2j}\right]^2 \left[\frac{e^{j20\pi t} + e^{-j20\pi t}}{2}\right]^2 + \frac{e^{j10\pi t} - e^{-j10\pi t}}{2j}\,\frac{e^{j5\pi t} - e^{-j5\pi t}}{2j}$$

$$= -\frac{1}{16}e^{-j120\pi t} - \frac{1}{8}e^{-j80\pi t} + \frac{1}{16}e^{-j40\pi t} - \frac{1}{4j}e^{-j15\pi t} - \frac{1}{4j}e^{-j5\pi t}$$

$$+ \frac{1}{4} + \frac{1}{4j}e^{j5\pi t} + \frac{1}{4j}e^{j15\pi t} + \frac{1}{16}e^{j40\pi t} - \frac{1}{8}e^{j80\pi t} - \frac{1}{16}e^{j120\pi t}$$

Problem 3-14

(a) Write $x(t)$ in terms of its Fourier series and replace t by $t - \tau$:

$$y(t) = x(t - \tau) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0(t - \tau)} = \sum_{n=-\infty}^{\infty} X_n e^{-jn\omega_0\tau} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} Y_n e^{jn\omega_0 t}$$

Comparing the two sums on the right, we see that $Y_n = X_n e^{-jn\omega_0\tau}$

(b) Using a series of steps similar to those used in part (a), we have

$$y(t) = x(t)e^{j2\pi f_0 t} = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi(n+1)f_0 t}$$

In the last sum, let $m = n + 1$:

$$y(t) = \sum_{m=-\infty}^{\infty} X_{m-1} e^{j2\pi m f_0 t} = \sum_{m=-\infty}^{\infty} Y_m e^{j2\pi m f_0 t}$$

comparing the two series, we find that $Y_m = X_{m-1}$.

Problem 3-15

Given

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t + \theta_n) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

Let

$$\cos(n\omega_0 t + \theta_n) = \cos\theta_n \cos(n\omega_0 t) - \sin\theta_n \sin(n\omega_0 t)$$

Thus, the trigonometric Fourier series can be written as

$$\begin{aligned}
x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\theta_n \cos(n\omega_0 t) - \sum_{n=1}^{\infty} a_n \sin\theta_n \sin(n\omega_0 t) \\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos\theta_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \right) - \sum_{n=1}^{\infty} a_n \sin\theta_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right) \\
&= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} (\cos\theta_n + j\sin\theta_n) e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{a_n}{2} (\cos\theta_n - j\sin\theta_n) e^{-jn\omega_0 t} \\
&= \equiv X_0 + \sum_{n=1}^{\infty} X_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} X_n e^{jn\omega_0 t} \\
&= X_0 + \sum_{n=1}^{\infty} X_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} X_{-n} e^{-jn\omega_0 t}
\end{aligned}$$

Matching coeffients in the series in the third and last lines, we obtain

$$a_0 = X_0;\ X_n = \frac{a_n}{2}(\cos\theta_n + j\sin\theta_n);\ X_{-n} = \frac{a_n}{2}(\cos\theta_n - j\sin\theta_n),\ n > 0$$

Solve these for a_n and θ_n to get

$$a_n = 2|X_n|;\ \theta_n = \tan^{-1}\left(\frac{\operatorname{Im} X_n}{\operatorname{Re} X_n} \right)$$

(b) This is obvious from the fact that cosine is an even function and sine is an odd function.

Problem 3-16

(a) The integral for the Fourier coefficients can be written as

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)\,e^{-jn\omega_0 t}\,dt = \frac{1}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt - \frac{j}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

Assume $x(t)$ is real and even. In the first integral on the right-hand side above, the integrand is even because the product of two even functions is even. Thus this integral is not 0 in general. In the second integral on the right-hand side above, the integrand is odd because the product of an even and an odd function is odd. Thus this integral is 0 if integrated over symmetric limits about $t = 0$ and, indeed, over any T_0-second interval by periodicity of the integrand. Hence, we conclude that the X_n's are real and, since the dependence on n is through the cosine which is an even function, X_n is an even function of n.

(b) In the expression given above, let $x(t)$ be real and odd. Now the first integral on the right-hand side is 0 because an odd times an even function is odd. The second integral on the right-hand side is not 0 in general because the product of two odd functions is even. Thus the X_n's are imaginary and odd since the dependence on n is through the sine which is an odd function.

(c) In the Fourier series for $x(t)$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

replace t by $t \pm T_0/2$ to get

$$x(t \pm T_0/2) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0(t \pm T_0/2)} = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} e^{\pm jn(2\pi/T_0)(T_0/2)}$$

$$= \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} e^{\pm jn\pi} = -\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \quad \text{(by hypothesis)}$$

The last two sums, when placed on the same side of the equation, produce

$$\sum_{n=-\infty}^{\infty} X_n\left[1 + e^{\pm jn\omega_0 t}\right] \equiv 0$$

For n odd, the bracketed term is 0, but for n even it equals 2. For the sum above to be identically 0 requires that the even-indexed X_n's be 0. For n even, the bracketed term makes these terms 0.

Problem 3-17

Property	a	b	c	d	e	f
Real coefficients		X			X	
Imaginary coefficients	X			X		
Complex coefficients			X			X
Even-indexed coefficient = 0	X	X	X	X	X	X
$X_0 = 0$	X	X	X	X	X	X

Problem 3-18

(a) $a_0 = 0$ because the average value of the waveform is 0;
(b) $b_n = 0$ for all n because the waveform is even;
(c) the X_n's are real because $x(t)$ is even;
(d) yes, so the even-indexed Fourier coefficients are 0.

Problem 3-19

Expand the second term of the given expression using a trigonometric identity to get

$$\begin{aligned} x(t) &= \cos(\omega_0 t) + \cos\Delta\omega t \cos(\omega_0 t) - \sin\Delta\omega t \sin(\omega_0 t) \\ &= (1 + \cos\Delta\omega t)\cos(\omega_0 t) - \sin\Delta\omega t \sin(\omega_0 t) \end{aligned}$$

Note that

$$A\cos(\omega_0 t + \theta) = A\cos\theta \cos\omega_0 t - A\sin\theta \sin\omega_0 t$$

Set this equal to the last equation of the first set and match multipliers of sine and cosine to get

$$A\cos\theta = 1 + \cos\Delta\omega t;\ A\sin\theta = \sin\Delta\omega t$$

Square and add these equations to get

$$A(t) = \sqrt{[1 + \cos\Delta\omega t]^2 + \sin^2\Delta\omega t} = \sqrt{2 + 2\cos\Delta\omega t} = 2|\cos(\Delta\omega t/2)|$$

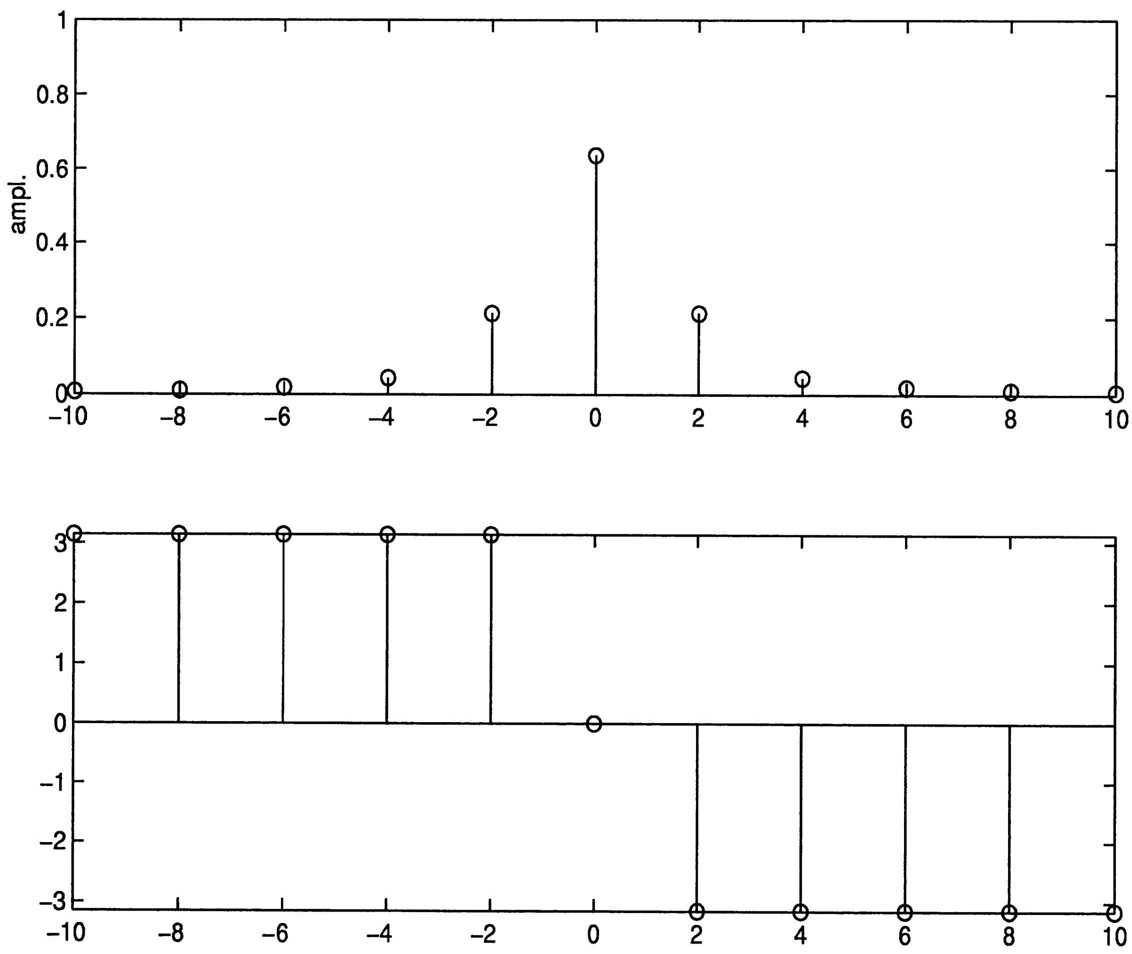
Divide the second equation by the first to get

$$\theta(t) = \tan^{-1}\left[\frac{\sin\Delta\omega t}{1 + \cos\Delta\omega t}\right]$$

Thus, the sum of the two cosines becomes

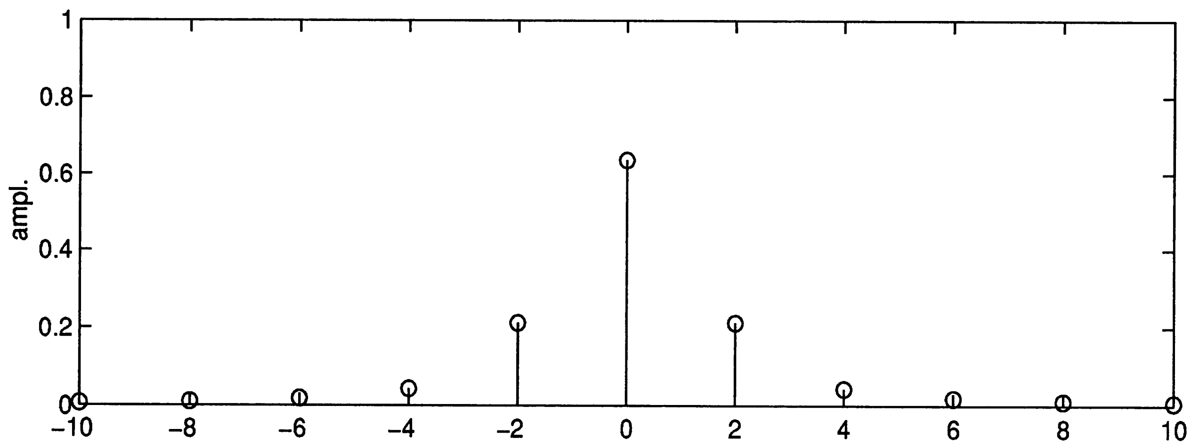
$$x(t) = 2|\cos(\Delta\omega t/2)|\cos\left[\omega_0 t + \tan^{-1}\left(\frac{\sin\Delta\omega t}{1 + \cos\Delta\omega t}\right)\right]$$

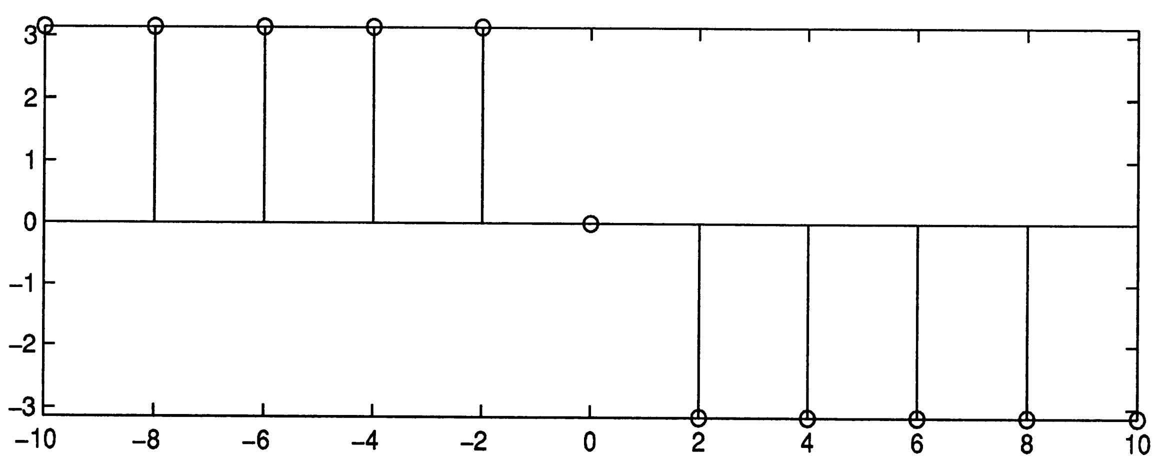
A plot is shown below for two different time scales. The ear will hear alternate reinforcements and cancellations.



Problem 3-20

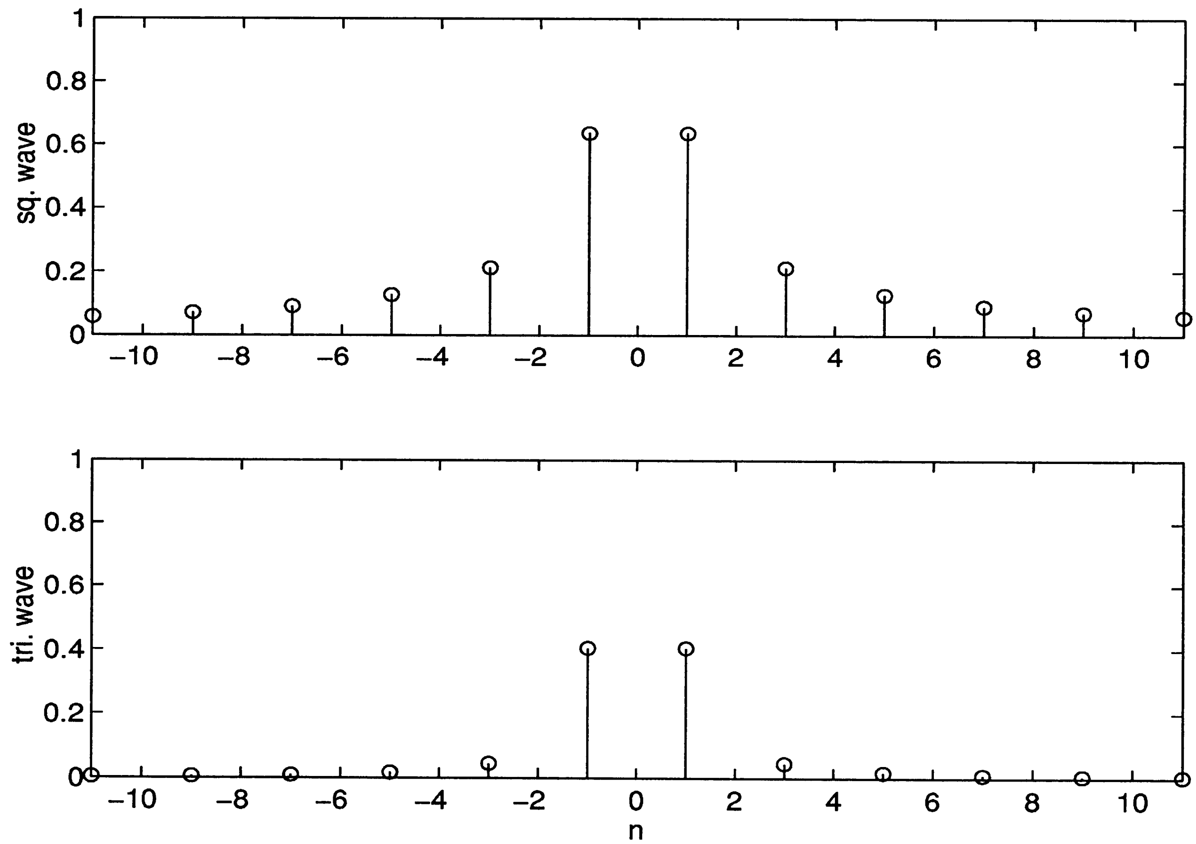
The double-sided amplitude spectra are shown below. To get the single-sided amplitude spectrum, take the portion of the double-sided spectrum for $f \geq 0$, leaving the line at $f = 0$ alone and doubling lines for $f > 0$. For the single-sided phase spectrum, simply take the positive frequency portion of the double-sided spectrum.





Problem 3-21

The two amplitude spectra are compared below. The spectrum for the triangular wave goes to 0 with n fastest, and therfore requires the least bandwidth.



Problem 3-22

(a) See Fig. 3-11(a), except that the nulls of the sinc function are at multiples of 500 Hz and the lines are spaced by 125 Hz.
(b) See Fig. 3-11(b), except that the nulls of the sinc function are at multiples of 1000 Hz and the lines are spaced by 125 Hz.
(c) See Fig. 3-11(c), except that the nulls of the sinc function are at multiples of 500 Hz and the lines are spaced by 62.5 Hz.

Problem 3-23

Start with (3-69) after expressing the transfer function in terms of amplitude and phase:

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n| e^{j\underline{/X_n}} |H(n\omega_0)| e^{j\underline{/H(n\omega_0)}} e^{jn\omega_0 t}$$

Recall that for $x(t)$ and $h(t)$, the magnitude functions are even and the phase functions are odd. Thus, we have the series of steps given below:

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{-1} |X_n|\,|H(n\omega_0)|\, e^{j[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}]} + X_0 + \sum_{n=1}^{\infty} |X_n|\,|H(n\omega_0)|\, e^{j[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}]} \\ &= \sum_{m=1}^{\infty} |X_{-m}|\,|H(-m\omega_0)|\, e^{j[-m\omega_0 t + \underline{/X_{-m}} + \underline{/H(-m\omega_0)}]} + X_0 + \sum_{n=1}^{\infty} |X_n|\,|H(n\omega_0)|\, e^{j[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}]} \\ &= X_0 + \sum_{n=1}^{\infty} 2|X_n|\,|H(n\omega_0)| \frac{e^{j[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}]}}{2} \\ &= X_0 + \sum_{n=1}^{\infty} 2|X_n|\,|H(n\omega_0)| \cos[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}] \end{aligned}$$

Problem 3-24

Apply (3-73). The transfer function of the system is

$$H(j\omega) = \frac{1}{1 + j\omega L/R} = \frac{1}{\sqrt{1 + (\omega L/R)^2}} e^{-j\tan^{-1}(\omega L/R)}$$

From Table 3-1, the exponential Fourier series coefficients for a triangular signal are

$$X_n = \frac{4A}{(n\pi)^2}, \; n \text{ odd}$$

Thus, (3-73) becomes

$$y(t) = \sum_{n=-\infty,\, n \text{ odd}}^{\infty} \left(\frac{8A}{(n\pi)^2 \sqrt{1 + n^2}} \right) \cos[n\omega_0 t - \tan^{-1}(n)]$$

Problem 3-25

(a) Rewrite the output as

$$y(t) = \cos[10\pi(t - 1/40)] + 5\cos[20\pi(t - 1/40)]$$

The delays in both components are the same, so there is no phase or delay distortion. However, the amplitudes are in a different ratio in the output than the input, so there is amplitude distortion.
(b) Rewrite the output as

$$y(t) = \cos[10\pi(t - 1/40)] + 2\cos[20\pi(t - 1/80)]$$

There is phase or delay distortion, but there is amplitude distortion.
(c) Rewrite the output as

$$y(t) = \cos[10\pi(t - 1/40)] + 2\cos[20\pi(t - 1/40)]$$

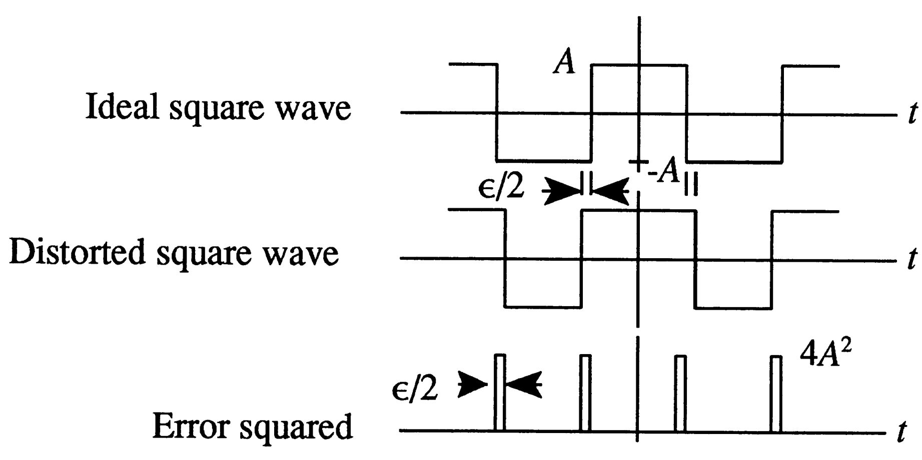
Comparing this with the input, it is clear that there is no distortion.
(d) Rewrite the output as

$$y(t) = 2\cos[10\pi(t - 1/40)] + 2\cos[20\pi(t - 1/160)]$$

This shows that there is amplitude distortion, but no phase distortion.

Problem 3-26

The ideal square wave, distorted square wave, and error-squared function are shown below:



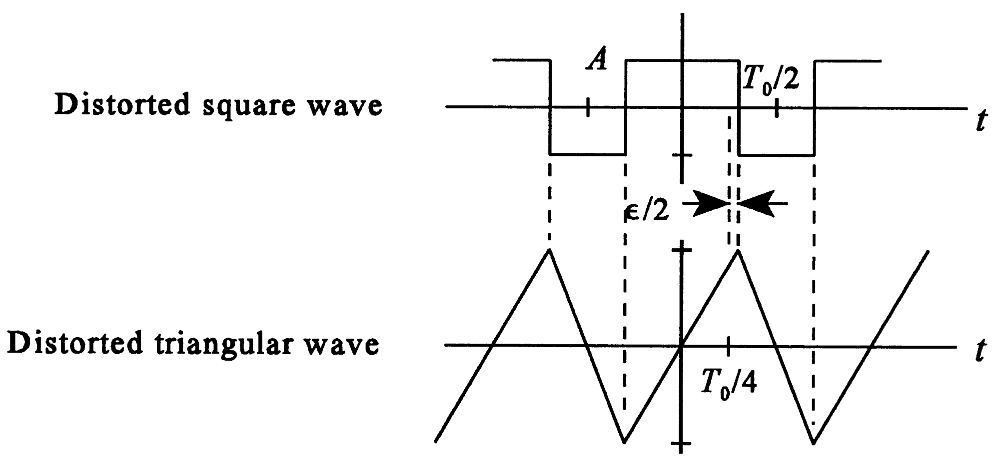
From the figure, the integral-squared error over one period is

$$\text{ISE} = (4A^2)(\epsilon/2)(2) = 4A^2\epsilon$$

The mean-squared error is

$$\text{Mean-squared error} = \frac{\text{ISE}}{T_0} = 4A^2\frac{\epsilon}{T_0}$$

(b) Assume ac coupling between the multivibrator and the RC integrator. The integrator will have a long positive-slope ramp and a short negative-slope ramp from integrating the unequal distorted square wave half cycles. The figure below illustrates this.



(Note that strictly speaking the triangular wave should be inverted due to the operational amplifier.)
(c) For the sinusoidal mode, find the Fourier series of the square wave. Since the waveform is even (see the distorted square wave above), we use a Fourier cosine series. The dc component is

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt = A\epsilon/T_0$$

For $n > 0$,

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{4}{T_0}\left[\int_0^{T_0/4+\epsilon/2} A\cos(n\omega_0 t)\,dt + \int_{T_0/4+\epsilon/2}^{T_0/2} -A\cos(n\omega_0 t)\,dt\right]$$

$$= \frac{4A}{n\pi}\sin[n\pi(\epsilon/T_0 + 0.5)]$$

The output power for the nth harmonic is

$$P_{n,\ \text{out}} = \frac{1}{2}|H(n\omega_0)|^2 a_n^2$$

where the transfer function of the filter is given in the problem statement.

(I) The ratio of 2nd harmonic power to fundamental power in the output is

$$\frac{P_{2,\ \text{out}}}{P_{1,\ \text{out}}} = \text{HC}_2 = \left[\frac{\sin[2\pi(\epsilon/T_0 + 0.5)]}{2\sin[\pi(\epsilon/T_0 + 0.5)]}\right]^2 \frac{1}{1 + 9Q^2/4}$$

(II) The ratio of 3rd harmonic power to fundamental power in the output is

$$\frac{P_{3,\ \text{out}}}{P_{1,\ \text{out}}} = \text{HC}_3 = \left[\frac{\sin[3\pi(\epsilon/T_0 + 0.5)]}{3\sin[\pi(\epsilon/T_0 + 0.5)]}\right]^2 \frac{1}{1 + 64Q^2/9}$$

If one solves $\text{HC}_2 = 0.001$ for Q, the result is 3.23, while the solution of $\text{HC}_3 = 0.0005$ is $Q = 5.029$. Thus, the latter is the most stringent condition.

Problem 3-27

For (c), one derivative produces impulses so the spectral lines approach 0 as $1/n$ as $n \to \infty$. For (d), two derivatives produce impulses, so the spectral lines approach 0 as $1/n^2$ as $n \to \infty$.

Problem 3-28

Two derivatives produce impulses, so the spectral lines approach 0 as $1/n^2$ as $n \to \infty$.

Problem 3-29

Expand the last term of (3-92) to get

$$\sum_{n=1}^{\infty}\left|d_n - \int_T x(t)\phi_n^*(t)\,dt\right|^2 = \sum_{n=1}^{\infty}\left[|d_n|^2 - d_n\int_T x^*(t)\phi_n(t)\,dt - d_n^*\int_T x(t)\phi_n^*(t)\,dt + \left|\int_T x(t)\phi_n^*(t)\,dt\right|^2\right]$$

Sum each term individually and substitute into (3-92):

$$\epsilon_N = \int_T |x(t)|^2 dt - \sum_{n=1}^{N}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2 + \sum_{n=1}^{N}|d_n|^2 - \sum_{n=1}^{N} d_n^*\int_T x(t)\phi_n^*(t)\,dt$$
$$- \sum_{n=1}^{N} d_n\int_T x^*(t)\phi_n(t)\,dt + \sum_{n=1}^{N}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2$$

It is seen that the 2nd and last terms on the right hand side cancel, giving (3-90).

Problem 3-30

(a) Normalize $f_1(t)$ to get $\phi_1(t)$:

$$\int_0^{\infty}(A_1 e^{-t})^2 dt = A_1^2\int_0^{\infty} e^{-2t}dt = A_1^2/2 = 1 \text{ or } A_1 = \sqrt{2}$$

Thus

$$\phi_1(t) = \sqrt{2}e^{-t}u(t)$$

(b) Let

$$\overline{\phi_2}(t) = f_2(t) - B\phi_1(t) = A_2 e^{-2t} - B\sqrt{2}e^{-t},\ t \ge 0$$

We require that

$$\int_0^{\infty}\overline{\phi_2}(t)\phi_1(t)\,dt = 0 \text{ or } \int_0^{\infty}\left(A_2 e^{-3t} - B\sqrt{2}e^{-2t}\right)dt = 0 \text{ or } \frac{\sqrt{2}}{3}A_2 - B = 0 \text{ or } B = \frac{\sqrt{2}}{3}A_2$$

Thus

$$\overline{\phi_2}(t) = A_2\left[e^{-2t} - \frac{2}{3}e^{-t}\right]u(t)$$

To find A_2, evaluate the normalization integral:

$$\int_0^\infty \phi_2^2(t)\,dt = 1 \text{ which gives } A_2 = 6$$

The second orthonormal function is, therefore,

$$\phi_2(t) = \left(6e^{-2t} - 4e^{-t}\right)u(t)$$

(c) Carrying out the steps of orthogonalization on the third function as suggested in the problem statement results in

$$\phi_3(t) = \frac{10\sqrt{6}}{17}\left[e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t}\right]u(t)$$

There is no obvious generalization for general n.

Problem 3-31

(a) From (3-94) and Table 3-1,

$$\text{ISE}_1 = \left[A^2 - 2\left(\frac{2A}{\pi}\right)^2\right]T_0 = A^2T_0\left(1 - \frac{8}{\pi^2}\right]$$

(b) Again using (3-94) and Table 3-1, we obtain the result

$$\text{ISE}_2 = \left[A^2 - 2\left(\frac{2A}{\pi}\right)^2 - 2\left(\frac{2A}{3\pi}\right)^2\right]T_0 = A^2T_0\left(1 - \frac{8}{\pi^2} - \frac{8}{9\pi^2}\right) = A^2T_0\left(1 - \frac{80}{9\pi^2}\right)$$

CHAPTER 4

Problem 4-1

(a) The Fourier transform of this signal is

$$X_a(f) = \int_{-\infty}^{\infty} Ae^{-\alpha t}u(t)e^{-j2\pi ft}\,dt = \int_{0}^{\infty} Ae^{-(\alpha + j2\pi f)t}\,dt = A\left.\frac{e^{-(\alpha + j2\pi f)t}}{-(\alpha + j2\pi f)}\right|_0^{\infty} = \frac{A}{\alpha + j2\pi f}$$

where the evaluation of the upper limit gives 0 because the problem statement says that $\alpha > 0$.
(b) The evaluation of this Fourier transform gives

$$X_b(f) = \int_{-\infty}^{\infty} Ae^{\alpha t}u(-t)e^{-j2\pi ft}\,dt = \int_{-\infty}^{0} Ae^{(\alpha - j2\pi f)t}\,dt = A\left.\frac{e^{(\alpha - j2\pi f)t}}{(\alpha - j2\pi f)}\right|_{-\infty}^{0} = \frac{A}{\alpha - j2\pi f}$$

(c) By direct evaluation, or using the results of the previous two parts noting that $x_c(t) = x_a(t) + x_b(t)$, gives

$$X_c(f) = A\left[\frac{1}{\alpha + j2\pi f} + \frac{1}{\alpha - j2\pi f}\right] = \frac{2A\alpha}{\alpha^2 + (2\pi f)^2}$$

(d) This signal is the difference between the signals of parts (a) and (b), and it follows that its Fourier transform is

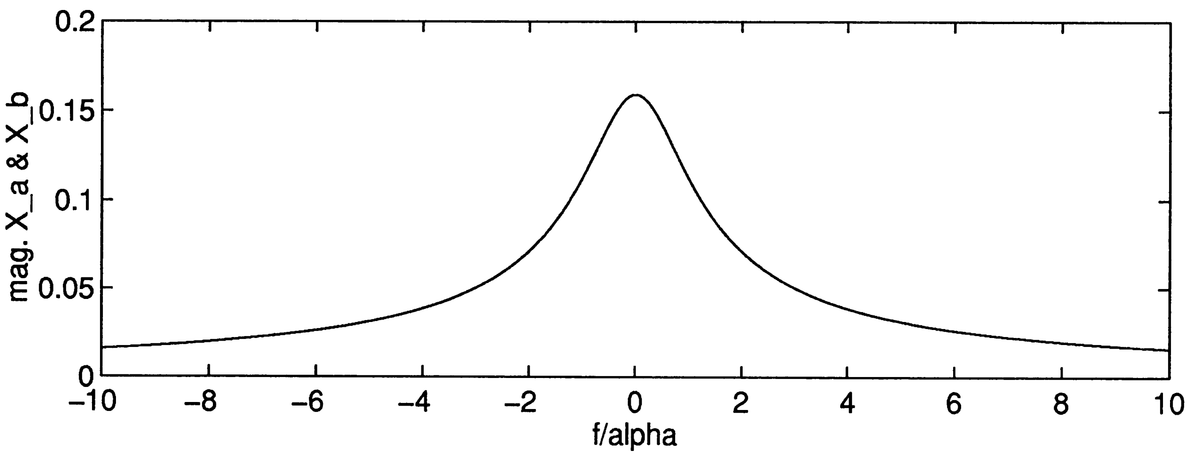
$$X_d(f) = A\left[\frac{1}{\alpha + j2\pi f} - \frac{1}{\alpha - j2\pi f}\right] = \frac{-j4A\pi f}{\alpha^2 + (2\pi f)^2}$$

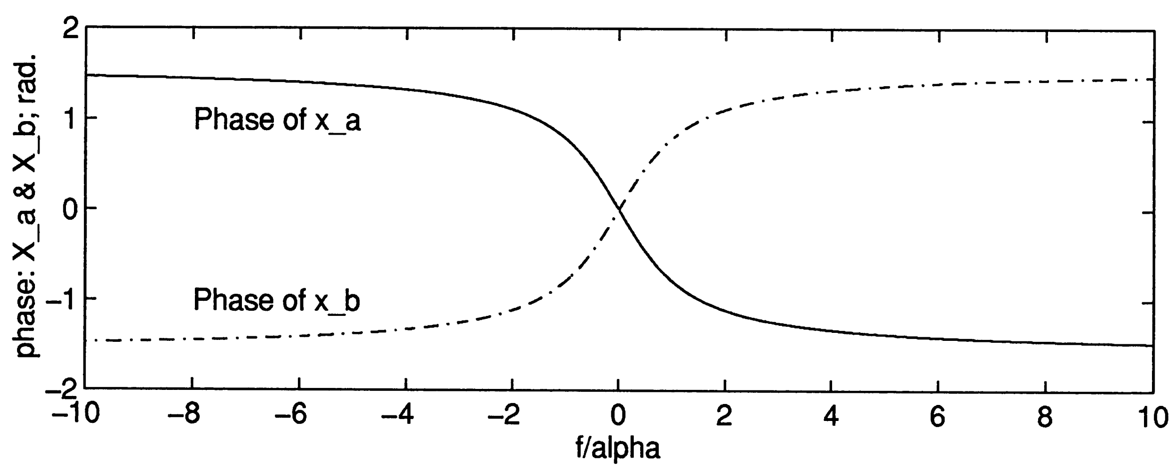
Problem 4-2

The Fourier transforms in terms of magnitude and phase are

$$X_a(f) = \frac{A}{\sqrt{\alpha^2 + (2\pi f)^2}} e^{-j\tan^{-1}(2\pi f/\alpha)} \text{ and } X_b(f) = \frac{A}{\sqrt{\alpha^2 + (2\pi f)^2}} e^{j\tan^{-1}(2\pi f/\alpha)}$$

Thus, both magnitude functions are the same and the phase functions are negations of each other. Plots are shown below:



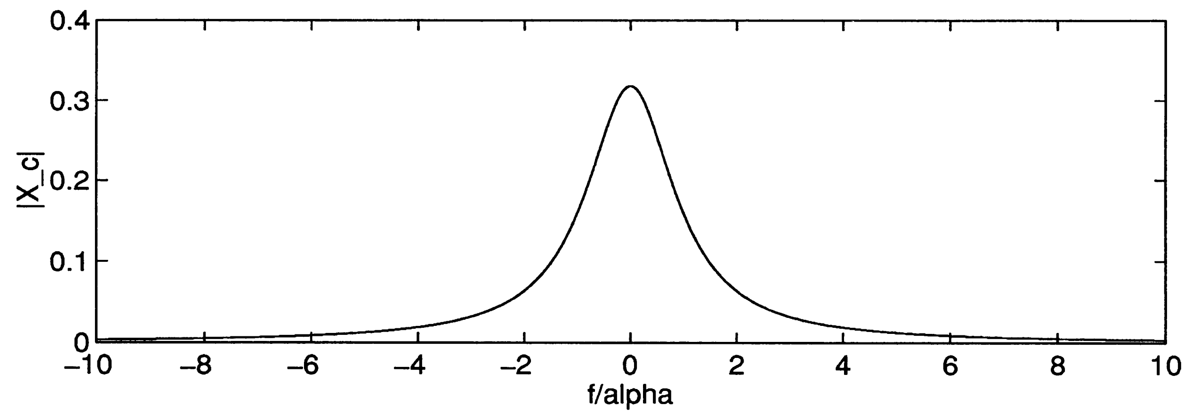


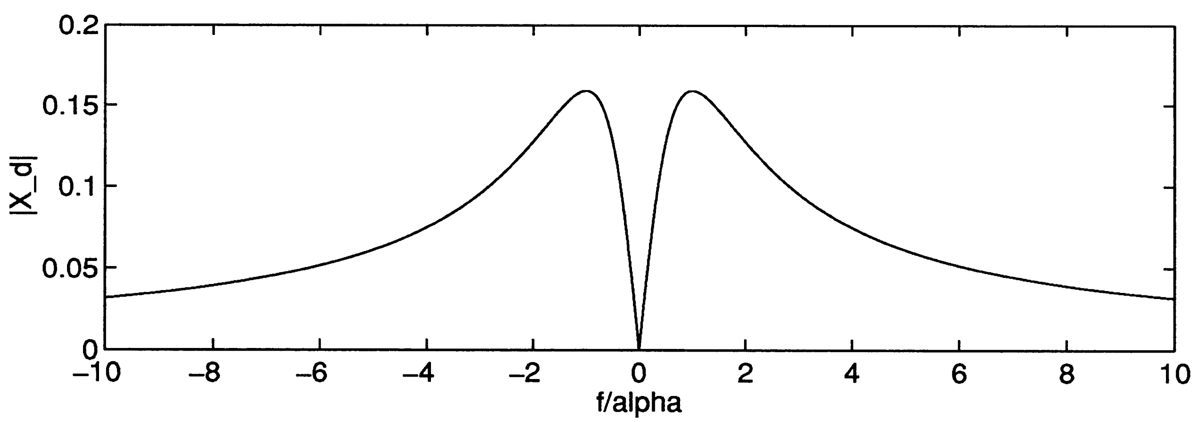
Problem 4-3

The phase function for $x_c(f)$ is identically 0. The phase function for $x_d(f)$ is $\pi/2$ for $f < 0$ and $-\pi/2$ for $f > 0$ or vice versa. The magnitude functions are given by

$$X_c(f) = \frac{2A\alpha}{\alpha^2 + (2\pi f)^2} \text{ and } X_d(f) = \frac{4A\pi|f|}{\alpha^2 + (2\pi f)^2}$$

Plots are given below ($A = 1$):





Problem 4-4

(a) Write the integral in terms of its real and imaginary parts, where $x(t)$ is assumed real and even here:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)\, dt - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\, dt$$

The second integral is 0 because its integrand is odd (an even times an odd function) and is integrated over symmetric limits about $f = 0$. The first integral is not 0 in general because its integrand is even, being the product of two even functions. Thus, the Fourier integral for $x(t)$ is purely real. Furthermore it is an even function of f, since this dependence comes through the cosine which is an even function. Since the integrand of the first integral is even, the integral can be evaluated by integrating from 0 to ∞ and doubling the result.
(b) Again write the integral in terms of its real and imaginary parts, where $x(t)$ is assumed real and odd. Now the first integral is 0 because its integrand is odd (an even times an odd function) and is integrated over symmetric limits about $f = 0$. The second integral is not 0 in general because its integrand is even, being the product of two odd functions. Thus, the Fourier integral for $x(t)$ is purely imaginary. Furthermore it is an odd function of f, since this dependence comes through the sine which is an odd function. Since the integrand of the second integral is even, the integral can be evaluated by integrating from 0 to ∞ and doubling the result.

Problem 4-5

(a) The integrand of the Fourier integral can be written in this case as

$$\begin{aligned} x(t)\, e^{-j2\pi ft} &= [x_R(t) + jx_I(t)]\,[\cos(2\pi ft) + j\sin(2\pi ft)] \\ &= x_R(t)\cos(2\pi ft) + x_I(t)\sin(2\pi ft) + j[x_I(t)\cos(2\pi ft) - x_R(t)\sin(2\pi ft)] \end{aligned}$$

Substitution of this into the Fourier transform integral and separating the integral into a sum of integrals gives the result of the problem statement.
(b) Under the conditions given, the second and last integrals of the problem statement are 0 since their integrands are odd and the integrals are over intervals symmetric about $t = 0$. Hence, only the first and third integrals are left. When put together as one integral, the result is that given in the problem statement.
(c) In this case, the first and third integrals are 0 since their integrands are the products of an odd function and cosine, which is even thus giving an odd integrand. When the second and fourth integrals are put together as one, the given result follows.

Problem 4-6

(a) The Fourier transform integral for this signal becomes

$$X_a(f) = \int_{-\infty}^{\infty} te^{-\alpha t}u(t)e^{-j2\pi ft}dt = \int_0^{\infty} te^{-\alpha t}e^{-j2\pi ft}dt$$

$$= \int_0^{\infty} te^{-(\alpha + j2\pi f)t}dt = \left.\frac{te^{-(\alpha + j2\pi f)t}}{-(\alpha + j2\pi f)}\right|_0^{\infty} + \frac{1}{(\alpha + j2\pi f)}\int_0^{\infty} e^{-(\alpha + j2\pi f)t}dt$$

$$= -\left.\frac{e^{-(\alpha + j2\pi f)t}}{(\alpha + j2\pi f)^2}\right|_0^{\infty} = \frac{1}{(\alpha + j2\pi f)^2}$$

(b) To evaluate this Fourier transform integral, we need the tabulated integral

$$\int t^2 e^{at}dt = \frac{e^{at}}{a^3}(a^2t^2 - 2at + 2)$$

For this signal, the Fourier transform integral is

$$X_b(t) = \int_{-\infty}^{\infty} t^2 u(t)u(1 - t)e^{-j2\pi ft}dt = \int_0^1 t^2 e^{-j2\pi ft}dt$$

$$= \left\{\frac{e^{-j2\pi ft}}{(-j2\pi f)^3}[(-j2\pi f)^2t^2 - 2(-j2\pi f)t + 2]\right\}_0^1$$

$$= \left\{\frac{1}{j2\pi f}\left[\frac{1}{j\pi f} - 1\right]e^{-j2\pi f} + \frac{2}{(j2\pi f)^3}\left[1 - e^{-j2\pi f}\right]\right\}$$

(c) The integral for this Fourier transform becomes

$$X_c(f) = \int_0^1 e^{-(\alpha + j2\pi f)t}dt = -\left.\frac{e^{-(\alpha + j2\pi f)t}}{\alpha + j2\pi f}\right|_0^1 = \frac{1 - e^{-(\alpha + j2\pi f)}}{\alpha + j2\pi f}$$

Problem 4-7

Use the fact that the signal is even to reduce the Fourier transform integral to the form

$$X(f) = 2\int_0^{\infty} x(t)\cos(2\pi f t)\,dt$$

Thus, the Fourier transform of this signal is given by the integral

$$X(f) = 2\left[\int_0^{(\tau-\epsilon)/2} A\cos(2\pi f t)\,dt + \int_{(\tau-\epsilon)/2}^{(\tau+\epsilon)/2} A\left\{1 + \cos\left[\frac{\pi}{\epsilon}\left(t - \frac{\tau-\epsilon}{2}\right)\right]\right\}\cos(2\pi f t)\,dt\right]$$

Break up the second integral, using the trigonometric identities

$$\cos\left(\frac{\pi}{\epsilon}t - \frac{\pi\tau}{2\epsilon} + \frac{\pi}{2}\right) = -\sin\left[\frac{\pi}{\epsilon}(t - \tau/2)\right]$$

$$\text{and } \cos(x)\cos(y) = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$$

Integrate and use trigonometric identities and much algebra to produce

$$X(f) = A\tau\,\mathrm{sinc}(f\tau)\frac{\cos(\pi f\epsilon)}{1 - (2\epsilon f)^2}$$

A plot is shown on the next page.

with convolution in the frequency domain to produce

$$X_2(f) = \Pi(f) * \left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] = \frac{1}{2}\Pi(f) + \int_{-\infty}^{\infty} \Pi(\lambda)\frac{d\lambda}{j2\pi(\lambda - f)}$$

$$= \frac{1}{2}\Pi(f) + \frac{1}{j2\pi}\int_{-1/2}^{1/2} \frac{d\lambda}{\lambda - f} = \frac{1}{2}\Pi(f) - \frac{j}{2\pi}\ln\left|\frac{2f-1}{2f+1}\right|$$

(c) Note that sgn(t) = 2 u(t) - 1, so

$$x_3(t) = 2\,\mathrm{sinc}(t)\,u(t) - \mathrm{sinc}(t) = 2x_2(t) - \mathrm{sinc}(t)$$

Using the results of part (b), we obtain

$$X_3(f) = \Pi(f) - \frac{j}{\pi}\ln\left|\frac{2f-1}{2f+1}\right| - \Pi(f) = -\frac{j}{\pi}\ln\left|\frac{2f-1}{2f+1}\right| = \frac{j}{\pi}\ln\left|\frac{2f+1}{2f-1}\right|$$

(d) The unit step makes the part of this signal to the left of $t = 0$ zero, so we have

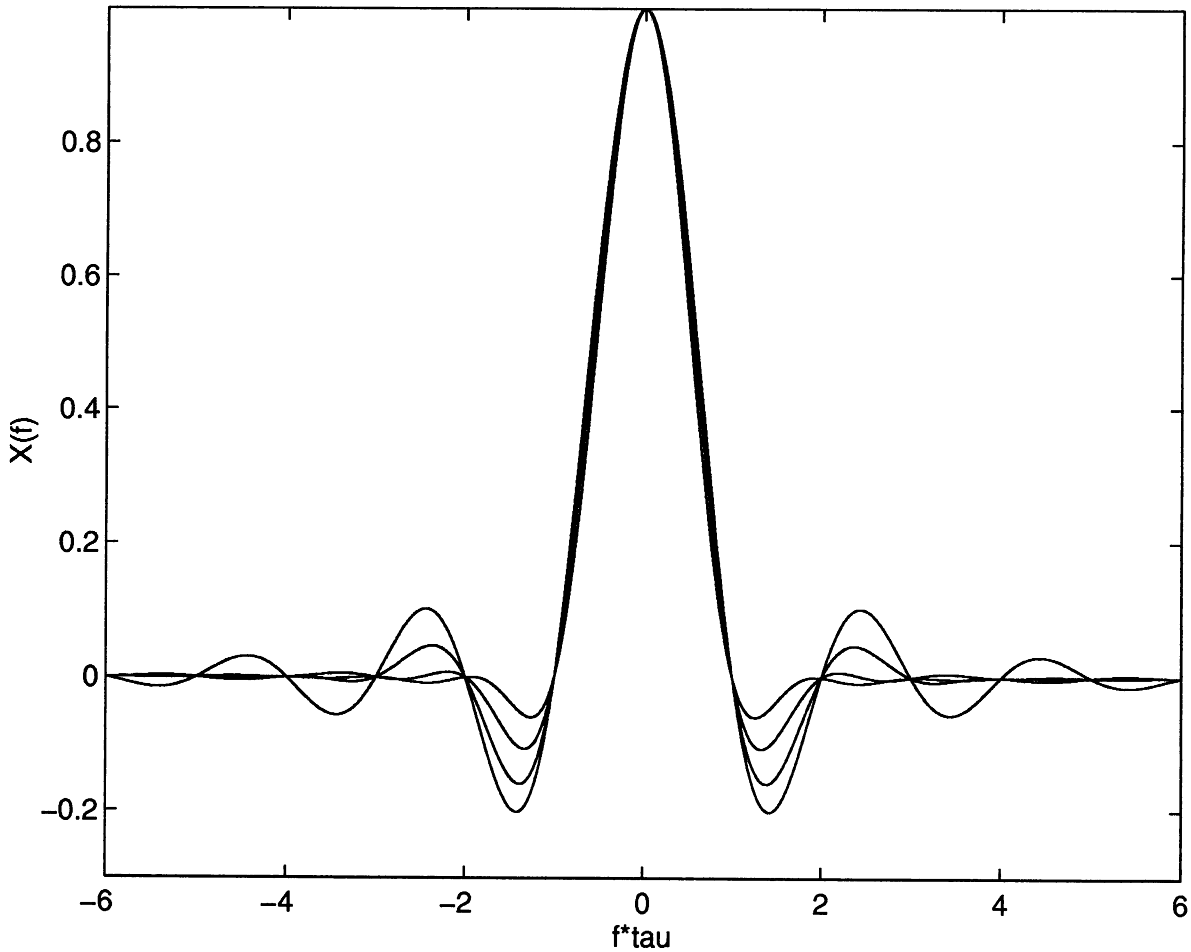
$$X_4(f) = \mathscr{F}[\exp(-t)u(t)] = \frac{1}{1 + j2\pi f}$$

(e) Write this signal as

$$x_5(t) = \exp(-|t|)[2u(t) - 1] = 2x_4(t) - \exp(-|t|)$$

Thus, its Fourier transform is

$$X_5(f) = \frac{2}{1 + j2\pi f} - \frac{1}{1 + j2\pi f} - \frac{1}{1 - j2\pi f} = -\frac{j4\pi f}{1 + (2\pi f)^2}$$



Spectrum for Problem 4-7: $\epsilon = 0.8$, largest oscillations; 0.6; 0.4; 0.2, smallest oscillations.

Problem 4-8

(a) Using the time delay and superposition theorems, we have

$$X_1(f) = \frac{1}{2}\left[e^{j2\pi f} + e^{j\pi f} + e^{-j\pi f} + e^{-j2\pi f}\right] = \cos(2\pi f) + \cos(\pi f)$$

(b) Use the transform pairs

$$2W\operatorname{sinc}(2Wt) \Leftrightarrow \Pi\left(\frac{f}{2W}\right) \text{ and } u(t) \Leftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

Problem 4-9

For $x(t)$ real

$$X(f) = \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)\,dt - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\,dt = X_R(f) + jX_I(f)$$

where the first integral is the real part and the second integral is the imaginary part. The magintude is

$$|X(f)| = \sqrt{X_R^2(f) + X_I^2(f)} = |X(-f)|$$

where the eveness of the maginitude is obvious because the real and imaginary parts are squared. The phase function is

$$\theta(f) = \underline{/X(f)} = \tan^{-1}\left[\frac{X_I(f)}{X_R(f)}\right] = -\tan^{-1}\left[\frac{X_I(-f)}{X_R(-f)}\right] = -\theta(-f)$$

The last equation holds because the imaginary part of $X(f)$ is an odd function of f (the dependence is through the sine function) and the real part is an even function. Thus, the ratio of imaginary and real parts is odd. Furthermore, the arctangent is an odd function of its argument, making the overall result odd.

Problem 4-10

(a) Use duality and the transform pair

$$A\exp(-\alpha|t|) \leftrightarrow \frac{2A\alpha}{\alpha^2 + (2\pi f)^2}$$

(see Problem 4-1(c)) to obtain

$$\mathcal{F}^{-1}[A\exp(-\alpha|f|)] = \frac{2A\alpha}{\alpha^2 + (-2\pi t)^2} = \frac{2A\alpha}{\alpha^2 + (2\pi t)^2}$$

(b) Write this signal as

$$X_2(f) = e^{-j\pi/2}e^{-\alpha f}u(f) + e^{j\pi/2}e^{\alpha f}u(-f)$$

Use the transform pair

$$e^{-\alpha t}u(t) \Leftrightarrow \frac{1}{\alpha + j2\pi f}$$

and duality to show that

$$\mathcal{F}^{-1}\left[e^{-\alpha f}u(f)\right] = \frac{1}{\alpha - j2\pi t}$$

Also note that

$$\mathcal{F}^{-1}[e^{\alpha f}u(-f)] = \int_{-\infty}^{0} e^{\alpha f}e^{j2\pi f t}\,dt = \int_{-\infty}^{0} e^{(\alpha + j2\pi t)f}\,df = \left.\frac{e^{(\alpha + j2\pi t)f}}{\alpha + j2\pi t}\right|_{-\infty}^{0} = \frac{1}{\alpha + j2\pi t}$$

Thus

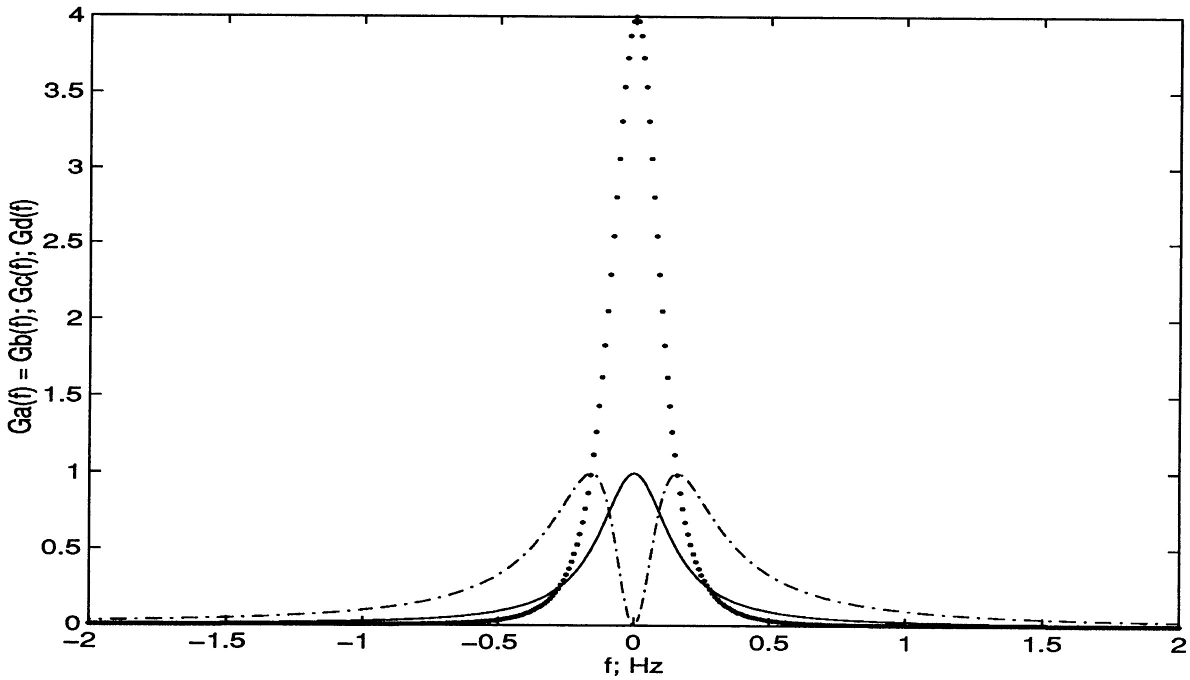
$$\mathcal{F}^{-1}\left[e^{-j\pi/2}e^{-\alpha f}u(f) + e^{j\pi/2}e^{\alpha f}u(-f)\right] = \frac{e^{-j\pi/2}}{\alpha - j2\pi t} + \frac{e^{j\pi/2}}{\alpha + j2\pi t} = \frac{4\pi t}{\alpha^2 + (2\pi t)^2}$$

Problem 4-11

The energy spectral densities are given below. Plots follow.

$$\text{(a) } G_a(f) = |X_a(f)|^2 = \frac{A^2}{\alpha^2 + (2\pi f)^2}; \text{ (b) } G_b(f) = G_a(f);$$

$$\text{(c) } G_c(f) = \frac{(2A\alpha)^2}{[\alpha^2 + (2\pi f)^2]^2}; \text{ (d) } G_d(f) = \frac{(4A\pi f)^2}{[\alpha^2 + (2\pi f)^2]^2}$$



Energy spectral densities: solid - G_a and G_b; dotted - G_c; dot/dash - G_d.

Problem 4-12

(a) See Problem 4-11 for $G_a(f)$. The energy for $|f| \le \alpha/\pi$ Hz is

$$E[|f| \le \alpha/\pi] = 2\int_0^{\alpha/\pi} \frac{A^2}{\alpha^2 + (2\pi f)^2} df = \frac{A^2}{\alpha\pi}\int_0^2 \frac{dx}{1+x^2} = \frac{A^2}{\alpha\pi}\tan^{-1}(2)$$

The total energy is $A^2/2\alpha$, so the fraction of total energy is

$$\text{Fraction of total} = \frac{2}{\pi}\tan^{-1}(2) = 70.5\%$$

(b) For this case,

$$E[|f| \le \alpha/2\pi] = 2\int_0^{\alpha/2\pi} \frac{A^2}{\alpha^2 + (2\pi f)^2} df = \frac{A^2}{\alpha\pi}\int_0^1 \frac{dx}{1+x^2} = \frac{A^2}{\alpha\pi}\tan^{-1}(1) = \frac{A^2}{4\alpha}$$

The fraction of total power is $(2/\pi)\tan^{-1}(1) = 50\%$.

Problem 4-13

(a) Note that

$$\lim_{\alpha \to 0} [x_d(t)] = u(t) - u(-t) = \text{sgn}(t)$$

for $A = 1$. Thus

$$\lim_{\alpha \to 0} [X_d(f)] = \mathscr{F}[\text{sgn}(t)] = j\frac{4\pi f}{4\pi^2 f^2} = \frac{j}{\pi f}$$

(b) Using superposition, the result of (a), and $\mathscr{F}[\delta(t)] = 1$, we obtain

$$\mathscr{F}[u(t)] = \mathscr{F}\left[\frac{1}{2}[\text{sgn}(t) + 1]\right] = \frac{j}{2\pi f} + \frac{1}{2}\delta(t)$$

Problem 4-14

Using the Fourier transform pair found in Problem 1c and the modulation theorem, we obtain

$$\mathscr{F}\left[e^{-a|t|}\cos(2\pi f_0 t)\right] = \frac{a}{a^2 + [2\pi(f - f_0)]^2} + \frac{a}{a^2 + [2\pi(f + f_0)]^2}$$

Consider

$$U(f) = \frac{a}{a^2 + [2\pi(f \pm f_0)]^2}$$

For $f = \pm f_0$, $\lim_{a \to 0} U(f) = \lim_{a \to 0} 1/a = \infty$. For $f \neq \pm f_0$, $\lim_{a \to 0} U(f) = \lim_{a \to 0} U(f) = 0$. Also note that

$$\int_{-\infty}^{\infty} \frac{a}{a^2 + [2\pi(f \pm f_0)]^2} df = \int_{-\infty}^{\infty} \frac{a}{a^2 + u^2}\frac{du}{2\pi} = \frac{1}{2\pi}\tan^{-1}(u/a)\Big|_{-\infty}^{\infty} = \frac{1}{2}$$

Thus, $U(f)$ has the properties of a delta function of area 1/2, centered at $\pm f_0$, as $a \to \infty$. We therefore conclude that

$$\mathscr{F}\left[e^{-a|t|}\cos(2\pi f_0 t)\right] = 0.5\,\delta(f - f_0) + 0.5\,\delta(f + f_0)$$

Problem 4-15

Note that

$$\tau^{-1}\Pi(t/\tau) \to \begin{cases} \infty, & t = 0,\ \tau \to 0 \\ 0, & t \neq 0,\ \tau \to 0 \end{cases}$$

Also, the area under this function is 1 no matter what the value of τ. In the limit, therefore, it is a unit impulse. Is Fourier transform is $\mathrm{sinc}(f\tau)$ which has the limit of 1 as $\tau \to 0$. This demonstrates that the Fourier transform of an impulse function is a constant equal to the area of the impulse function.

Problem 4-16

(a) The signal may be written as

$$x_a(t) = \Pi(t - 1.5) + \Pi(t + 1.5)$$

Using superposition, time delay, and the Fourier transform pair given, we get

$$X_a(f) = \mathrm{sinc}(f)\,e^{-j2\pi f(1.5)} + \mathrm{sinc}(f)\,e^{j2\pi f(1.5)} = 2\mathrm{sinc}(f)\cos(3\pi f)$$

The second signal may be written as

$$x_b(t) = \Pi(t - 1.5) - \Pi(t + 1.5)$$

Thus, its Fourier transform is

$$X_a(f) = \mathrm{sinc}(f)\,e^{-j2\pi f(1.5)} - \mathrm{sinc}(f)\,e^{j2\pi f(1.5)} = -2j\,\mathrm{sinc}(f)\sin(3\pi f)$$

The third signal may be written as

$$x_c(t) = \Pi(t/4) + \Pi(t/2)$$

Using superposition and the transform pair given, its Fourier transform is

$$X_c(f) = 4\,\mathrm{sinc}(4f) + 2\,\mathrm{sinc}(2f)$$

The fourth signal may be written as

$$x_d(t) = \Pi[(t - 1.5)/3] - \Pi(t - 1.5)$$

Its Fourier transform is

$$X_d(f) = [3\,\mathrm{sinc}(3f) + \mathrm{sinc}(f)]\,e^{-j3\pi f}$$

(b) $x_a(t)$ is even which implies that $X_a(f)$ is real and even;
$x_b(t)$ is odd which implies that $X_b(f)$ is imaginary and odd;
$x_c(t)$ is even which implies that $X_c(f)$ is real and even;
$x_d(t)$ is neither which says that its Fourier transform is complex and has no special symmetry properties.

Problem 4-17

The amplitude spectrum of the recorded signal is

$$|X(f)| = A\,\Pi\left(\frac{f - 5000}{9900}\right)$$

It is played back at half the speed it is recorded. Thus, the playback time is twice the record time, and $a = 0.5$ in the scale change theorem. Thus, the playback spectrum is

$$|X(f)| = \frac{A}{0.5}\Pi\left(\frac{f/0.5 - 5000}{9900}\right) = 2A\,\Pi\left(\frac{f - 2525}{4950}\right)$$

The spectrum of the playback signal is half as wide as that of the record signal.

Problem 4-18

From the solution to Problem 4-16(c)

$$X_c(f) = 4\,\mathrm{sinc}(4f) + 2\,\mathrm{sinc}(2f)$$

Using the duality theorem, we have

$$x(t) = X_c(-t) = 4\,\mathrm{sinc}(4t) + 2\,\mathrm{sinc}(2t)$$

which results because the spectrum is even.

Problem 4-19

(a) Using the modulation theorem and the result of Problem 4-18(a), we obtain

$$y_a(t) = x_a(t)\cos(20\pi t) \leftrightarrow \text{sinc}(f-10)\cos[3\pi(f-10)] + \text{sinc}(f+10)\cos[3\pi(f+10)]$$

(b) Again using the modulation theorem and the results of Problem 4-18(b), we have

$$y_b(t) = x_b(t)\cos(20\pi t) \leftrightarrow -j\,\text{sinc}(f-10)\sin[3\pi(f-10)] + j\,\text{sinc}(f+10)\sin[3\pi(f+10)]$$

(c) The same procedure as used for parts (a) and (b) gives

$$y_c(t) = x_c(t)\cos(20\pi t) \leftrightarrow 2\,\text{sinc}[4(f-10)] + 2\,\text{sinc}[4(f+10)] + \text{sinc}[2(f-10)] + \text{sinc}[2(f+10)]$$

(d) The modulation theorem and result of Problem 4-18(d) gives

$$y_d(t) = x_d(t)\cos(20\pi t) \leftrightarrow \frac{1}{2}[3\,\text{sinc}[3(f-10)] + \text{sinc}(f-10)]e^{-j3\pi(f-10)} + \frac{1}{2}[3\,\text{sinc}[3(f+10)] + \text{sinc}(f+10)]e^{-j3\pi(f+10)}$$

Problem 4-20

(a) This signal may be expressed as

$$x_a(t) = \Lambda(t+1) + \Lambda(t-1)$$

Its second derivative is

$$\frac{dx_a^2(t)}{dt^2} = \delta(t+2) - 2\delta(t+1) + 2\delta(t) - 2\delta(t-1) + \delta(t-2)$$

Now use superposition, delay, and the transform pair given in the problem statement to obtain

$$\mathcal{F}\left\{\frac{dx_a^2(t)}{dt^2}\right\} = (j2\pi f)^2 X_a(f) = e^{j4\pi f} - 2e^{j2\pi f} + 2 - 2e^{-j2\pi f} + e^{-j4\pi f} = -8\sin^2(\pi f)\cos(2\pi f)$$

Thus

$$X_a(f) = -\frac{8\sin^2(\pi f)\cos(2\pi f)}{-4\pi^2 f^2} = 2\,\mathrm{sinc}^2(f)\cos(2\pi f)$$

(b) This signal may be written as

$$x_b(t) = -\Lambda(t+1) + \Lambda(t-1)$$

Its second derivative is

$$\mathscr{F}\left\{\frac{dx_b^2(t)}{dt^2}\right\} = (j2\pi f)^2 X_b(f) = -e^{j4\pi f} + 2e^{j2\pi f} - 2e^{-j2\pi f} + e^{-j4\pi f} = -2j\sin(4\pi f) + 4j\sin(2\pi f)$$

The Fourier transform of $x_b(t)$ is therefore

$$X_b(f) = -\frac{j}{\pi f}[2\,\mathrm{sinc}(2f) - 2\,\mathrm{sinc}(4f)]$$

Clearly, the first transform is real and even, as it should be, because of the eveness of the signal. The second is imaginary and odd, as it should be, because of the oddness of the signal.

Problem 4-21

The Fourier transform of $x_0(t)$ may be obtained by direct evaluation or using the differentiation theorem. From a sketch, its first derivative is

$$\frac{dx_0(t)}{dt} = -\Pi(t - 1/2)$$

Thus, by the differentiation theorem,

$$\mathscr{F}[dx_0(t)/dt] = j2\pi f X_0(f) = -\mathrm{sinc}(f)\, e^{-j\pi f}$$

so

$$X_0(f) = -\frac{\mathrm{sinc}(f)}{j2\pi f} e^{-j\pi f}$$

(a) This signal is obviously $\Lambda(t)$, but in keeping with the problem statement, we note that it may be written as

$$x_1(t) = x_0(t) + x_0(-t)$$

By the time reversal theorem and superposition, we have

$$X_1(f) = X_0(f) + X^*(f) = -\frac{\text{sinc}(f)}{j2\pi f} e^{-j\pi f} + \frac{\text{sinc}(f)}{j2\pi f} e^{j\pi f} = \frac{\text{sinc}(f)}{j\pi f} \frac{e^{j\pi f} - e^{-j\pi f}}{j2} = \text{sinc}^2(f)$$

(b) This signal may be written as

$$x_2(t) = x_0(t) - x_0(-t)$$

Its Fourier transform is

$$\begin{aligned} X_2(f) &= X_0(f) - X^*(f) = -\frac{\text{sinc}(f)}{j2\pi f} e^{-j\pi f} - \frac{\text{sinc}(f)}{j2\pi f} e^{j\pi f} \\ &= -\frac{\text{sinc}(f)}{j\pi f} \frac{e^{j\pi f} + e^{-j\pi f}}{2} = -\frac{\text{sinc}(f)}{j\pi f} \cos(\pi f) \end{aligned}$$

(c) This signal is

$$x_3(t) = \Pi(t/2) - \Lambda(t)$$

Using the Fourier transform of the square pulse and the triangle together with superposition, we have

$$X_3(f) = 2\,\text{sinc}(2f) - \text{sinc}^2(f)$$

(d) This signal may be written as

$$x_4(t) = x_0[-(t+1)] + x_0(-t) + x_0[-(t-1)] + x_0[-(t-2)]$$

Thus, the Fourier transform, using time reversal, time delay, and superposition, is

$$X_4(f) = \frac{\text{sinc}(f)}{j2\pi f} e^{j3\pi f} + \frac{\text{sinc}(f)}{j2\pi f} e^{j\pi f} + \frac{\text{sinc}(f)}{j2\pi f} e^{-j\pi f} + \frac{\text{sinc}(f)}{j2\pi f} e^{-j3\pi f}$$

Problem 4-22

Let

$$X_1(f) = \frac{1}{3 + j2\pi f} \leftrightarrow e^{-3t}u(t) = x_1(t) \text{ and } X_2(f) = 10\,\mathrm{sinc}(2f) \leftrightarrow 5\Pi(t/2) = x_2(t)$$

Thus, by the convolution theorem

$$x(t) = \mathscr{F}^{-1}[X_1(f)X_2(f)] = \int_{-\infty}^{\infty} 5\Pi(\lambda/2)e^{-3(t-\lambda)}u(t-\lambda)\,d\lambda = \begin{cases} 0, & t < -1 \\ \frac{5}{3}\left[1 - e^{-3(t+1)}\right], & |t| \le 1 \\ \frac{5}{3}\left[e^{3} - e^{-3}\right]e^{-3t}, & t > 1 \end{cases}$$

Problem 4-23

(a) To prove that

$$x_1(t)x_2(t) \leftrightarrow X_1(f)X_2(f)$$

Write out the Fourier transform integral for the product:

$$\mathscr{F}[x_1(t)x_2(t)] = \int_{-\infty}^{\infty} x_1(t)x_2(t)e^{-j2\pi ft}dt$$

Represent $x_1(t)$ by its inverse Fourier transform:

$$x_1(t) = \mathscr{F}^{-1}[X_1(f)] = \int_{-\infty}^{\infty} X_1(\lambda)e^{-j2\pi f\lambda}d\lambda$$

Note the use of λ for the variable of integration so that we won't get confused with f when substituted into the Fourier transform integral. This substitution gives

$$\mathscr{F}[x_1(t)x_2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1(\lambda)x_2(t)e^{-j2\pi ft}e^{j2\pi\lambda t}d\lambda\,dt$$

$$= \int_{-\infty}^{\infty} X_1(\lambda)\left\{\int_{-\infty}^{\infty} x_2(t)e^{j2\pi(\lambda - f)t}dt\right\}d\lambda = \int_{-\infty}^{\infty} X_1(\lambda)X_2(\lambda - f)\,d\lambda$$

(b) Use the Fourier transform pair

$$2W\,\mathrm{sinc}(2Wt) \leftrightarrow \Pi(f/2W)$$

Note that

$$\Pi\left(\frac{f}{2W}\right) * \Pi\left(\frac{f}{2W}\right) = 2AW\Lambda\left(\frac{f}{2W}\right)$$

In the case at hand, $2W = 3$ and $A = 5$. Therefore,

$$x(t) = \mathscr{F}^{-1}[15\,\Lambda(f/3)] = 5[3\mathrm{sinc}(3t)][3\mathrm{sinc}(3t)] = 45\,\mathrm{sinc}^2(3t)$$

Problem 4-24

(a) For the rectangular pulse, the energy is $E_R = A^2\tau$. For the half-cosine pulse, it is

$$E_{HC} = 2\int_0^{\tau/2} C^2\cos^2\left(\frac{\pi t}{\tau}\right)dt = \frac{1}{2}C^2\tau$$

For the raised cosine pulse, the energy is

$$E_{RC} = 2\int_0^{\tau/2} B^2[1 + \cos(2\pi t/\tau)]^2 dt = \frac{3}{2}B^2\tau$$

(b) Using the results of (a), we have

$$A^2\tau = \frac{3}{2}B^2\tau = \frac{1}{2}C^2\tau$$

Therefore, choose the amplitudes according to the relationships

$$B = \sqrt{\frac{2}{3}}A \text{ and } C = \sqrt{2}A$$

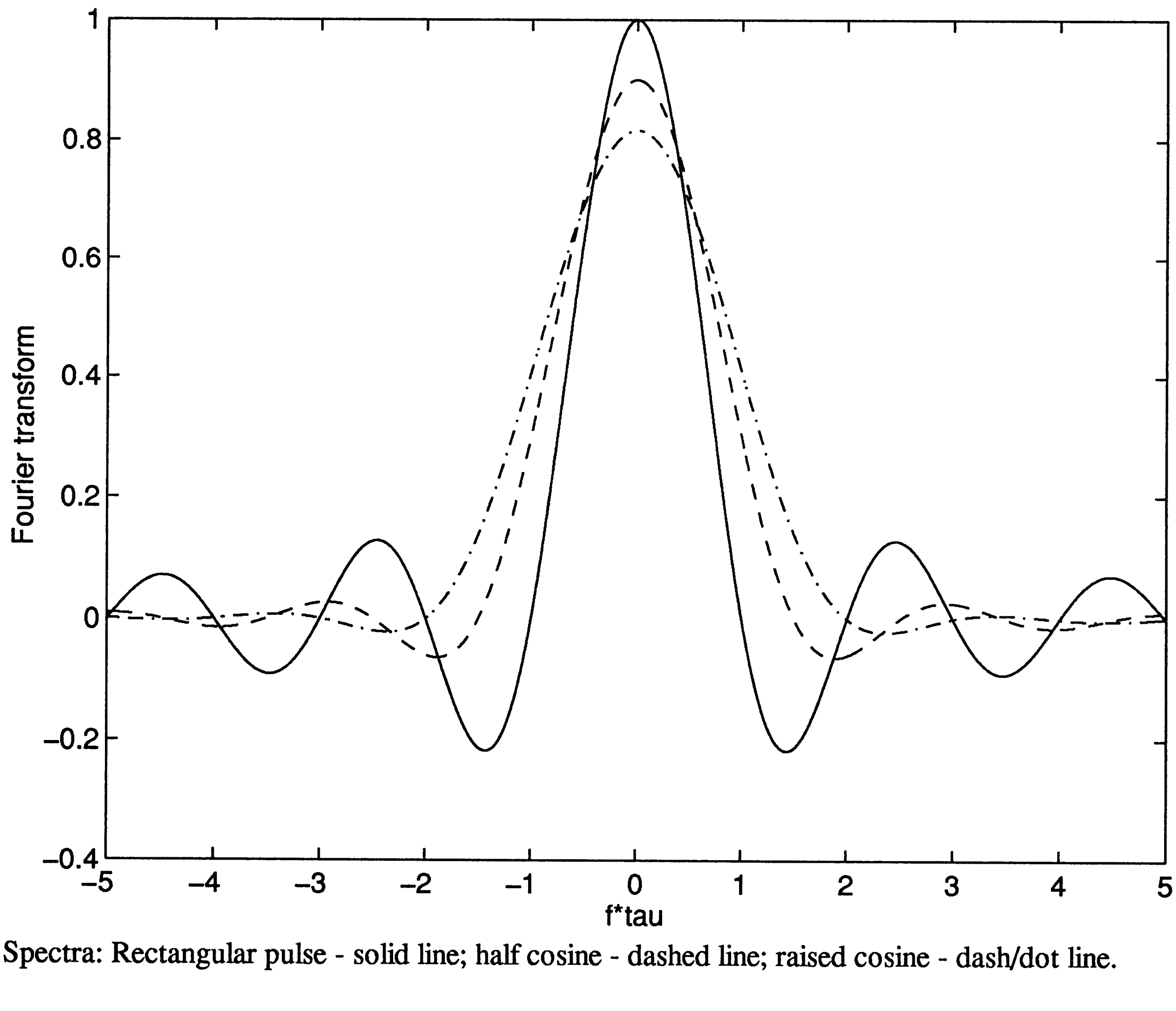
Express all three pulses in terms of A. Use the Fourier transform of a square pulse and the modulation theorem to finally get

$$X_R(f) = A\tau\,\mathrm{sinc}(f\tau)$$

$$X_{RC}(f) = \sqrt{\frac{2}{3}}A\tau\left[\mathrm{sinc}(f\tau) + \frac{1}{2}\mathrm{sinc}(f\tau - 1) + \frac{1}{2}\mathrm{sinc}(f\tau + 1)\right]$$

$$X_{HC}(f) = \frac{A\tau}{\sqrt{2}}[\mathrm{sinc}(f\tau - 0.5) + \mathrm{sinc}(f\tau + 0.5)]$$

Their spectra are compared below. Note that since all three are real, there is no need to plot both amplitude and phase spectra.



Spectra: Rectangular pulse - solid line; half cosine - dashed line; raised cosine - dash/dot line.

Problem 4-25

(a) Use the Fourier transform pairs

$$x_1(t) \Leftrightarrow \frac{1}{(\alpha + j2\pi f)^2} \quad \text{(see Prob. 4-6a)}$$

$$\text{and } h_1(t) \Leftrightarrow \frac{1}{\beta + j2\pi f}$$

Thus, the Fourier transform of the convolution of the two signals is

$$x_1(t) * h_1(t) = \mathcal{F}^{-1}\left[\frac{1}{(\alpha + j2\pi f)^2}\frac{1}{\beta + j2\pi f}\right]$$

Expand in partial fractions to get

$$x_1(t) * h_1(t) = \mathcal{F}^{-1}\left\{\frac{1}{(\beta - \alpha)^2}\left[\frac{1}{\beta + j2\pi f} - \frac{1}{\alpha + j2\pi f} + \frac{\beta - \alpha}{(\alpha + j2\pi f)^2}\right]\right\}$$

$$= \frac{1}{\beta - \alpha}\left[\frac{1}{\beta - \alpha}e^{-\beta t} - \left(\frac{1}{\beta - \alpha} - t\right)e^{-\alpha t}\right]u(t)$$

(b) For this case, just interchange α and β.

(c) In this case

$$x_3(t) * h_3(t) = \mathcal{F}^{-1}\left[\frac{1}{\alpha + j2\pi f}\frac{1}{-\beta + j2\pi f}\right]$$

$$= \frac{1}{\alpha - \beta}\left[\frac{1}{\alpha + j2\pi f} + \frac{1}{-\beta + j2\pi f}\right] = \frac{1}{\alpha - \beta}\left[e^{-\alpha t}u(t) + e^{\beta t}u(-t)\right]$$

Problem 4-26

(a) Using ac sinusoidal steadystate circuit analysis and voltage division, the transfer function is

$$H_{RC}(j\omega) = \frac{R}{R + 1/j\omega C} = \frac{j(f/f_3)}{1 + j(f/f_3)} \text{ where } f_3 = \frac{1}{2\pi RC}$$

(b) If

$$y(t) = \frac{dx(t)}{dt}$$

then

$$Y(j\omega) = j\omega X(j\omega)$$

and

$$H_{\text{diff}}(j\omega) = j\omega = j2\pi f$$

We want the high-pass RC filter transfer function of part (a) to approximate this. If

$$|j\omega RC| \ll 1 \text{ or } |\omega| \ll \frac{1}{RC}$$

then the denominator of the RC filter transfer function is approximately 1, so

$$H_{RC}(j\omega) = j\omega RC$$

and we have the desired approximate differentiator transfer function.

Problem 4-27

Use trigonometric identities to write

$$x(t) = 2\,\text{sinc}(2t)[1 + \cos(8\pi t)]$$

Use the transform pair $2W\,\text{sinc}(2Wt) \leftrightarrow \Pi(f/2W)$ with the modulation theorem to get

$$x(t) \leftrightarrow \Pi(f/2) + 0.5\Pi[(f-4)/2] + 0.5\Pi[(f+4)/2]$$

(a) Pass the signal through a lowpass filter with gain of 2 to eliminate the frequency components centered at $f = \pm 4$ Hz so that the output spectrum is $2\Pi(f/2)$. Thus the output signal will be $y(t) = 4\,\mathrm{sinc}(2t)$.
(b) Pass the signal through a bandpass filter with gain 3 so that the output spectrum is $1.5\Pi[(f - 4)/2] + 1.5\Pi[(f + 4)/2]$ so that the output signal is $3\,\mathrm{sinc}(2t)\cos(8\pi t)$.

Problem 4-28

The integrator output and spectrum is

$$y(t) = A\int_{-\infty}^{t} x(\lambda)\,d\lambda \leftrightarrow A\left[\frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\,\delta(f)\right] = \frac{A}{j2\pi f}X(f) \text{ if } X(0) = 0$$

For the lowpass RC filter,

$$H_{RC}(f) = \frac{1}{1 + j2\pi fRC}$$

If $|j2\pi fRC| >> 1$, then

$$H_{RC}(f) \approx \frac{1}{j2\pi fRC}$$

and the RC lowpass filter acts like an ideal integrator.

Problem 4-29

(a) The output is

$$\begin{aligned} y(t) &= \cos\left(2\pi\times 10t - \frac{10}{60}\frac{\pi}{2}\right) + \cos\left(2\pi\times 30t - \frac{30}{60}\frac{\pi}{2}\right) \\ &= \cos\left[20\pi\left(t - \frac{1}{240}\right)\right] + \cos\left[60\pi\left(t - \frac{1}{240}\right)\right] \end{aligned}$$

Since both amplitudes come out of the system in the same ratio as they went in, and since the delays of the two components are the same at the output, there is no distortion.
(b) For this case, the output is

$$y(t) = \cos\left(2\pi\times10t - \frac{10}{60}\frac{\pi}{2}\right) + \cos\left(2\pi\times70t - \frac{\pi}{2}\right)$$

$$= \cos\left[20\pi\left(t - \frac{1}{240}\right)\right] + \cos\left[140\pi\left(t - \frac{1}{280}\right)\right]$$

There is no amplitude distortion, but there is phase (delay) distortion because the delays of the two components differ.

(c) The output is

$$y(t) = \cos\left(2\pi\times10t - \frac{10}{60}\frac{\pi}{2}\right) + 2\cos\left(2\pi\times110t - \frac{\pi}{2}\right)$$

$$= \cos\left[20\pi\left(t - \frac{1}{240}\right)\right] + 2\cos\left[220\pi\left(t - \frac{1}{440}\right)\right]$$

There is both amplitude and phase (delay) distortion in this case.

(d) The output in this case is

$$y(t) = \cos\left(2\pi\times65t - \frac{\pi}{2}\right) + 2\cos\left(2\pi\times110t - \frac{\pi}{2}\right)$$

$$= \cos\left[130\pi\left(t - \frac{1}{260}\right)\right] + 2\cos\left[220\pi\left(t - \frac{1}{440}\right)\right]$$

In this case, there is clearly both amplitude and phase distortion.

Problem 4-30

(a) For the input

$$x(t) = A\Pi\left(\frac{t - \tau/2}{\tau}\right)$$

the energy spectrum is

$$G_x(f) = |X(f)|^2 = (A\tau)^2 \operatorname{sinc}^2(f\tau)$$

The total energy in the pulse is $A^2\tau$, and the fraction of the total energy in the main lobe is

$$E_{x,\,\mathrm{ML}} = \frac{1}{A^2\tau}\int_{-1/\tau}^{1/\tau} (A\tau)^2 \operatorname{sinc}^2(f\tau)\,df = 2\int_0^1 \operatorname{sinc}^2(u)\,du = 90.28\% \text{ by numerical integration}$$

(b) The input signal is

$$x_2(t) = A\Pi\left(\frac{t - \tau/2}{\tau}\right) + A\Pi\left(\frac{t - T - \tau/2}{\tau}\right)$$

The spectrum of this two-pulse sequence is

$$X_2(f) = A\tau \operatorname{sinc}(f\tau)\left[e^{-j\pi f\tau} + e^{-j2\pi f(T + \tau/2)}\right] = 2A\tau \operatorname{sinc}(f\tau)\cos(\pi f\tau)\,e^{-j\pi f(T+\tau)}$$

The energy spectrum of this signal is

$$G_{X_2}(f) = 4(A\tau)^2 \operatorname{sinc}^2(f\tau)\cos^2(\pi f\tau)$$

A plot of this energy spectrum is provided at the end of the problem.
(c) For the N-pulse signal given by

$$x_N(t) = \sum_{n=0}^{N-1} A\Pi\left(\frac{t - nT - \tau/2}{\tau}\right)$$

we obtain the Fourier transform

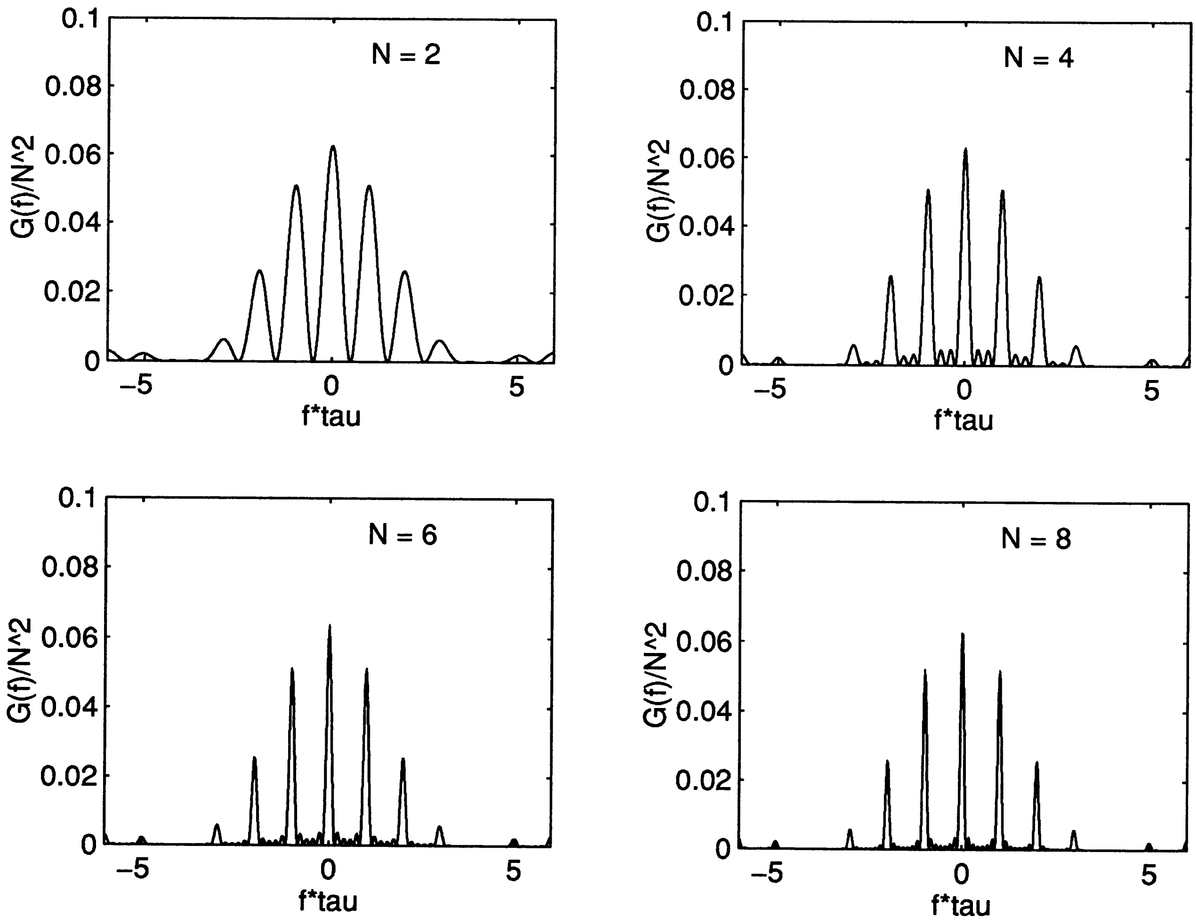
$$X_N(f) = \sum_{n=0}^{N-1} A\tau\,\mathrm{sinc}(f\tau)\,e^{-j2\pi(nT+\tau/2)f} = A\tau\,\mathrm{sinc}(f\tau)\sum_{n=0}^{N-1}\left(e^{-j2\pi fT}\right)^n e^{-j\pi f\tau}$$

$$= A\tau\,\mathrm{sinc}(f\tau)\,\frac{\sin(\pi fNT)}{\sin(\pi fT)}\,e^{-j\pi f[(N-1)T+\tau]}$$

The energy spectral density for this signal is

$$G_N(f) = (A\tau)^2\mathrm{sinc}^2(f\tau)\,\frac{\sin^2(\pi fNT)}{\sin^2(\pi fT)}$$

Plots are provided below for $G_N(f)$. It is apparent that as N increases the comb filter is able to do a better job of filtering the signal for background noise.



Problem 4-31

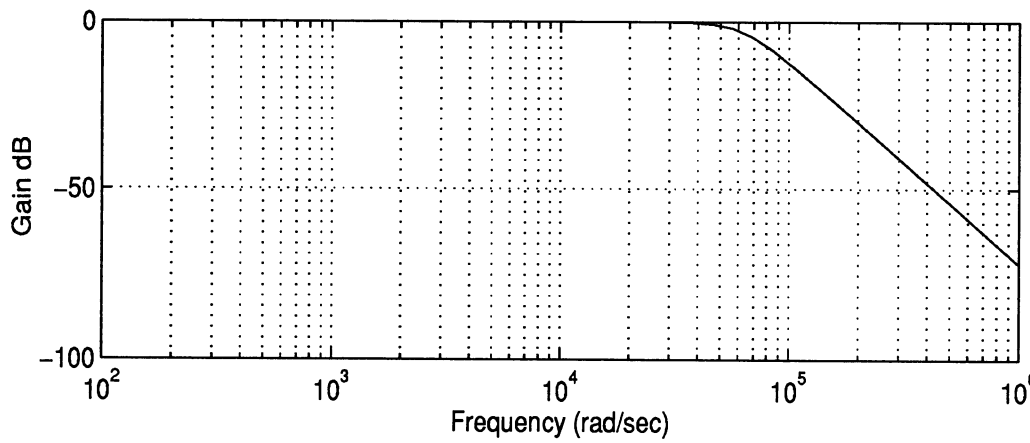
Using sinusoidal steadystate analysis, the transfer function may be shown to be

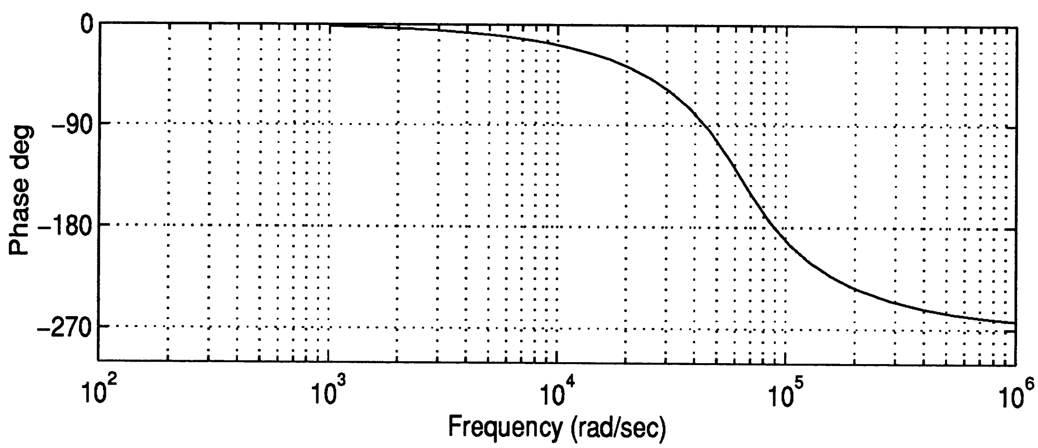
$$H(j\omega) = \frac{1}{-j\omega^3 C_1 RLC_2 - \omega^2 LC_2 + j\omega R(C_1 + C_2) + 1}$$

When the values given in the problem statement are substituted, we obtain

$$H(j\omega) = \frac{1}{1 + 2\left(\frac{j\omega}{2\pi\times10^4}\right) + 2\left(\frac{j\omega}{2\pi\times10^4}\right)^2 + \left(\frac{j\omega}{2\pi\times10^4}\right)^3}$$

This is a third-order Butterworth filter frequency response function with 3-dB cutoff frequency of 10 kHz. The amplitude and phase response functions are plotted below.





Problem 4-32

(a) Use the transform pair

$$2W\,\text{sinc}(2Wt) \leftrightarrow \Pi(f/2W)$$

Also write

$$\cos(\pi f) = 0.5e^{j\pi f} + 0.5e^{-j\pi f}$$

and use the time delay theorem to get

$$x(t) = 0.5[\text{sinc}(t - 0.5) + \text{sinc}(t + 0.5)]$$

(b) Using the modulation theorem along with the result obtained in part (a), we obtain

$$y(t) = 2x(t)\cos(2\pi f_0 t) = [\text{sinc}(t - 0.5) + \text{sinc}(t + 0.5)]\cos(2\pi f_0 t)$$

(c) Obviously $f_2 = f_0$, $A = 2$, and $f_1 \geq f_0 + 1$.

Problem 4-33

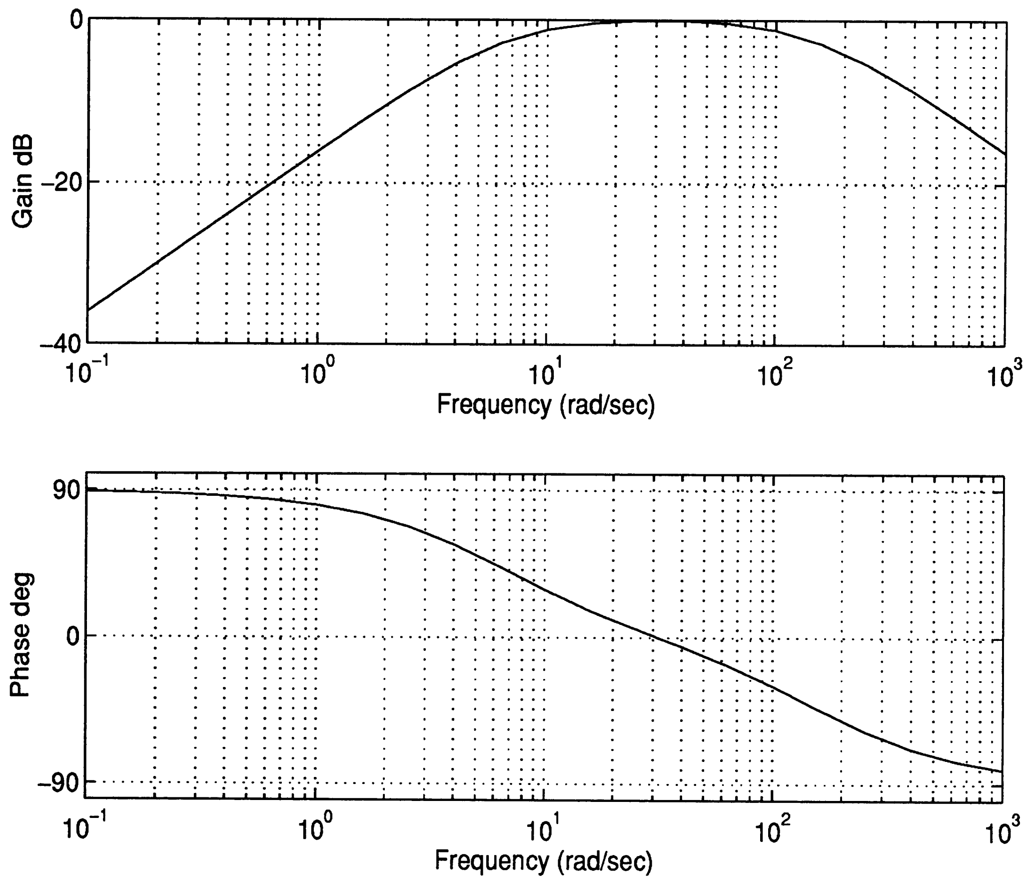
(a) Using the admittance of the parallel combinations, we have

$$H(j\omega) = \frac{1}{Y_{adm}(j\omega)} = \frac{1}{\frac{1}{j\omega L} + \frac{1}{R} + j\omega C} = R\frac{j\omega\frac{L}{R}}{1 - \omega^2 LC + \frac{L}{R}j\omega}$$

Further rearrangement after substituting $f = \omega/2\pi$ gives

$$H(f) = R\frac{j(f/f_b)}{1 + j(f/f_b) - (f/f_0)^2} \text{ where } f_0 = \frac{1}{2\pi\sqrt{LC}} \text{ and } f_b = \frac{R}{2\pi L}$$

The amplitude and phase response functions are shown below for $f_0 = 5$ Hz and $f_b = 1$ Hz.



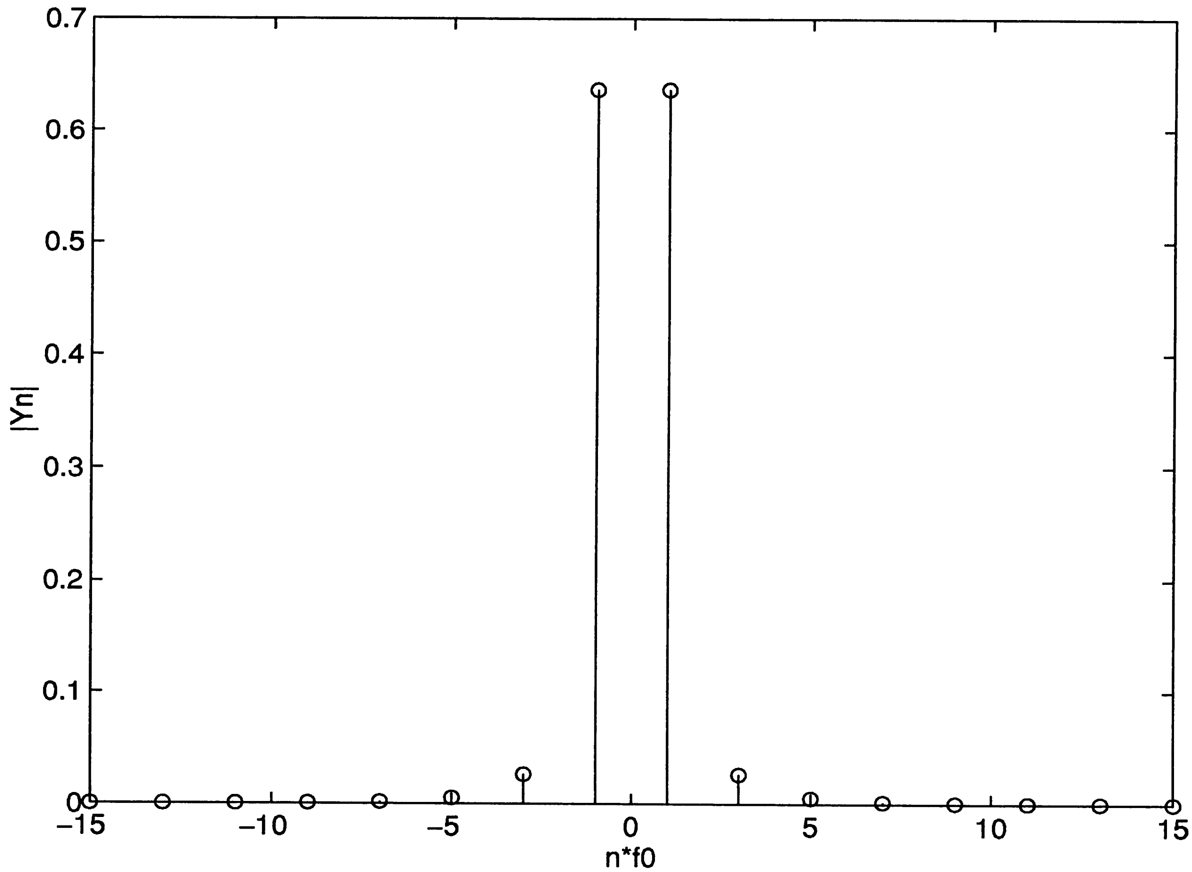
(b) For the square wave input,

$$X_n = \begin{cases} \dfrac{2A}{|n|\pi}, & n = \pm 1, \pm 5, \ldots \\ -\dfrac{2A}{|n|\pi}, & n = \pm 3, \pm 7, \ldots \end{cases}$$

The output spectrum of the filter with this input is

$$Y(f) = \sum_{n=-\infty}^{\infty} X_n H(nf_0)\delta(f - nf_0) = \sum_{n=-\infty}^{\infty} Y_n \delta(f - nf_0)$$

where $f_0 = 1/T$. The amplitude spectrum of the output is shown below for $f_0 = f_b$.



(c) The maginitude of the nth spectral line of the output is

$$|Y_n| = \frac{\left(\frac{2A}{|n|\pi}\right)\left(\frac{nf_0}{f_b}\right)}{\sqrt{\left[1 - \left(\frac{nf_0}{f_b}\right)^2\right]^2 + \left(\frac{nf_0}{f_b}\right)^2}}, \; n \text{ odd}$$

The power in the nth harmonic is proportional to the square of the magnitude of the corresponding line. Thus, the ration of the powers in the fundamental component to that in the nth harmonic is

$$\frac{P_{\text{fund}}}{P_{n\text{th har}}} = \frac{\left[1 - \left(\frac{nf_0}{f_b}\right)^2\right]^2 + \left(\frac{nf_0}{f_b}\right)^2}{\left[1 - \left(\frac{f_0}{f_b}\right)^2\right]^2 + \left(\frac{f_0}{f_b}\right)^2}$$

If $f_0 = f_b$, this simplifies to

$$\frac{P_{\text{fund}}}{P_{n\text{th har}}} = (1 - n^2)^2 + n^2$$

(d) We solve the above equation for f_b with $n = 3$ and the left-hand side equal to $10^3 = 1000$:

$$1000 = \frac{\left[1 - \left(\frac{3f_0}{f_b}\right)^2\right]^2 + \left(\frac{3f_0}{f_b}\right)^2}{\left[1 - \left(\frac{f_0}{f_b}\right)^2\right]^2 + \left(\frac{f_0}{f_b}\right)^2}$$

The solution is $f_0/f_b = 1.92$. To get the 3rd harmonic 40 dB down (left-hand side of the above equation equals 10,000), this ratio needs to be 3.36.

Problem 4-34

This follows easily if the Fourier coefficients of the input and the transfer function are expressed in polar form in (3-68).

Problem 4-35

The impulse response is the inverse Fourier of the transfer function:

$$h_{HP}(t) = \mathcal{F}^{-1}\left[H_0 e^{-j2\pi f t_0} - H_0 \Pi(f/2B)\, e^{-j2\pi f t_0}\right]$$

Use superposition, time delay, and the transform pair

$$\text{sinc}(2Bt) \leftrightarrow \frac{1}{2B}\Pi\left(\frac{f}{2B}\right)$$

to get

$$h_{HP}(t) = H_0 \delta(t - t_0) - 2BH_0 \text{sinc}[2B(t - t_0)]$$

Problem 4-36

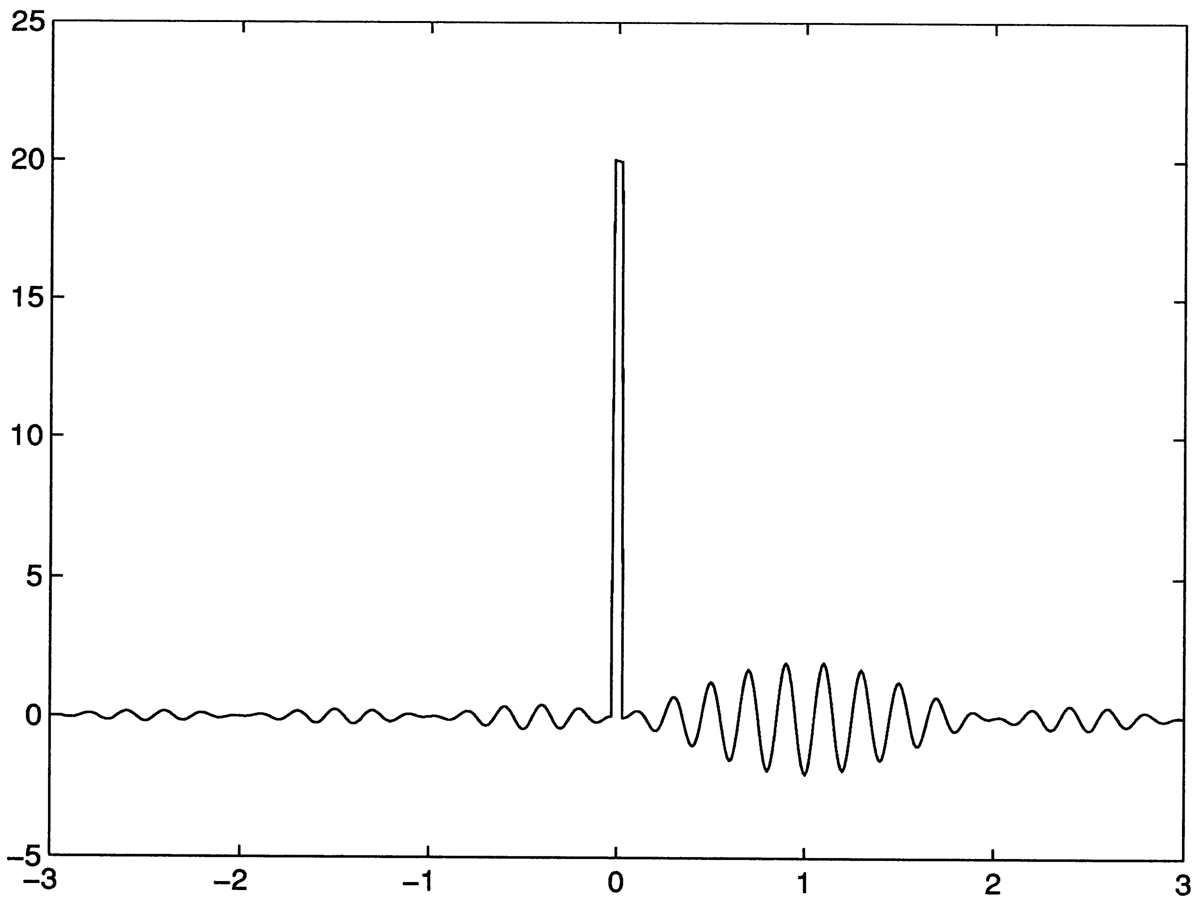
Write the transfer function as

$$H(f) = H_0\left[1 - \Pi\left(\frac{f - f_0}{B}\right) - \Pi\left(\frac{f + f_0}{B}\right)\right]e^{-j2\pi f_0}$$

Use the transform pair of the previous problem along with the superposition and modulation theorems to get

$$h_{HP}(t) = H_0\delta(t - t_0) - 2BH_0\operatorname{sinc}[2B(t - t_0)]\cos[2\pi f_0(t - t_0)]$$

A plot is provided below with $B = H_0 = t_0 = 1$ and $f_0 = 5$ (rectangular pulse of height 20 represents an impulse).



Problem 4-37

Generalize the filter transfer function to 3-dB cutoff frequency ω_c and expand in partial fractions.

$$H(\omega) = \frac{\omega_c^3}{(j\omega)^3 + 2\omega_c(j\omega)^2 + 2\omega_c^2(j\omega) + \omega_c^3}$$
$$= \frac{\omega_c}{j\omega + \omega_c} - \frac{\omega_c(1 - j/\sqrt{3})/2}{j\omega + \omega_c(1 + j\sqrt{3})/2} - \frac{\omega_c(1 + j/\sqrt{3})/2}{j\omega + \omega_c(1 - j\sqrt{3})/2}$$

Inverse Fourier transform using the transform pair

$$e^{-\alpha t}u(t) \leftrightarrow \frac{1}{\alpha + j\omega}$$

to obtain the impulse response

$$h(t) = \omega_c e^{-\alpha t} - 2\mathrm{Re}\left[\omega_c(1 - j/\sqrt{3})/2\, e^{-\omega_c(1 - j/\sqrt{3})t/2}\right]$$
$$= \omega_c e^{-\omega_c t} - \omega_c e^{-0.5\omega_c t}\left[\cos(\sqrt{3}\omega_c t/2) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}\omega_c t/2)\right],\ t \ge 0$$

The step response is found by integrating the impulse response:

$$a(t) = \int_{-\infty}^{t} h(\lambda)\,d\lambda = 1 - e^{-\omega_c t} - \frac{2}{\sqrt{3}}e^{-\omega_c t/2}\sin\left(\frac{\sqrt{3}}{2}\omega_c t\right),\ t \ge 0$$

Now set the step response equal to 0.1 and 0.9, respectively, and solve for the corresponding values of t. The difference is the 10% to 90% rise time. The results are

$$\omega_c t|_{10\%} = 1.006 \text{ and } \omega_c t|_{90\%} = 3.296$$

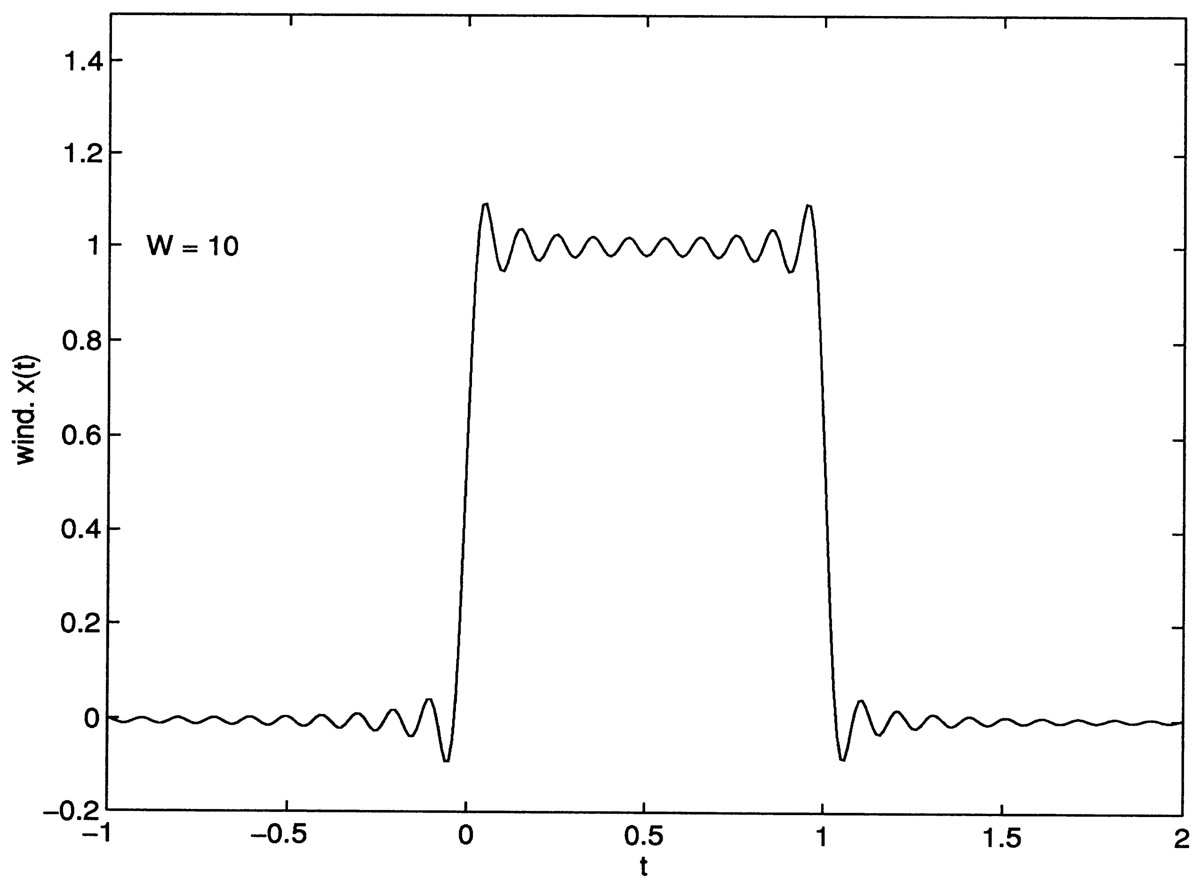
Thus, the rise time is

$$T_R = \frac{2.29}{\omega_c} = \frac{0.36}{f_{3\text{-dB}}}$$

Problem 4-38

A MATLAB program is given below for performing the plot:

```
%      Plot for Problem 4-38
%
t_max = 2;
t = -t_max+1:.01:t_max;
L = length(t);
W = 10;
tp = [2*t(1):.01:2*t(L)];
w_r = 2*W*sinc(2*W*t);
x = pls_fn(t-.5);
x_tilde_r = .01*conv(w_r, x);                    % Convolve rectangular window with pulse
plot(tp, x_tilde_r,'-w'),xlabel('t'),ylabel('wind. x(t)'),...
        text(-.9,1, ['W = ',num2str(W)]),axis([-t_max+1 t_max -.2 1.5])
```



Problem 4-39

(a) We may write this waveform as

$$x_a(t) = 2\Lambda(t/3) * \sum_{n=-\infty}^{\infty} \delta(t - 6n)$$

The Fourier transform of this signal, using (4-111), is

$$X_a(f) = \mathscr{F}[2\Lambda(t/3)] \times \frac{1}{6}\sum_{n=-\infty}^{\infty} \delta(f - n/6) = 6\,\mathrm{sinc}^2(3f) \times \frac{1}{6}\sum_{n=-\infty}^{\infty} \delta(f - n/6)$$

$$= \sum_{n=-\infty}^{\infty} \mathrm{sinc}^2(n/2)\,\delta(f - n/6)$$

A plot is provided at the end of the problem.

(b) Write one period of this triangular waveform as

$$x_b(t) = 4\Lambda(t/6) - 2\Pi(t/12), \; -6 \le t \le 6$$

Therefore, using (4-111), the spectrum is

$$X_b(f) = [4 \times 6\,\mathrm{sinc}^2(6f) - 2 \times 12\,\mathrm{sinc}(12f)] \times \frac{1}{12}\sum_{n=-\infty}^{\infty} \delta(f - n/12)$$

$$= 2 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \mathrm{sinc}^2(n/2)\,\delta(f - n/12)$$

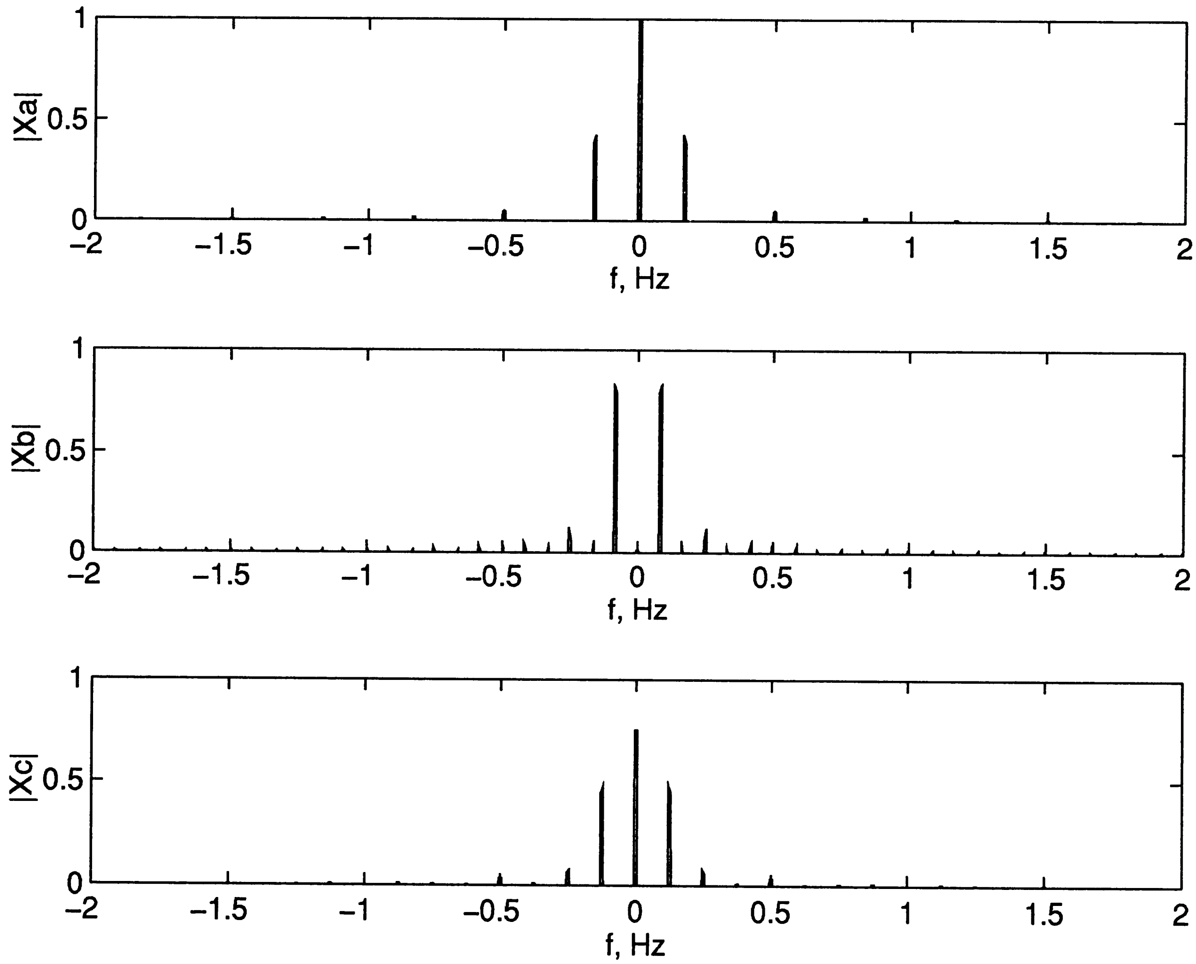
(c) Write one period of this waveform as

$$x_c(t) = 2\Lambda(t/3)$$

Again using (4-111), the spectrum is

$$X_c(f) = \frac{3}{4}\sum_{n=-\infty}^{\infty} \mathrm{sinc}^2(3n/8)\,\delta(f - n/8)$$

Plots are given below for all three cases. The delta functions are approximated as square pulses and their heights are equal to their weights.



Problem 4-40

The basic pulse may be written as

$$p(t) = \frac{A}{2}\left[1 + \cos\left(\frac{2\pi t}{\tau}\right)\right]\Pi\left(\frac{t}{\tau}\right)$$

and the waveform may be written as

$$x(t) = p(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \leftrightarrow P(f) \times \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(n - nf_0),\ f_0 = \frac{1}{T_0}$$

Using the modulation theorem

$$P(f) = \frac{A\tau}{2}\operatorname{sinc}(f\tau) + \frac{A\tau}{4}\left[\operatorname{sinc}(f\tau - 1) + \operatorname{sinc}(f\tau + 1)\right]$$

Thus, the Fourier transform of the periodic waveform, from (4-111), is

$$X(f) = (1/T_0)\sum_{n=-\infty}^{\infty}(A\tau/2)\left[\operatorname{sinc}(n\tau/T_0) + \operatorname{sinc}(n\tau/T_0 - 1) + \operatorname{sinc}(n\tau/T_0 + 1)\right]\delta(f - nf_0)$$

For example, if $\tau = T_0$, then

$$X(f) = (A/2)\sum_{n=-\infty}^{\infty}\left[\operatorname{sinc}(n) + 0.5\operatorname{sinc}(n-1) + 0.5\operatorname{sinc}(n+1)\right]\delta(f - nf_0)$$

The first sinc function is 0 except for $n = 0$, the second sinc function is 0 except for $n = 1$, and the third sinc function is 0 except for $n = -1$. Thus, the spectrum consists of three impulses: one at $f = -f_0$ of weight $A/2$, one at $f = 0$ of weight A, and one at $f = f_0$ of weight $A/2$. This what one would expect since the periodic waveform is a cosine raised up by $A/2$.

Problem 4-41

(a) The Hilbert transform is

$$\mathrm{HT}[\cos(2\pi f_0 t)] = \cos(2\pi f_0 t - \pi/2) = \sin(2\pi f_0 t)$$

(b) For this case,

$$\mathrm{HT}[\sin(2\pi f_0 t)] = \sin(2\pi f_0 t - \pi/2) = -\cos(2\pi f_0 t)$$

(c) As suggested in the problem statement, use pair 16 of Table 4-2 along with (4-119) to write

$$\mathrm{HT}[\Pi(t/\tau)] = \frac{1}{\pi t}*\Pi(t/\tau) = \int_{-\infty}^{\infty}\frac{\Pi(\lambda/\tau)}{\pi(t-\lambda)}d\lambda = \frac{1}{\pi}\int_{-\tau/2}^{\tau/2}\frac{d\lambda}{t-\lambda} = \frac{1}{\pi}\ln\left(\frac{|t-\tau/2|}{|t+\tau/2|}\right)$$

Problem 4-42

See Example 4-21. You might experiment with other types of pulse signals, such as a raised cosine.

CHAPTER 5

Problem 5-1

(a) Use superposition and the Laplace transforms of the unit step and decaying exponential to get

$$X(s) = \frac{1}{s} - \frac{1}{s+2} = \frac{2}{s(s+2)}$$

(b) Use superposition and the Laplace transform of an exponential to get

$$X(s) = \frac{1}{s+2} - \frac{1}{s+10} = \frac{8}{(s+2)(s+10)}$$

(c) Use superposition, time delay, and the Laplace transform of a unit step to get

$$X(s) = \frac{1}{s} - \frac{1}{s}e^{-10s} = \frac{1 - e^{-10s}}{s}$$

(d) Use superposition, time delay, and the Laplace transform of a unit impulse to get

$$X(s) = 1 - e^{-10s}$$

Problem 5-2

(a) Use superposition, time delay, and the Laplace transform of a unit impulse to get

$$X_1(s) = 1 - e^{-s} + e^{-2s}, \text{ all } s$$

(b) Use superposition, the Laplace transform of a unit step, and the Laplace transform of an exponential to get

$$X_2(s) = \frac{1}{s} + \frac{1}{s+3} = \frac{2s+3}{s(s+3)}, \; \text{Re}(s) > 0$$

(c) Use superposition, the Laplace transform of an exponential, and time delay:

$$X_3(s) = \frac{1}{s+4} - \frac{e^{-s}}{s+4}, \; \text{Re}(s) > -4$$

Problem 5-3

(a) Use superposition and the Laplace transforms of a step and exponential:

$$x_1(t) = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{1}{s+10}\right] = u(t) + e^{-10t}u(t) = \left(1 + e^{-10t}\right)u(t)$$

(b) Use the same approach as part (a):

$$x_2(t) = \mathcal{L}^{-1}\left[1 + \frac{1}{s} + \frac{1}{s+2}\right] = \delta(t) + u(t) + e^{-2t}u(t)$$

(c) The result is obtained as

$$x_3(t) = \mathcal{L}^{-1}\left[\frac{1}{s+5} - \frac{1}{s+10}\right] = e^{-5t}u(t) - e^{-10t}u(t) = \left(e^{-5t} - e^{-10t}\right)u(t)$$

Problem 5-4

The Fourier transforms of (b) and (d) do not exist. In both cases, the path of integration for (5-3) cannot be chosen as the $j\omega$ axis since there are singularities on it. In the case of (c), the Laplace transform does not represent $x(t)$ for $t < 0$ since $x(t) \neq 0$ for $t < 0$.

Problem 5-5

(a) Write the cosine as

$$\cos(200\pi t) = \frac{1}{2}e^{j200\pi t} + \frac{1}{2}e^{-j200\pi t}$$

and use superposition and the transform pair for an exponential to get

$$X_a(s) = \frac{1/2}{s + j200\pi} + \frac{1/2}{s - j200\pi} = \frac{s}{s^2 + (200\pi)^2}$$

(b) Write the sine as

$$\sin(200\pi t) = \frac{1}{2j}e^{j200\pi t} - \frac{1}{2j}e^{-j200\pi t}$$

and use superposition and the transform pair for an exponential to get

$$X_b(s) = \frac{1/2j}{s + j200\pi} - \frac{1/2j}{s - j200\pi} = \frac{200\pi}{s^2 + (200\pi)^2}$$

(c) Write the given signal as

$$x_c(t) = \cos(200\pi t) + \sin(200\pi t) = x_a(t) + x_b(t)$$

Use superposition along with the results of the previous two parts to get

$$X_c(s) = \frac{s}{s^2 + (200\pi)^2} + \frac{200\pi}{s^2 + (200\pi)^2} = \frac{s + 200\pi}{s^2 + (200\pi)^2}$$

(d) It will not give the same result as (c) because the time delay theorem assumes a delayed function multiplied by $u(t - t_0)$ where t_0 is the time delay.

Problem 5-6

From a sketch, it follows that

$$\frac{dx(t)}{dt} = \frac{d\Lambda(t - 1)}{dt} = u(t) - 2u(t - 1) + u(t - 2)$$

The differentiation theorem gives

$$sX(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$

Thus

$$X(s) = \frac{1}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right] = \frac{1}{s^2}\left[1 - e^{-s}\right]^2$$

Problem 5-7

(a) Laplace transform the differential equation to get

$$\left[s^2X(s) - \frac{dx}{dt}\bigg|_{t=0-} - sx(0-)\right] + 6[sX(s) - x(0-)] + 5X(s) = \frac{1}{s+7}$$

The given initial conditions are 0, giving

$$X(s) = \frac{1}{(s+7)(s+5)(s+1)} = \frac{A}{s+7} + \frac{B}{s+5} + \frac{C}{s+1}$$

The unknown coefficients may be found as follows:

$$A = (s+7)X(s)\big|_{s=-7} = 1/12;\ B = (s+5)X(s)\big|_{s=-5} = -1/8;$$

$$C = (s+1)X(s)\big|_{s=-1} = 1/24$$

Using these in the expansion for $X(s)$ and superposition to do the inverse Laplace transform, we get

$$x(t) = \left[\frac{1}{12}e^{-7t} - \frac{1}{8}e^{-5t} + \frac{1}{24}e^{-t}\right]u(t)$$

(b) Laplace transform both differential equations, using the given initial conditions, to get

$$(s+3)X(s) + 2Y(s) = \frac{1}{s};\ sY(s) - X(s) = 0$$

Solve simultaneously for $Y(s)$ to get

$$Y(s) = \frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

Find the unknown coefficients as in part (a). The results are $A = 1/2$, $B = 1/2$, and $C = -1$. Thus $y(t)$, by inverse Laplace transformation, is

$$y(t) = \left[\frac{1}{2}(1 + e^{-2t}) - e^{-t}\right]u(t)$$

We may solve for $X(s)$ from the second equation as $X(s) = sY(s)$. Thus, $x(t) = dy(t)/dt = (e^{-t} - e^{-2t})u(t)$.

Problem 5-8

Find the integro-differential equation for the current by applying KVL:

$$(2+3)\,i(t) + \frac{1}{10}\int_{-\infty}^{t} i(\lambda)\,d\lambda = 0,\ t > 0 \text{ or } 5\,i(t) + \frac{1}{10}\int_{-\infty}^{t} i(\lambda)\,d\lambda = 0,\ t > 0$$

Laplace transform this equation using the initial condition $v(0\text{-}) = -10$ V to get

$$5I(s) + \frac{I(s)}{10s} = \frac{10}{s} \text{ or } I(s) = \frac{2}{s + 1/50}$$

Therefore, the current is

$$i(t) = 2e^{-t/50}u(t)$$

Problem 5-9

By KVL

$$2\frac{di(t)}{dt} + 6\,i(t) + 4\int_{-\infty}^{t} i(\lambda)\,d\lambda = \delta(t)$$

Laplace transform to get

$$\left(2s + 6 + \frac{4}{s}\right)I(s) = 1$$

or

$$I(s) = \frac{s}{2(s+2)(s+1)} = \frac{1}{s+2} - \frac{1/2}{s+1}$$

where the solution to Problem 5-7 can be followed for the steps in the partial fraction expansion. Thus, the current is

$$i(t) = \left[e^{-2t} - \frac{1}{2}e^{-t}\right]u(t)$$

Problem 5-10

(a) Expand as shown below, using s-shift and pairs 4 and 5 of Table 5-5:

$$X_1(s) = \frac{s+10}{s^2+8s+20} = \frac{s+10}{(s+4)^2+4} = \frac{s+4}{(s+4)^2+4} + 3\frac{2}{(s+4)^2+4}$$

Thus,

$$x_1(t) = [\cos(2t) + 3\sin(2t)]e^{-4t}u(t)$$

(b) Follow the same procedure as in part (a):

$$X_2(s) = \frac{s+3}{s^2+4s+5} = \frac{s+3}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} + \frac{1}{(s+2)^2+1}$$

The inverse Laplace transform is

$$x_2(t) = [\cos(t) + \sin(t)]e^{-2t}u(t)$$

(c) This rational fraction may be expanded as

$$X_3(s) = \frac{s}{s^2+6s+18} = \frac{s}{(s+3)^2+9} = \frac{s+3}{(s+3)^2+9} - \frac{3}{(s+3)^2+9}$$

The time domain waveform is

$$x_3(t) = [\cos(3t) - \sin(3t)]e^{-3t}u(t)$$

(d) This function of s may be written as

$$X_1(s) = \frac{10}{s^2+10s+34} = \frac{10}{3}\frac{3}{(s+5)^2+9}$$

When inverse Laplace transformed, this gives the time domain waveform $x_4(t) = (10/3)\sin(3t)e^{-5t}u(t)$.

Problem 5-11

(a) Using time shift and superposition, the Laplace transform of the series expression given in the problem statement is

$$X(s) = \sum_{n=0}^{\infty} P(s)e^{-snT_0} = P(s)\sum_{n=0}^{\infty} e^{-snT_0} = \frac{P(s)}{1 - e^{-sT_0}}$$

(b) Using the result of Problem 5-6 and the time scale theorem, we have

$$P(s) = \mathcal{L}[\Lambda(t/\tau - 1)] = \frac{\tau}{(\tau s)^2}[1 - e^{-\tau s}]^2 = \frac{1}{\tau s^2}[1 - e^{-\tau s}]^2$$

Applying the result of par(a), we have

$$X_\Delta(s) = \frac{1}{\tau s^2}\frac{[1 - e^{-\tau s}]^2}{1 - e^{-sT_0}}$$

(c) The Laplace transform of the basic pulse shape in this case is

$$\begin{aligned} P(s) &= \mathcal{L}\{\sin(\omega_0 t)[u(t) - u(t - T_0/2)]\} \\ &= \mathcal{L}\{\sin(\omega_0 t)u(t) - \sin(\omega_0 t)u(t - T_0/2)]\} \\ &= \mathcal{L}\{\sin(\omega_0 t)u(t) + \sin(\omega_0(t - T_0/2))u(t - T_0/2)]\},\ \omega_0 T_0/2 = \pi \end{aligned}$$

Using the Laplace transform of a sine wave and the delay theorem, we obtain

$$P(s) = \frac{\omega_0}{s^2 + \omega_0^2}\left[1 + e^{-sT_0/2}\right]$$

Using the result of part (a), we have

$$x_{HR}(t) = \frac{\omega_0}{s^2 + \omega_0^2}\frac{\left[1 + e^{-sT_0/2}\right]}{1 - e^{-sT_0}}$$

(d) The result of part (c) can be used, but using the general result from part (a), the period is now $T_0/2$ so the result is similar to that obtained in (c) except that the denominator repeat factor is 1 - exp($-sT_0/2$) rather than 1 - exp($-sT_0$).

Problem 5-12

(a) Write the part of the waveform for $t \geq 2$ as

$$x_2(t) = e^2 e^{-(t-2)}[u(t-2) - u(t-4)] = e^2 e^{-(t-2)} u(t-2) - e^{-(t-4)} u(t-4)]$$

Using superposition, the Laplace transform for an exponential, and time delay, we obtain

$$X_2(s) = e^2 \frac{e^{-2s}}{s+2} - \frac{e^{-4s}}{s+4}$$

The first half of the waveform may be written as

$$x_1(t) = e^t[u(t) - u(t-2)] = e^t u(t) - e^t u(t-2) = e^t u(t) - e^2 e^{t-2} u(t-2)$$

Using superposition, the Laplace transform for an exponential, and time delay, we obtain

$$X_1(s) = \frac{1}{s-1} - \frac{e^2 e^{-2s}}{s-1} = \frac{1}{s-1}[1 - e^{-2(s-1)}]$$

The Laplace transform of the composite waveform is the sum of these two transforms:

$$X(s) = \frac{1}{s-1}[1 - e^{-2(s-1)}] + e^2 \frac{e^{-2s}}{s+2} - \frac{e^{-4s}}{s+4}$$

(b) Use the Laplace transform of part (a) with the general result of Problem 5-11(a) with $T_0 = 4$.

Problem 5-13

(a) Initial value:

$$\lim_{t\to 0^+} x_1(t) = \lim_{s\to\infty} s\frac{s+10}{s^2+3s+2} = 1$$

Final value:

$$\lim_{t\to\infty} x_1(t) = \lim_{s\to 0} s\frac{s+10}{s^2+3s+2} = 0$$

(b) Initial value:

$$\lim_{t\to 0^+} x_2(t) = \lim_{s\to\infty} s\frac{5}{s^3+s^2+9s+9} = 0$$

Final value - factor the denominator as

$$X_2(s) = \frac{5}{(s+1)(s^2+9)}$$

The factor $s^2 + 9$ corresponds to a sinusoid in the time domain. As a result, the final value does not exist due to the oscillatory term.

(c) The initial value does not exist. The reason that the limit of $sX_3(s)$ doesn't exist as $s \to \infty$ is because of the impulse in the inverse Laplace transform of $X_3(s)$.

Final value:

$$\lim_{t\to\infty} x_3(t) = \lim_{s\to 0} s\frac{s^2+5s+7}{s^2+3s+2} = 0$$

(d) Initial value and final values, respectively:

$$\lim_{t\to 0^+} x_4(t) = \lim_{s\to\infty} s\frac{s+3}{s^2+2s} = 1;\ \lim_{t\to\infty} x_4(t) = \lim_{s\to 0} s\frac{s+3}{s^2+2s} = \frac{3}{2}$$

Problem 5-14

Given that

$$X(s) = \frac{s+2}{s^2+4s+5}$$

Find Laplace transforms of the following signals.
(a) $y_1(t) = x(2t - 1)u(2t - 1) = x[2(t - 1/2)]u[2(t - 1/2)]$. Use time delay:

$$\mathcal{L}[x(t-t_0)u(t-t_0)] = X(s)e^{-st_0}$$

and time scaling

$$\mathcal{L}[x(at)] = \frac{1}{a}X(s/a)$$

to get

$$Y_1(s) = \frac{1}{2}X(s/2)e^{-s/2} = \frac{s+4}{s^2+8s+20}e^{-s/2}$$

(b) $y_2(t) = tx(t)$. Note that if

$$X(s) = \int_0^\infty x(t)e^{-st}dt \text{ then } \frac{dX(s)}{ds} = \int_0^\infty x(t)[-te^{-st}dt]$$

In general

$$\frac{d^nX(s)}{ds^n} = (-1)^n\int_0^\infty x(t)t^ne^{-st}dt = (-1)^n\mathcal{L}[t^nx(t)] \text{ or } \mathcal{L}[t^nx(t)] = (-1)^n\frac{d^nX(s)}{ds^n}$$

For the given $X(s)$, we find that

$$Y_2(s) = \mathcal{L}[tx(t)] = -\frac{dX(s)}{ds} = -\frac{(s^2+4s+5)-(s+2)(2s+4)}{(s^2+4s+5)^2} = \frac{s^2+4s+3}{(s^2+4s+5)^2}$$

(c) $y_3(t) = e^{-3t}x(t)$. Use s-shift to get

$$Y_3(s) = X(s+3) = \frac{s+5}{s^2+10s+26}$$

(d) $y_4(t) = x(t)*x(t)$. Use the convolution theorem to get

$$Y_4(s) = X(s)X(s) = \frac{(s+2)^2}{(s^2+4s+5)^2}$$

(e) $y_5(t) = dx(t)/dt$. Use the differentiation theorem along with the initial condition

$$x(0) = \lim_{s\to\infty} \frac{s(s+2)}{s^2+4s+5} = 1$$

to get

$$Y_5(s) = s\frac{s+2}{s^2+4s+5} - 1 = -\frac{2s+5}{s^2+4s+5}$$

(f) Use time scaling and superposition to get

$$\begin{aligned} Y_6(s) &= 2(4)\frac{4s+2}{(4s)^2+4(4s)+5} + \frac{3}{5}\frac{s/5+2}{(s/5)^2+4(s/5)+5} \\ &= 2\frac{s+1/2}{s^2+s+5/16} + 3\frac{s+10}{s^2+20s+125} \end{aligned}$$

(g) Use s-shift to get

$$\begin{aligned} Y_7(s) &= \frac{1}{2}X(s+j7) + \frac{1}{2}X(s-j7) \\ &= \frac{1}{2}\frac{s+2+j7}{(s+j7)^2+4(s+j7)+5} + \frac{1}{2}\frac{s+2-j7}{(s-j7)^2+4(s-j7)+5} \end{aligned}$$

Problem 5-15

(a) The separate Laplace transforms are

$$X_1(s) = \frac{1}{s+2} \text{ and } X_2(s) = \frac{e^{-5s}}{s}$$

The convolution theorem of Laplace transforms gives for $Y(s)$

$$Y(s) = \frac{1}{s+2}\frac{e^{-5s}}{s} = \frac{e^{-5s}}{s(s+2)} = \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right]e^{-5s}$$

Using superposition and time delay, we get

$$y(t) = \frac{1}{2}\left[1 - e^{-2(t-5)}\right]u(t-5)$$

(b) The Laplace transforms of the given signals are

$$X_1(s) = \frac{s}{s^2+5^2} \text{ and } X_2(s) = \frac{3}{s^2+3^2}$$

The Laplace transform of their convolution is

$$Y(s) = \frac{s}{s^2+5^2}\frac{3}{s^2+3^2} = \frac{3}{16}\frac{s}{s^2+9} - \frac{3}{16}\frac{s}{s^2+25}$$

Inverse Laplace transformation gives

$$y(t) = \frac{3}{16}\left[\cos(3t) - \cos(5t)\right]u(t)$$

(c) The Laplace transformations of the given signals are

$$X_1(s) = \frac{1}{s+2} \text{ and } X_2(s) = \frac{s}{s^2+5^2}$$

The Laplace transform of their convolution is

$$Y(s) = \frac{1}{s+2}\frac{s}{s^2+5^2} = \frac{2}{29}\left[\frac{s+25/2}{s^2+5^2} - \frac{1}{s+2}\right]$$

Inverse Laplace transformation gives

$$y(t) = \frac{2}{29}\left[\cos(5t) + \frac{5}{2}\sin(5t) - e^{-2t}\right]u(t)$$

(d) The Laplace transforms of the given signals are

$$X_1(s) = \frac{3}{s^2+3^2} \text{ and } X_2(s) = \frac{e^{-5s}}{s}$$

The Laplace transform of their convolution is

$$Y(s) = \frac{3}{s^2+3^2}\frac{e^{-5s}}{s} = \frac{1}{3}\left[\frac{1}{s} - \frac{s}{s^2+3^2}\right]e^{-5s}$$

Inverse Laplace transformation gives

$$y(t) = \frac{1}{3}\{1 - \sin[3(t-5)]\}\,u(t-5)$$

which follows by superposition and time delay.

Problem 5-16

(a) Use partial fraction expansion to get

$$X_1(s) = \frac{9}{s+1} - \frac{8}{s+2}$$

This gives the time domain signal

$$x_1(t) = (9e^{-t} - 8e^{-2t})\,u(t)$$

The initial and final values of 1 and 0, respectively, aree with those found previously.

(b) Using partial fraction expansion, we get

$$X_2(s) = \frac{1/2}{s+1} - \frac{s/2}{s^2+3^2} + \frac{1}{6}\frac{3}{s^2+3^2}$$

The time domain waveform is

$$x_2(t) = \frac{1}{2}\left[e^{-t} - \cos(3t) + \frac{1}{3}\sin(3t)\right]u(t)$$

The initial value agrees. The final value doesn't exist.
(c) First use long division to produce a proper rational fraction:

$$X_3(s) = 1 + \frac{2s+5}{s^2+3s+2} = 1 + \frac{3}{s+1} - \frac{1}{s+2}$$

The corresponding time-domain waveform is

$$x_3(t) = \delta(t) + (3e^{-t} - e^{-2t})u(t)$$

The final value agrees. The initial value doesn't exist due to the impulse.
(d) Using partial fraction expansion, we obtain

$$X_4(s) = \frac{3}{2s} - \frac{1}{2}\frac{1}{s+2}$$

The corresponding time-domain waveform is

$$x_4(t) = \left(\frac{3}{2} - \frac{1}{2}e^{-2t}\right)u(t)$$

The final value of 3/2 and the initial value of 1 both agree with what was obtained previously.

Problem 5-17

The Laplace-transformed differential equations assuming a step input and 0 initial conditions, from (2-130), are

$$(R + sL)I(s) + K\Omega(s) = \frac{1}{s} \text{ and } (sJ + B)\Omega(s) = KI(s)$$

Elimination of $I(s)$ gives

$$\Omega(s) = \frac{K}{s[(R + sL)(B + sJ) + K^2]} = \frac{K}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

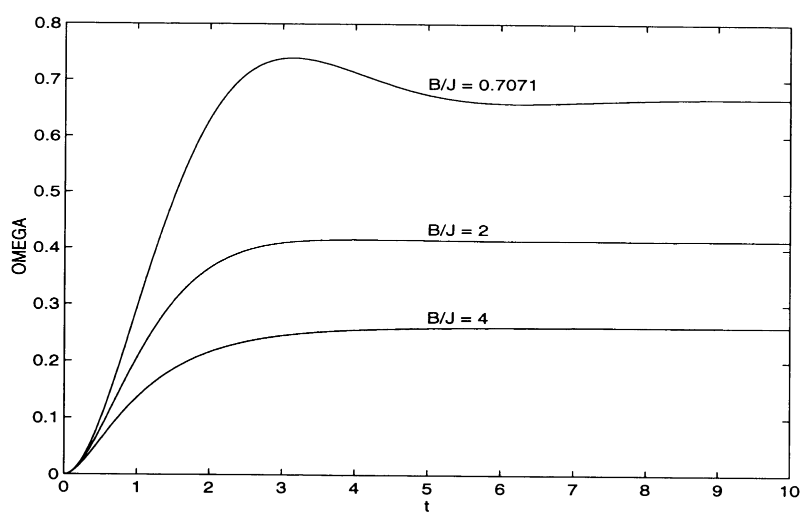
We may have an underdamped or overdamped system depending on whether $\zeta < 1$ or $\zeta > 1$. Assume it is underdamped. Then $\Omega(s)$ is best rewritten as

$$\Omega(s) = \frac{K}{s[s^2 + 2\zeta\omega_n s + (\zeta\omega_n)^2 + (1 - \zeta^2)\omega_n^2]} = \frac{K}{s[(s + \zeta\omega_n)^2 + (1 - \zeta^2)\omega_n^2]}$$
$$= \frac{K}{\omega_n^2}\left[1 - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + (1 - \zeta^2)\omega_n^2}\right]$$

Using s-shift and superposition, the inverse Laplace transform is

$$\Omega(t) = \frac{K}{\omega_n^2}\left\{1 - \left[\cos(\sqrt{1 - \zeta^2}\,\omega_n t) + \frac{\zeta\omega_n}{\sqrt{1 - \zeta^2}\,\omega_n}\sin(\sqrt{1 - \zeta^2}\,\omega_n t)\right]e^{-\zeta\omega_n t}\right\}u(t)$$

The overdamped case will consist of two decaying exponentials for the last two terms. Plots shown to the right are the same as obtained by simulation in Chapter 2.



Problem 5-18

(a) Expand in partial fractions as

$$X(s) = \frac{5}{s+2} + \frac{2}{s+3} + \frac{1}{s^2+4}$$

The time-domain waveform is

$$x(t) = \left[5e^{-2t} + 2e^{-3t} + \frac{1}{2}\sin(2t)\right]u(t)$$

(b) The partial fraction expansion for this Laplace transform is

$$X(s) = \frac{1}{s+1} + \frac{1}{s+3} + \frac{1}{(s+1)^2+4}$$

Use superposition and appropriate transform pairs to get

$$x(t) = \left[e^{-t} + e^{-3t} + \frac{1}{2}e^{-t}\sin(2t)\right]u(t)$$

Problem 5-19

The partial fraction expansion is

$$X(s) = \frac{1}{(s+1)^2} + \frac{7}{s+3}$$

Use superposition and appropriate transform pairs to get

$$x(t) = \left[te^{-t} + 7e^{-3t}\right]u(t)$$

Problem 5-20

The partial fraction expansion is

$$X(s) = \frac{1}{s+1} - \frac{27}{(s+3)^3}$$

and the time-domain waveform is

$$x(t) = \left[e^{-t} - \frac{27}{2}t^2 e^{-3t}\right]u(t)$$

Problem 5-21

(a) The partial fraction expansion is

$$X(s) = \frac{1}{s+1} + \frac{1}{(s^2+2^2)^2}$$

Use pair 9 of Table 5-3:

$$x(t) = \left[e^{-t} + \frac{1}{16}\sin(2t) - \frac{1}{8}t\cos(2t)\right]u(t)$$

(b) Expand in partial fractions as

$$X(s) = \frac{1/4}{s+1} - \frac{1/4(s+1)}{(s+1)^2+2^2} - \frac{2}{[(s+1)^2+2^2]^2}$$

Use pair 9 of Table 5-3 and s-shift to get

$$x(t) = \left\{\frac{1}{4}e^{-t} - \frac{1}{4}e^{-t}\left[\cos(2t) + \frac{1}{2}\sin(2t) - t\cos(2t)\right]\right\}u(t)$$

Problem 5-22

Pair 1: Use (1-66) to integrate the Laplace transform integral:

$$\mathcal{L}[\delta^{(n)}(t)] = \int_{0-}^{\infty}\delta^{(n)}(t)e^{-st}dt = (-1)^n\frac{d^n e^{-st}}{dt^n}\bigg|_{s=0} = (-1)^n(-1)^n s^n e^{-st}\big|_{s=0} = s^n$$

Pair 3:

$$\mathcal{L}[t^n e^{-\alpha t}u(t)/n!] = \int_0^{\infty}\frac{t^n}{n!}e^{-\alpha t}e^{-st}dt = \int_0^{\infty}\frac{t^n}{n!}e^{-(\alpha+s)t}dt$$

Use integration by parts or look up the integral in a table to get

$$\mathcal{L}[t^n e^{-\alpha t}u(t)/n!] = \frac{1}{(s+\alpha)^{n+1}}$$

Problem 5-23

(a) First consider the convolution of sine and cosine:

$$\begin{aligned}\sin(\omega_0 t)u(t)*\cos(\omega_0 t)u(t) &= \int_0^{\infty}\sin(\omega_0\lambda)\cos[\omega_0(t-\lambda)]u(t-\lambda)\,d\lambda \\ &= \int_0^{t}\frac{1}{2}[\sin(2\omega_0\lambda-\omega_0 t)+\sin(\omega_0 t)]\,d\lambda = \frac{t}{2}\sin(\omega_0 t),\ t>0\end{aligned}$$

But, by the convolution theorem of Laplace transforms, we hav

$$\sin(\omega_0 t)u(t)*\cos(\omega_0 t)u(t) = \frac{t}{2}\sin(\omega_0 t)\,u(t) \leftrightarrow \frac{\omega_0 s}{(s^2+\omega_0^2)^2}$$

Problem 5-24

(a) Differentiate the Laplace transform integral:

$$\frac{dX(s)}{ds} = -\int_0^\infty tx(t)e^{-st}dt \text{ and } \frac{d^2X(s)}{ds^2} = \int_0^\infty t^2x(t)e^{-st}dt$$

Thus we have

$$X(0) = \int_0^\infty x(t)\,dt; \quad \left.\frac{dX(s)}{ds}\right|_{s=0} = -\int_0^\infty tx(t)\,dt; \quad \left.\frac{d^2X(s)}{ds^2}\right|_{s=0} = X''(0) = \int_0^\infty t^2x(t)dt$$

Using these relationships in the definitions given in the problem statement, we get the expressions for the delay and duration given in the problem statement. The last expression follows by expanding the numerator integral to get

$$\tau^2 = \frac{\int_0^\infty t^2x(t)\,dt}{\int_0^\infty x(t)\,dt} - 2t_0\frac{\int_0^\infty tx(t)\,dt}{\int_0^\infty x(t)\,dt} + t_0^2\frac{\int_0^\infty x(t)\,dt}{\int_0^\infty x(t)\,dt}$$

Using the relationships derived in part (a), we obtain

$$\tau^2 = \frac{X''(0)}{X(0)} - 2t_0\frac{X'(0)}{X(0)} + t_0^2 = \frac{X''(0)}{X(0)} - \left[\frac{X'(0)}{X(0)}\right]^2$$

where the last relationship follows because $t_0 = -X'(0)/X(0)$.

(b) Let t_0, t_{01}, t_{02}, τ, τ_1, and τ_2 be delays and durations of the signals $x(t)$, $x_1(t)$, and $x_2(t)$, respectively. Let $X(s)$, $X_1(s)$, and $X_2(s)$ be their respective Laplace transforms. Using the convolution theorem of Laplace transforms, we obtain

$$t_0 = -\frac{X'(0)}{X(0)} = -\left.\frac{1}{X(0)}\frac{d}{ds}[X_1(s)X_2(s)]\right|_{s=0} = -\frac{X_1'(0)}{X_1(0)} - \frac{X_2'(0)}{X_2(0)} = t_{01} + t_{02}$$

which follows by using the chain rule for differentiation. A similar derivation follows for the duration.

(c) In this case

$$X_1(s) = \frac{A}{s+\alpha} \text{ and } X_2(s) = \frac{B}{s+\beta}$$

The various quantities need are

$$X_1(0) = \frac{A}{\alpha};\ X_2(0) = \frac{B}{\beta};\ X_1'(0) = -\frac{A}{\alpha^2};\ X_2'(0) = -\frac{B}{\beta^2};\ X_1''(0) = \frac{2A}{\alpha^3};\ X_2''(0) = \frac{2B}{\beta^3}$$

These give the following expressions for delays and durations:

$$t_{01} = \frac{1}{\alpha};\ t_{02} = \frac{1}{\beta};\ \tau_1 = \frac{1}{\alpha^2};\ \tau_2 = \frac{1}{\beta^2}$$

Thus,

$$t_0 = \frac{1}{\alpha} + \frac{1}{\beta};\ \tau^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

Problem 5-25

(a) By definition

$$\mathcal{L}[x(t/a)] = \int_0^\infty x(t/a)\, e^{-st}\, dt$$

Let $u = t/a$. This gives

$$\mathcal{L}[x(t/a)] = a\int_0^\infty x(u)\, e^{-sau}\, du \doteq aX(as),\ a > 0$$

(b) Consider

$$\frac{dX(s)}{ds} = \frac{d}{ds}\int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty x(t)\frac{d}{ds}e^{-st}\, dt = -\int_0^\infty tx(t)\, e^{-st}\, dt = -\mathcal{L}[tx(t)]$$

(c) Continue to differentiate the Laplace transform integral with respect to s under the integral. At the nth differentiation we have

$$\frac{dX^n(s)}{ds^n} = \int_0^\infty x(t)\frac{d^n}{ds^n}e^{-st}dt = \int_0^\infty (-t)^n x(t)e^{-st}dt = (-1)^n \mathcal{L}[t^n x(t)]$$

Thus

$$\mathcal{L}[t^n x(t)] = (-1)^n \frac{d^n X(s)}{ds^n}$$

(d) Apply the theorem proved in (c) and the transform pair

$$\mathcal{L}[e^{-\alpha t}u(t)] = \frac{1}{s+\alpha}$$

We have

$$\mathcal{L}[t^n e^{-\alpha t}u(t)] = (-1)^n \frac{d^n}{ds^n}(s+\alpha)^{-1} = (-1)^n \frac{d^{n-1}}{ds^{n-1}}(-1)(s+\alpha)^{-2}$$
$$= (-1)^n \frac{d^{n-2}}{ds^{n-2}}(-1)(-2)(s+\alpha)^{-3} = \dots = \frac{n!}{(s+\alpha)^{n+1}}$$

Problem 5-26

(a) Use of the residue function in MATLAB gives the following:

```
EDU» den = [1 0 9 0];
EDU» [R,P,k]=residue(num, den)

R =
     0 - 0.3333i
     0 + 0.3333i
  1.0000
P =
     0 + 3.0000i
     0 - 3.0000i
     0
k =
   []
```

R gives the residues, or expansion coefficients, P gives the roots (poles), and k gives the remainder (empty in this case). Thus, the partial fraction expansion is

$$X_1(s) = \frac{-j/3}{s - j3} + \frac{j/3}{s + j3} + \frac{1}{s} = \frac{2}{s^2 + 9} + \frac{1}{s}$$

Using appropriate transform pairs, the time-domain signal is

$$x_1(t) = \left[\frac{2}{3}\sin(3t) + 1\right]u(t)$$

(b) The partial fraction expansion for this rational fraction is

$$X_2(s) = \frac{2}{s} - \frac{3}{s^2 + 9} + \frac{1}{(s^2 + 9)^2}$$

This is the most convenient form for inverse Laplace transformation. Using the residue function in MATLAB puts everything in terms of simple factors, so is not so useful here. The above may be derived by writing the imaginary poles as $s + j3$ and $s - j3$, using Heaviside's expansion technique, and then putting pole pairs back together. Using the Laplace transform table, the time-domain signal is

$$x_2(t) = \left\{2 - \sin(3t) + \frac{1}{54}\left[\sin(3t) - 3t\cos(3t)\right]\right\}u(t)$$

The moral of the story on this part is that paper and pencil is sometimes more useful than a computer, or perhaps that the student should be fluent with both means.

(c) Again, the residue function of MATLAB doesn't do so well because it expands in terms of simple factors. The general form of the expansion is

$$X_3(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 9} + \frac{Es + F}{(s^2 + 9)^2}$$

Find B and A using Heaviside's technique as

$$B = \left.\frac{18}{(s^2 + 9)^2}\right|_{s=0} = \frac{2}{9} \text{ and } A = \left.\frac{d}{ds}\frac{18}{(s^2 + 9)^2}\right|_{s=0} = 0$$

To find the other constants, put everything over a common denominator and match coefficients of like powers of s. The results are $C = 0$, $D = -2/9$, $E = 0$, and $F = 2$. Thus

$$X_3(s) = \frac{2/9}{s^2} - \frac{2/9}{s^2 + 9} + \frac{2}{(s^2 + 9)^2}$$

Using Laplace transform tables, the time-domain signal is found to be

$$x_3(t) = \left[t - \frac{1}{27}\sin(3t) + \frac{t}{9}\cos(3t)\right]u(t)$$

(d) Using long division, expand the factor involving the exponential as

$$\frac{1}{1 + e^{-2s}} = 1 - e^{-2s} + e^{-4s} - e^{-6s} + \dots$$

Thus

$$X_5(s) = \frac{1}{s + 3}\left[1 - e^{-2s} + e^{-4s} - e^{-6s} + \dots\right]$$

Now use the transform pair for an exponential, superposition, and the delay theorem to get

$$x_4(t) = e^{-3t}u(t) - e^{-3(t-2)}u(t-2) + e^{-3(t-4)}u(t-4) - e^{-3(t-6)}u(t-6) + \dots$$

(e) Use the same approach as in the previous part after factoring out the s in the denominator:

$$X_5(s) = \frac{2}{s}\left[1 - e^{-3s} + e^{-6s} - e^{-9s} + \dots\right]$$

The inverse Laplace transform is

$$x_5(t) = 2[u(t) - u(t-3) + u(t-6) - u(t-9) + \dots]$$

This is a square wave starting at $t = 0$ of amplitudes 2 and 0, and of period 6.

Problem 5-27

(a) By KVL, with $v_c(0-) = -V$ with respect to the current reference, we have

$$\frac{1}{C}\int_0^t i(\lambda)\,d\lambda - V + Ri(t) = 0 \text{ or } \frac{1}{sC}I(s) - \frac{V}{s} + RI(s) = 0 \text{ or } I(s) = \frac{V/R}{s + 1/RC}$$

Using the Laplace transform pair for an exponential, we obtain

$$i(t) = \frac{V}{R}e^{-t/RC}u(t)$$

(b) The initial conditions are $I(0-) = 0$ and $v_c(0-) = -V$ with respect to the current reference. Using KVL, we obtain

$$\frac{1}{C}\int_0^t i(\lambda)\,d\lambda - V + L\frac{di(t)}{dt} = 0 \text{ or } \frac{1}{sC}I(s) - \frac{V}{s} + sLI(s) = 0 \text{ or } I(s) = \frac{(V/L)\sqrt{LC}}{(s^2 + 1/LC)\sqrt{LC}}$$

Using the inverse Laplace transform of a sine, we obtain

$$i(t) = V\sqrt{\frac{C}{L}}\sin\left(\frac{t}{\sqrt{LC}}\right)u(t)$$

Problem 5-28

(a) Write KVL around the loop in the direction of the current reference, noting that $i(0-) = V/R_1$:

$$0 = (R_1 + R_2)I(s) + sLI(s) - L\left(\frac{V}{R_1}\right) \text{ or } I(s) = \frac{V/R_1}{s + \dfrac{R_1 + R_2}{L}}$$

Thus, the time-domain current is

$$i(t) = \frac{V}{R_1}e^{-(R_1 + R_2)t/L}u(t)$$

(b) Write KVL around the loop noting that $i(0-) = V/R$ and $v_C(0-) = 0$:

$$0 = \frac{I(s)}{sC} + L[sI(s) - V/R] \text{ or } I(s) = \frac{V}{R}\frac{s}{s^2 + 1/LC}$$

Using the Laplace transform of a cosine, we obtain

$$i(t) = \frac{V}{R}\cos\left(\frac{t}{\sqrt{LC}}\right)u(t)$$

Problem 5-29

(a) Write the given transform as

$$X(s) = \frac{1}{s+1}\frac{1}{s+1}$$

Thus, the time domain signal is

$$x(t) = x_1(t)*x_1(t) = [e^{-t}u(t)]*[e^{-t}u(t)]$$
$$= \int_{-\infty}^{\infty} e^{-\lambda}u(\lambda)e^{-(t-\lambda)}u(t-\lambda)\,d\lambda = e^{-t}\int_0^t d\lambda = te^{-t},\ t \geq 0$$

(b) use the transform pair $1/s \leftrightarrow u(t)$. By the integration theorem,

$$\frac{1}{s}\frac{1}{s} \leftrightarrow \int_{-\infty}^{\infty} u(\lambda)u(t-\lambda)\,d\lambda = \int_0^t d\lambda = t,\ t \geq 0 \text{ or } \frac{1}{s}\frac{1}{s} \leftrightarrow tu(t) = r(t)$$

Using the s-shift theorem, we obtain

$$\frac{1}{s+1}\frac{1}{s+1} \leftrightarrow te^{-t}u(t)$$

which is the same as obtained in part (a).

Problem 5-30

(a) Use s-shift:

$$X_1(s) = \frac{s+5}{(s+5)^2+9}$$

Therefore

$$x_1(t) = e^{-5t}\cos(3t)\,u(t)$$

(b) Use time delay:

$$X_2(s) = \frac{1}{s+4} - \frac{e^{-2s}}{s+4}$$

The time domain waveform is therefore

$$x_2(t) = e^{-4t}u(t) - e^{-4(t-2)}u(t-2)$$

(c) Use s-shift along with the Laplace transform of a ramp:

$$x_3(t) = te^{-t}u(t)$$

Problem 5-31

(a) Initial value:

$$\lim_{t \to 0+} x_1(t) = \lim_{s \to \infty} s\frac{s+5}{s^2+10s+34} = 1$$

Final value:

$$\lim_{t \to \infty} x_1(t) = \lim_{s \to 0} s\frac{s+5}{s^2+10s+34} = 0$$

(b) Initial value:

$$\lim_{t \to 0^+} x_2(t) = \lim_{s \to \infty} s \frac{1 - e^{-2s}}{s + 4} = 1$$

Final value:

$$\lim_{t \to \infty} x_2(t) = \lim_{s \to 0} s \frac{1 - e^{-2s}}{s + 4} = 0$$

(c) Initial value:

$$\lim_{t \to 0^+} x_3(t) = \lim_{s \to \infty} \frac{s}{(s + 1)^2} = 0$$

Final value:

$$\lim_{t \to \infty} x_3(t) = \lim_{s \to 0} \frac{s}{(s + 1)^2} = 0$$

All results correspond to the limits of the time functions of Problem 5-30.

<u>Problem 5-32</u>

(a) Expand as

$$X_1(s) = \frac{A}{s} + \frac{B}{s + 1} + \frac{Cs + D}{s^2 + 1}$$

where

$$A = sX_1(s)|_{s = 0} = 1 \text{ and } B = (s + 1)X_1(s)|s = -1 = -1/2$$

Find the last term by subtraction:

$$\frac{Cs + D}{s^2 + 1} = \frac{1}{s(s + 1)(s^2 + 1)} - \frac{1}{s} + \frac{1/2}{s + 1} = -\frac{1}{2}\frac{s + 1}{s^2 + 1}$$

Thus,

$$X_1(s) = \frac{1}{s} - \frac{1/2}{s+1} - \frac{s/2}{s^2+1} - \frac{1/2}{s^2+1}$$

Using appropriate Laplace transform pairs, the time-domain signal is found to be

$$x_1(t) = \left[1 - \frac{1}{2}e^{-t} - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t)\right]u(t)$$

(b) Expand as

$$X_2(s) = \frac{A}{s} + \frac{F(s)}{(s+1)^2}$$

A is given by

$$A = sX_2(s)\big|_{s=0} = 1$$

The last term is found by subtraction:

$$\frac{F(s)}{(s+1)^3} = X_2(s) - \frac{1}{s} = -\frac{s^2+3s+3}{(s+1)^3} = -\frac{s^2}{(s+1)^3} - \frac{3}{(s+1)^2}$$

Therefore,

$$X_2(s) = \frac{1}{s} - \frac{s^2}{(s+1)^3} - \frac{3}{(s+1)^2}$$

and the time-domain signal is

$$x_2(t) = u(t) - \frac{d^2}{dt^2}\left[\frac{t^2}{2}e^{-t}u(t)\right] - 3te^{-t}u(t) = \left[1 - \left(1 - t - \frac{t^2}{2}\right)e^{-t}\right]u(t)$$

Problem 5-33

(a) Write this time function as

$$x_1(t) = 2e^{-5t}u(t) + 2e^{3t}u(-t) = x_a(t) + x_b(t)$$

The double-sided Laplace transform is then

$$X_1(s) = X_a(s) + X_b(-s)$$

where

$$X_a(s) = \mathcal{L}_s[x_a(t)] \text{ and } X_b(s) = \mathcal{L}_s[x_b(-t)]$$

The subscript denotes the single-sided Laplace transform. For these two transforms, we obtain

$$X_a(s) = \frac{2}{s+5},\ \text{Re}(s) > -5 \text{ and } X_b(s) = \frac{2}{s+3},\ \text{Re}(s) > -3$$

Thus, the double-sided Laplace transform of $x_1(t)$ is

$$X_1(s) = \frac{2}{s+5} + \frac{2}{-s+3} = -\frac{16}{s^2+2s-15},\ -5 < \text{Re}(s) < 3$$

(b) Following the same procedure as in (a), we have

$$X_a(s) = \frac{1}{s+1},\ \text{Re}(s) > -1 \text{ and } X_b(s) = \frac{1}{s},\ \text{Re}(s) > 0$$

The double-sided Laplace transform of the given signal is

$$X_2(s) = \frac{1}{s+1} - \frac{1}{s} = -\frac{1}{s(s+1)},\ -1 < \text{Re}(s) < 0$$

(c) As before, we have

$$X_a(s) = \frac{1}{s+1},\ \text{Re}(s) > -1 \text{ and } X_b(s) = \frac{s}{s^2+25},\ \text{Re}(s) > 0$$

The double-sided Laplace transform of the given signal is

$$X_2(s) = \frac{1}{s+1} - \frac{s}{s^2+25} = \frac{-s+25}{(s+1)(s^2+25)}, \quad -1 < \mathrm{Re}(s) < 0$$

Problem 5-34

Expand as

$$X_1(s) = \frac{A}{s+5} + \frac{B}{s+1} = \frac{1}{4}\left[-\frac{1}{s+5} + \frac{1}{s+1}\right]$$

The first term has a pole at $s = -5$ to the left of the given region of convergence; hence, it corresponds to the positive-time portion. The second term has a pole at $s = -1$ to the right of the given region of convergence; hence, it corresponds to the negative-time portion. Therefore,

$$x_1(t) = \frac{1}{4}\left[-5e^{-5t}u(t) + e^{t}u(-t)\right]$$

(b) Expand as

$$X_2(s) = \frac{As+B}{s^2+1} + \frac{C}{s+1} = -\frac{s-1}{s^2+1} + \frac{1}{s+1}$$

The first term corresponds to the negative-time portion because the poles are to the right of the given region of convergence. The second term corresponds to the positive-time portion because the pole is to the left of the given region of convergence. Thus, the desired time-domain waveform is given by

$$x_2(t) = \mathcal{L}_s^{-1}\left(-\frac{-s+1}{s^2+1}\right)_{t\to -t} + e^{-t}u(t)$$
$$= [\cos(t) + \sin(t)]\,u(t)\big|_{t\to -t} + e^{-t}u(t) = [\cos(t) - \sin(t)]\,u(-t) + e^{-t}u(t)$$

Problem 5-35

With parameter values given, we have

$$I(s) = \frac{sX(s) - v_C(0-)}{L\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} = \frac{sX(s)}{s^2 + 2s + 2} = \frac{sX(s)}{(s+1)^2 + 1}$$

For a square wave input,

$$X(s) = \frac{1}{s}\frac{1 - e^{-sT_0/2}}{1 + e^{-sT_0/2}}$$

Therefore, the Laplace transform of the current is

$$I(s) = \frac{1}{(s+1)^2 + 1}\left[\frac{1 - e^{-sT_0/2}}{1 + e^{-sT_0/2}}\right]$$

Use the sum

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to expand the term in brackets as

$$\frac{1 - e^{-sT_0/2}}{1 + e^{-sT_0/2}} = \left(1 - e^{-sT_0/2}\right)\sum_{n=0}^{\infty} (-1)^n e^{-snT_0/2}$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n e^{-snT_0/2} - \sum_{n=0}^{\infty} (-1)^n e^{-s(n+1)T_0/2} = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-snT_0/2}$$

Hence, the Laplace transform of the current is

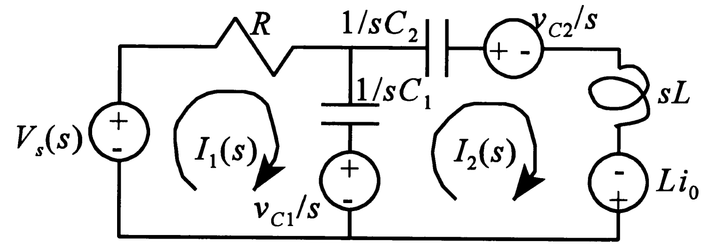
$$I(s) = \frac{1}{(s+1)^2 + 1}\left[1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-snT_0/2}\right]$$

Using pair 7 of Table 5-3 and the time delay theorem, we obtain the result for $i(t)$ given in the problem statement.

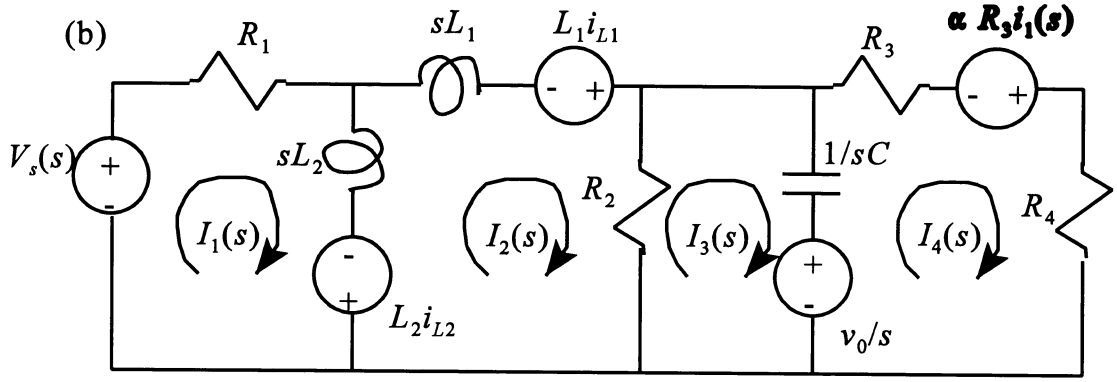
CHAPTER 6

Problem 6-1

(a)



(b)



Problem 6-2

(a) The loop equations are:

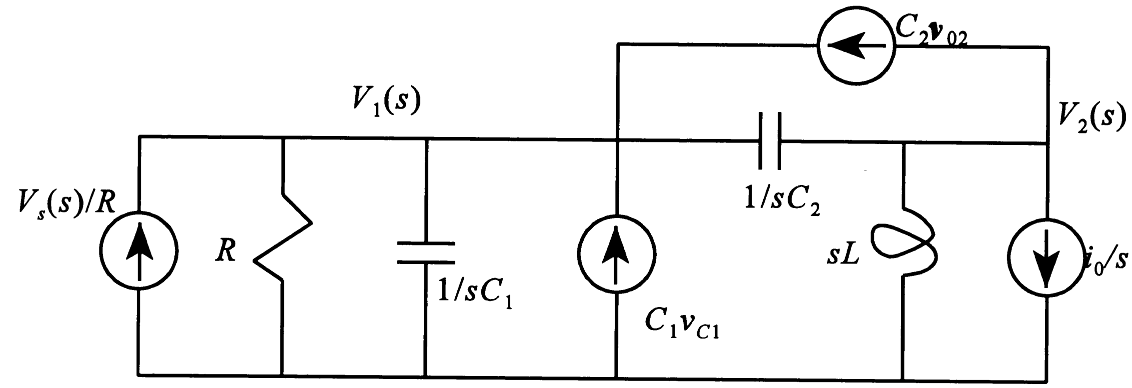
$$\begin{bmatrix} R + 1/sC_1 & -1/sC_1 \\ -1/sC_1 & 1/sC_1 + 1/sC_2 + sL \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_s(s) - v_{C_1}/s \\ v_{C_1}/s - v_{C_2}/s + Li_0 \end{bmatrix}$$

(b) The loop equations for this case are:

$$\begin{bmatrix} R_1 + sL_2 & -sL_2 & 0 & 0 \\ -sL_2 & sL_1 + sL_2 + R_2 & -R_2 & 0 \\ 0 & -R_2 & R_2 + 1/sC & -1/sC \\ -\alpha R_3 & 0 & -1/sC & 1/sC + R_3 + R_4 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \end{bmatrix} = \begin{bmatrix} V_s(s) + L_2 i_{L_2} \\ -L_2 i_{L_2} + L_1 i_{L_1} \\ -v_0/s \\ v_0/s \end{bmatrix}$$

Problem 6-3

Do a Thevinin to Norton transformation of all voltage sources. For example, the circuit of part (a) becomes:



Problem 6-4

(a) The matrix equations for the nodal analysis are:

$$\begin{bmatrix} 1/R + sC_1 + sC_2 & -sC_2 \\ -sC_2 & sC_2 + 1/sL \end{bmatrix}\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} V_s(s)/R + C_1 v_{C_1} + C_2 v_{C_2} \\ -C_2 v_{C_2} - i_0/s \end{bmatrix}$$

(b) The nodal analysis matrix equations are:

$$\begin{bmatrix} 1/R_1 + 1/sL_1 + 1/sL_2 & -1/sL_{1_1} & 0 \\ -1/sL_1 & 1/sL_1 + sC + 1/R_2 + 1/R_3 & -1/R_3 \\ 0 & -1/R_3 & 1/R_3 + 1/R_4 \end{bmatrix}\begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix} = \begin{bmatrix} V_s(s)/R_1 - i_{L_1}/s - i_{L_2}/s \\ i_{L_1}/s + Cv_0 - \alpha I_1(s) \\ \alpha I_1(s) \end{bmatrix}$$

Note that

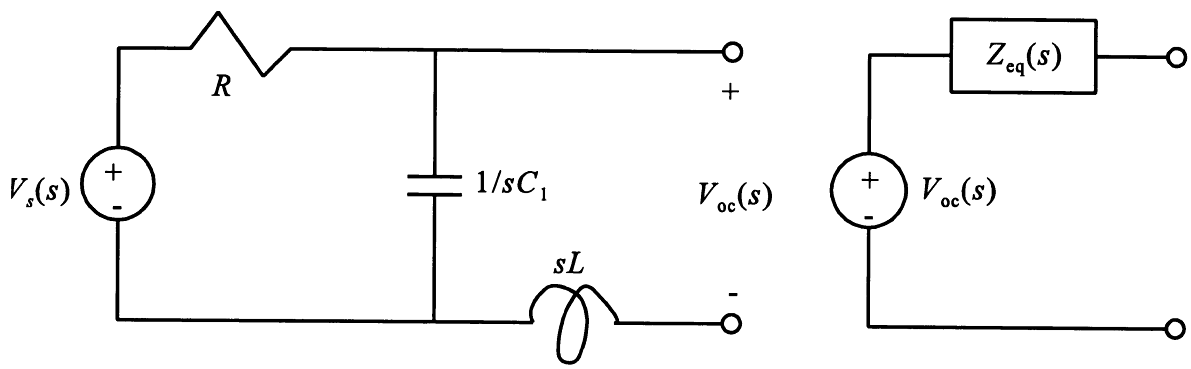
$$I_1(s) = [V_s(s) - V_1(s)]/R_1$$

When substituted for $I_1(s)$ on the right-hand side and rearranging, we obtain the following matrix equation:

$$\begin{bmatrix} 1/R_1 + 1/sL_1 + 1/sL_2 & -1/sL_{1_1} & 0 \\ -1/sL_1 - \alpha/R_1 & 1/sL_1 + sC + 1/R_2 + 1/R_3 & -1/R_3 \\ \alpha/R_1 & -1/R_3 & 1/R_3 + 1/R_4 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix} = \begin{bmatrix} V_s(s)/R_1 - i_{L_1}/s - i(0-)/s \\ i_{L_1}/s + Cv_0 - \alpha V_s(s)/R_1 \\ \alpha V_s(s)/R_1 \end{bmatrix}$$

Problem 6-5

The circuit is redrawn between the terminals a-b below:



By voltage division, the open circuit voltage is

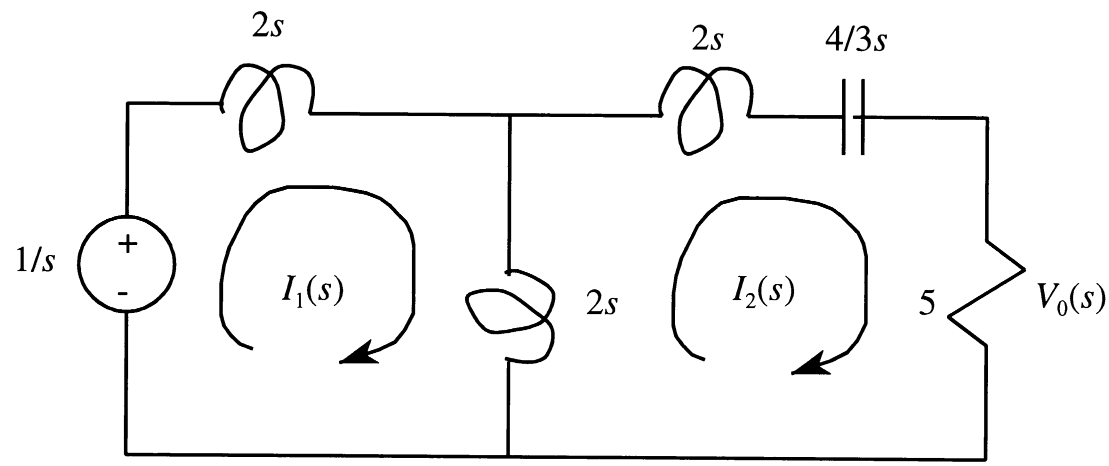
$$V_{oc}(s) = \frac{1/sC_1}{R + 1/sC_1} V_s(s) = \frac{V_s(s)}{1 + sRC_1}$$

Set $V_s(s) = 0$ and find the impedance looking back into the terminals a-b:

$$Z_{eq} = sL + \frac{1}{sC_1} \| R = \frac{s^2 LC_1 R + sL + R}{sC_1 R + 1}$$

Problem 6-6

Replace the coupled coils with a T-equivalent circuit:



Writing KVL equations around each loop, we obtain

$$\begin{bmatrix} 4s & -2s \\ -2s & 4s + \dfrac{4}{3s} + 5 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} 1/s \\ 0 \end{bmatrix}$$

Solve for $I_2(s)$ and then find the output voltage:

$$V_0(s) = 5I_2(s) = \frac{5}{6\left(s^2 + \frac{5}{3}s + \frac{4}{9}\right)} = \frac{5}{6(s + 1/3)(s + 4/3)} = \frac{A}{s + 1/3} + \frac{B}{s + 4/3}$$

The expansion coeffiients are

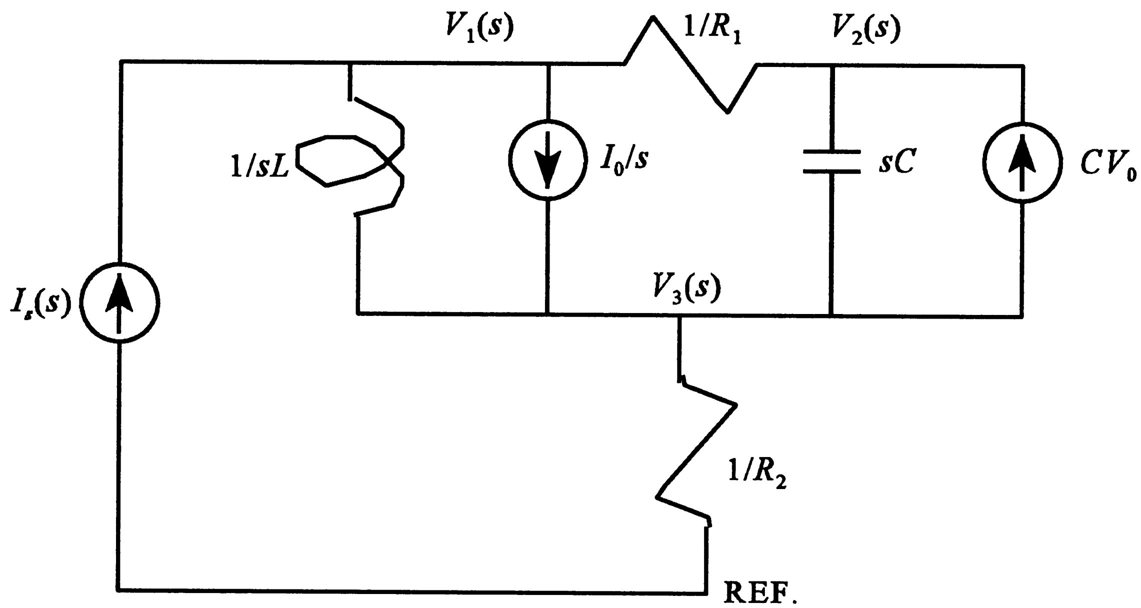
$$A = (s + 1/3)\,V_0(s)|_{s = -1/3} = \frac{5}{6} \text{ and } B = (s + 4/3)\,V_0(s)|_{s = -4/3} = -\frac{5}{6}$$

Thus the output current is

$$v_0(t) = \frac{5}{6}\left[e^{-t/3} - e^{-4t/3}\right]u(t)$$

Problem 6-7

The Laplace transform equivalent circuit with admittances and current sources for initial conditions is shown below. This form is appropriate for writing KCL equations.



By inspection, the KCL equations written at each node are given by the matrix equation

$$\begin{bmatrix} \frac{1}{sC} + \frac{1}{R_1} & -\frac{1}{R_1} & -\frac{1}{sL} \\ -\frac{1}{R_1} & \frac{1}{R_1} + sC & -sC \\ -\frac{1}{sL} & -sC & \frac{1}{sL} + \frac{1}{R_2} + sC \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix} = \begin{bmatrix} I_s(s) - \frac{I_0}{s} \\ CV_0 \\ \frac{I_0}{s} - CV_0 \end{bmatrix}$$

(b) Replace the initial condition generators with series voltage sources. Let the voltage across the current source be $V_s(s)$. Let the mesh currents be $I_1(s)$ and $I_2(s)$. Note that $I_1(s) = I_s(s)$, so $I_1(s)$ is known. Thus the first KVL equation determines $V_s(s)$. The KVL equations are

$$\begin{bmatrix} sL + R_2 & -sL \\ -sL & sL + R_1 + \frac{1}{sC} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_s(s) + LI_0 \\ -LI_0 - \frac{V_0}{s} \end{bmatrix}$$

Problem 6-8

(a) Assume clockwise mesh currents and 0 initial conditions. Then the KVL equations are

$$R_1 i_1 + R_2(i_1 - i_2) + L_1\frac{d}{dt}(i_1 - i_2) + M_{12}\frac{d}{dt}(i_2 - i_3) + M_{13}\frac{di_3}{dt} = v(t) \text{ (mesh 1)}$$

$$L_1\frac{d}{dt}(i_2 - i_1) + R_2(i_2 - i_1) + L_2\frac{d}{dt}(i_2 - i_3) + \frac{1}{C}\int_{-\infty}^{t}(i_2 - i_3)d\lambda$$

$$+ M_{12}\frac{d}{dt}(i_1 - i_2) + M_{23}\frac{d(-i_3)}{dt} + M_{13}d\frac{(-i_3)}{dt} + M_{12}\frac{d(i_3 - i_2)}{dt} = 0 \text{ (mesh 2)}$$

and $$L_2\frac{d}{dt}(i_3 - i_2) + \frac{1}{C}\int_{-\infty}^{t}(i_3 - i_2)d\lambda + M_{12}\frac{d}{dt}(i_2 - i_1)$$

$$+ M_{23}\frac{di_3}{dt} + L_3\frac{di_3}{dt} + M_{13}\frac{d}{dt}(i_1 - i_2) + M_{23}\frac{d}{dt}(i_3 - i_2) + R_3 i_3 = 0 \text{ (mesh 3)}$$

(b) The Laplace-transformed mesh equations are (the dependence of the currents and voltage on s has been suppressed to conserve space):

$$(R_1 + R_2 + sL_1)I_1 - (R_2 + sL_1 - sM_{12})I_2 + s(M_{13} - M_{12})I_3 = V$$

$$(-sL_1 + sM_{12} - R_2)I_1 + (sL_1 + R_2 + sL_2 - 2sM_{12} + \frac{1}{sC})I_2 + (sM_{12} - sM_{13} - sL_2 - \frac{1}{sC} - sM_{23})I_3 = 0$$

$$-(sM_{12} - sM_{13})I_1 - (sL_2 + \frac{1}{sC} - sM_{12} + sM_{13} + sM_{23})I_2 + (sL_2 + \frac{1}{sC} + sL_3 + R_3 + 2sM_{23})I_3 = 0$$

(c) It is impossible to use an equivalent-T circuit model for more than two coils because of the multiple coupling between coils.

Problem 6-9

Treat the $\alpha I_1(s)$ current source as an independent source initially so that the matrix equations may be written down by inspection. They are

$$\begin{bmatrix} 1/R_1 + 1/sL & -1/sL & 0 \\ -1/sL & 1/sL + 1/R_2 + sC + 1/R_3 & -1/R_3 \\ 0 & -1/R_3 & 1/R_3 + 1/R_4 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix} = \begin{bmatrix} V_s(s)/R_1 - i_0/s \\ i_0/s + Cv_0 - \alpha I_1(s) \\ \alpha I_1(s) \end{bmatrix}$$

Now substitute

$$I_1(s) = [V_1(s) - V_2(s)]/sL + i_0$$

[see Fig. 6-11 for the definition of $I_1(s)$.] Take the voltage-dependent terms to the left-hand side to obtain the modified matrix equation:

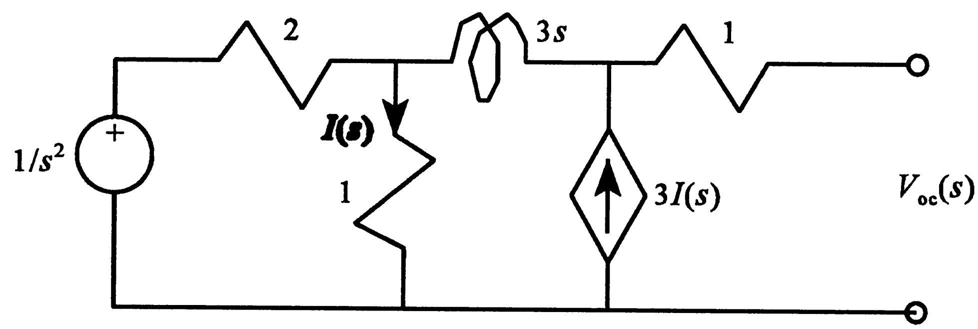
$$\begin{bmatrix} 1/R_1 + 1/sL & -1/sL & 0 \\ (\alpha - 1)/sL & (1-\alpha)/sL + 1/R_2 + sC + 1/R_3 & -1/R_3 \\ -\alpha/sL & -1/R_3 + \alpha/sL & 1/R_3 + 1/R_4 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix} = \begin{bmatrix} V_s(s)/R_1 - i_0/s \\ i_0/s + Cv_0 - \alpha i_0 \\ \alpha i_0 \end{bmatrix}$$

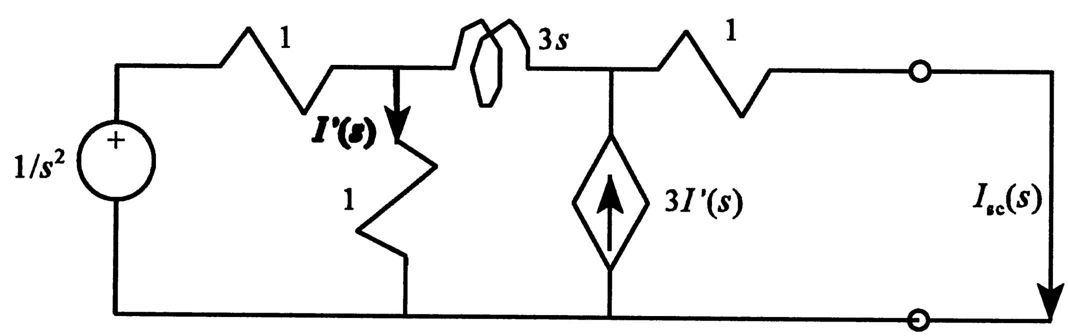
Problem 6-10

(a) Use the fact that

$$Z_{eq}(s) = V_{oc}(s)/I_{sc}(s)$$

The first circuit below may be used to find $V_{os}(s)$ and the second circuit to find $I_{sc}(s)$.





The KCL equations for the first circuit are

$$\frac{1}{2}\left(V_a - \frac{1}{s^2}\right) + I + \frac{V_a - V_b}{3s} = 0, \; V_a = I$$

$$\frac{V_b - V_a}{3s} = 3I = 3V_a$$

Solve the second equation for V_b in terms of V_a:

$$V_b = (9s + 1)V_a$$

Substitute into the first equation to get

$$V_b = V_{oc} = (9s + 1)V_a = \frac{3(9s + 1)}{s^2}$$

Now work with the second circuit to get the short circuit current. Note that all voltages and currents are different than for the first circuit. The node voltage equations are

$$\frac{1}{2}\left(V_a - \frac{1}{s^2}\right) + I' + \frac{V_a - V_b}{3s} = 0, \; V_a = I'$$

$$\frac{V_b - V_a}{3s} - 3I' + V_b = 0$$

Solve the second for V_a in terms of V_b:

$$V_a = \frac{1 + 3s}{1 + 9s} V_b$$

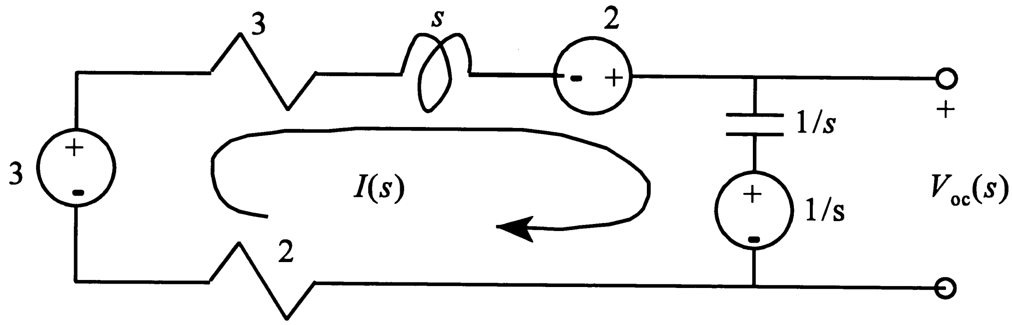
Substitute into the first equation and solve for V_b. Note that $I_{sc} = V_b/1$:

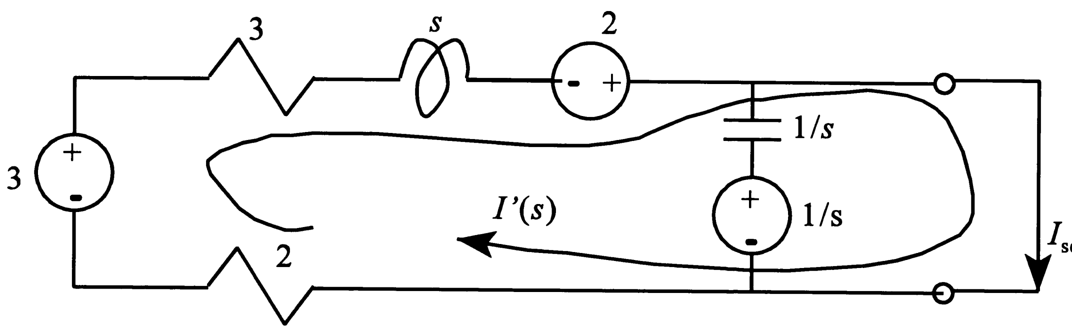
$$I_{sc} = \frac{V_b}{1} = \frac{9s + 1}{9s - 1}\frac{1}{2s^2}$$

Divide V_{oc} by I_{sc} to get Z_T. These results may then be used in the circuits of Fig. 6-9.

$$Z_T = 6(9s - 1)$$

(b) First do a Thevenin-to-Norton equivalent of the left-hand current source and parallel resistor. The resulting open-circuit and short-circuit Laplace-transform euqivalent circuits are shown below:





Use the top circuit to get open circuit voltage:

$$\left(2+3+5+\frac{1}{s}\right)I(s) = 3+2-\frac{1}{s} \text{ or } I(s) = \frac{5s-1}{s^2+5s+1}$$

Use Ohm's law and KVL to get the open circuit voltage:

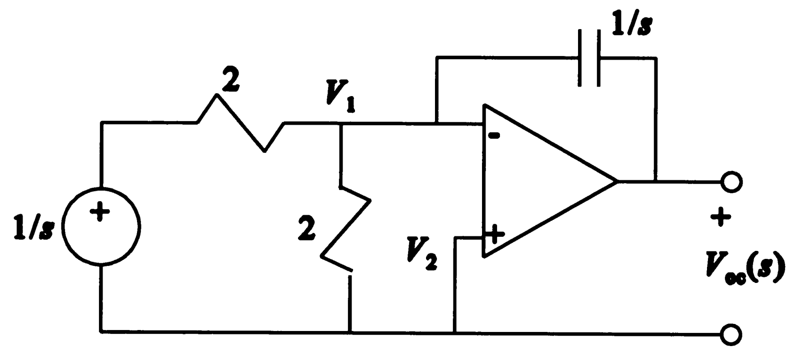
$$V_{oc}(s) = I(s)/s + 1/s = (s+10)/(s^2+5s+1)$$

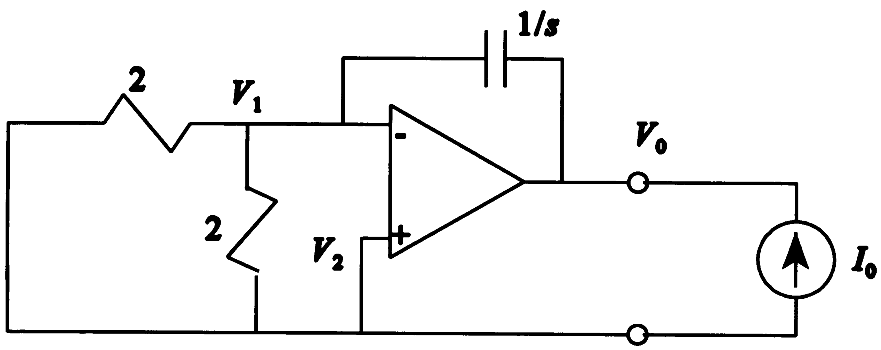
Use the lower circuit to get the short circuit current:

$$(2+3+s)I'(s) = 3+2 \text{ or } I'(s) = I_{sc}(s) = 5/(s+5)$$

The Thevenin equivalent impedance is $Z_T(s) = V_{oc}/I_{sc} = (s+5)(s+10)/[5(s^2+5s+1)$. These results may then be used in the circuits of Fig. 6-9.

(c) Find the open circuit voltage for this circuit and then use the test source at the output method. The circuits shown below are gemane.





For the top circuit, KCL at node 1 gives

$$\frac{V_1 - 1/s}{2} + \frac{V_1}{2} + \frac{V_1 - V_{oc}}{1/s} = 0$$

By the high gain/infinite input impedance properties of the operational amplifier $V_1 = V_2 = 0$, so

$$V_{oc}(s) = -\frac{1}{2s^2}$$

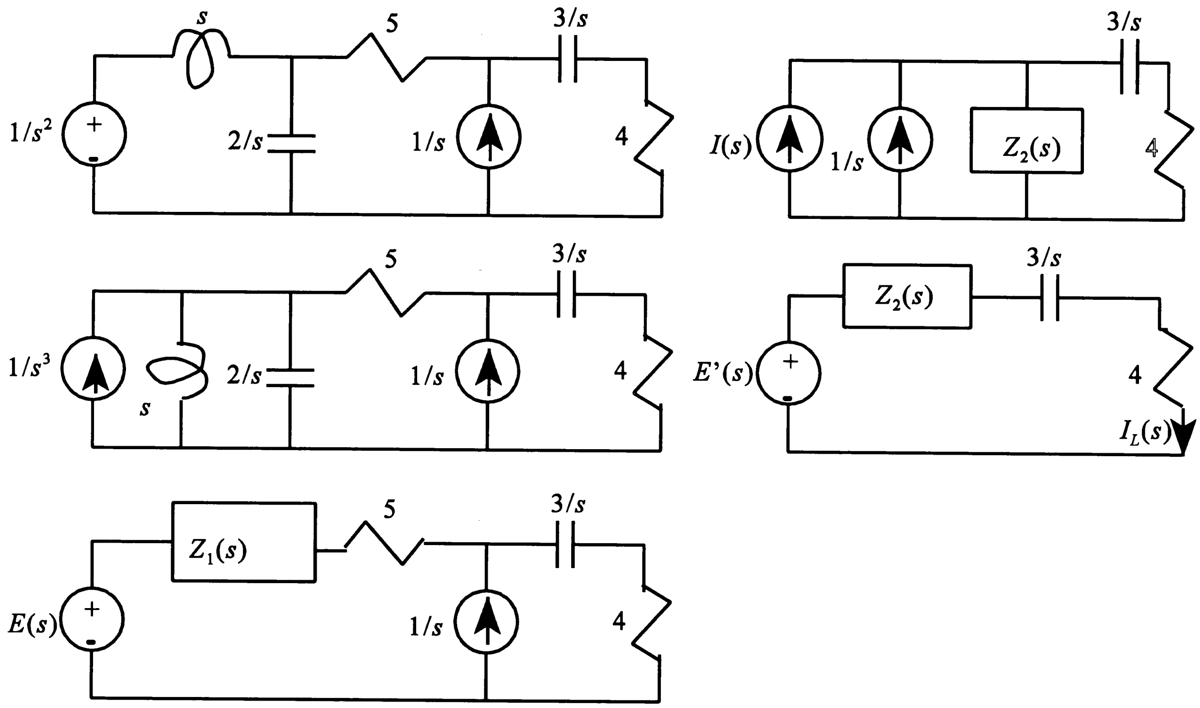
To find the equivalent impedance, use the lower circuit, remembering that the voltages are different than for the first circuit. The KCL equation at the input is

$$\frac{V_1}{2} + \frac{V_2}{2} + \frac{V_1 - V_2}{1/s} = 0$$

Again use the fact that the voltages at the two op-amp inputs are approximately equal and 0. From this and from the above equation, we deduce that $V_0 = 0$ for arbitrary I_0. Therefore, $Z_T = 0$ and the Thevenin equivalent circuit consists solely of a voltage source of value V_{oc}. There is no Norton equivalent.

Problem 6-11

See the series of Thevenin-Norton source transformations below:



To get from the first to the second cirucit on the left, we use a Thevenin-to-Norton source transformation. To get from the second to the third, we use a Norton-to-Thevenin source transformation. The voltage and impedance expressions for the third ciruit are, respectively,

$$E(s) = \frac{2}{s^2(s^2+2)} \quad \text{and} \quad Z_1(s) = \frac{2s}{s^2+2}$$

To get to the top circuit on the right, use another Thevenin-to-Norton transformation. The source current and impedance are, respectively,

$$I(s) = \frac{E(s)}{Z_1(s) + 5} = \frac{2}{s^2(5s^2 + 2s + 10)} \text{ and } Z_2(s) = 5 + \frac{2s}{s^2 + 2} = \frac{5s^2 + 2s + 10}{s^2 + 2}$$

To get to the last circuit, we combine the parallel current sources and do another Norton-to-Thevenin transformation. The voltage source in this last circuit is given by

$$E'(s) = I'(s)Z_2(s) = \frac{5s^3 + 2s^2 + 10s + 2}{s^2(s^2 + 2)}$$

Writing a KVL around the loop, we have

$$I_L(s) = \frac{\dfrac{5s^3 + 2s^2 + 10s + 2}{s^2(s^2 + 2)}}{\dfrac{3}{s} + \dfrac{5s^2 + 2s + 10}{s^2 + 2} + 4} = \frac{5s^3 + 2s^2 + 10s + 2}{s(9s^3 + 5s^2 + 18s + 6)}$$

Problem 6-12

Laplace transforming (6-66) with initial conditions assumed 0 results in

$$[a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0]Y(s) = [b_m^m + b_{m-1} s^{m-1} + \cdots + b_0]X(s)$$

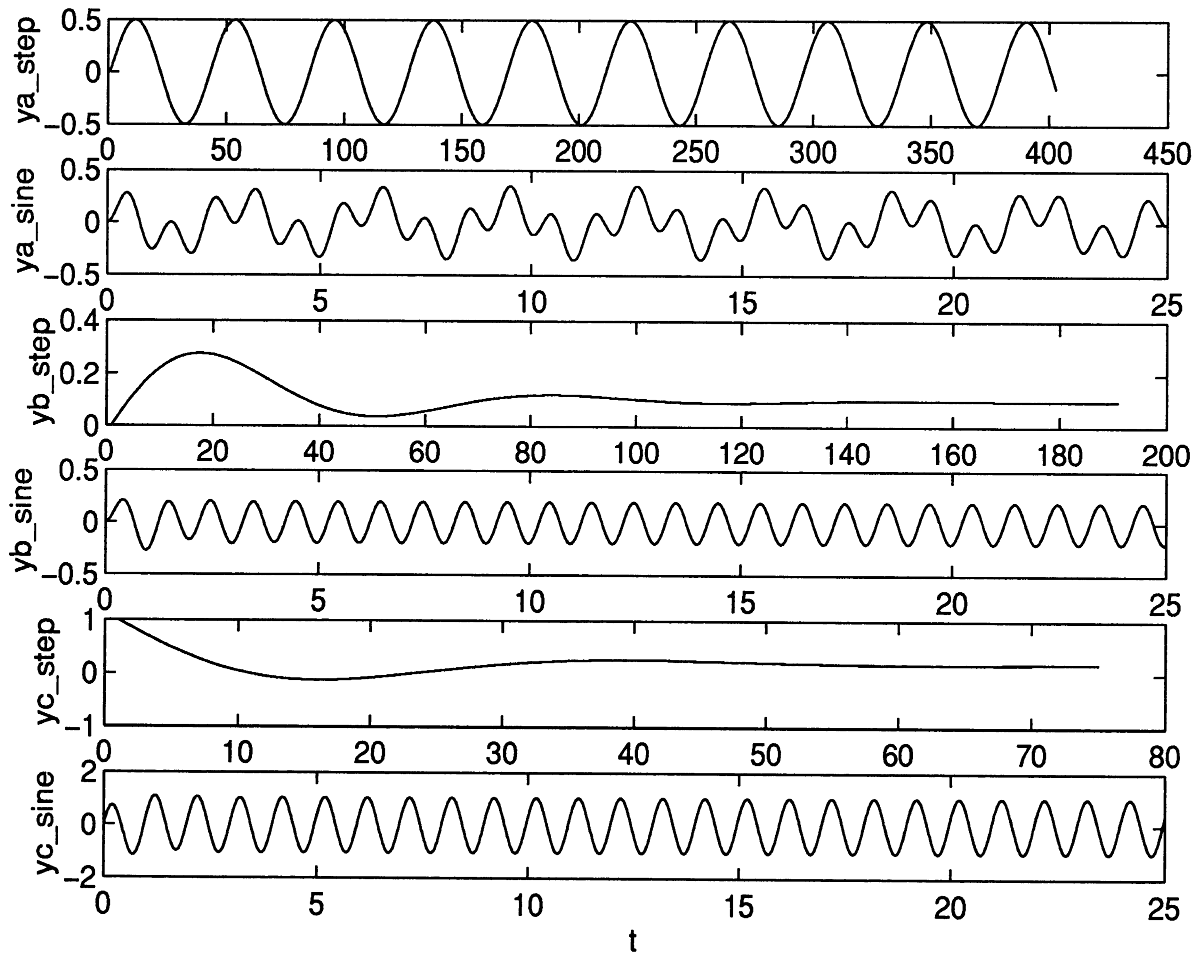
Solving for the ratio $Y(s)/X(s)$ gives (6-65).

Problem 6-13

The first three are satisfactory. The fourth is not because the degree of the numerator exceeds that of the denominator. The fifth is not because it has a pole in the right-half plane. A MATLAB program for doing the plots is given below. Plots follow on the next page.

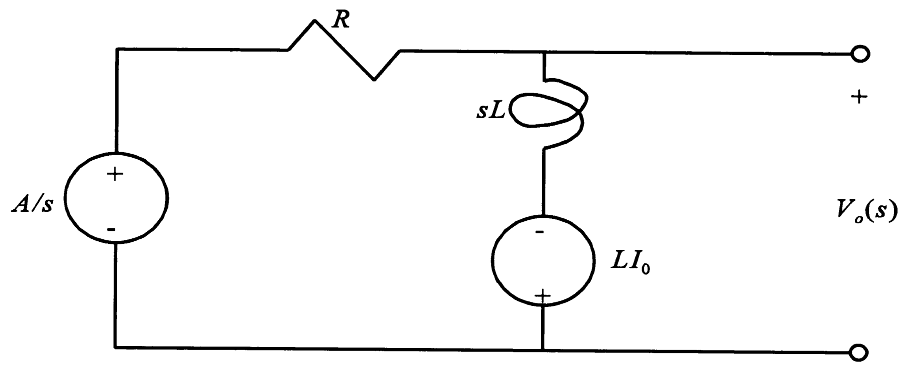
```
%      Plots for Problem 6-13
%
clg
t = 0:.01:25;
na = [0 1 0];
da = [1 0 4];
x = sin(2*pi*t);
ya_step = step(na, da);
```

```
ya_sine = lsim(na, da, x, t);
nb = [0 1 1];
db = [1 2 10];
yb_step = step(nb, db);
yb_sine = lsim(nb, db, x, t);
nc = [1 0 1];
dc = [1 2 5];
yc_step = step(nc, dc);
yc_sine = lsim(nc, dc, x, t);
subplot(6,1,1),plot(ya_step),ylabel('ya_step')
subplot(6,1,2),plot(t,ya_sine),ylabel('ya_sine')
subplot(6,1,3),plot(yb_step),ylabel('yb_step')
subplot(6,1,4),plot(t,yb_sine),ylabel('yb_sine')
subplot(6,1,5),plot(yc_step),ylabel('yc_step')
subplot(6,1,6),plot(t,yc_sine),xlabel('t'),ylabel('yc_sine')
```



Problem 6-14

The Laplace-transform equivalent circuit is shown below:



Use superposition. Due to the source, the output, by voltage division, is

$$V_s(s) = \frac{sL}{R + sL}\frac{A}{s} = \frac{A}{s + R/L}$$

Due to the initial condition source, the output is

$$V_{IC}(s) = \frac{sLI_0}{s + R/L} - LI_0 = -\frac{RI_0}{s + R/L}$$

Thus, the Laplace transform of the total output voltage is

$$V_o(s) = \frac{A}{s + R/L} - \frac{RI_0}{s + R/L}$$

and the time-domain output is

$$v_o(t) = Ae^{-Rt/L}u(t) - RI_0e^{-Rt/L}u(t)$$

The entire output is transient (goes away as $t \to \infty$). The first term in the above result is the ZSR (depends only on the input), and the second term is the ZIR (depends only on the initial state of the circuit, not on the input).

Problem 6-15

From the figure,

$$V_2(s) = -Z_L(s)I_2(s) \quad (1); \quad V_0(s) = R_0I_1(s) + V_1(s) \quad (2)$$
$$I_1(s) = Y_{11}(s)V_1(s) + Y_{12}(s)V_2(s) \quad (3)$$
$$I_2(s) = Y_{21}(s)V_1(s) + Y_{22}(s)V_2(s) \quad (4)$$

Substitute (3) into the (2) to get

$$V_0 = R_0(Y_{11}V_1 + Y_{12}V_2) + V_1 \qquad (5)$$

Solve (5) for V_1:

$$V_1 = \frac{V_0 - R_0Y_{12}V_2}{R_0Y_{11} + 1} \qquad (6)$$

Substitute (6) into (4) to solve for $I_2(s)$ and substitute that result into (1):

$$V_2(1 + Z_LY_{22}) = Z_LY_{21}\left[\frac{-V_0 + R_0Y_{12}V_2}{R_0Y_{11} + 1}\right] \quad \text{or} \quad V_2 = -\frac{Z_LY_{21}(V_0 - R_0Y_{12}V_2)}{1 + R_0Y_{11}} - Z_LY_{12}V_2$$

Solve this equation for the ratio of V_2 to V_0 get the overall transfer function, including the effect of source resistance and load impedance. The result is

$$H(s) = -\frac{Z_LY_{21}}{R_0Y_{11} + 1 + Z_LY_{22}R_0Y_{11} + Z_LY_{22} - Z_LY_{21}Y_{12}R_0}$$

This may be rearranged to give the result of the problem statement.

Problem 6-16

(a) With the open circuit, $I_2 = 0$ and

$$Y_{21}V_1 = -Y_{22}V_2$$

from the equation defining the two-port parameters (see the statement for Problem 6-15). Thus,

$$H(s) = \frac{V_2}{V_1} = -\frac{Y_{21}}{Y_{22}}$$

But

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0} \quad \text{and} \quad Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0}$$

The former equation means that we short circuit the input and apply V_2 to the output; the latter equation means that we short circuit the output and apply V_1 to the input. Using the former equation (i.e., short circuiting the input and driving the output by V_2), the nodal equations become

$$sCV_2 + sC(V_3 - V_2) + 2V_3/R = 0 \quad (1)$$
$$V_4/R + (V_4 - V_2)/R + 2CsV_4 = 0 \quad (2)$$
$$sC(V_2 - V_1) + (V_2 - V_4)/R = I_2 \quad (3)$$

where the definitions for V_3 and V_4 given in the problem statement are used. Solve (1) and (2) for V_3 and V_4 and substitute into (3). The result is

$$V_2\left[\frac{1}{R} + sC - \frac{(sC)^2}{2/R + 2Cs} - \frac{(1/R)^2}{2/R + 2Cs}\right] = I_2$$

Solve for the ratio I_2/V_2 (recall that this is for $V_1 = 0$):

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0} = \frac{1/R^2 + 4Cs/R + (sC)^2}{2/R + 2Cs} \quad \text{(A)}$$

Now assume the output is short circuited with the input being driven by V_1. The nodal equations are

$$sC(V_3 - V_1) + sCV_3 + 2V_3/R = 0 \quad (4)$$
$$(V_4 - V_1)/R + 2CsV_4 + V_4/R = 0 \quad (5)$$
$$sCV_3 + V_4/R = I_2 \quad (6)$$

Solve (4) and (5) for V_3 and V_4 in terms of V_1 and substitute into (6). The result is

$$I_2 = \left[\frac{(sC)^2}{2Cs + 2/R} + \frac{(1/R)^2}{2Cs + 2/R} \right] V_1$$

(Recall that this is for $V_2 = 0$.) Now solve for the ratio I_2/V_1:

$$Y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2 = 0} = \frac{(sC)^2 + 1/R^2}{2Cs + 2/R} \quad \text{(B)}$$

(b) Use (A) and (B) in $H(s) = -Y_{21}/Y_{22}$. This gives the transfer function

$$H(s) = \frac{s^2 + 1/(RC)^2}{s^2 + \frac{4}{RC}s + \frac{1}{(RC)^2}}$$

The zeros are at

$$z_{1,2} = \pm j(1/RC)$$

and the poles are given by

$$p_{1,2} = -\frac{2}{RC} \pm \sqrt{\frac{3}{(RC)^2}} = -\frac{3.73}{RC}, -\frac{0.27}{RC}$$

(c) The amplitude response is given by

$$|H(j\omega)| = \frac{\left| \omega^2 - \frac{1}{(RC)^2} \right|}{\sqrt{\left(\frac{1}{(RC)^2} - \omega^2 \right)^2 + \frac{16\omega^2}{(RC)^2}}}$$

Note that $H(0) = 1$. We want the amplitude response at frequency ω_c to be 0.707, or the amplitude response squared to be 0.5. That is,

$$\frac{1}{2} = \frac{\left|\omega_c^2 - \frac{1}{(RC)^2}\right|}{\sqrt{\left(\frac{1}{(RC)^2} - \omega_c^2\right)^2 + \frac{16\omega_c^2}{(RC)^2}}}$$

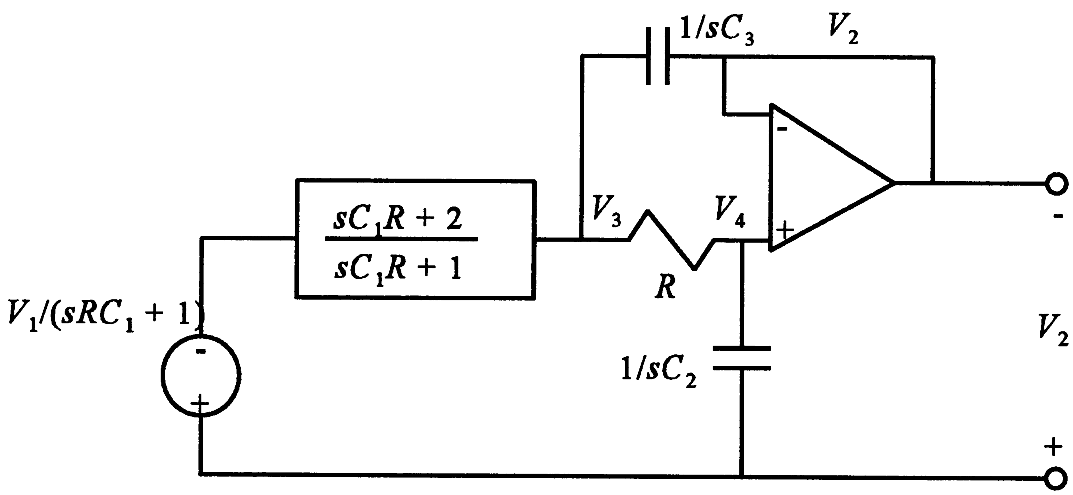
The solutions of this equation are

$$\omega_c = \frac{2}{RC} \pm \frac{\sqrt{5}}{RC} \quad \text{or} \quad (\omega_c RC)^2 = (\sqrt{5} \pm 2)^2$$

(d) We want the zero at $\omega_z = (2\pi)(60) = 1/RC$. For $C = 10^{-6}$ F, we get $R = 10^6/120\pi = 2.65$ kΩ.

Problem 6-17

(a) Do a Thevenin-to-Norton equivalent of the input, and a parallel combination of R and $1/sC_1$; then combine $R \| 1/sC_1$ and the second R in series to finally end up with the Laplace-equivalent circuit shown below:



Write a KCL equation at node 3:

$$\frac{V_3 - V_1/(sC_1R + 1)}{(sC_1R + 2)/(sC_1R + 1/R)} + \frac{V_3 - V_2}{R} + sC_3(V_3 - V_2) = 0 \qquad (1)$$

Write a KCL equation at node 4:

$$\frac{V_4 - V_3}{R} + sC_2V_4 = 0 \tag{2}$$

Because of the ideal operational amplifier, $V_2 = V_4$. We may use (3) to replace V_4 by V_2 in (1) and (2). Then from (2) with the substitution, we obtain

$$V_3 = (1 + sC_2R)\,V_2 \tag{3}$$

Replace V_3 in (1) using (3). We then solve for V_2/V_1 to get

$$H(s) = \frac{1}{R^3C_1C_2C_3s^3 + 2R^2C_2(C_1 + C_3)s^2 + R(C_1 + 3C_2)s + 1}$$

or

$$H(s) = \frac{1}{R^3C_1C_2C_3}\frac{1}{s^3 + \frac{2}{R}\left(\frac{1}{C_1} + \frac{1}{C_3}\right) + \frac{1}{R^2C_3}\left(\frac{3}{C_1} + \frac{1}{C_2}\right)s + \frac{1}{R^3C_1C_2C_3}}$$

With the parameter values given, the following MATLAB program (echo on) gives the values of the denominator polynomial coefficients, the roots or poles, and compares the complex poles obtained with what should be the case for the complex poles of a Butterworth filter with the real root matching.

```
%      Solution for Problem 6-17
%
C1 = 0.0022*10^(-6);
C2 = 0.00033*10^(-6);
C3 = 0.0056*10^(-6);
R = 10000;
A = (2/R)*(1/C1+1/C3)
A =
 1.2662e+005
B = (1/(R^2*C3))*(3/C1+1/C2)
B =
 7.8463e+009
C = 1/(R^3*C1*C2*C3)
```

```
C =
  2.4597e+014
p = [1 A B C];
s_poles = roots(p)
s_poles =
  1.0e+004 *
  -6.4078
  -3.1273 + 5.3484i
  -3.1273 - 5.3484i
s_poles_butter = s_poles(1)*(cos(pi/3)+i*sin(pi/3))
s_poles_butter =
-3.2039e+004- 5.5493e+004i
```

Note that the poles of the designed filter and the true Butterworth filter match up fairly well. One could adjust the *R* and *C* values to give a closer match.

Problem 6-18

Using the ciruit diagram given in the problem statement, we find the following KCL equations at nodes 3, 4, and 5:

$$\frac{V_3 - V_1}{R_1} + sC_1(V_3 - V_2) + \frac{V_3 - V_4}{R_2} = 0 \text{ (node 3)}$$

$$\frac{V_4 - V_3}{R_2} + sC_2V_4 = 0 \text{ (node 4)}$$

$$\frac{V_5 - V_2}{(1 - \beta)R} + \frac{V_5}{\beta R} = 0 \text{ (node 5)}$$

By the infinite gain property of the ideal operational amplifier, we have $V_4 = V_5$. From the node 5 equation, we have

$$V_5 = \beta V_2 = V_4 \qquad \text{(A)}$$

Combine the node 4 equation and (A) to get

$$V_3 = \beta(1 + sC_2R_2)V_2 \qquad \text{(B)}$$

Use $V_4 = \beta V_2$ in the node 3 equation to get

$$V_3\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_1\right) - V_2\left(sC_1 + \frac{\beta}{R_2}\right) = \frac{V_1}{R_1}$$

Substitute for V_3 from (B) and take the ratio of V_2/V_1 to get

$$H(s) = \frac{V_2}{V_1} = \frac{1/(R_1C_1R_2C_2\beta)}{s^2 + \left[\frac{1}{R_1C_1} + \frac{1}{R_2C_1} + \left(1 - \frac{1}{\beta}\right)\frac{1}{R_2C_2}\right]s + \frac{1}{R_1C_1R_2C_2}}$$

Using the definitions of ω_0 and α given in the problem statement, it is seen that this may be put into the form given there. Note that R does not appear in the final result.

(b) The poles are given by

$$s_{1,2} = -\alpha \pm 0.5\sqrt{4\alpha^2 - 4\omega_0^2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} \text{ if } \omega_0 > \alpha$$

Suppose that we want poles at

$$s_{1,2} = -\omega_3[\cos\theta \pm j\sin\theta]$$

where ω_3 is the radius of a semicircle in the complex plane centered at the origin and θ is the angle up from the $-\sigma$ axis. Thus,

$$-\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\omega_3[\cos\theta \pm j\sin\theta]$$

Equating real and imaginary parts and solving for ω_3 and α, we obtain

$$\omega_0 = \omega_3 \text{ and } \alpha = \omega_3\cos\theta$$

Now assume that $R_1 = R_2$ and $C_1 = C_2$. Thus, from the definitions of α and ω_0 and the coefficents found for $H(s)$, we have

$$2\alpha = \frac{1}{RC}\left[3 - \frac{1}{\beta}\right] \text{ and } \omega_0^2 = \frac{1}{RC}$$

Combine to obtain

$$\beta = \frac{1}{3 - 2\cos\theta}$$

For a two-pole filter, the poles are at ±45 degrees with respect to the -σ axis (θ = ±45°). Thus,

$$\beta = \frac{1}{3 - 2\cos\pi/4} = 0.631$$

Also, for a 3-dB frequency of 10000 Hz, we have

$$\omega_3 = \omega_0 = \frac{1}{R_1C_1} \quad (R_1C_1 = R_2C_2)$$

or

$$R_1C_1 = \frac{1}{\omega_3} = \frac{10^{-4}}{2\pi} = 1.59\times10^{-5} \text{ seconds}$$

(c) For a four-pole Butterworth filter, cascade two stages with θ = ±22.5° for the first stage and θ = ±67.5° for the second stage. The design is summarized in the table below:

Stage	θ (degrees)	β
1	22.5	0.868
2	67.5	0.448

Problem 6-19

To determine the RHP poles, use the Routh array.
(a) The polynomial is $s^3 + 6s^2 + 3s$ 2, and the Routh array is

$$\begin{array}{llll} s^3: & 1 & 3 & 0 \\ s^2: & 6 & 2 & 0 \\ s^1: & 8/3 & 0 & \\ s^0: & 2 & 0 & \end{array}$$

Since there are no sign changes in the left-hand column, there are no RHP roots.

(b) The polynomial is $s^4 + 3s^3 + 12s^2 + 12s + 36$, and the Routh array is

$$\begin{array}{lcccc} s^4: & 1 & 12 & 36 & 0 \\ s^3: & 3 & 12 & 0 & 0 \\ s^2: & 8 & 36 & 0 & \\ s^1: & -3/2 & 0 & 0 & \\ s^0: & 36 & 0 & & \end{array}$$

Since there are two sign changes in the first column, there are two RHP roots.
(c) The polynomial is $s^4 + s^3 - s - 1$, and the Routh array is

$$\begin{array}{lccccl} s^4: & 1 & 0 & -1 & 0 & \\ s^3: & 1 & -1 & 0 & 0 & \\ s^2: & 1 & -1 & 0 & & \text{(aux. poly.)} \\ s^1: & 0(2) & 0 & 0 & & \\ s^0: & -1 & 0 & & & \end{array}$$

The 4th row was obtained by differentiating the auxilary polynomial obtained from the row directly above. It is $s^2 - 1$. Since there is one sign change in the first column, there is one RHP root.
(d) The polynomial is $4s^5 + s^4 + 4s^3 + s^2 + 15s + 10$, and the Routh array is

$$\begin{array}{lcccc} s^5: & 4 & 4 & 15 & 0 \\ s^4: & 1 & 1 & 10 & 0 \\ s^3: & \epsilon & -25 & 0 & 0 \\ s^2: & \dfrac{\epsilon + 25}{\epsilon} & 10 & 0 & \\ s^1: & -\left(25 + \dfrac{10\epsilon^2}{\epsilon + 25}\right) & 0 & 0 & \\ s^0: & 10 & 0 & & \end{array}$$

As $\epsilon \to 0$, the 5th entry of the 1st column becomes negative. There are two sign changes in the 1st column of the Routh array, which means that there are two right-half plane roots.

Problem 6-20

(a) The denominator of the transfer function gives the Routh array

$$\begin{array}{llll} s^3: & 1 & 11 & 0 \\ s^2: & 7 & 5+K & 0 \\ s^1: & 11 - \dfrac{5+K}{7} & 0 & 0 \\ s^0: & 5+K & 0 & 0 \end{array}$$

All elements of the first column must be positive, so we require that

$$11 - \frac{5+K}{7} > 0 \text{ and } 5 + K > 0$$

The first inequality gives $K < 72$ and the second inequality gives $K > -5$, or

$$-5 < K < 72$$

(b) The denominator polynomial gives the Routh array

$$\begin{array}{lllll} s^4: & 1 & 2 & K & 0 \\ s^3: & 2 & 1 & 0 & 0 \\ s^2: & 3/2 & K & 0 & \\ s^1: & 1 - 4K/3 & 0 & 0 & \\ s^0: & K & 0 & 0 & \end{array}$$

All elements of the 1st column must be positive, giving

$$1 - 4K/3 > 0 \text{ and } K > 0$$

Solution of this pair of inequalities gives

$$0 < K < 3/4$$

Problem 6-21

The Routh array is

$$\begin{array}{llll} s^3: & 1 & b & 0 \\ s^2: & a & c & 0 \\ s^1: & \dfrac{ab-c}{a} & 0 & 0 \\ s^0: & c & 0 & 0 \end{array}$$

Requiring all elements in the first column to be positive results in

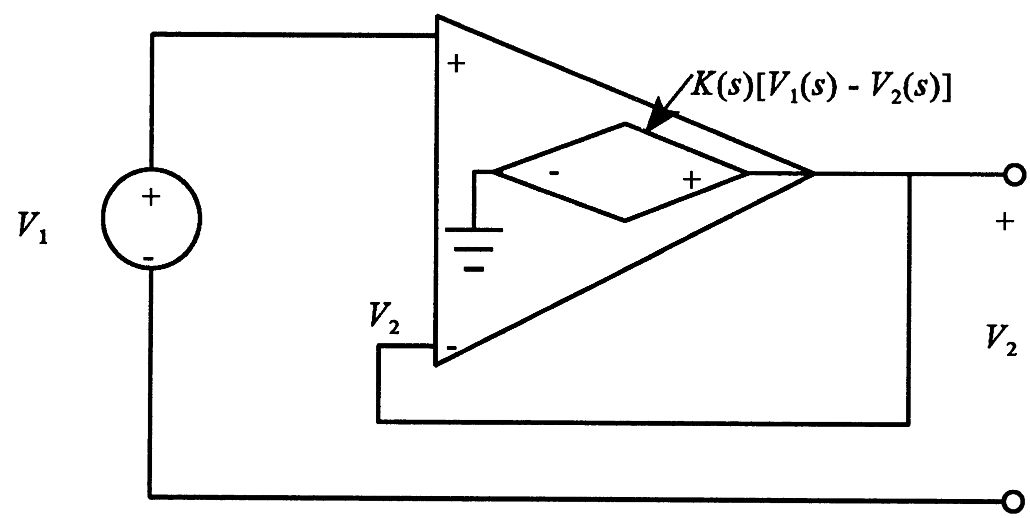
$$a > 0;\ ab - c > 0;\ c > 0$$

Combining the inequalities results in

$$b > \frac{c}{a} > 0$$

Problem 6-22

The Laplace-transformed equivalent circuit diagram is



Since

$$V_2(s) = K(s)[V_1(s) - V_2(s)]$$

we have

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{K(s)}{K(s) + 1}$$

where

$$K(s) = \frac{A_0}{(1 + s/\omega_{p_2})(1 + \omega_{p_2})^2}$$

Substituting into the expression for $H(s)$, we have

$$H(s) = \frac{A_0 \omega_{p_1} \omega_{p_2}{}^2}{s^3 + (\omega_{p_1} + 2\omega_{p_2})s^2 + (2\omega_{p_1}\omega_{p_2} + \omega_{p_2}{}^2)s + \omega_{p_1}\omega_{p_2}{}^2(1 + A_0)}$$

The Routh array is

$$\begin{array}{llll}
s^3: & 1 & 2\omega_{p_1}\omega_{p_2} + \omega_{p_2}{}^2 & 0 \\
s^2: & \omega_{p_1} + 2\omega_{p_2} & \omega_{p_1}\omega_{p_2}{}^2(1 + A_0) & 0 \\
s^1: & \omega_{p_1}\left[2\omega_{p_1} + \omega_{p_2} - \dfrac{\omega_{p_1}\omega_{p_2}(1 + A_0)}{\omega_{p_1} + 2\omega_{p_2}}\right] & 0 & 0 \\
s^0: & \omega_{p_1}\omega_{p_2}(1 + A_0) & 0 & 0
\end{array}$$

We require that each element of the first column to be greater than zero. All cutoff frequencies and the dc gain are assumed greater than zero. Therefore, the only element in question is the third row one. For it to be greater than zero, we require that

$$2\omega_{p_1} + \omega_{p_2} - \frac{\omega_{p_1}\omega_{p_2}(1 + A_0)}{\omega_{p_1} + 2\omega_{p_2}} > 0$$

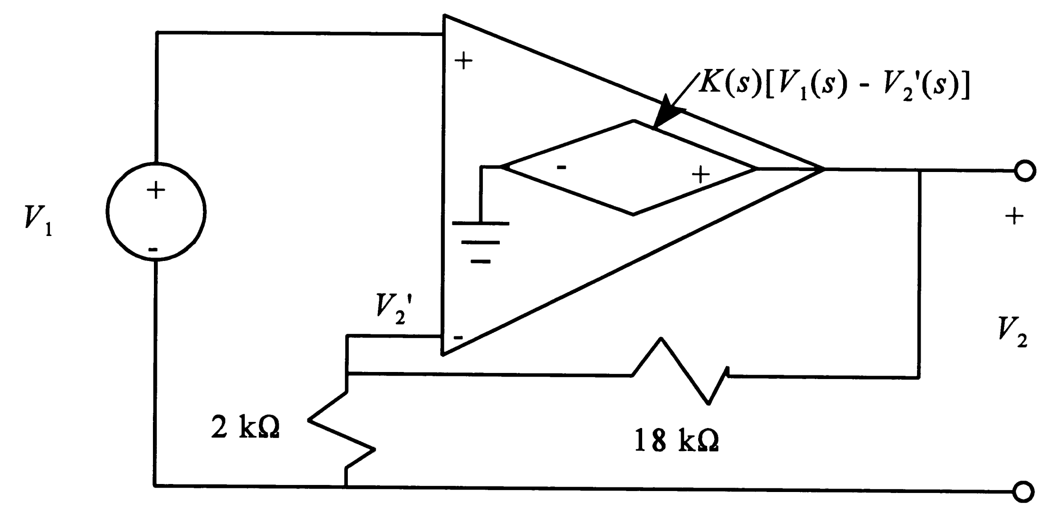
The parameter values are given in the following table:

Type	A_0	ω_{p1}	ω_{p2}	$2\omega_{p_1} + \omega_{p_2} - \dfrac{\omega_{p_1}\omega_{p_2}(1 + A_0)}{\omega_{p_1} + 2\omega_{p_2}}$
741	10^5	20π	$2\pi\times10^6$	$\approx \pi \times10^6$
748	10^5	200π	$2\pi\times10^6$	$\approx -8\pi \times10^6$

From the last column of the above table, it is seen that the type 748 results in an unstable amplifier configuration.

Problem 6-23

The Laplace-transformed equivalent circuit is shown below:



The KCL equation at the negative-reference input terminal of the operational amplifier is

$$\frac{V_2' - V_2}{18\times10^3} + \frac{V_2'}{2000} = 0 \text{ or } V_2' = \frac{V_2}{10}$$

Thus

$$V_2 = K(V_1 - V_2') = K\left(V_1 - \frac{V_2}{10}\right)$$

and the transfer function is

$$H(s) = \frac{V_2}{V_1} = \frac{K}{1 + K/10} = \frac{10K}{K + 10}$$

But

$$K(s) = \frac{A_0}{(1 + s/\omega_{p_2})(1 + \omega_{p_2})^2}$$

so the transfer function becomes

$$H(s) = \frac{A_0 \omega_{p_1} \omega_{p_2}{}^2}{s^3 + (\omega_{p_1} + 2\omega_{p_2})s^2 + (2\omega_{p_1}\omega_{p_2} + \omega_{p_2}{}^2)s + \omega_{p_1}\omega_{p_2}{}^2(1 + A_0/10)}$$

If we now do a Routh array, we see that the only difference from the condition for stability derived in Problem 6-22 is that A_0 is replaced by $A_0/10$. Thus, the condition is

$$2\omega_{p_1} + \omega_{p_2} - \frac{\omega_{p_1}\omega_{p_2}(1 + A_0/10)}{\omega_{p_1} + 2\omega_{p_2}} > 0$$

With the dc gain divided by 10, it is now found that both configurations (using the type 741 and 748 operational amplifiers) are stable.

Problem 6-24

(a) Rewrite the transfer function as

$$H(s) = \frac{5(s/10 + 1)}{(s + 1)(s/2 + 1)}$$

The corner frequencies for the poles are $\omega = 1$ and $\omega = 2$ radians per second. There is a corner frequency for the zero at $\omega = 10$ radians per second. The gain is $20 \log_{10}(5) = 14$ dB. Bode plots for amplitude and phase are given at the end of the problem.
(b) Rewrite the transfer function as

$$H(s) = \frac{5/9}{[(s/3)^2 + 1)](s + 1)}$$

The corner frequency for the simple pole factor is at $\omega = 1$ radians per second and the corner frequency for the quadratic pole factor is at $\omega = 3$ radians per second with $\varsigma = 0$. The gain is factor is $20 \log_{10}(5/9) = -5.1$ dB. Bode plots for amplitude and phase are given at the end of the problem.
(c) Rewrite the transfer function as

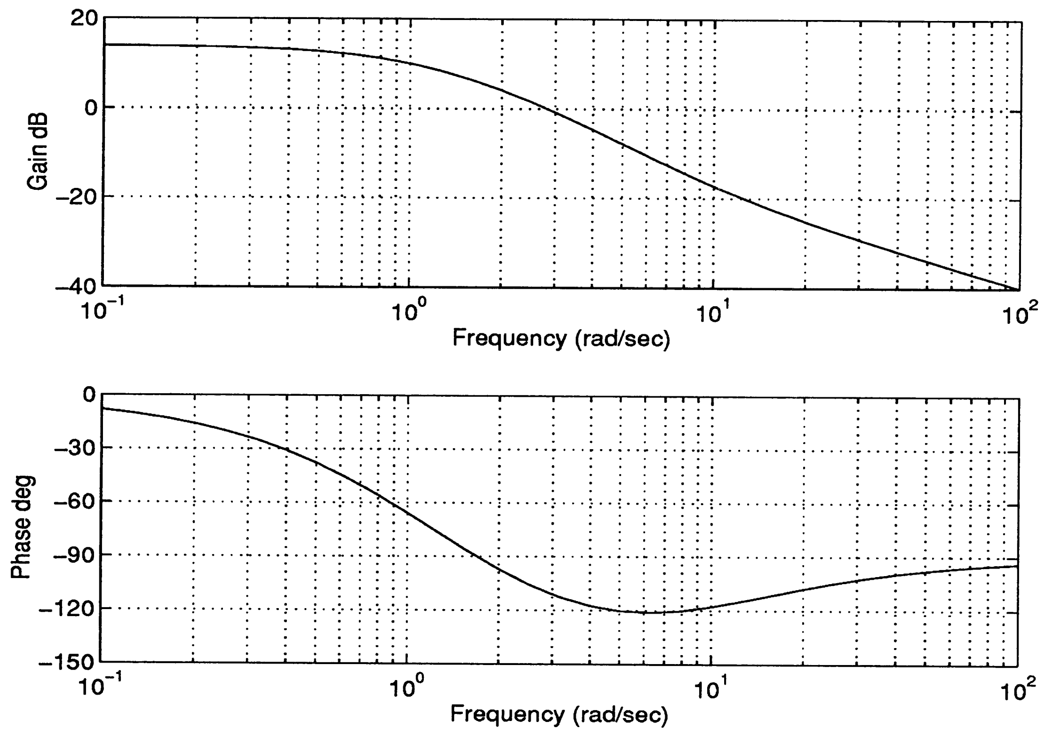
$$H(s) = \frac{3.5[(s/\sqrt{7})^2 + 2(2.5/\sqrt{7})(s/\sqrt{7}) + 1]}{(s/2 + 1)(s + 1)}$$

The corner frequency for the pole factors are at $\omega = 1$and $\omega = 2$ radians per second. The corner frequency for the quadratic zero factor is at $\omega = 7^{1/2} = 2.65$ radians per second with $\varsigma = 2.5/7^{1/2} = 0.94$. The gain is factor is $20 \log_{10}(3.5) = 10.9$ dB. Bode plots for amplitude and phase are given at the end of the problem.
(d) Rewrite the transfer function as

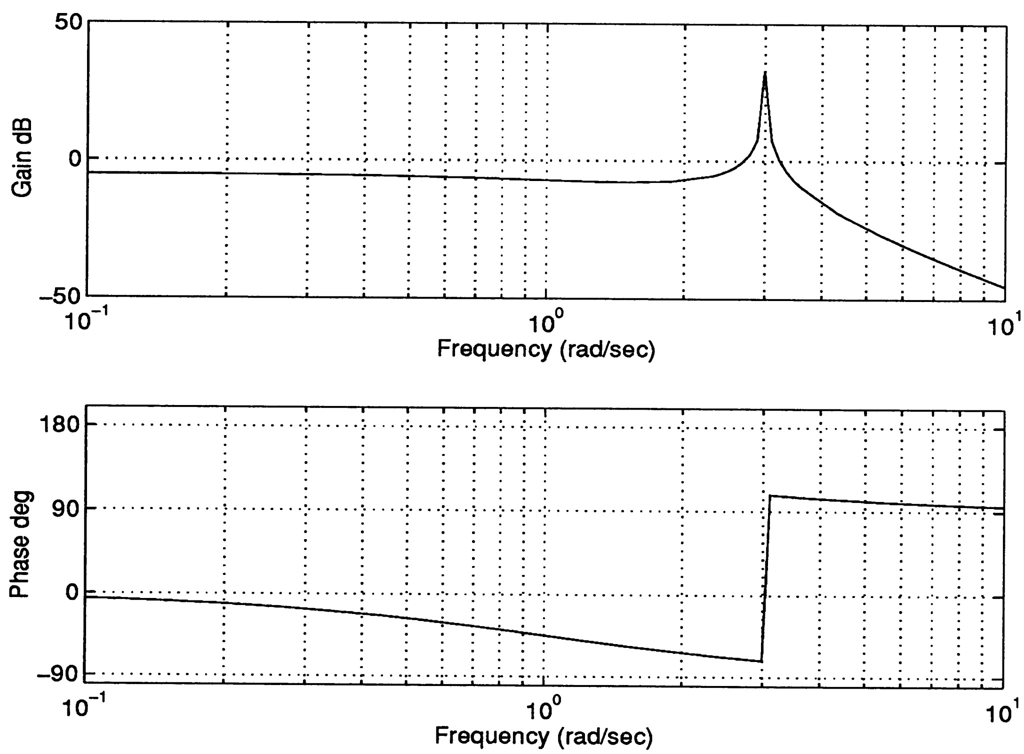
$$H(s) = \frac{1.5(s/3 + 1)}{s(s/2 + 1)}$$

The corner frequencies for the poles are $\omega = 0$ and $\omega = 2$ radians per second. There is a corner frequency for the zero at $\omega = 3$ radians per second. The gain is $20 \log_{10}(1.5) = 3.5$ dB. Bode plots for amplitude and phase are given below. MATLAB statements to do the first one are given below.

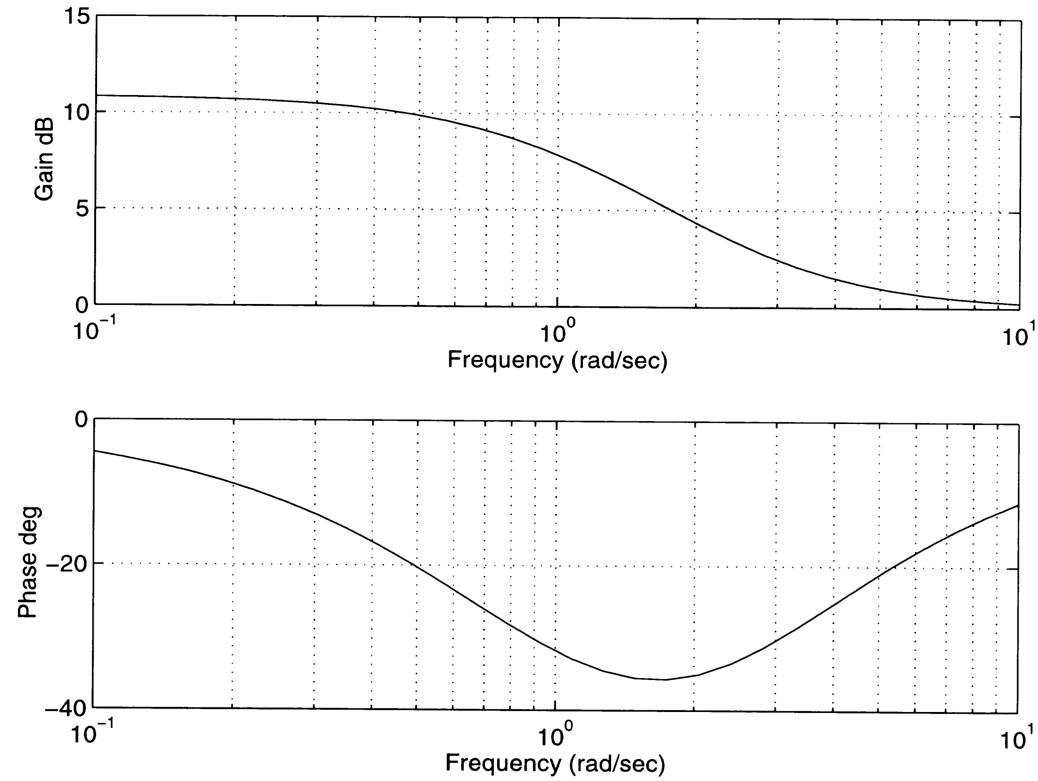
```
na = [0 1 10];
da = [1 3 2];
bode(na, da)
```



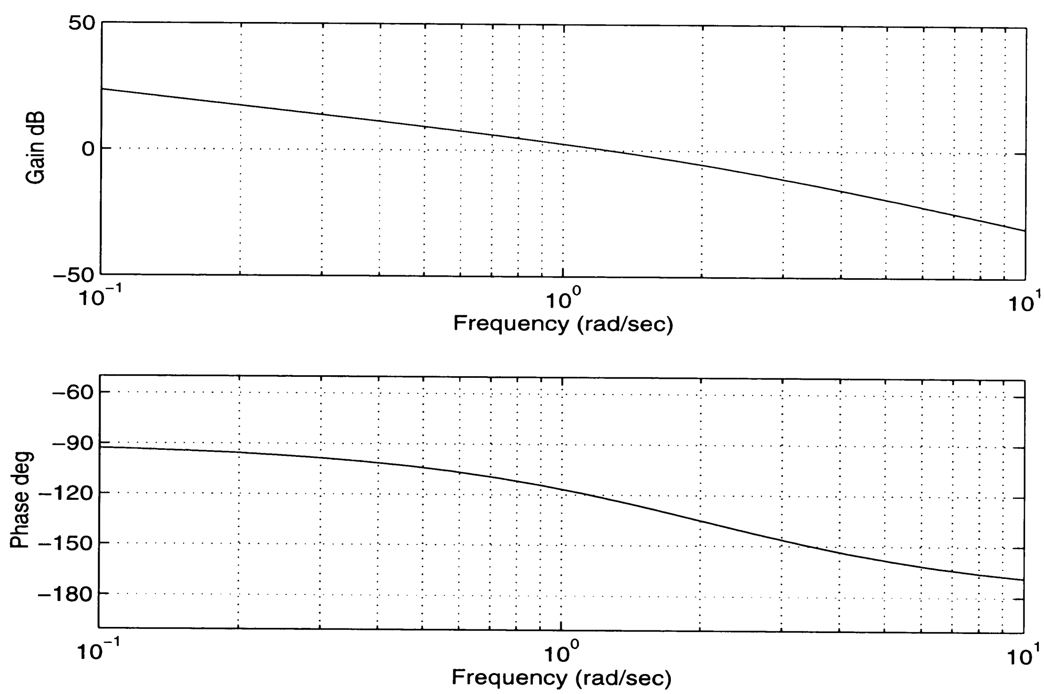
Bode plots for (a)



Bode plots for (b)



Bode plots for (c)



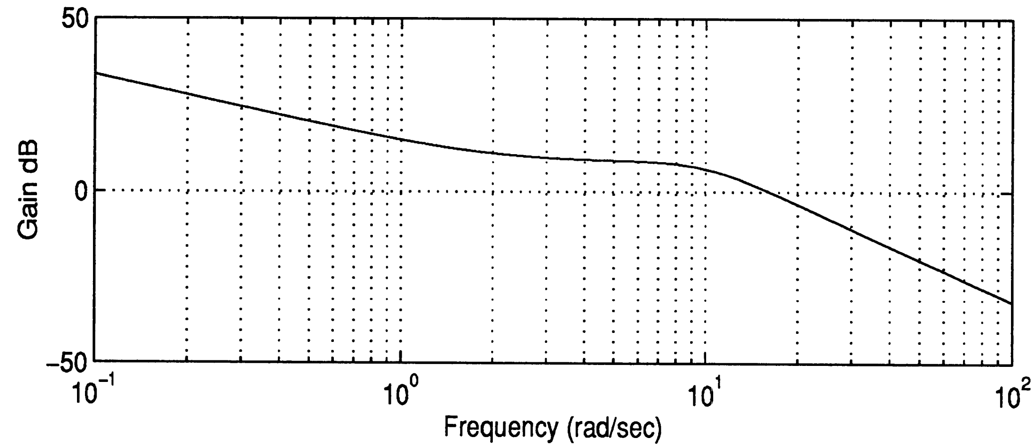
Bode plots for (d)

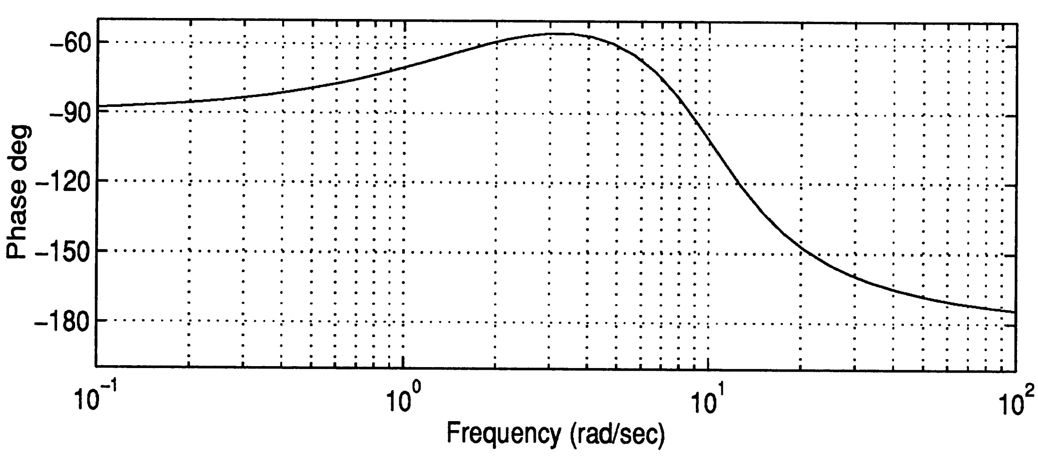
Problem 6-25

Write the transfer function as

$$H(s) = \frac{5(s/2+1)}{s[(s/10)^2 + 2(0.6)(s/10) + 1]}$$

There is a simple zero factor with corner frequency at $\omega = 2$ radians per second, a simple pole factor at the origin (crosses 0 dB at $\omega = 0$), and a complex pole factor with corner frequency at $\omega = 10$ radians per second with $\varsigma = 0.6$. The gain is $20\log_{10}(5) = 14$ dB. The magnitude and phase Bode plots are shown below.



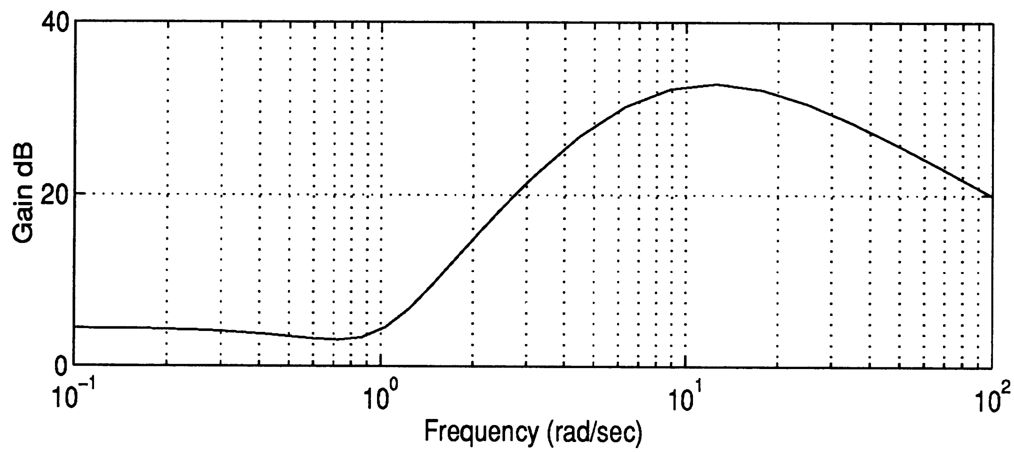


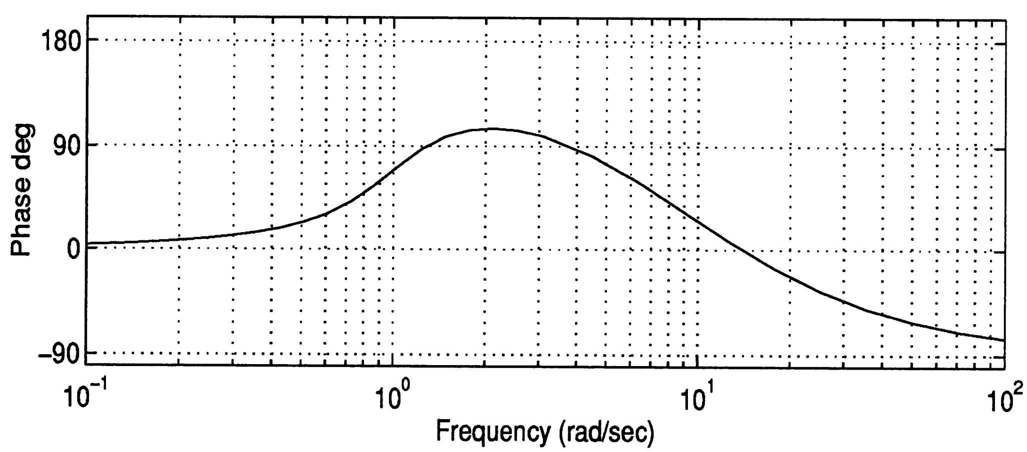
Problem 6-26

Write the transfer function as

$$H(s) = \frac{(5/3)[s^2 + 2(0.5)s + 1]}{(s/6 + 1)(s/10 + 1)^2}$$

There are complex zero factors with corner frequency at $\omega = 1$ radians per second and $\varsigma = 0.5$, a simple pole factor at $\omega = 6$ radian per second, and double pole factors with corner frequency at $\omega = 10$ radians per second. The gain is $20 \log_{10}(5/3) = 4.4$ dB. The magnitude and phase Bode plots are shown below.





Problem 6-27

Consider

$$G(s) = \frac{1}{T^2 s^2 + 2\zeta T s + 1}$$

Then

$$G(j\omega) = \frac{1}{(1 - T^2\omega^2) + j2\zeta T\omega}$$

and

$$|G(j\omega)|^2 = \frac{1}{(1 - T^2\omega^2)^2 + 4\zeta^2 T^2\omega^2} = \frac{1}{1 + (4\zeta^2 - 2)T^2\omega^2 + T^4\omega^4}$$

If a maximum exists, it will be when the derivative of the above expression is 0. Thus compute

$$\frac{d|G(j\omega)|^2}{d\omega} = -\frac{4T^4\omega^3 + 2(4\zeta^2 - 2)T^2\omega}{[1 + (4\zeta^2 - 2)T^2\omega^2 + T^4\omega^4]^2}$$

The derivative is 0 when the numerator is zero, or when

$$4T^2\omega[T^2\omega^2 + 2\zeta^2 - 1] = 0$$

Thus one root is $\omega = 0$, which is of no interest. For the remaining factor, a positive real root exists if

$$2\zeta^2 - 1 < 0$$

That is, a peak in the amplitude response exists. On the other hand, a positive, real root does not exist if

$$2\zeta^2 - 1 > 0 \text{ or if } \zeta > \frac{\sqrt{2}}{2} = 0.707$$

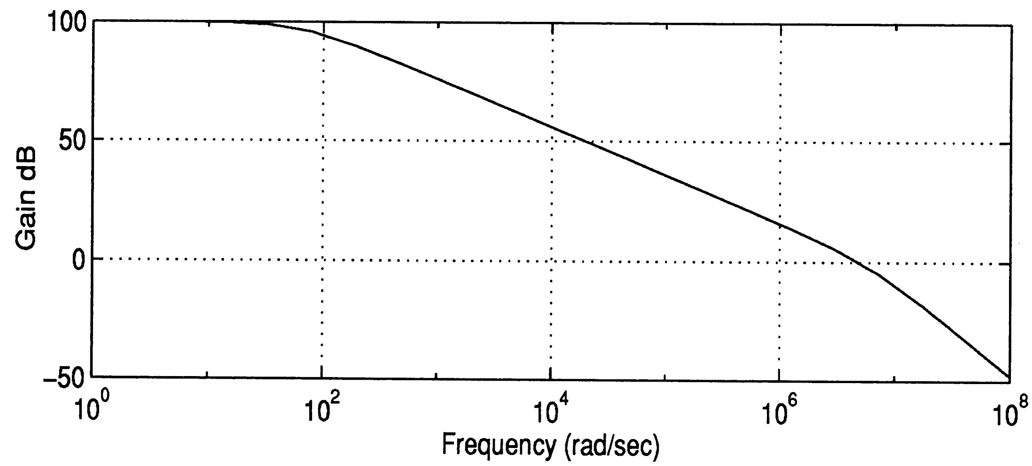
For $\varsigma > 0.707$, therefore, no peak exists.

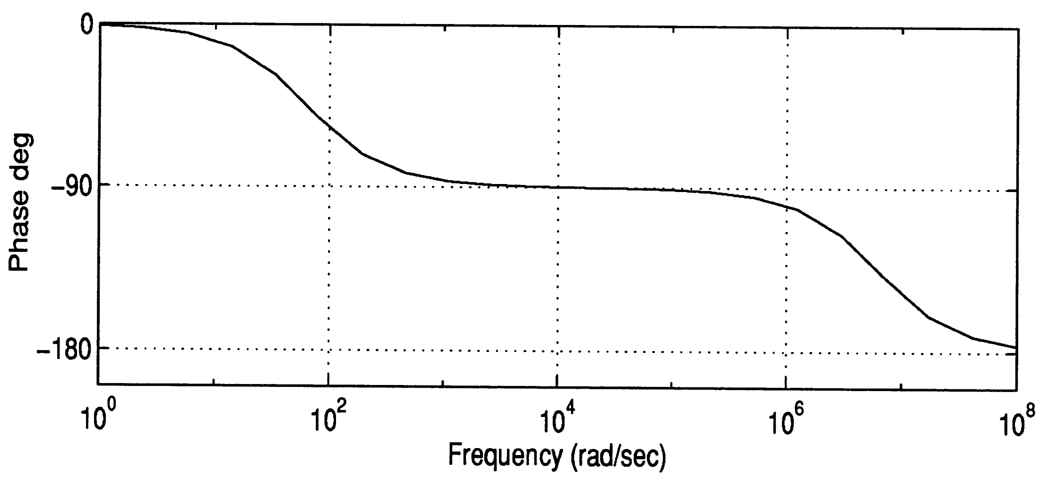
Problem 6-28

(a) For the type 741, the transfer function is

$$K(s) = \frac{10^5}{(s/20\pi + 1)(s/2\pi\times10^6 + 1)} = \frac{4\pi^2\times10^{12}}{s^2 + 2\pi\times1.00001\times10^6 s + 4\pi^2\times10^7}$$

The magnitude and phase bode plots are given below.





(b) The Bode plots for the type 748 are similar, except that the lowest corner frequency is at $\omega = 200\pi$ radians per second.

Problem 6-29

The summer output is

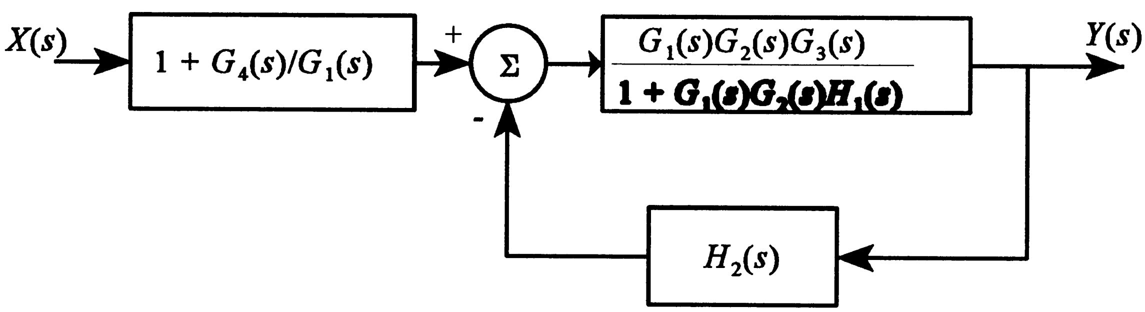
$$W(s) = G_1(s)X_1(s) - G_2(s)X_2(s) \text{ and } Y(s) = G_3W(s)$$

Therefore

$$Y(s) = G_1(s)G_3(s)X_1(s) - G_2(s)G_3(s)X_2(s)$$

Problem 6-30

First, move the output of $G_4(s)$ to the first summer. Then simplify the inner loop and combine with $G_2(s)$. At this point, we have the block diagram shown below:



Now simplify the loop and combine the resulting blocks in cascade to get the following:

$$H(s) = \frac{Y(s)}{X(s)} = (1 + G_4/G_1)\frac{G_1G_2G_3/(1 + G_1G_2H_1)}{1 + G_1G_2G_3H_2/(1 + G_1G_2H_1)} = \frac{(G_1 + G_4)G_2G_3}{1 + G_1G_2H_1 + G_1G_2G_3H_2}$$

Problem 6-31

For the system shown

$$Y(s) = \frac{G(s)D(s) + KG(s)R(s)}{1 + KG(s)} = \frac{\dfrac{6b}{s(s+1)(s+3)} + \dfrac{6aK}{s(s+1)(s+3)}}{1 + \dfrac{6K}{(s+1)(s+3)}} = \frac{6b + 6aK}{s[(s+1)(s+3) + 6K]}$$

Apply the final value theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \frac{6b + baK}{3 + 6K}$$

Further,

$$\lim_{K \to \infty}[\lim_{t \to \infty} y(t)] = a$$

Problem 6-32

(a) Using the last entry of Table 6-2, we obtain

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\dfrac{s}{(s+1)(s+a)}}{1 + \dfrac{s}{(s+1)(s+a)}\dfrac{b}{s}} = \frac{s}{s^2 + (a+1)s + a + b}$$

We want

$$H(s) = \frac{s}{s^2 + 9s + 20}$$

Comparing denominators of the above two expressions, we find $a = 8$ and $b = 12$.
(b) Use the Routh array on the denominator polynomial, $D(s) = s^2 + 4s + (b + 3)$. It is

$$\begin{array}{llll} s^2: & 1 & b+3 & 0 \\ s^1: & 4 & 0 & 0 \\ s^0: & b+3 & 0 & 0 \end{array}$$

For stability, all elements of the first column must be greater than 0, which yields $b > -3$.
(c) Let $X(s) = 1/s^2$ (ramp input) to get the Laplace transform of the output as

$$Y(s) = \frac{1}{s(s+4)(s+5)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+5}$$

The expansion coefficients are

$$A = sY(s)|_{s=0} = \frac{1}{20};\ B = (s+4)Y(s)|_{s=-4} = -\frac{1}{4};\ C = (s+5)Y(s)|_{s=-5} = \frac{1}{5}$$

Thus,

$$Y(s) = \frac{1}{20}\left[\frac{1}{s} - \frac{5}{s+4} + \frac{4}{s+5}\right]$$

The output as a function of time is

$$y(t) = \frac{1}{20}[1 - 5e^{-4t} + 4e^{-5t}]u(t)$$

Problem 6-33

Use the last entry of Table 6-2 to get

$$Y(s) = \frac{H(s)X(s)}{1+H(s)} \text{ and } \mathrm{E}(s) = \frac{X(s)}{1+H(s)}$$

(a) For a ramp input, $X(s) = 1/s^2$ and

$$\mathrm{E}(s) = \frac{1}{s^2}\frac{1}{1+1/s} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Thus, the time-domain error is

$$\epsilon(t) = (1 - e^{-t})u(t) \rightarrow 0 \text{ as } t \rightarrow 0$$

(b) For this $H(s)$, we obtain

$$\mathrm{E}(s) = \frac{1}{s^2}\frac{1}{1+1/s^2} = \frac{1}{(s^2+1)}$$

The time-domain error is

$$\epsilon(t) = \sin(t)\, u(t)$$

Obviously this does not approach 0 as t goes to infinity.
(c) For the assumed form for $H(s)$, we obtain

$$\mathrm{E}(s) = \frac{1}{s^2 + A(s)}$$

Let $A(s) = bs + c$ such that we have two real poles. This means that $s\mathrm{E}(s) \to 0$ as $s \to \infty$ and $\epsilon(t)$ is not oscillatory. For example,

$$A(s) = 3s + 2$$

will work. For this choice

$$\mathrm{E}(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s + 1)(s + 2)} = \frac{1}{s + 2} - \frac{1}{s + 1}$$

and

$$\epsilon(t) = (e^{-2t} - e^{-t})\, u(t)$$

which approaches 0 at t increases. For this $A(s)$,

$$Y(s) = \frac{3s + 2}{s^2(s + 1)(s + 2)} = \frac{1}{s^2} - \frac{1}{s + 1} + \frac{1}{s + 2}$$

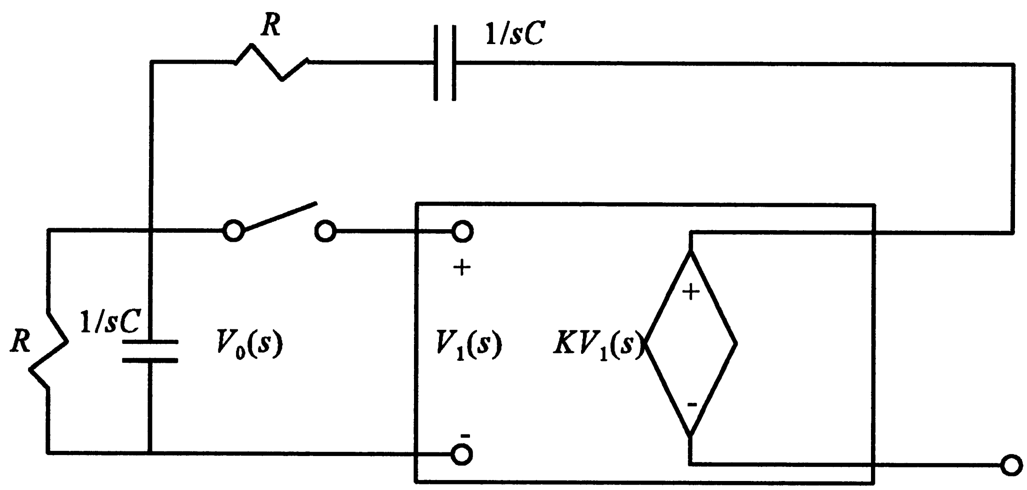
This gives the time-domain response

$$y(t) = r(t) - (e^{-t} - e^{-2t})\, u(t)$$

This obviously approaches a ramp as time gets large.

Problem 6-34

The Laplace-transform equivalent circuit is shown below:



(a) Write a KCL equation at the *RC* junction:

$$\frac{V_0}{R} + sCV_0 + \frac{V_0 - KV_1}{R + 1/sC} = 0$$

Solve for the transfer function:

$$H(s) = \frac{V_0(s)}{V_1(s)} = \frac{KRCs}{(RCs)^2 + 3(RCs) + 1}$$

We want $H(s) = 1$ or

$$(RCs)^2 + (3 - K)RCs + 1 = 0$$

Use the Routh array to examine stability:

$$
\begin{array}{llll}
s^2: & (RC)^2 & 1 & 0 \\
s^1: & 3-K & 0 & 0 \\
s^0: & 1 & 0 & 0
\end{array}
$$

We need 3 - $K < 0$ or $K > 3$ for instability (oscillation). With $K = 3$, we have

$$s^2 = -\frac{1}{(RC)^2} \text{ or } s = \pm\frac{j}{RC}$$

Therefore, the frequency of oscillation is

$$\omega = \frac{1}{RC} \text{ radians/second}$$

For example, $C = 0.159$ μF gives a frequency range of 10 Hz to 10 kHz as R ranges from 10^5 to 100 ohms.

Problem 6-35

Because of the isolation stage between the RC circuits, the overall transfer function is the product of the separate transfer functions. Using a Laplace-transform equivalent circuit for each and voltage division, the transfer function of each RC stage is

$$H_{RC}(s) = \frac{Y(s)}{X(s)} = \frac{1/sC}{R + 1/sC} = \frac{1}{1 + sRC}$$

Therefore, for the cascade of two,

$$H_2(s) = [H_{RC}(s)]^2 = \frac{1}{(1 + sRC)^2}$$

(b) Because of the isolation amplifiers, it is obvious that the overall transfer function is the product of the transfer functions of the RC stages in the cascade. Thus, the given result follows.
(c) Since

$$\mathcal{L}^{-1}[H_i(s)H_j(s)] = h_i(t)*h_j(t) \text{ where } h_i(t) = \mathcal{L}^{-1}[H_i(s)] \text{ etc.}$$

it follows that $h_i(t) = \mathcal{L}^{-1}[1/(1 + sRC)] = \exp(-t/RC)u(t)/RC$, and

$$h_n(t) = \frac{1}{RC}e^{-t/RC}u(t) * \frac{1}{RC}e^{-t/RC}u(t) * \cdots * \frac{1}{RC}e^{-t/RC}u(t)$$

(d) From Problem 5-24

$$\text{Delay} = t_0 = t_{01} + t_{02} + \cdots + t_{0n}$$

where

$$t_{0i} = -H_i'(0)/H_i(0)$$

with the prime denoting differentiation with respect to s. But

$$H_i(s) = \frac{1}{1 + sRC} \text{ and } H_i'(s) = -\frac{RC}{(1 + sRC)^2}$$

Thus $H_i(0) = 1$ and $H_i'(0) = -RC$. Therefore, the delay of the impulse response of the cascade is

$$\text{Delay} = t_0 = RC + RC + \cdots + RC = nRC$$

Also, from Problem 5-24, recall that the square of the impulse response duration is

$$\text{Duration}^2 = \tau_0^2 = \tau_{01}^2 + \tau_{02}^2 + \cdots + \tau_{0n}^2$$

where

$$\tau_i^2 = \frac{H_i''(0)}{H_i(0)} - \left[\frac{H_i'(0)}{H_i(0)}\right]^2$$

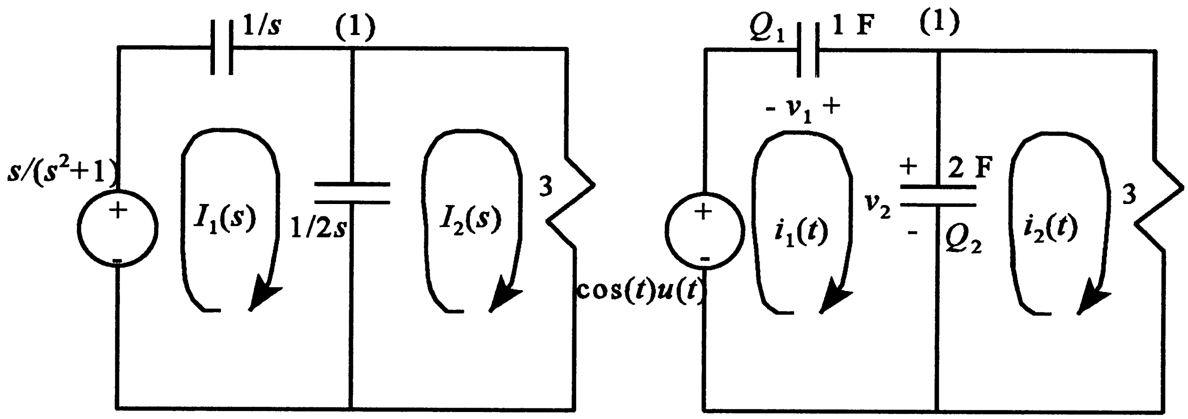
From above,

$$H_i''(0) = \frac{d}{ds}\left[-\frac{RC}{(1 + sRC)^2}\right]_{s=0} = \left.\frac{2(RC)^2}{(1 + sRC)^3}\right|_{s=0} = 2(RC)^2$$

Therefore, $\tau_i^2 = 2(RC)^2 - (RC)^2 = (RC)^2$ and $\tau^2 = n(RC)^2$ or $\tau = n^{1/2}RC$.

Problem 6-36

The Laplace-transform equivalent circuit is shown below. Also shown is the time-domain circuit with voltages and charges defined for use in part (b).



(a) The KVL equations around the two meshes are, by inspection,

$$\begin{bmatrix} \dfrac{3}{2s} & -\dfrac{1}{2s} \\ -\dfrac{1}{2s} & 3+\dfrac{1}{2s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s}{s^2+1} \\ 0 \end{bmatrix}$$

The inverse of the impedance matrix is

$$Z^{-1}(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 3+\dfrac{1}{2s} & \dfrac{1}{2s} \\ \dfrac{1}{2s} & \dfrac{3}{2s} \end{bmatrix} \quad \text{where } \Delta(s) = \det[Z(s)] = \frac{9s+1}{2s^2}$$

Use $I(s) = Z^{-1}(s)V_s(s)$ to get the loop currents:

$$I_1(s) = \frac{6s^3+s^2}{(9s+1)(s^2+1)} \quad \text{and} \quad I_2(s) = \frac{s^2}{(9s+1)(s^2+1)}$$

Use long division to put $I_1(s)$ in proper form:

$$I_1(s) = \frac{2}{3} + R(s) = \frac{2}{3} + \frac{s^2 - 18s - 2}{27(s + 1/9)(s^2 + 1)} = \frac{2}{3} + \frac{1}{82}\left[\frac{1/27}{s + 1/9} - 3\frac{s}{s^2 + 1} - \frac{55}{s^2 + 1}\right]$$

where the partial fraction expansion follows by finding the 1/27 and subtracting the first term from the full result. The time-domain current for mesh 1 is

$$i_1(t) = \frac{2}{3}\delta(t) + \frac{1}{82}\left[\frac{1}{27}e^{-t/9} + 3\cos(t) - 55\sin(t)\right]u(t)$$

The Laplace-transformed current for mesh 2 may be expanded as

$$I_2(s) = \frac{1}{82}\left[\frac{1}{9}\frac{1}{s + 1/9} + 9\frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}\right]$$

Inverse Laplace transformation gives

$$i_2(t) = \frac{1}{82}\left[\frac{1}{9}e^{-t/9} + 9\cos(t) - \sin(t)\right]u(t)$$

(b) Use the right-hand circuit shown on the previous page. Conservation of charge through $t = 0$ at the capacitor junctions is used to establish $t = 0+$ initial conditions. At node (1), we have

$$C_1v_1(0-) + C_2v_2(0-) = C_1v_1(0+) + C_2v_2(0+)$$

Since the initial voltages at $t = 0-$ are 0, we have the right-hand side of the above equation equal to 0, or

$$v_1(0+) + 2v_2(0+) = 0$$

KVL around mesh 1 gives

$$1 + v_1(0+) - v_2(0+) = 0$$

Solve these two equations for the 0+ initial conditions. The results are $v_1(0+) = 1/3$ V and $v_2(0+) = -2/3$ V. Now use the Laplace transform circuit with 0+ initial conditions to find the mesh currents for $t > 0$. With proper initial condition voltage generators based on 0+ initial conditions, we obtain the matix KVL equations given below:

$$\begin{bmatrix} \frac{3}{2s} & -\frac{1}{2s} \\ -\frac{1}{2s} & 3 + \frac{1}{2s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+1} - \frac{3}{2s} - \frac{1}{3s} \\ \frac{1}{3s} \end{bmatrix}$$

Using the inverse of the impedance matrix found previously with the new right-hand side matrix of voltages, we find that

$$I_1(s) = \frac{s^2 - 18s - 2}{27(s + 1/9)(s^2 + 1)} \text{ and } I_2(s) = \frac{s^2}{27(s + 1/9)(s^2 + 1)}$$

The result for $I_1(s)$ is the same as obtained for $R(s)$ in part (a) - i.e., all except the term giving the delta function, which isn't surprising since we used 0+ initial conditions and the delta function has already occurred. The result for $I_2(s)$ is the same here as obtained in part (a), which again is not surprising since no delta function is present in $i_2(t)$. Thus, the two approaches are consistent.

Problem 6-37

(a) The solution of part (a) is analogous to that of Problem 6-36. Now we write node voltage equations (use KCL) to get

$$\begin{bmatrix} 1 + \frac{1}{s} & -1 \\ -1 & 1 + \frac{2}{s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}$$

The matrix to the left is the admittance matrix. Its inverse is

$$\begin{bmatrix} 1 + \frac{2}{s} & 1 \\ 1 & 1 + \frac{1}{s} \end{bmatrix}$$

The solution for the node voltages is given by multiplying the right-hand side (source matrix) of the KCL equations by the inverse of the admittance matrix. The results are

$$V_1(s) = \frac{s+2}{3s+2} = \frac{1}{3} + \frac{4}{9}\frac{1}{s+2/3} \text{ and } V_2(s) = \frac{s}{3s+2} = \frac{1}{3} - \frac{2}{9}\frac{1}{s+2/3}$$

Inverse Laplace transformation gives the time-domain voltages as

$$v_1(t) = \frac{1}{3}\delta(t) + \frac{4}{9}e^{-2t/3}u(t) \text{ and } v_2(t) = \frac{1}{3}\delta(t) - \frac{2}{9}e^{-2t/3}u(t)$$

(b) Use the time domain current together with continuity of flux linkages through $t = 0$ to get 0+ initial conditions. The 0- currents are assumed to be 0, so continuity of flux linkages reduces to

$$-L_1 i_1(0+) + L_2 i_2(0+) = 0$$

which follows because, as far as the mesh containing the two inductors is concerned, i_1 and i_2 are in opposite directions (clockwise and counterclockwise). Using the given inductance values, this equation becomes

$$-i_1(0+) + \frac{1}{2}i_2(0+) = 0$$

Conservation of charge at the REF node gives

$$i_1(0+) + i_2(0+) = 1$$

Hence, the initial currents are $i_1(0+) = 1/3$ A and $i_2(0+) = 2/3$ A. Now use the Laplace-transformed equivalent circuit, including initial condition current generators, to obtain the 0+ node equations

$$\begin{bmatrix} 1 + \frac{1}{s} & -1 \\ -1 & 1 + \frac{2}{s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s} - \frac{1}{3s} \\ -\frac{2}{3s} \end{bmatrix}$$

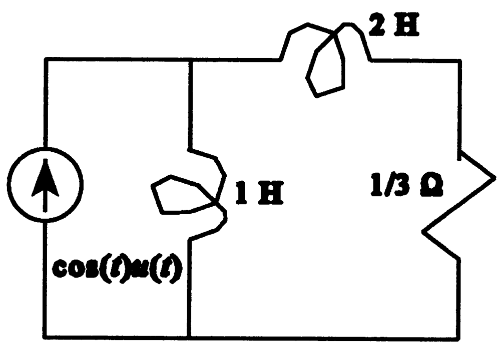
The inverse of the admittance matrix was found above. Multiply it times the matrix on the right-hand side in the above matrix equation to get

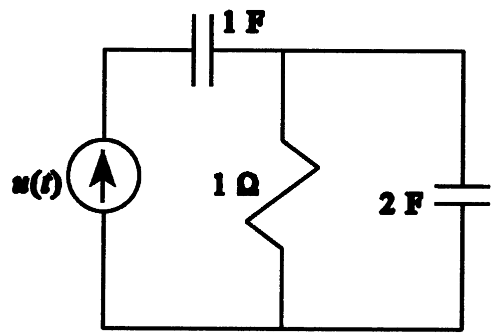
$$V_1(s) = \frac{4}{9}\frac{1}{s+2/3} \text{ and } V_2(s) = -\frac{2}{9}\frac{1}{s+2/3}$$

The time-domain voltages are the inverse Laplace transforms of these equations, which are the non-delta function terms found previously.

Problem 6-38

The duals are shown below:





Problem 6-39

The impedance of $Z_5(s)$ in series with $Y_6(s)$ is

$$Z_5(s) + \frac{1}{Y_6(s)}$$

and the admittance of the combination of $Y_4(s)$ in parallel with the series combination of $Z_5(s)$ and $Y_6(s)$ is

$$Y_4(s) + \frac{1}{Z_5(s) + \frac{1}{Y_6(s)}}$$

Continue this procedure to get the result given in the problem statement.

CHAPTER 7

Problem 7-1

(a) From the given equations we can write

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ -4 \end{bmatrix} u$$

and

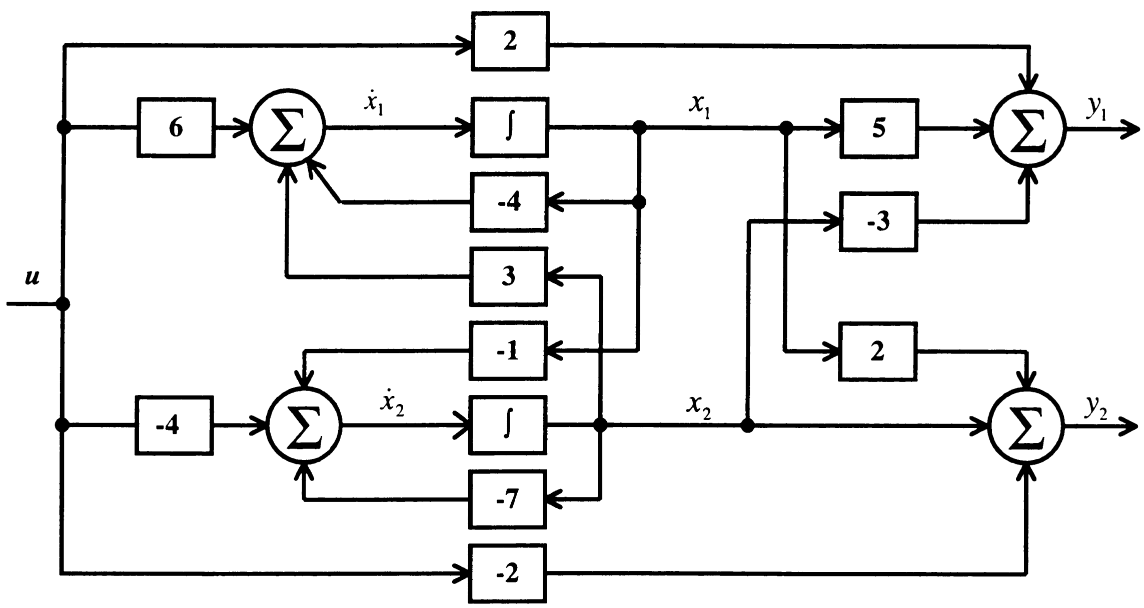
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} u$$

Thus

$$\mathbf{A} = \begin{bmatrix} -4 & 3 \\ -1 & -7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

(b) The simulation diagram follows:



Problem 7-2

(a) From the given equations we can write

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 1 \\ 3 & -2 & 1 \\ 0 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and

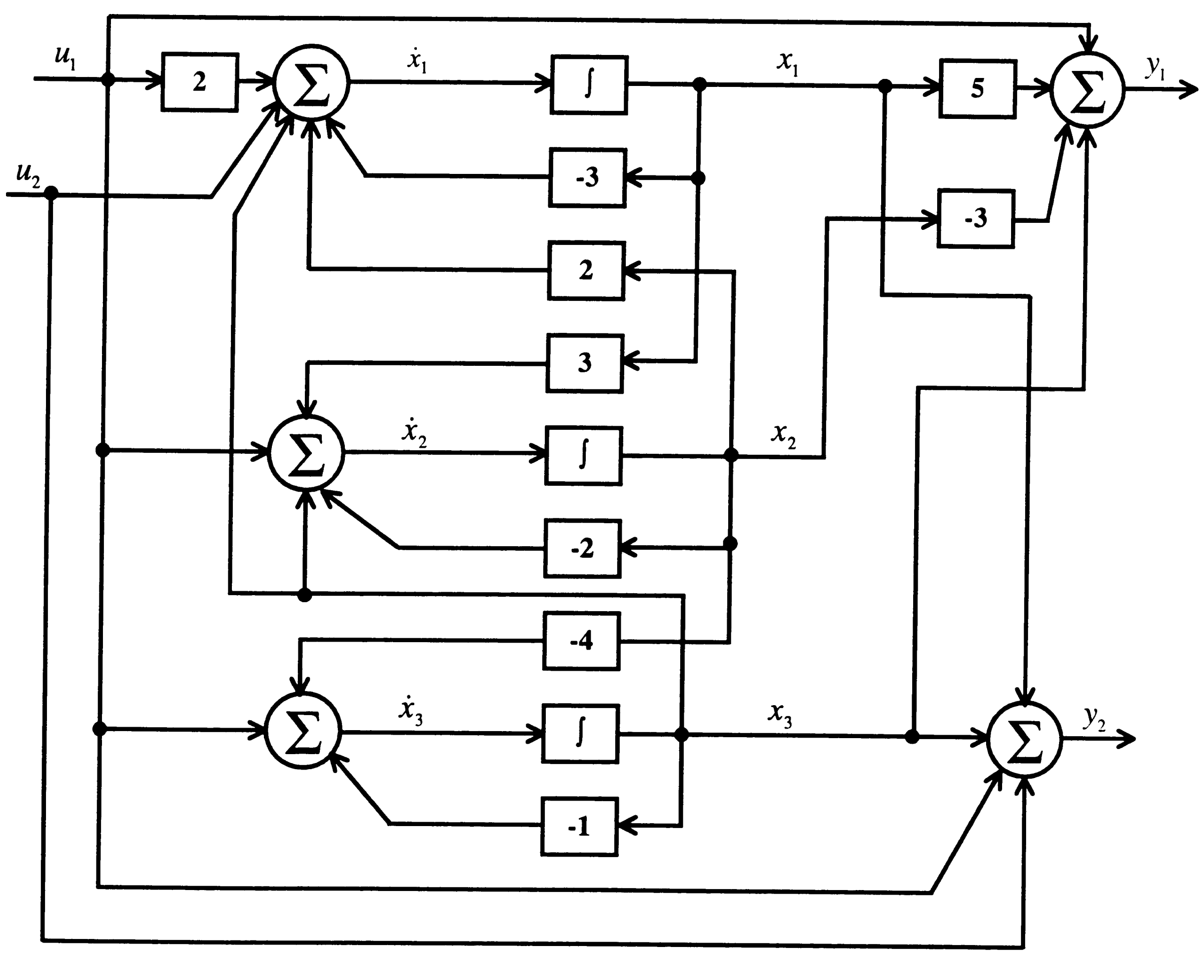
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Thus

$$\mathbf{A} = \begin{bmatrix} -3 & 2 & 1 \\ 3 & -2 & 1 \\ 0 & -4 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 5 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(b)



Problem 7-3

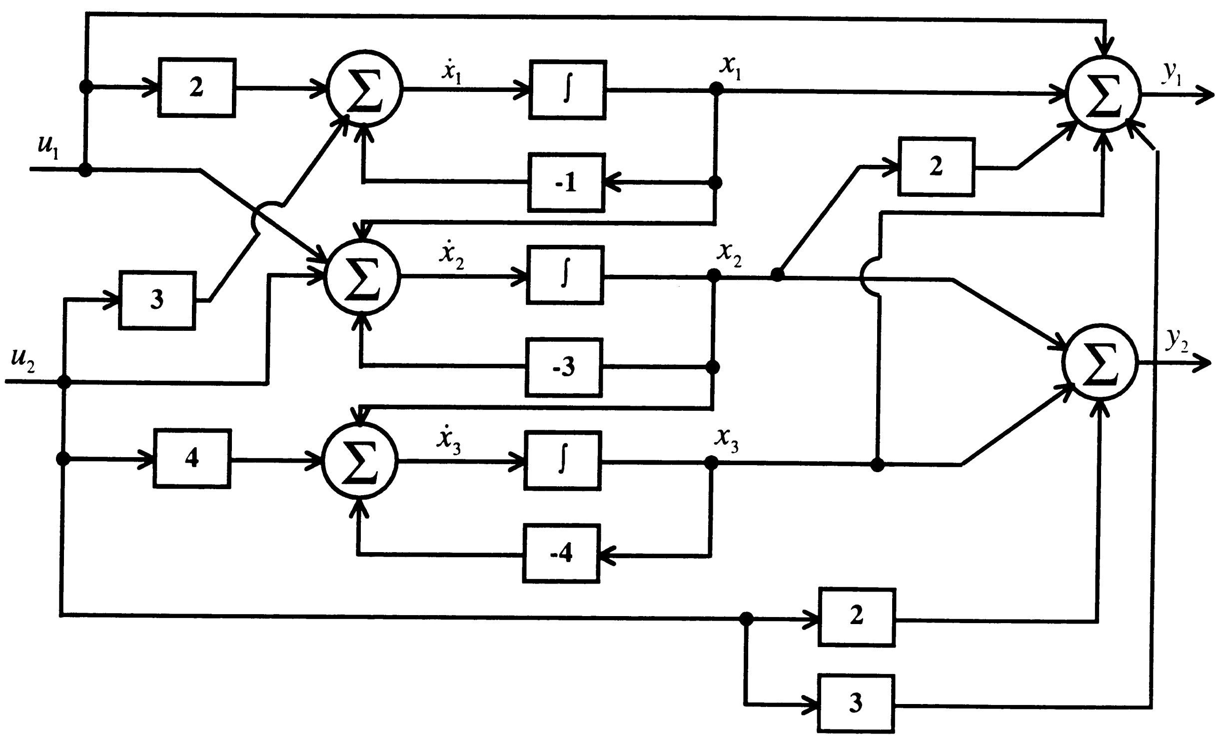
Observation of the A, B, C and D matrices clearly show that the system has three state variables, two inputs and two outputs. The state equations are

$$\dot{x}_1 = -x_1 + 0x_2 + 0x_3 + 2u_1 + 3u_2$$
$$\dot{x}_2 = x_1 - 3x_2 + 0x_3 + u_1 + u_2$$
$$\dot{x}_3 = 0x_1 + x_2 - 4x_3 + 0u_1 + 4u_2$$

The output equations are

$$y_1 = x_1 + 2x_2 + x_3 + u_1 + 3u_2$$
$$y_2 = 0x_1 + x_2 + x_3 + 0u_1 + 2u_2$$

The simulation diagram is



Problem 7-4

The state equations are written directly by observation of the network. The state equations are

$$\dot{x}_1 = -2x_1 + 2x_2 + 2u_1 - u_2$$
$$\dot{x}_2 = 5x_1 - 5x_2 + 4u_2$$

The output equation is

$$y = 2x_1 + 3x_2 + 7u_2$$

In matrix form these equations become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Problem 7-5

Since $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

we can write

$$\mathbf{A}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

and

$$\mathbf{A}^3 = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -64 \end{bmatrix}$$

and so forth. Since

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2 t^2 + \frac{1}{6}\mathbf{A}^3 t^3 + \ldots$$

we can write

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \ldots & 0 \\ 0 & 1 - 4t + \frac{16}{2}t^2 - \frac{64}{6}t^3 + \ldots \end{bmatrix}$$

Recognizing the series expansion for $e^{\mathbf{A}t}$ allows us to write

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix}$$

Problem 7-6

With

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix}$$

The matrix $(s\mathbf{I} - \mathbf{A})$ is

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+2 & 0 \\ 0 & s+5 \end{bmatrix}$$

The state-transition matrix is

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+2)(s+5)}\begin{bmatrix} s+5 & 0 \\ 0 & s+2 \end{bmatrix}$$

which can be written

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+5} \end{bmatrix}$$

Taking the inverse Laplace transform yields, for $t \geq 0$,

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-5t} \end{bmatrix}$$

We can check this result using series expansion. First we compute

$$\mathbf{A}^2 = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 25 \end{bmatrix}$$

and

$$\mathbf{A}^3 = \begin{bmatrix} 4 & 0 \\ 0 & 25 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} -8 & 0 \\ 0 & -125 \end{bmatrix}$$

Thus

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 - 2t + \frac{4}{2}t^2 - \frac{8}{6}t^3 + \ldots & 0 \\ 0 & 1 - 5t + \frac{25}{2}t^2 - \frac{125}{6}t^3 + \ldots \end{bmatrix}$$

Recognizing the series expansion for $e^{\mathbf{A}t}$ allows us to write

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-5t} \end{bmatrix}$$

The results agree.

Problem 7-7

Note that for the given matrix **A**

$$\mathbf{A}^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \lambda_n^k \end{bmatrix} \qquad k = 0,\ 1,\ 2,\ \dots$$

Using the series definition of $e^{\mathbf{A}t}$

$$e^{\mathbf{A}t} = \begin{bmatrix} \left(1+\lambda_1 t+\frac{1}{2}\lambda_1^2 t^2\dots\right) & 0 & \dots & 0 \\ 0 & \left(1+\lambda_2 t+\frac{1}{2}\lambda_2^2 t^2\dots\right) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \left(1+\lambda_n t+\frac{1}{2}\lambda_n^2 t^2\dots\right) \end{bmatrix}$$

which is

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

Problem 7-8

With $t_0 = 0$, we can apply (7-25) to determine $x(t)$. This gives

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-\lambda)\mathbf{B}u(\lambda)d\lambda$$

In Problem 7-5 we showed that, for $t \geq 0$,

$$\Phi(t) = e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix}$$

Thus, the second term in the expression for $x(t)$ is, since $u(\lambda) = 1$ over the range of integration,

$$\int_0^t \Phi(t-\lambda)\mathbf{B}u(\lambda)d\lambda = \int_0^t \begin{bmatrix} e^{-(t-\lambda)} & 0 \\ 0 & e^{-4(t-\lambda)} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} d\lambda$$

$$= \int_0^t \begin{bmatrix} e^{-t}e^{\lambda} \\ 2e^{-4t}e^{4\lambda} \end{bmatrix} d\lambda$$

$$= \begin{bmatrix} e^{-t}\left(e^{t}-1\right) \\ \frac{2}{4}e^{-4t}\left(e^{4t}-1\right) \end{bmatrix} = \begin{bmatrix} 1-e^{-t} \\ \frac{1}{2}\left(1-e^{-4t}\right) \end{bmatrix}$$

Thus

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1-e^{-t} \\ \frac{1}{2}\left(1-e^{-4t}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1-2e^{-t} \\ \frac{1}{2}+\frac{1}{2}e^{-4t} \end{bmatrix}$$

Problem 7-9

For $t_0 = 0$ and $u = 0$ equation (7-13) becomes

$$\begin{aligned} x\left[(k+1)\Delta t\right] &= x(k\Delta t) + \Delta t\mathbf{A}\ x(k\Delta t) \\ &= \left(I + \Delta t\ \mathbf{A}\right)\ x(k\Delta t) \end{aligned}$$

Letting k run from 0 forward yields the expressions

$$\begin{aligned} x(\Delta t) &= \left(I + \Delta t\ \mathbf{A}\right)x(0) \\ x(2\Delta t) &= \left(I + \Delta t\ \mathbf{A}\right)x(\Delta t) = \left(I + \Delta t\ \mathbf{A}\right)^2 x(0) \\ &\vdots \\ x(n\Delta t) &= \left(I + \Delta t\ \mathbf{A}\right)^n\ x(0) \end{aligned}$$

(a) For $\Delta t = \frac{1}{2}$, to compute $x(1)$ we have $n = 2$ so that

$$x(1) = \left(I + \frac{1}{2}\mathbf{A}\right)^2 x(0)$$

This gives

$$x(1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix}^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.49 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.98 \end{bmatrix}$$

(b) For $\Delta t = \frac{1}{10}$, to compute $x(1)$ the value of n is given by $n = 10$. Thus

$$x(1) = \begin{bmatrix} 0.90 & 0 \\ 0 & 0.94 \end{bmatrix}^{10} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

which gives

$$x(1) = \begin{bmatrix} 0.349 & 0 \\ 0 & 0.539 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.046 \\ 1.077 \end{bmatrix}$$

(c) For

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -0.6 \end{bmatrix}$$

We have

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-0.6t} \end{bmatrix}$$

and

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-0.6t} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^{-t} \\ 2e^{-0.6t} \end{bmatrix}$$

Evaluating at $t = 1$ yields

$$x(1) = \begin{bmatrix} 3e^{-1} \\ 2e^{-0.6} \end{bmatrix} = \begin{bmatrix} 1.104 \\ 1.098 \end{bmatrix}$$

Clearly the solution for part (b) is more accurate than the solution found in part (a). This was to be expected because of the smaller sampling interval Δt. Note that in part (b) Δt is small compared to the system time constants of the system, which are $\frac{1}{1} = 1$, and $\frac{1}{0.6} = 1.67$ seconds.

Problem 7-10

Since

$$e^{\mathbf{A}t} = (s\mathbf{I} - \mathbf{A})^{-1}$$

we first write $(s\mathbf{I} - \mathbf{A})$ as

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+1 & 0 \\ 0 & s+4 \end{bmatrix}$$

Therefore

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+4)} \begin{bmatrix} s+4 & 0 \\ 0 & s+1 \end{bmatrix}$$

Thus

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+4} \end{bmatrix}$$

Taking the inverst-Laplace transform yields, for $t \geq 0$,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix}$$

Problem 7-11

With

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

we can write

$$(s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}$$

which yields

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s^3+6s+11s+6}\begin{bmatrix} s^2+6s+11 & s+6 & 1 \\ -6 & s(s+6) & s \\ -6s & -(11s+6) & s^2 \end{bmatrix}$$

We can show that

$$s^3+6s+11s+6 = (s+1)(s+2)(s+3)$$

Representing

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \mathbf{A}_{11}(s) & \mathbf{A}_{12}(s) & \mathbf{A}_{13}(s) \\ \mathbf{A}_{21}(s) & \mathbf{A}_{22}(s) & \mathbf{A}_{23}(s) \\ \mathbf{A}_{31}(s) & \mathbf{A}_{32}(s) & \mathbf{A}_{33}(s) \end{bmatrix}$$

We can inverse each $\mathbf{A}_{ij}(s)$ using partial fraction expansion. This yields

$$e^{\mathbf{A}t} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}$$

where

$$a_{11}(t) = \left(3e^{-t} - 3e^{-2t} + e^{-3t}\right)u(t)$$
$$a_{12}(t) = \left(\frac{5}{2}e^{-t} - 4e^{-2t} + \frac{3}{2}e^{-3t}\right)u(t)$$
$$a_{13}(t) = \left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right)u(t)$$
$$a_{21}(t) = \left(3e^{-t} + 6e^{-2t} - 3e^{-3t}\right)u(t)$$
$$a_{22}(t) = \left(-\frac{5}{2}e^{-t} + 8e^{-2t}0 - \frac{9}{2}e^{-3t}\right)u(t)$$
$$a_{23}(t) = \left(-\frac{1}{2}e^{-t} + 2e^{-2t} - \frac{3}{2}e^{-3t}\right)u(t)$$
$$a_{31}(t) = \left(3e^{-t} - 12e^{-2t} + 9e^{-3t}\right)u(t)$$
$$a_{32}(t) = \left(\frac{5}{2}e^{-t} - 16e^{-2t} + \frac{27}{2}e^{-3t}\right)u(t)$$
$$a_{33}(t) = \left(\frac{1}{2}e^{-t} - 4e^{-2t} + \frac{9}{2}e^{-3t}\right)u(t)$$

Problem 7-12

The first step is to determine the eignvalues of the matrix $\mathbf{A}$. We write

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda+5 \end{bmatrix} = \lambda^2 + 5\lambda + 4 = (\lambda+1)(\lambda+4) = 0$$

Thus, the eignvalues are $\lambda = -1$ and $\lambda = -4$. This gives

$$e^{-t} = \alpha_0(t) - \alpha_1(t)$$
$$e^{-4t} = \alpha_0(t) - 4\alpha_1(t)$$

Subtracting gives

$$e^{-t} - e^{-4t} = 3\alpha_1(t)$$

so that

$$\alpha_1(t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right)$$

Also

$$\alpha_0(t) = \alpha_1(t) + e^{-t} = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$$

From the Caley-Hamilton theorem

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A}$$

Thus

$$e^{\mathbf{A}t} = \begin{bmatrix} \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} & 0 \\ 0 & -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{bmatrix} + \frac{1}{3}\left(e^{-t} - e^{-4t}\right)\begin{bmatrix} 0 & 1 \\ 4 & -5 \end{bmatrix}$$

Combining terms gives

$$e^{\mathbf{A}t} = \begin{bmatrix} \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} \\ \frac{4}{3}e^{-t} - \frac{4}{3}e^{-4t} & -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{bmatrix}$$

Problem 7-13

Using

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x_0 + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\ \mathbf{U}(s)$$

we first must determine $(s\mathbf{I} - \mathbf{A})^{-1}$. For the given system

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+3)}\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

Thus

$$\mathbf{Y}(s) = \frac{1}{(s+1)(s+3)}\left\{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\left(\frac{1}{s}\right)\right\}$$

which is

$$\mathbf{Y}(s) = \frac{1}{(s+1)(s+3)}\left\{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+5 \\ s-3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{2}{s} \\ 2 \end{bmatrix}\right\}$$

or

$$\mathbf{Y}(s) = \frac{1}{(s+1)(s+3)}\left\{s+5+\frac{2}{s}\right\} = \frac{s^2+5s+2}{s(s+1)(s+3)}$$

Partial fraction expansion yields

$$\mathbf{Y}(s) = \frac{2}{3}\frac{1}{s} + \frac{1}{s+1} - \frac{2}{3}\frac{1}{s+3}$$

so that

$$y(t) = \left(\frac{2}{3} + e^{-t} - \frac{2}{3}e^{-3t}\right)\, u(t)$$

Problem 7-14

The transfer function matrix is given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

for the given system

$$(s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}$$

so that

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{(s+1)(s+3)}\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

Since $\mathbf{C} = \mathbf{I}$, the identity matrix,

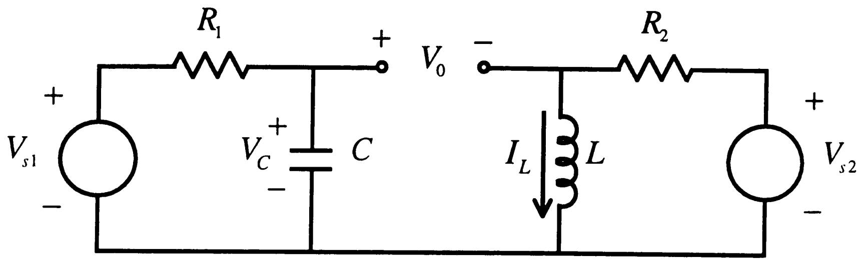
$$\mathbf{H}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \frac{1}{(s+1)(s+4)}\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{(s+1)(s+4)}\begin{bmatrix} 2 \\ 2s \end{bmatrix}$$

Thus, the transfer function matrix is

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{2}{(s+1)(s+4)} \\ \dfrac{2s}{(s+1)(s+4)} \end{bmatrix}$$

Problem 7-15

For the circuit given we define the inductor current and the capacitor voltage as the state variables. This yields the circuit diagram



This gives the equations

$$C\dot{V}_C = \frac{1}{R_1}(V_{s1} - V_C)$$

$$L\dot{I}_L = V_{s2} - I_L R_2$$

The output equation is

$$V_0 = V_c + R_2 I_L - V_{s2}$$

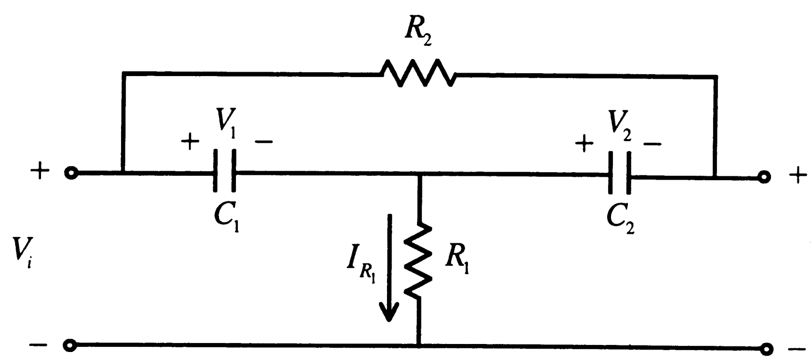
In matrix form these expressions become

$$\begin{bmatrix} \dot{V}_C \\ \dot{I}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} V_C \\ I_L \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C} & 0 \\ 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} V_{s1} \\ V_{s2} \end{bmatrix}$$

and

$$V = \begin{bmatrix} 1 & R_2 \end{bmatrix} \begin{bmatrix} V_c \\ I_L \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} V_{s1} \\ V_{s2} \end{bmatrix}$$

Problem 7-16



First we define the state variables as V_1 and V_2. The circuit equations then become

$$C_1\dot{V}_1 = I_{R_1} + C_2\dot{V}_2$$

$$C_2\dot{V}_2 = -\frac{1}{R_2}(V_1 + V_2)$$

Next we must eliminate I_{R_1}. Clearly

$$I_{R_1} = \frac{1}{R_1}(V_i - V_1)$$

Thus

$$\dot{V}_1 = \frac{V_1}{R_1C_1} - \frac{V_1}{R_2C_1} - \frac{V_2}{R_2C_1} + \frac{V_i}{R_1C_1}$$

$$\dot{V}_2 = \frac{V_1}{R_2C_2} - \frac{V_2}{R_2C_2}$$

The output equation is

$$V_0 = V_i - V_1 - V_2$$

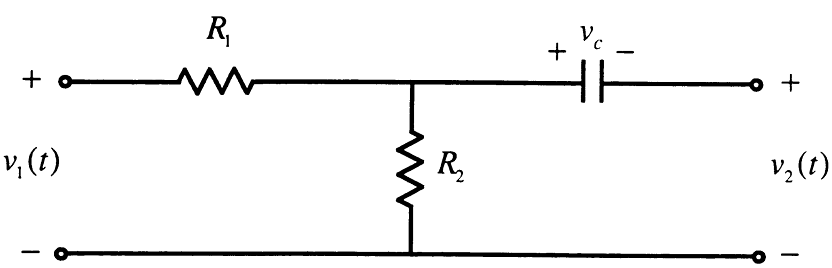
In matrix form these become

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2}\right) & -\frac{1}{R_2C_1} \\ -\frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_i$$

$$V_0 = \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + V_0$$

Problem 7-17

For this circuit we choose the capacitor voltage as the (only) state variable.



Note that $\dot{v}_c = 0$. Thus the state equation is

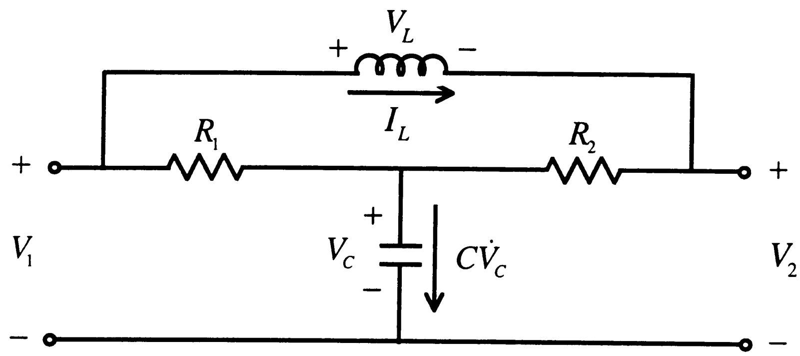
$$\dot{v}_c = 0$$

and the output equation is

$$v_2(t) = -v_c + \frac{R_2}{R_1 + R_2} v_1(t)$$

Problem 7-18

For the given circuit we choose the capacitor voltage and the inductor current as the state variables. The system input is assumed to be $v_1(t)$ and the system output is assumed to be $v_2(t)$.



The state equations are

$$V_1 = R_1\left(C\dot{V}_C - I_L\right) + V_C$$

and

$$L\dot{I}_L = V_1 - V_2 = V_1 - R_2 I_L - V_C$$

Thus

$$\dot{V}_C = -\frac{1}{R_1 C}V_C + \frac{1}{R_1 C}I_L + \frac{1}{R_1 C}V_1$$

$$\dot{I}_L = \frac{1}{L}V_C - \frac{R_2}{L}I_L + \frac{1}{L}V_1$$

and the output equation is

$$V_2 = V_C + R_2 I_C$$

In matrix form these become

$$\begin{bmatrix}\dot{V}_C \\ \dot{I}_L\end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C} & \frac{1}{R_1 C} \\ -\frac{1}{L} & -\frac{R_2}{L}\end{bmatrix}\begin{bmatrix} V_C \\ I_L\end{bmatrix} + \begin{bmatrix}\frac{1}{R_1 C} \\ \frac{1}{L}\end{bmatrix} V_1$$

and

$$V_2 = \begin{bmatrix}1 & R_2\end{bmatrix}\begin{bmatrix}V_C \\ I_L\end{bmatrix}$$

Problem 7-19

In order to obtain the state model we first write

$$\frac{Y(s)}{U(s)} = H(s) = \frac{s+1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

where

$$A = \frac{1}{3}, \qquad B = \frac{2}{3}$$

Thus

$$Y(s) = \frac{1}{3}\frac{U(s)}{s} + \frac{2}{3}\frac{U(s)}{s+3} = \frac{1}{3}X_1(s) + \frac{2}{3}X_2(s)$$

We therefore let

$$X_1(s) = \frac{U(s)}{s} \Rightarrow \dot{x}_1 = u$$

and

$$X_2(s) = \frac{U(s)}{s+3} \Rightarrow \dot{x}_2 = -3x_2 + u$$

The state model is therefore defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

<u>Problem 7-20</u>

(a) First we write

$$\frac{Y(s)}{U(s)} = \frac{3}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

where

$$A = \frac{3}{2}, \qquad B = -\frac{3}{2}$$

Thus

$$Y(s) = \frac{3}{2}\frac{U(s)}{s+1} - \frac{3}{2}\frac{U(s)}{s+3} = \frac{3}{2}X_1(s) - \frac{3}{2}X_2(s)$$

where we have defined

$$X_1(s) = \frac{U(s)}{s+1} \Rightarrow \dot{x}_1 = -x_1 + u$$

and

$$X_2(s) = \frac{U(s)}{s+3} \Rightarrow \dot{x}_2 = -3x_2 + u$$

The state model is therefore defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) Without canceling the $s+1$ terms we write

$$\frac{Y(s)}{U(s)} = \frac{s+1}{(s+1)(s+5)(s+6)} = \frac{A}{s+1} + \frac{B}{s+5} + \frac{C}{s+6}$$

where

$$A = 0, \qquad B = 1, \qquad C = -1$$

Thus

$$Y(s) = 0\frac{U(s)}{s+1} + \frac{U(s)}{s+5} - \frac{U(s)}{s+6}$$

$$= 0X_1(s) + X_2(s) - X_3(s)$$

where we have defined

$$X_1(s) = \frac{U(s)}{s+1} \Rightarrow \dot{x}_1 = -x_1 + u$$

$$X_2(s) = \frac{U(s)}{s+5} \Rightarrow \dot{x}_2 = -5x_2 + u$$

$$X_3(s) = \frac{U(s)}{s+6} \Rightarrow \dot{x}_3 = -6x_3 + u$$

The state model is therefore defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note that we have a state that is not observable.

Problem 7-21

(a) The function

$$\frac{Y(s)}{U(s)} = \frac{s^2+4s+1}{(s+1)(s+2)} = \frac{s^2+4s+1}{s^2+3s+2}$$

is not expressed as a proper rational fraction. Using long division we can write

$$\frac{Y(s)}{U(s)} = 1 + \frac{s-1}{(s+1)(s+2)}$$

Now

$$\frac{s-1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

where

$$A = -2, \qquad B = 3$$

Thus

$$Y(s) = U(s) - 2\frac{U(s)}{s+1} + 3\frac{U(s)}{s+2} = U(s) - 2X_1(s) + 3X_2(s)$$

where we have defined

$$X_1(s) = \frac{U(s)}{s+1} \Rightarrow \dot{x}_1 = -x_1 + u$$

and

$$X_2(s) = \frac{U(s)}{s+2} \Rightarrow \dot{x}_2 = -2x_2 + u$$

The state model is therefore defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

(b) First we write

$$\frac{Y(s)}{U(s)} = \frac{s^2+2s+2}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

where

$$A = \frac{2}{3}, \qquad B = -\frac{1}{2}, \qquad C = \frac{5}{6}$$

Thus

$$\begin{aligned} Y(s) &= \frac{2}{3}\frac{U(s)}{s} - \frac{1}{2}\frac{U(s)}{s+1} + \frac{5}{6}\frac{U(s)}{s+3} \\ &= \frac{2}{3}X_1(s) - \frac{1}{2}X_2(s) + \frac{5}{6}X_3(s) \end{aligned}$$

where

$$X_1(s) = \frac{U(s)}{s} \Rightarrow \dot{x}_1 = u$$

$$X_2(s) = \frac{U(s)}{s+1} \Rightarrow \dot{x}_2 = -x_2 + u$$

and

$$X_3(s) = \frac{U(s)}{s+3} \Rightarrow \dot{x}_3 - 3x_3 + u$$

The state model is therefore given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Problem 7-22

(a) Since

$$\frac{Y(s)}{U(s)} = \frac{s^2 + s + 3}{s(s+1)} = \frac{s^2 + s + 3}{s^2 + s}$$

is not a proper rational fraction, long division is applied to yield

$$\frac{Y(s)}{U(s)} = 1 + \frac{3}{s(s+1)} = 1 + \frac{A}{s} + \frac{B}{s+1}$$

where

$$A = 3, \qquad B = -3$$

Thus

$$Y(s) = U(s) + 3\frac{U(s)}{s} - 3\frac{U(s)}{s+1} = U(s) + 3X_1(s) - 3X_2(s)$$

where

$$X_1(s) = \frac{U(s)}{s} \Rightarrow \dot{x}_1 = u$$

and

$$X_2(s) = \frac{U(s)}{s+1} \Rightarrow \dot{x}_2 = -x_2 + u$$

The state model is therefore defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

(b) First we write

$$\frac{Y(s)}{U(s)} = \frac{s^2 + 3s + 2}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

where

$$A = \frac{2}{3}, \qquad B = 0, \qquad C = \frac{1}{3}$$

Thus

$$Y(s) = \frac{2}{3}\frac{U(s)}{s} + 0\frac{U(s)}{s+1} + \frac{1}{3}\frac{U(s)}{s+3}$$

or

$$Y(s) = \frac{2}{3}X_1(s) + 0X_2(s) + \frac{1}{3}X_3(s)$$

where we have defined

$$X_1(s) = \frac{U(s)}{s} \Rightarrow \dot{x}_1 = u$$

$$X_2(s) = \frac{U(s)}{s+1} \Rightarrow \dot{x}_2 = -x_2 + u$$

and

$$X_3(s) = \frac{U(s)}{s+3} \Rightarrow \dot{x}_3 = -3x_3 + u$$

Thus the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

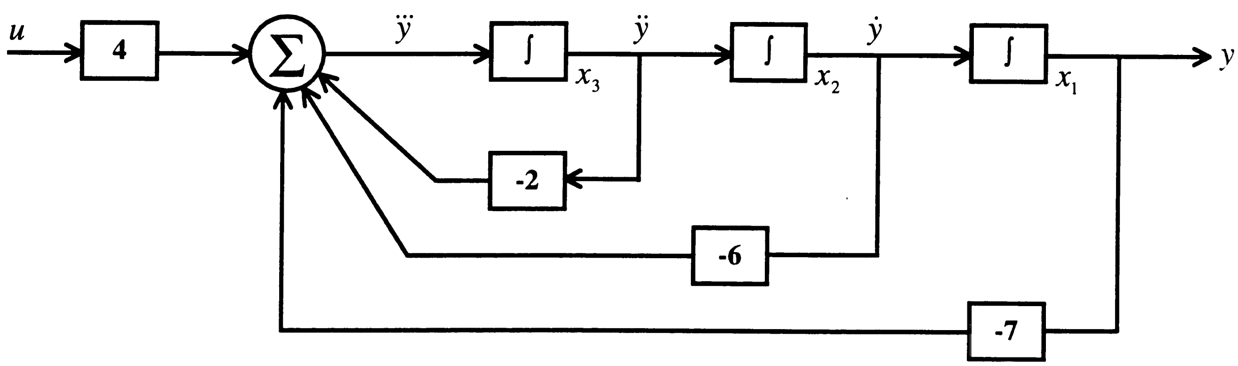
Note that the state defined by x_2 is not observable on the output.

Problem 7-23

(a) A simple method for obtaining a state model from the differential equation is to draw a simulation diagram, define the state variables as the integrator outputs and write the resulting equation. Solving the differential equation for $\dddot{y}$ yields

$$\dddot{y} = 4u - 2\ddot{y} - 6\dot{y} - 7y$$

The simulation diagram is



With the satate variables defined as shown on the diagram we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -7x_1 - 6x_2 - 2x_3 + 4u \end{aligned}$$

In matrix form these are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} u$$

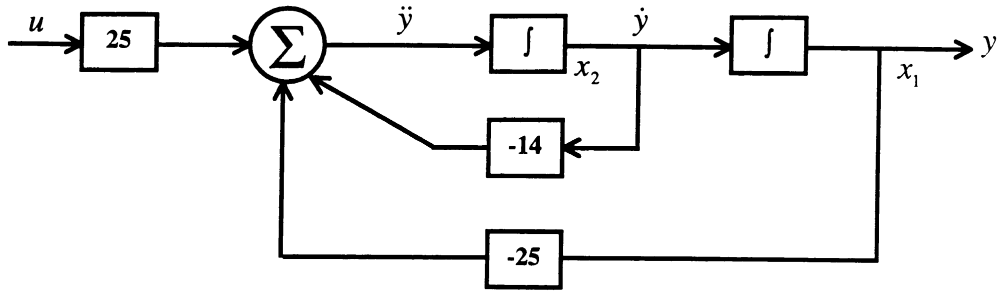
and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) The differential equation can be written

$$\ddot{y} = 25u - 14\dot{y} - 25y$$

which yields the simulation diagram



Clearly

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -25x_1 - 14x_2 + 25u$$

The state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note: Since the differential equation is second-order, it is easily factored and the techniques used in Problems 7-19 through 7-22 can be used. We will see that a different, but equally valid, state model results. Laplace-transforming the differential equation yields

$$Y(s)\left[s^2 + 14s + 25\right] = 25U(s)$$

Thus

$$\frac{Y(s)}{U(s)} = \frac{25}{s^2 + 14s + 25} = \frac{25}{(s + 2.101)(s + 11.899)}$$

Using partial-fraction expansion gives

$$Y(s) = 2.552\frac{U(s)}{s + 2.101} - 2.552\frac{U(s)}{s + 11.899}$$

which can be expressed

$$Y(s) = 2.552X_1(s) - 2.552X_1(s)$$

We have defined

$$X_1(s) = \frac{U(s)}{s + 2.101} \Rightarrow \dot{x}_1 = -2.101x_1 + u$$

and

$$X_2(s) = \frac{U(s)}{s + 11.899} \Rightarrow \dot{x}_2 = -11.899x_2 + u$$

The state model is therefore defined by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2.101 & 0\\ 0 & -11.899\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

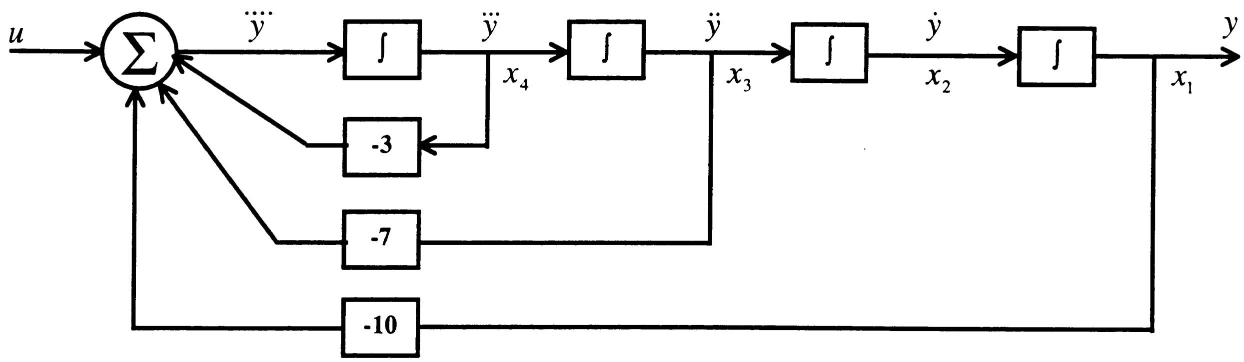
and

$$y=\begin{bmatrix}2.552 & -2.552\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

(c) This problem is easily solved using a simulation diagram. First we write

$$\ddddot{y}=3\dddot{y}-7\ddot{y}-10\dot{y}+u$$

The simulation diagram is



Thus

$$\dot{x}_1=x_2,\qquad \dot{x}_2=x_3,\qquad \dot{x}_3=x_4,\qquad \dot{x}_4=\ddddot{y}$$

The state model is therefore

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -10 & 0 & -7 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}u$$

and

$$y=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

Problem 7-24

(a) Using partial fraction expansion

$$\frac{Y(s)}{U(s)}=\frac{s+2}{(s+1)(s+3)}=\frac{1}{2}\frac{1}{s+1}+\frac{1}{2}\frac{1}{s+3}$$

so that

$$Y(s) = \frac{1}{2}\frac{U(s)}{s+1} + \frac{1}{2}\frac{U(s)}{s+3}$$

or

$$Y(s) = \frac{1}{2}X_1(s) + \frac{1}{2}X_2(s)$$

where we have defined

$$X_1(s) = \frac{U(s)}{s+3} \Rightarrow \dot{x}_1 = -x_1 + u$$

and

$$X_1(s) = \frac{U(s)}{s+3} \Rightarrow \dot{x}_2 = -3x_2 + u$$

Thus the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The MATLAB verification follows:

```
» A = [-1 0; 0 -3];
» B = [1 1]';
» C = [1/2 1/2];
» D = [0];
» [num,den] = ss2tf(A,B,C,D);
» num

num =

     0     1     2

» den

den =

     1     4     3
```

This agrees with the original transfer function.

(b) Using partial fraction expansion

$$\frac{Y(s)}{U(s)} = \frac{s+4}{s(s+2)(s+3)(s+6)} = \frac{1}{9}\frac{1}{s} - \frac{1}{4}\frac{1}{s+2} + \frac{1}{9}\frac{1}{s+3} + \frac{1}{36}\frac{1}{s+6}$$

Using the same technique as before, we get the following state equations:

$$\dot{x}_1 = u$$
$$\dot{x}_2 = -2x_2 + u$$
$$\dot{x}_3 = -3x_3 + u$$
$$\dot{x}_4 = -6x_4 + u$$

This yields the state model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

The output equation is

$$y = \begin{bmatrix} \frac{1}{9} & -\frac{1}{4} & \frac{1}{9} & \frac{1}{36} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The MATLAB verification follows. Note that the results are in agreement and that convolution was used to insure that the denominations are correct when expressed in polynomial form.

```
» A = [0 0 0 0; 0 -2 0 0; 0 0 -3 0; 0 0 0 -6];
» B = [1 1 1 1]';
» C = [1/9 -1/4 1/9 1/36];
» D = [0];
» [num,den] = ss2tf(A,B,C,D);
» num

num =

         0    0.0000    0.0000    1.0000    4.0000

» den

den =

     1    11    36    36     0

» a = conv([1 0],[1 2]);
» b = conv([1 3],[1 6]);
» c = conv(a,b);
» c

c =

     1    11    36    36     0
```

(c) For

$$\frac{Y(s)}{U(s)} = \frac{s^2+2s+7}{(s+1)(s+7)} = \frac{1}{s} - \frac{1}{s+1} + \frac{1}{s+7}$$

we have the state equations

$$\dot{x}_1 = u$$
$$\dot{x}_2 = -x_2 + u$$
$$\dot{x}_3 = -7x_3 + u$$

The state model then becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The MATLAB verification follows.

```
» A = [0 0 0; 0 -1 0; 0 0 -7];
» B = [1 1 1]';
» C = [1 -1 1];
» D = [0];
» [num,den] = ss2tf(A,B,C,D);
» num

num =

         0    1.0000    2.0000    7.0000

» den

den =

     1     8     7     0
```

Problem 7-25

(a) We have

$$x_{k+1} = \mathbf{F}x_k + \mathbf{G}u_k$$

where

$$\mathbf{F} = e^{\mathbf{A}t}, \qquad \mathbf{G} = \left(e^{\mathbf{A}t} - 1\right)\mathbf{A}^{-1}\mathbf{B}$$

since $\mathbf{A}^{-1}$ exists. Now

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \qquad e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

For $T = 0.24$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-0.25} & 0 \\ 0 & e^{-0.5} \end{bmatrix} = \begin{bmatrix} 0.7788 & 0 \\ 0 & 0.6065 \end{bmatrix} = \mathbf{F}$$

From the definition of $\mathbf{G}$ we have

$$\mathbf{G} = \begin{bmatrix} -0.2212 & 0 \\ 0 & -0.3935 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4424 \\ -0.1968 \end{bmatrix}$$

Thus

$$\begin{bmatrix} x_{1_{k+1}} \\ x_{2_{k+1}} \end{bmatrix} = \begin{bmatrix} 0.7788 & 0 \\ 0 & 0.6065 \end{bmatrix} \begin{bmatrix} x_{1_k} \\ x_{2_k} \end{bmatrix} + \begin{bmatrix} 0.4424 \\ -0.1968 \end{bmatrix} u_k$$

(b) First we must find $e^{\mathbf{A}t}$. This is perhaps best accomplished using Laplace transform techniques. First we compute

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

Using partial fraction expansion yields

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \frac{2}{s+1}-\frac{1}{s+2} & \frac{1}{s+1}-\frac{1}{s+2} \\ \frac{-2}{s+1}+\frac{2}{s+2} & \frac{-1}{s+1}+\frac{2}{s+2} \end{bmatrix}$$

so that

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

At $T = 0.3$ seconds $e^{\mathbf{A}t}$ becomes

$$e^{\mathbf{A}t} = \begin{bmatrix} 0.9328 & 0.1920 \\ -0.3840 & 0.3568 \end{bmatrix} = \mathbf{F}$$

Thus

$$\mathbf{G} = \begin{bmatrix} -0.0672 & 0.1920 \\ -0.03840 & -0.6432 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0672 & 0.1920 \\ -0.03840 & -0.6432 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0672 \\ 0.3840 \end{bmatrix}$$

Now that both $\mathbf{F}$ and $\mathbf{G}$ are known we can write the final result

$$\begin{bmatrix} x_{1_{k+1}} \\ x_{2_{k+1}} \end{bmatrix} = \begin{bmatrix} 0.9328 & 0.1920 \\ -0.3840 & 0.3568 \end{bmatrix} \begin{bmatrix} x_{1_k} \\ x_{2_k} \end{bmatrix} + \begin{bmatrix} 0.0672 \\ 0.3840 \end{bmatrix} u_k$$

Problem 7-26

Since $u_{1_k} = 0$ and $u_{2_k} = 1$ for $k \geq 0$, $(7-92)$ becomes, for $k \geq 0$,

$$\begin{bmatrix} x_{1_{k+1}} \\ x_{2_{k+1}} \end{bmatrix} = \begin{bmatrix} 0.975 & 0.078 \\ -0.467 & 0.585 \end{bmatrix} \begin{bmatrix} x_{1_k} \\ x_{2_k} \end{bmatrix} + \begin{bmatrix} 0.103 \\ 0.052 \end{bmatrix}$$

$$y_k = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_{1_k} \\ x_{2_k} \end{bmatrix}$$

Beginning with $x_0 = 0$ we can compute the states recursively, and then compute y_k. This yields

k	x_{1_k}	x_{2_k}	y_k
0	0	0	0
1	0.103	0.052	0.207
2	0.207	0.034	0.276
3	0.307	-0.025	0.257
4	0.400	-0.106	0.188
5	0.485	-0.197	0.091
6	0.561	-0.290	-0.018
7	0.627	-0.380	-0.132
8	0.685	-0.463	-0.241
9	0.735	-0.539	-0.343
10	0.778	-0.607	-0.435

CHAPTER 8

Problem 8-1

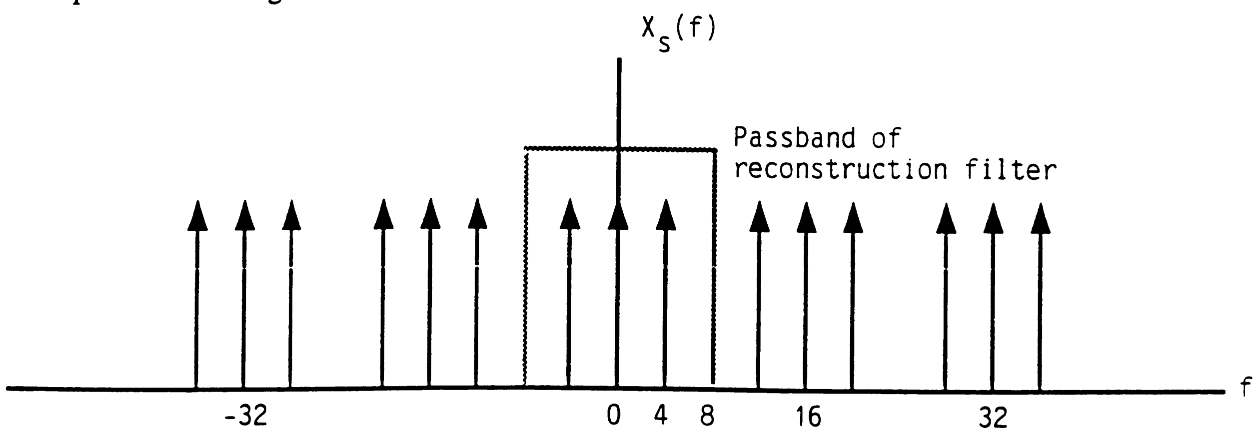
With

$$x(t) = 4 + 8\cos 8\pi t$$

and

$$X(f) = 4\delta(f) + 4\delta(f-4) + 4\delta(f+4)$$

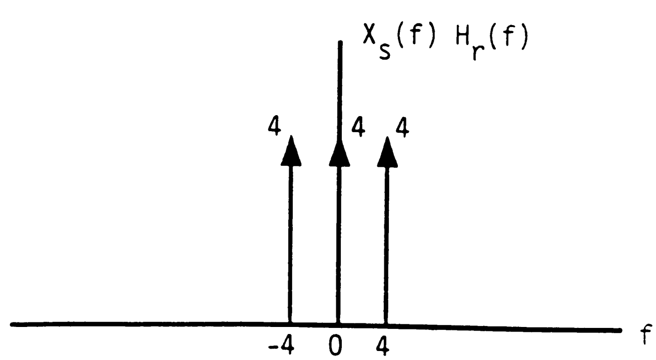
the specturm of the sampled signal appears as shown below. For a sampling frequency of 16 Hz all impulses have wieght 64.



The reconstruction filter has the transfer function

$$H_r(f) = \begin{cases} \frac{1}{16}, & |f| \le 8 \\ 0, & \text{otherwise} \end{cases}$$

The spectrum at the output of the reconstruction filter is



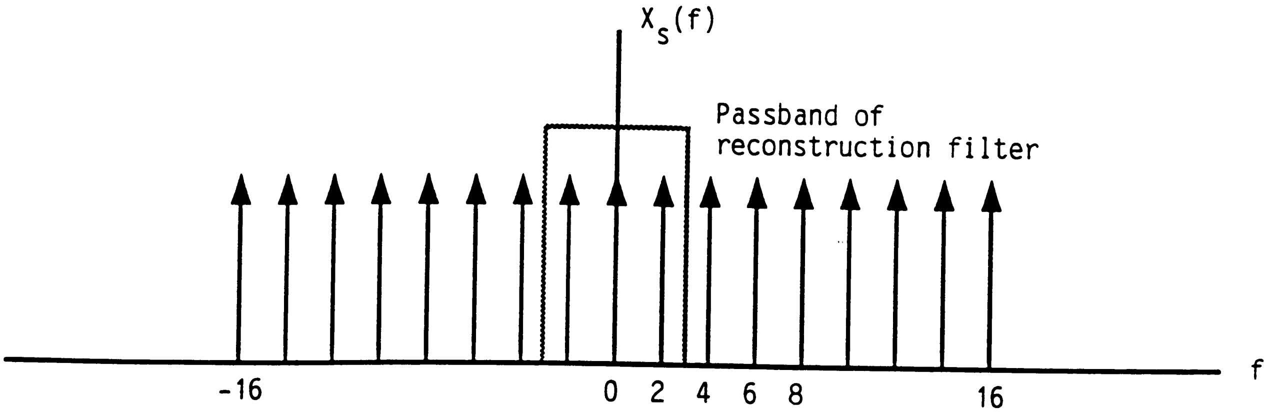
Clearly

$$X_s(f)\, H_r(f) = X(f)$$

which corresponds exactly to the time domain signal $x(t)$.

Problem 8-2

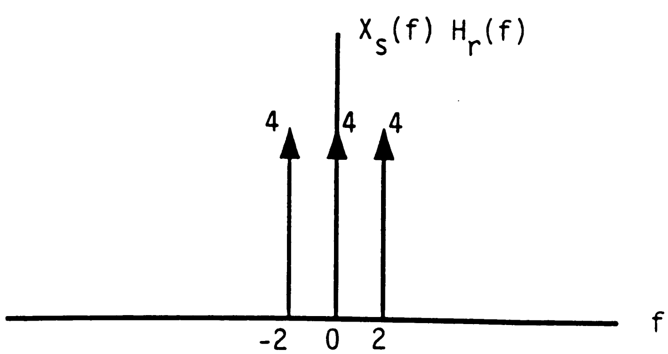
With $x(t)$ and $X(f)$ as defined in the previous problem, and with a sampling frequency of 6 Hz, the spectrum of the sampled signal appears as shown below. All impulses have weight $4f_s = 24$.



The reconstruction filter has the transfer function

$$H_r(f) = \begin{cases} \frac{1}{6}, & |f| < 3 \\ 0, & \text{otherwise} \end{cases}$$

The spectrum at the output of the reconstruction filter is



Clearly

$$X_s(f)\, H_r(f) \neq X(f)$$

and the original signal cannot be recovered from the samples.

Problem 8-3

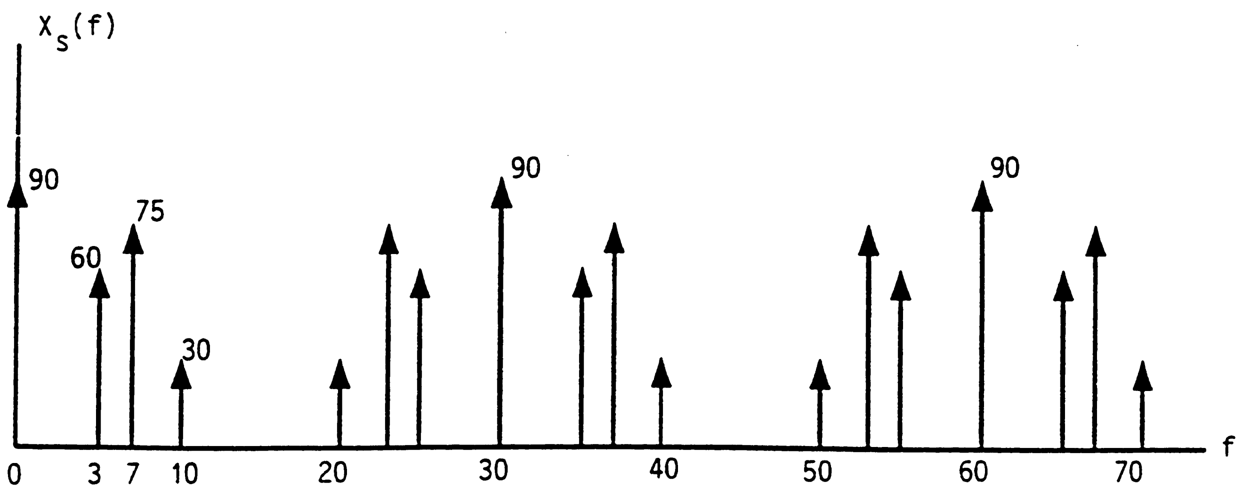
For

$$x(t) = 3 + 4\cos 10\pi t + 5\cos 14\pi t + 2\cos 20\pi t$$

we have

$$\begin{aligned} X(f) &= 3\delta(f) + 2\delta(f+5) + 2\delta(f-5) \\ &+ \frac{5}{2}\delta(f+7) + \frac{5}{2}\delta(f-7) \\ &+ \delta(f+10) + \delta(f-10) \end{aligned}$$

The dc component and the positive-frequency portion of the spectrum of the sampled signal is shown below for $f \leq 70$.

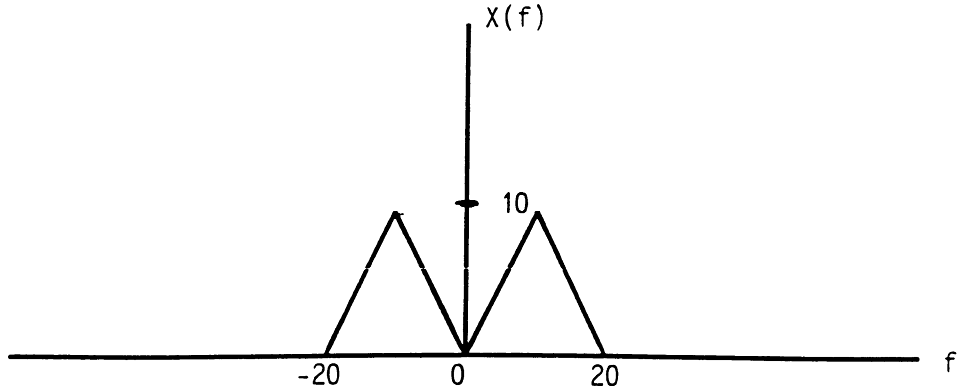


The original signal, $x(t)$, can be reconstructed from the sampled signal, $x_s(t)$, using a lowpass filter having the transfer function

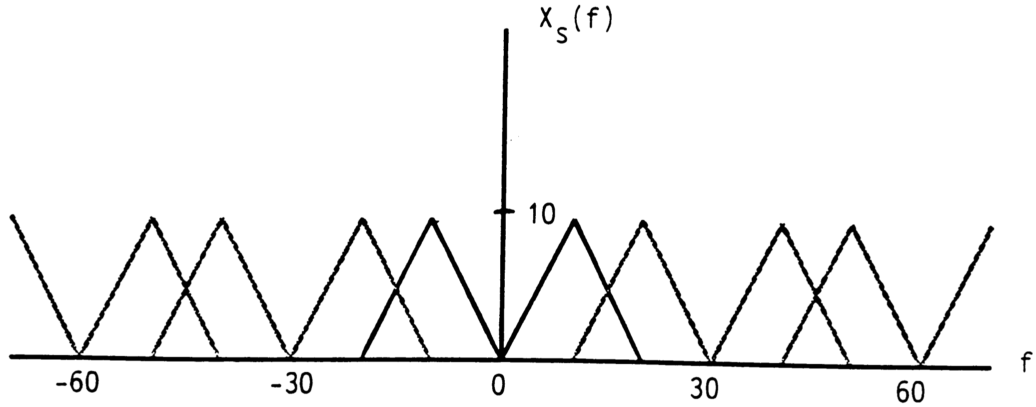
$$H_r(f = \begin{cases} \frac{1}{30}, & |f| < 15 \\ 0, & \text{otherwise} \end{cases}$$

Problem 8-4

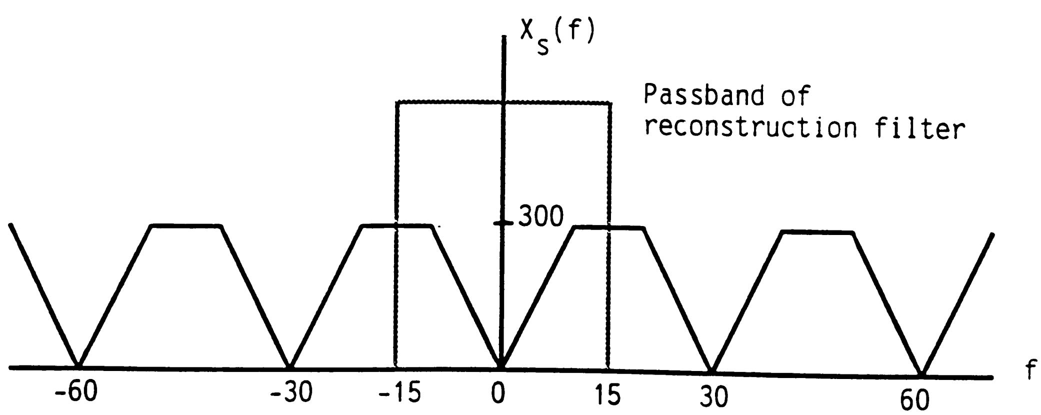
The spectrum $X(f)$ is shown below



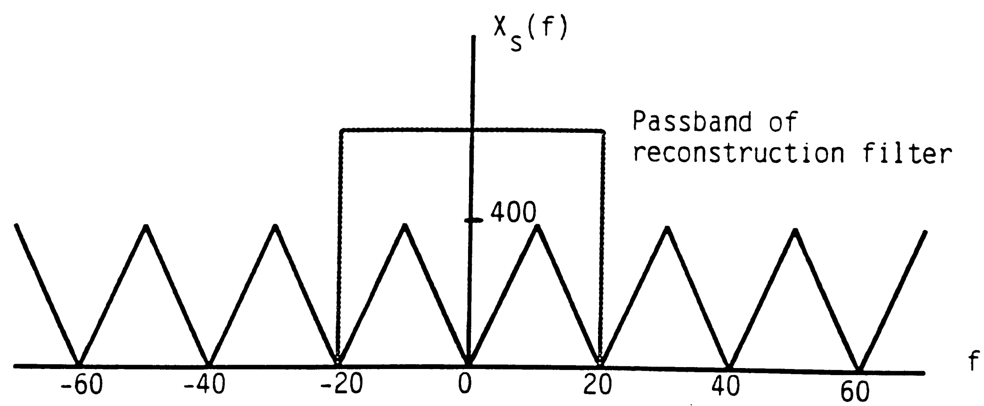
with a sampling frequency of 30 Hz the original spectrum and the translated spectrum are as shown.



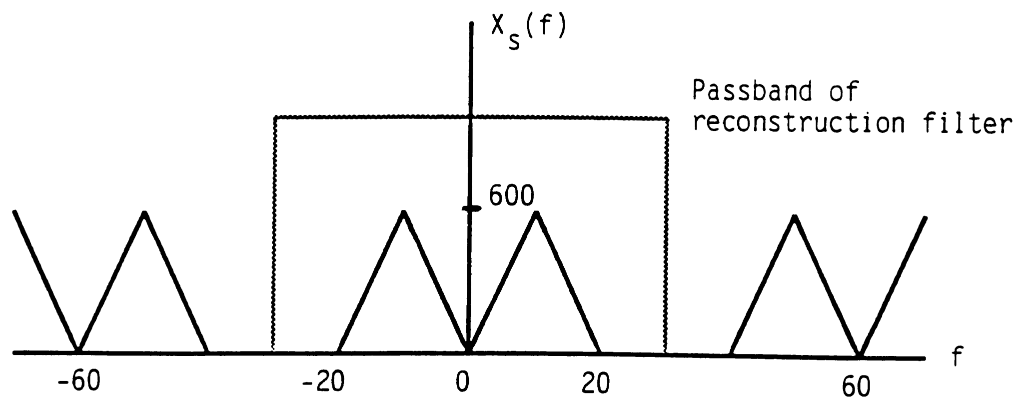
Adding these spectra yields the following result. Clearly it is not possible to recover the original signal from the sampled signal.



With a sampling frequency of 40 Hz, the following spectrum results.



With a sampling frequency of 60 Hz, the following spectrum results.



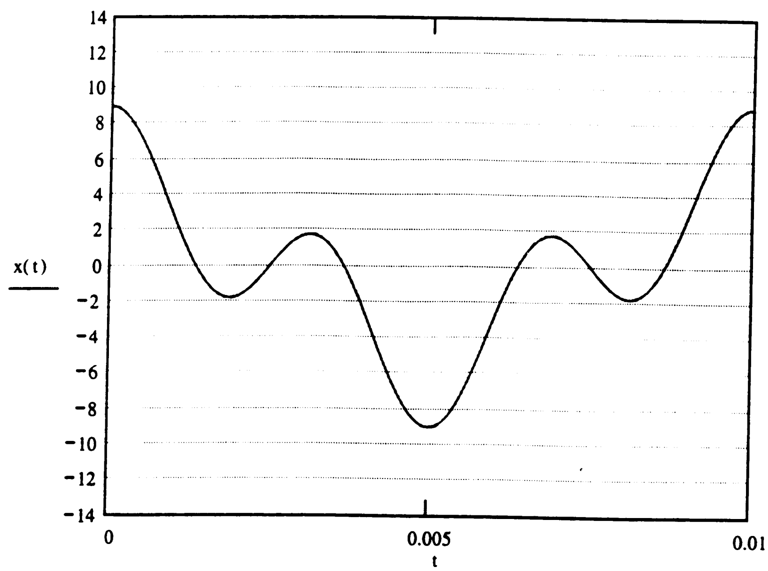
Of the three sampling frequencies, 40 Hz and 60 Hz are acceptable while 30 Hz is not acceptable.

Problem 8-5

(a) If

$$x(t) = 5\cos 200\pi t + 4\cos 600\pi t$$

The maximum value of $x(t)$ is *9* and the minimum value of $x(t)$ is *–9* as shown in the following illustration.



The dynamic range of $x(t)$ is therefore

$$D = 9 - (-9) = 18$$

(b) For the given signal, the signal power is

$$P_s = \frac{1}{T_0}\int_0^{T_0} (5\cos 200\pi t + 4\cos 600\pi t)^2\, dt$$

where $T_0 = 0.01$ is the period. The integral evaluates to

$$P_s = \frac{1}{2}(25 + 16) = \frac{41}{2} = 20.5$$

Equation (8-51) therefore yields

$$\begin{aligned}(SNR)_{dB} &= 10.79 + 6.02n + 10\log_{10}(20.5) - 20\log_{10}(18) \\ &= -1.20 + 6.02n\end{aligned}$$

The *SNR* for $x(t)$ is *2.96dB* inferior to the *SNR* for the sinusoidal signal of Example 8-3.

Problem 8-6

(a) For the signal shown the dynamic range is

$$D = 10 - (-10) = 20$$

(b) By symmetry the signal power is

$$P_s = \frac{1}{T/4}\int_0^{T/4}\left(\frac{10t}{T/4}\right)^2 dt = \frac{100}{3}$$

Equation (8-51) yields

$$\begin{aligned}(SNR)_{dB} &= 10.79 + 6.02n + 10\log_{10}\left(\frac{100}{3}\right) - 20\log_{10}(20) \\ &= 6.02n\end{aligned}$$

The output *SNR* is 1.76 dB inferior to the *SNR* computed in Example 8-3 for the sinusoidal signal.

Problem 8-7

By Parseval's theorem

$$\left\langle x^2(t)\right\rangle = \sum_{n=-\infty}^{\infty}|X_n|^2$$

Therefore

$$SNR = \frac{\sum_{n=-\infty}^{\infty}|X_n|^2}{\frac{D^2}{12}2^{-2n}} = \frac{12}{D^2}\left(2^{2n}\right)\sum_{n=-\infty}^{\infty}|X_n|^2$$

Problem 8-8

The signal

$$x(t) = 4 + 8\cos 8\pi t$$

has power

$$P = \frac{1}{T_o}\int_0^{T_o} (4 + 8\cos 8\pi t)^2 dt$$

where T_o is the period of the waveform. Performing the integration yields

$$P = 16 + \frac{1}{2}(8)^2 = 48$$

The dynamic range of the signal is

$$D = 2(8) = 16$$

Using (8-51) the *SNR* is

$$SNR = 10.79 + 6.02n + 10\log_{10}(48) - 10\log_{10}(16)$$

This gives

$$SNR = 15.56 + 6.02n$$

For $n = 8$

$$SNR = 15.56 + 6.02(8) = 63.72\ dB$$

For $n = 10$

$$SNR = 15.56 + 6.02(10) = 75.76\ dB$$

For $n = 12$

$$SNR = 15.56 + 6.02(12) = 87.80\ dB$$

Problem 8-9

Since

$$H(f) = \frac{1}{1 + j\left(\frac{f}{f_3}\right)^2}$$

it follows that the amplitude response is

$$|H(f)| = \frac{1}{\sqrt{1 + (f/f_3)^2}}$$

and the amplitude response, expressed in dB, is

$$20\log_{10}|H(f)| = -10\log_{10}\left(1+\left(\frac{f}{f_s}\right)^2\right)$$

(a) Using the given definitions of δ_1 and δ_2 gives

$$-10\log_{10}\left(1+\left(\frac{W}{f_3}\right)^2\right) = \delta_1$$

from which

$$\frac{W}{f_3} = \sqrt{10^{-0.1\delta_1}-1}$$

and

$$\frac{f_s - W}{f_3} = \sqrt{10^{-0.1\delta_2}-1}$$

Since

$$\frac{f_s - W}{f_3} = \frac{f_s}{f_3} - \frac{W}{f_3} = \frac{f_s}{W}\frac{W}{f_3} - \frac{W}{f_3}$$

Solving for f_s/W gives

$$\frac{f_s}{W} = \frac{\left(\frac{W}{f_3}\right) + \left(\frac{f_s - W}{f_3}\right)}{\left(\frac{W}{f_3}\right)}$$

Substituting for $\frac{(f_s - W)}{f_3}$ and $\frac{W}{f_3}$ yields

$$f_s = W\frac{\sqrt{10^{-0.1\delta_1}-1}+\sqrt{10^{-0.1\delta_2}-1}}{\sqrt{10^{-0.1\delta_1}-1}}$$

(b) For $\delta_1 = -3dB$ and $\delta_2 = -30dB$

$$\frac{f_s}{W} = \frac{\sqrt{10^{0.3}-1}+\sqrt{10^{3}-1}}{10^{0.3}-1} = 32.68$$

For $\delta_1 = -1dB$ and $\delta_2 = -30dB$

$$\frac{f_s}{W} = \frac{\sqrt{10^{0.1}-1}+\sqrt{10^{3}-1}}{\sqrt{10^{0.1}-1}} = 123.07$$

For $\delta_1 = -1dB$ and $\delta_2 = -40dB$

$$\frac{f_s}{W} = \frac{\sqrt{10^{0.1}-1}+\sqrt{10^{4}-1}}{\sqrt{10^{0.1}-1}} = 196.51$$

Note that reducing the attenuation at $f = W$ from *-3* to *-1dB* results in the necessity of increasing the sampling frequency by a factor of *4*, while increasing the attenuation at $f = f_s - W$ from –30 to –40dB requires the sampling frequency to be increased by 60%. Large values of f_s/W can be avoided by using a higher-order filter.

Problem 8-10

(a) From the definition of *H(f)* we have

$$H(f) = \frac{(2\pi f_3)^2}{(2\pi f_3)^2 - (2\pi f)^2 + j\sqrt{2}(2\pi)^2 f f_3}$$
$$= \frac{f_3^2}{\left(f_3^2 - f^2\right) + j\sqrt{2} f f_3}$$

The amplitude response is

$$|H(f)| = \frac{f_3^2}{\sqrt{\left(f_3^2 - f^2\right)^2 + 2f^2 f_3^2}} = \frac{f_3^2}{\sqrt{f_3^4 + f^4}}$$

and the square of the amplitude response is

$$|H(f)|^2 = \frac{f_3^4}{f_3^4 + f^4}$$

(b) In order to derive the required expression we first write

$$|H(f)|^2 = \frac{1}{1+\left(\dfrac{f}{f_3}\right)^4}$$

and

$$|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{f_3}\right)^4}}$$

This gives

$$20\log_{10}|H(f)| = -10\log_{10}\left(1+\left(\frac{f}{f_3}\right)^4\right)$$

Using the definitions given in Problem 8-9 yield

$$-10\log_{10}\left(1+\left(\frac{W}{f_3}\right)^4\right) = \delta_1$$

or

$$\frac{W}{f_3} = \left[10^{-0.1\delta_1}-1\right]^{1/4}$$

and

$$-10\log_{10}\left(1+\left(\frac{f_s-W}{f_3}\right)^4\right) = \delta_2$$

which gives

$$\frac{f_s-W}{f_3} = \left[10^{-0.1\delta_2}-1\right]^{1/4}$$

Once again we write

$$\frac{f_s-W}{f_3} = \frac{f_s}{W}\,\frac{W}{f_3} - \frac{W}{f_3}$$

This yields, as it did in the preceding problem

$$\frac{f_s}{W} = \frac{\frac{W}{f_3} - \left(\frac{f_s-W}{f_3}\right)}{\frac{W}{f_3}}$$

Substituting the previously determined values yields

$$\frac{f_s}{W} = \frac{\left[10^{-0.1\delta_1}-1\right]^{1/4} + \left[10^{-0.1\delta_2}-1\right]^{1/4}}{\left[10^{-0.1\delta_1}-1\right]^{1/4}}$$

(c) For $\delta_1 = -3dB$ and $\delta_2 = -20\ dB$

$$\frac{f_s}{W} = \frac{\left[10^{0.3} - 1\right]^{1/4} + \left[10^2 - 1\right]^{1/4}}{\left[10^{0.3} - 1\right]^{1/4}} = 4.16$$

For $\delta_1 = -1dB$ and $\delta_2 = -20dB$

$$\frac{f_s}{W} = \frac{\left[10^{0.1} - 1\right]^{1/4} + \left[10^2 - 1\right]^{1/4}}{\left[10^{0.1} - 1\right]^{1/4}} = 5.42$$

For $\delta_1 = -1dB$ and $\delta_2 = -40dB$

$$\frac{f_s}{W} = \frac{\left[10^{0.1} - 1\right]^{1/4} + \left[10^4 - 1\right]^{1/4}}{\left[10^{0.1} - 1\right]^{1/4}} = 15.02$$

As expected, higher-order reconstruction filters require much lower sampling frequencies.

Problem 8-11

The given transfer function can be written

$$H(s) = \frac{T\omega_3^2}{\left(s + \frac{\omega_3}{\sqrt{2}} + j\frac{\omega_3}{\sqrt{2}}\right) + \left(s + \frac{\omega_3}{\sqrt{2}} - j\frac{\omega_3}{\sqrt{2}}\right)}$$

$$= \frac{A}{s + \frac{\omega_3}{\sqrt{2}} + j\frac{\omega_3}{\sqrt{2}}} + \frac{A^*}{s + \frac{\omega_3}{\sqrt{2}} - j\frac{\omega_3}{\sqrt{2}}}$$

The value of A is

$$A = \frac{T\omega_3^2\sqrt{2}}{-j2\omega_3} = -\frac{\sqrt{2}T\omega_3}{j2}$$

Thus

$$H(s) = \sqrt{2}T\omega_3 e^{-(\omega_3/\sqrt{2})t}\ \frac{e^{j(\omega_3/\sqrt{2})t} - e^{-j(\omega_3/\sqrt{2})t}}{2j}$$

or

$$H(s) = \sqrt{2}\,T\,\omega_3 e^{-(\omega_3/\sqrt{2})t}\sin\left(\frac{\omega_3}{\sqrt{2}}\right)t$$

The interpolation formula then becomes

$$y(t) = \sqrt{2}\,T\,\omega_3 \sum_{k=-\infty}^{\infty} e^{-(\omega_3/\sqrt{2})(t-kT)}\sin\left(\frac{\omega_3}{\sqrt{2}}\right)(t-kT)$$

Problem 8-12

For an RC lowpass reconstruction filter the output is given by (8-40). With $x(nT) = a$ for all n

$$y(t) = \sum_{k=-\infty}^{\infty} a(2\pi f_s T)e^{-2\pi f_3(t-kT)}u(t-kT)$$

Since $u(t-kT) = 0$ for $k > \left[\frac{t}{T}\right]$, where $\left[\frac{t}{T}\right]$ is the integer part of t/T, we get

$$y(t) = a(2\pi f_s T)e^{-2\pi f_3 t}\sum_{k=-\infty}^{\left[\frac{t}{T}\right]} e^{2\pi f_3 kT}$$

or

$$y(t) = a(2\pi f_s T)\,e^{-2\pi f_3 t}\sum_{k=-\left[\frac{t}{T}\right]}^{\infty} e^{2\pi f_3 kT}$$

With the change of index $m = k + \left[\frac{t}{T}\right]$ we get

$$y(t) = a(2\pi f_s T)e^{-2\pi f_3 t}\sum_{m=0}^{\infty} e^{-2\pi f_3\left(m-\frac{t}{T}\right)T}$$

(Note that $\frac{t}{T}$ has been replaced by a continuous variable. We will later assume T small so that $\left[\frac{t}{T}\right]T \to t$.) Thus

$$y(t) = a(2\pi f_3 T)\sum_{m=0}^{\infty} e^{-2\pi f_3 mT}$$

Performing the summation gives

$$y(t) = a(2\pi f_3 T)\frac{1}{1 - e^{-2\pi f_3 T}}$$

For T small so that $2\pi f_3 T << 1$, which implies a high sampling frequency,

$$e^{-2\pi f_3 T} = 1 - 2\pi f_3 T$$

and

$$y(t) = a\frac{2\pi f_3 T}{1 - (1 - 2\pi f_3 T)} = a$$

Problem 8-13

From entry 3 in Table 8-1

$$\sum_{n=0}^{\infty} e^{-anT} z^{-n} = \frac{1}{1 - e^{-aT} z^{-1}}$$

Differentiating both sides with respect to a gives

$$\sum_{n=0}^{\infty} (-nT) e^{-anT} z^{-n} = \frac{(-1)T e^{-aT} z^{-1}}{\left(1 - e^{-aT} z^{-1}\right)^2}$$

Multiplying by -1 yields entry 5 in Table 8-1

$$\sum_{n=0}^{\infty} (nT) e^{-anT} z^{-1} = \frac{T e^{-aT} z^{-1}}{\left(1 - e^{-aT} z^{-1}\right)^2}$$

With $a = 0$ the preceeding expression becomes

$$\sum_{n=0}^{\infty} (nT) z^{-n} = \frac{T z^{-1}}{\left(1 - z^{-1}\right)^2}$$

which is entry 4 in Table 8-1.

Problem 8-14

With $\alpha = jb$ we get

$$\sum_{n=0}^{\infty} e^{-jbnT} z^{-n} = \frac{1}{1-e^{-jbT} z^{-1}}$$

Multiplying the numerator and denominator of the right-hand side (*RHS*) of the above expression yields

$$RHS = \frac{1-e^{jbT} z^{-1}}{\left(1-e^{-jbT} z^{-1}\right)\left(1-e^{jbT} z^{-1}\right)}$$

or

$$RHS = \frac{\left(1-\cos(bT)z^{-1}\right) - j\sin(bT)z^{-1}}{1-2\cos(bT)z^{-1}+z^{-2}}$$

The left-hand side (*LHS*) is

$$LHS = \sum_{n=0}^{\infty} \cos(bnT)z^{-n} - j\sum_{n=0}^{\infty} \sin(bnT)z^{-n}$$

Equating real parts gives

$$\sum_{n=0}^{\infty} \cos(bnT)z^{-n} = \frac{1-\cos(bT)z^{-1}}{1-2\cos(bT)z^{-1}+z^{-2}}$$

Equating imaginary parts gives

$$\sum_{n=0}^{\infty} \sin(bT)z^{-n} = \frac{\sin(bT)z^{-1}}{1-2\cos(bT)z^{-1}+z^{-2}}$$

Problem 8-15

Letting $\alpha = a + jb$ gives

$$\sum_{n=0}^{\infty} e^{-(a+jb)nT} z^{-n} = \frac{1}{1-e^{-(a+jb)T} z^{-1}}$$

The left-hand side (*LHS*) gives

$$LHS = \sum_{n=0}^{\infty} e^{-anT} \cos bnT\, z^{-n} - j\sum_{n=0}^{\infty} e^{-anT} \sin bnT\, z^{-n}$$

The right-hand side (*RHS*) gives

$$RHS = \frac{1-e^{-aT}e^{jbT}z^{-1}}{\left(1-e^{-aT}e^{-jbT}z^{-1}\right)\left(1-e^{-aT}e^{jbT}z^{-1}\right)}$$

$$= \frac{\left(1-e^{-aT}\cos bT\,z^{-1}\right)-je^{-aT}\sin bT\,z^{-1}}{1-2e^{-aT}\cos bT\,z^{-1}+e^{-2aT}z^{-2}}$$

Equating real parts of *LHS* and *RHS* yields

$$\sum_{n=0}^{\infty} e^{-anT}\cos bnT\,z^{-n} = \frac{1-e^{-aT}\cos bT\,z^{-1}}{1-2e^{-aT}\cos bT\,z^{-1}+e^{-2aT}z^{-2}}$$

Equating imaginary parts of *LHS* and *RHS* yields

$$\sum_{n=0}^{\infty} e^{-anT}\sin bnT\,z^{-n} = \frac{e^{-aT}\sin bT\,z^{-1}}{1-2e^{-aT}\cos bT\,z^{-1}+e^{-2aT}z^{-2}}$$

Problem 8-16

If

$$\sum_{n=0}^{\infty} x(nT)z^{-n} = X(z)$$

we can write

$$\sum_{n=0}^{\infty} a^n x(nT)z^{-n} = \sum_{n=0}^{\infty} x(nT)\left(\frac{z}{a}\right)^{-n} = X\left(\frac{x}{a}\right)$$

From Entry 2 in Table 8-1

$$\sum_{n=0}^{\infty} (1)z^{-n} = \frac{1}{1-z^{-1}}$$

Using the form given above yields

$$\mathfrak{Z}\left\{e^{-anT}\right\} = X\left(\frac{z}{e^{-aT}}\right)$$

$$= \frac{1}{1-\left(\frac{z}{e^{-aT}}\right)^{-1}} = \frac{1}{1-e^{-aT}z^{-1}}$$

Problem 8-17

By definition

$$X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n}$$

Differentiating with respect to z gives

$$\frac{d}{dz}X(z) = \sum_{n=0}^{\infty} x(nT)(-n)z^{-n-1}$$

Multiplying by $-z$ yields

$$-z\frac{d}{dz}X(z) = \sum_{n=0}^{\infty} [nx(nT)]z^{-n}$$

From Entry 3 in Table 8-1

$$\sum_{n=0}^{\infty} e^{-anT}z^{-n} = \frac{1}{1-e^{-aT}z^{-1}}$$

Differentiating with respect to z gives

$$\sum_{n=0}^{\infty} e^{-anT}(-n)z^{-n-1} = \frac{-e^{-aT}z^{-2}}{\left(1-e^{-aT}z^{-1}\right)^2}$$

Multiplying by $-z$ yields

$$\sum_{n=0}^{\infty} ne^{-anT}z^{-n} = \frac{e^{aT}z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^2}$$

This is a general result. Multiplying by T and letting $a = 0$ gives

$$\sum_{n=0}^{\infty} (nT)z^{-n} = \frac{Tz^{-1}}{\left(1-z^{-1}\right)^2}$$

Problem 8-18

The sample values are given by

$$x(nT) = 5\sin\left(\frac{20\pi n}{25}\right) + 2\prod\left(\frac{0.04n - 0.14}{0.16}\right)$$

which can be written

$$x(nT) = 5\sin\left(\frac{4\pi n}{5}\right) + 2\prod\left(\frac{n - 3.5}{4}\right)$$

Recognizing that

$$\prod\left(\frac{n - 3.5}{4}\right) = \begin{cases} 1, & 2 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$

yields

$$x(0) = 0$$

$$x(T) = 5\sin\frac{4\pi}{5} = 2.939$$

$$x(2T) = 5\sin\frac{8\pi}{5} + 2 = -4.755 + 2 = -2.755$$

$$x(3T) = 5\sin\frac{12\pi}{5} + 2 = 4.755 + 2 = 6.755$$

$$x(4T) = 5\sin\frac{16\pi}{5} + 2 = -2.939 + 2 = -0.939$$

$$x(5T) = 5\sin\frac{20\pi}{5} + 2 = \sin 4\pi + 2 = 2$$

$$x(6T) = 5\sin\frac{24\pi}{5} = 2.939$$

$$x(7T) = 5\sin\frac{28\pi}{5} = -4.755$$

$$x(8T) = 5\sin\frac{32\pi}{5} = 4.755$$

This gives

$$X(z) = 2.939z^{-1} - 2.755z^{-2} + 6.755z^{-3} - 0.939z^{-4} + 2z^{-5} + 2.939z^{-6} - 4.755z^{-7} + 4.755z^{-8}$$

Problem 8-19

(a) The z-transform of $x(nT) = \left(\frac{1}{5}\right)^n u(n)$ is

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^{n} z^{-u} = \sum_{n=0}^{\infty} \left(\frac{1}{5} z^{-1}\right)^{n}$$

which is

$$X(z) = \frac{1}{1 - \frac{1}{5} z^{-1}}, \quad |z| > \frac{1}{5}$$

(b) The z-transform of $x(nT) = \left(-\frac{1}{5}\right)^{n} u(n)$ is

$$X(z) = \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^{n} z^{-n} = \sum_{n=0}^{\infty} \left(-\frac{1}{5} z^{-1}\right)^{n}$$

which is

$$X(z) = \frac{1}{1 + \frac{1}{5} z^{-1}}, \quad |z| > \frac{1}{5}$$

(c) The z-transform of $x(nT) = u(n) + \left(\frac{3}{4}\right)^{n} u(n-4)$ is

$$X(z) = \sum_{n=0}^{\infty} z^{-n} + \sum_{n=4}^{\infty} \left(\frac{3}{4}\right)^{n} z^{-n}$$

which can be written

$$X(z) = \sum_{n=0}^{\infty} (z^{-1})^{n} + \sum_{n=4}^{\infty} \left(\frac{3}{4} z^{-1}\right)^{n}$$

With the change of index $k = n - 4$ in the second sum, this becomes

$$X(z) = \frac{1}{1 - z^{-1}} + \sum_{k=0}^{\infty} \left(\frac{3}{4} z^{-1}\right)^{k+4}$$

This gives

$$X(z) = \frac{1}{1 - z^{-1}} + \frac{\left(\frac{3}{4} z^{-1}\right)^{4}}{1 - \frac{3}{4} z^{-1}}, \quad |z| > 1$$

The region of convergence results by recognizing that the first term exists for $|z| > 1$ and the second term exists for $|z| > \frac{3}{4}$. The z-transform exists only when both terms are defined. This requires that $|z| > 1$.

(d) The z-transform of $x(nT) = 2u(n) - 2u(n-8)$ is

$$X(z) = \sum_{n=0}^{\infty} 2z^{-n} - \sum_{n=8}^{\infty} 2z^{-n}$$

Letting $k = n - 8$ in the second sum is

$$X(z) = \sum_{n=0}^{\infty} 2(z^{-1})^n - \sum_{k=0}^{\infty} 2(z^{-1})^{k+8}$$

This gives

$$X(z) = \frac{2}{1-z^{-1}} - \frac{2z^{-8}}{1-z^{-1}}$$

or

$$X(z) = \frac{2(1-z^{-8})}{1-z^{-1}}, \quad |z| \neq 0$$

The above expression gives $X(z)$ in closed form. The region of convergence is justified by recognizing that $X(z)$ can also be written in terms of the finite sum.

$$X(z) = 2 + 2z^{-1} + 2z^{-2} + 2z^{-3} + 2z^{-4} + 2z^{-5} + 2z^{-6} + 2z^{-7}$$

The terms in the above series of the form z^{-k} for $k > 0$ are clearly defined for all z except $z = 0$.

Problem 8-20

(a) The MATLAB script for solving Problem 8-19(a) is shown below along with the result.

```
syms n z                  % Make n and z symbolic
xn = (1/5)^n;             % Define x(n)
xz = ztrans(xn,n,z);      % z-transform
xz                        % Display results

xz =

5*z/(5*z-1)
```

We see that the result is in agreement with the previous problem.

We see that the result is in agreement with the previous problem.

(b) The MATLAB script for solving Problem 8-19(b) is shown below along with the result.

```
syms n z                  % Make n and z symbolic
xn = (-1/5)^n;            % Define x(n)
xz = ztrans(xn,n,z);      % z-transform
xz                        % Display results

xz =

5*z/(5*z+1)
```

Once again we see that the result is in agreement with the previous problem.

(c) The are a number of approaches that can be taken to this problem. We shall simply recognize that if the z-transform of $x(nT)$ is $X(z)$, the z-transform of $x((n-k)T)$ is $X(z)z^{-k}$. We also recognize that

$$\left(\frac{3}{4}\right)^n = \left(\frac{3}{4}\right)^4 \left(\frac{3}{4}\right)^{n-4}$$

Using these observations allows us to solve the problem using the following MATLAB script.

```
syms n z                          % Make n and z symbolic
x1n = 1;                          % Define first term of x(n)
x1z = ztrans(x1n,n,z);            % z-transform x(n)
x2ns = (3/4)^n;                   % Define 'shifted' second term
x2zs = ztrans(x2ns,n,z);          % z-transform
x2z = (x2zs)*z^(-4)*(3/4)^4;      % Remove 'shift'
xz = x1z+x2z;                     % Combine terms
xz                                % Display results
xz =

z/(z-1)+81/64/z^3/(4*z-3)
```

A little manipulation shows that the result is in agreement with the previous problem.

(d) This part of the problem is solved using the same approach as was used in part (c). The result is the following MATLAB script.

```
syms n z                    % Make n and z symbolic
xn = 2;                     % Define x(n) for first term
x1z = ztrans(xn,n,z);       % z-transform
x2z = x1z*z^(-8);           % Define second term
xz = x1z+x2z;               % Combine terms
xz                          % Display result

xz =

2*z/(z-1)+2/z^7/(z-1)
```

We see that the result is in agreement with Problem 8-19(d).

Problem 8-21

(a) The z-transform of $x(nT)=\left(\frac{2}{3}\right)^n u(n-4)$ is

$$X(z)=\sum_{n=4}^{\infty}\left(\frac{2}{3}\right)^n z^{-n}=\sum_{n=4}^{\infty}\left(\frac{2}{3}z^{-1}\right)^n$$

With the change of index $k=n-4$ we have

$$X(z)=\sum_{k=0}^{\infty}\left(\frac{2}{3}z^{-1}\right)^{k+4}$$

Thus

$$X(z)=\frac{\left(\frac{2}{3}\right)^4 z^{-4}}{1-\frac{2}{3}z^{-1}},\quad |z|>\frac{2}{3}$$

(b) The z-transform of $x(nT)=\left(\frac{2}{3}\right)^{n-4} u(n-4)$ is

$$X(z)=\sum_{n=4}^{\infty}\left(\frac{2}{3}\right)^{n-4} z^{-n}$$

We once again use the change of index $k=n-4$. This gives

$$X(z)=\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k z^{-(k+4)}=z^{-4}\sum_{k=0}^{\infty}\left(\frac{2}{3}z^{-1}\right)^k$$

Thus

$$X(z)=\frac{z^{-4}}{1-\frac{2}{3}z^{-1}},\quad |z|>\frac{2}{3}$$

Problem 8-22

The following two MATLAB scripts solve Problem 8-22. See the solution for Problem 8-20 for insight into the method used.

```
syms n z                            % Make n and z symbolic

%  Following is the script for part (a).

xna = (2/3)^n;                      % Define x(n) for first term
xza = ztrans(xna,n,z);              % z-transform
xza = ((2/3)^4)*(z^(-4))*xza;       % Shift and modify
xza                                 % Display result

xza =

16/27/z^3/(3*z-2)


%  Following is the script for part (b).

xnbs = (2/3)^n;                     % Define 'shifted' x(n)
xzbs = ztrans(xnbs,n,z);            % z-transform
xzb = (z^(-4))*xzbs;                % Shift to get the correct result
xzb                                 % Display result

xzb =

3/z^3/(3*z-2)
```

Problem 8-23

From $X(z)=\exp\{az^{-1}\}$ we have

$$X(z)=\sum_{n=0}^{\infty}\frac{(az^{-1})^n}{n!}=\sum_{n=0}^{\infty}\frac{a^n z^{-n}}{n!}$$

The value of $x(5T)$ is given by the coefficient of z^{-5} in the series expansion. this is the $n=5$ term. Thus

$$x(5T)=\frac{a^5}{120}$$

Problem 8-24

The z-transform of $x(nT) = a^n \sin\left(\frac{\pi}{2}n\right)$ for $n \geq 0$ and real a is

$$X(z) = \sum_{n=0}^{\infty} a^n \sin\left(\frac{\pi}{2}n\right) z^{-n}$$

Since $\sin\left(\frac{\pi}{2}n\right) = 0$ for n even and ± 1 for n odd, a little thought shows that $X(z)$ can be written in the form

$$X(z) = \sum_{k=0}^{\infty} a^{2k+1} (-1)^k z^{-2k-1}$$

which is

$$X(z) = az^{-1} \sum_{k=0}^{\infty} \left(-a^2 z^{-2}\right)^k$$

so that

$$X(z) = \frac{az^{-1}}{1 + a^2 z^{-2}}, \quad |z| > a$$

Problem 8-25

(a) The z-transform of $x(nT) = \left(\frac{1}{2}\right)^n [u(n) - u(n-10)]$ is

$$X(z) = \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{9} \left(\frac{1}{2} z^{-1}\right)^n$$

which is

$$X(z) = \frac{1 - \left(\frac{1}{2} z^{-1}\right)^{10}}{1 - \frac{1}{2} z^{-1}} = \frac{z^{10} - \frac{1}{1024}}{z^9 \left(z - \frac{1}{2}\right)}$$

Note that there is not a pole at $z = \frac{1}{2}$. There is a 9th order pole at $z = 0$. While this fact is not completely obvious from the closed-form expression for $X(z)$, it is obvious by writing $X(z)$ in the form

$$X(z) = 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + \left(\frac{1}{8}\right)\left(\frac{1}{z}\right)^3 + \cdots + \left(\frac{1}{512}\right)\left(\frac{1}{z}\right)^9$$

(b) The z-transform of $x(nT) = \left(\frac{1}{2}\right)^n [u(n) + u(n-10)]$ is

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=10}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

which is

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=10}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{k=0}^{10} \left(\frac{1}{2} z^{-1}\right)^{k+10}$$

Performing the summations yields

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \left[1 + \frac{z^{-1}}{1024}\right], \quad |z| > \frac{1}{2}$$

Problem 8-26

(a) The energy in the sequence

$$x(nT) = \left(\frac{1}{2}\right)^n u(n)$$

is

$$E = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

(b) The energy in the sequence

$$x(nT) = \left(\frac{1}{2}\right)^n u(n-6)$$

is

$$E = \sum_{n=6}^{\infty} \left(\frac{1}{2}\right)^{2n} = \sum_{n=6}^{\infty} \left(\frac{1}{4}\right)^n$$

With the change of index $k = n - 6$ this becomes

$$E = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k+6} = \left(\frac{1}{4}\right)^6 \frac{1}{1 + \frac{1}{4}} = \frac{4}{3} \left(\frac{1}{4}\right)^6$$

or

$$E = \frac{1}{3}\left(\frac{1}{4}\right)^5 = \frac{1}{3072}$$

(c) The energy in

$$x(n) = \left(\frac{1}{2}\right)^{n-6} u(n-6)$$

is

$$E = \sum_{n=6}^{\infty}\left(\left(\frac{1}{2}\right)^{n-6}\right)^2 = \sum_{n=6}^{\infty}\left(\frac{1}{4}\right)^{n-6}$$

With the change of index $k = n - 6$ this becomes

$$E = \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = \frac{1}{1+\frac{1}{4}} = \frac{4}{3}$$

With a little thought it is obvious that the energy in (c) is the same as (a). The signal in (c) is simply the signal in (a) shifted six sample periods.

(d) The energy in

$$x(nT) = \left[4\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{4}\right)^n\right] u(n)$$

is

$$E = \sum_{n=0}^{\infty}\left[4\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{4}\right)^n\right]^2$$

which can be written

$$E = \sum_{n=0}^{\infty}\left[16\left(\frac{1}{9}\right)^n - 24\left(\frac{1}{12}\right)^n + 9\left(\frac{1}{16}\right)^n\right]$$

Performing the summation yields

$$\begin{aligned} E &= 16\left(\frac{9}{8}\right) - 24\left(\frac{12}{11}\right) + 9\left(\frac{16}{15}\right) \\ &= \frac{23{,}760 - 34{,}560 + 12{,}672}{1320} = \frac{1872}{1320} \\ &= 1.41818 \end{aligned}$$

Problem 8-27

(a) Since

$$X(z) = 1 - 0.3z^{-1} + 0.7z^{-2} + 0.8z^{-3} - 0.3z^{-4}$$

it follows that

$$\begin{aligned} x(0) &= 1 \\ x(T) &= -0.3 \\ x(2T) &= 0.7 \\ x(3T) &= 0.8 \\ x(4T) &= -0.3 \\ x(nT) &= 0, \ n > 4 \end{aligned}$$

(b) Since

$$X(z) = \left(1 - z^{-1}\right)^3 = 1 - 3z^{-1} + 3z^{-2} - z^{-3}$$

it follows that

$$\begin{aligned} x(0) &= 1 \\ x(T) &= -3 \\ x(2T) &= 3 \\ x(3T) &= -1 \\ x(nT) &= 0, \ n > 3 \end{aligned}$$

(c) Since

$$X(z) = \left(1 - 0.8z^{-1}\right)^3 = 1 - 2.4^{-1} + 1.92z^{-2} - 0.512z^{-3}$$

it follows that

$$\begin{aligned} x(0) &= 1 \\ x(T) &= -2.4 \\ x(2T) &= 1.92 \\ x(3T) &= -0.512 \\ x(nT) &= 0, \ n > 3 \end{aligned}$$

(d) Since

$$X(z) = \left(1 - 0.9z^{-2}\right)^3 \left(1 + 0.9z^{-2}\right)^3$$

we can write $X(z)$ as

$$X(z) = \left(1 - 0.81z^{-4}\right)^3$$

or

$$X(z) = 1 - 0.243z^{-4} + 1.9683z^{-8} - 0.5314z^{-12}$$

Thus

$$\begin{aligned} x(0) &= 1 \\ x(4T) &= -0.243 \\ x(8T) &= 1.9683 \\ x(12T) &= -0.5314 \end{aligned}$$

and

$$x(nT) = 0, \quad n \neq 0,\ 4,\ 8 \text{ or } 12$$

Problem 8-28

(a) Since

$$X(z) = z^{-2}\left(1 + 0.3z^{-1} - 0.7z^{-2}\right)\left(1 + z^{-3}\right)$$

can be written

$$X(z) = z^{-2} + 0.3z^{-3} - 0.7z^{-4} + z^{-5} + 0.3z^{-6} - 0.72z^{-7}$$

we have

$$\begin{aligned} x(0) = x(T) &= 0 \\ x(2T) &= 1 \\ x(3T) &= 0.3 \\ x(4T) &= -0.7 \\ x(5T) &= 1 \end{aligned}$$

$$x(6T) = 0.3$$
$$x(7T) = -0.7$$
$$x(nT) = 0, \; n > 7$$

(b) Since

$$X(z) = z^{-2}\left(1 + 0.3z^{-1} - 0.7z^{-2}\right)^2\left(1 + z^{-3}\right)$$

it follows that $X(z)$ can be written

$$X(z) = z^{-2} + 0.6z^{-3} - 1.31z^{-4} + 0.58z^{-5} + 1.09z^{-6} - 1.31z^{-7} - 0.42z^{-8} + 0.49z^{-9}$$

it follows that

$$x(0) = x(T) = 0$$
$$x(2T) = 1$$
$$x(3T) = 0.6$$
$$x(4T) = -1.31$$
$$x(5T) = 0.58$$
$$x(6T) = 1.09$$
$$x(7T) = -1.31$$
$$x(8T) = -0.42$$
$$x(9T) = 0.49$$
$$x(nT) = 0, \; n > 9$$

Problem 8-29

(a) The long division is shown below

$$
\begin{array}{r l}
 & 1+0.30z^{-1}-0.21z^{-2}+0.347z^{-3}+0.3171z^{-4} \\
\hline
1-0.30z^{-1}+0.30z^{-2}-0.50z^{-3} \big) & 1 \\
 & 1-0.30z^{-1}+0.30z^{-2}-0.50z^{-3} \\
\hline
 & 0.30z^{-1}-0.30z^{-2}+0.50z^{-3} \\
 & 0.30z^{-1}-0.09z^{-2}+0.09z^{-3}-0.15z^{-4} \\
\hline
 & -0.21z^{-2}+0.41z^{-3}+0.15z^{-4} \\
 & -0.21z^{-2}+0.063z^{-3}-0.063z^{-4} \\
\hline
 & 0.347z^{-3}+0.213z^{-4} \\
 & 0.347z^{-3}-0.1041z^{-4} \\
\hline
 & 0.3171z^{-4}
\end{array}
$$

Therefore

$$x(0)=1$$

$$x(T)=0.3$$

$$x(2T)=-0.21$$

$$x(3T)=0.347$$

$$x(4T)=0.3171$$

(b) Since

$$X(z)=\frac{1-0.72^{-1}}{\left(1+0.5z^{-1}\right)^2}=\frac{1-0.72^{-1}}{1+z^{-1}+0.25z^{-2}}$$

we write the long division as

$$
\begin{array}{r l}
 & 1-1.70z^{-1}+1.45z^{-2}-1.025z^{-3}+0.6625z^{-4} \\
\hline
1+z^{-1}+0.25z^{-2} \big) & 1-0.70z^{-1} \\
 & 1+z^{-1}+0.25z^{-2} \\
\hline
 & -1.70z^{-1}-0.25z^{-2} \\
 & -1.70z^{-1}-1.70z^{-2}-0.425z^{-3} \\
\hline
 & 1.45z^{-2}+0.425z^{-3} \\
 & 1.45z^{-2}+1.45z^{-3}+0.3625z^{-4} \\
\hline
 & -1.025z^{-3}-0.3625z^{-4} \\
 & -1.025z^{-3}-1.025z^{-4} \\
\hline
 & 0.6625z^{-4}
\end{array}
$$

Thus

$$
\begin{aligned}
x(0) &= 1 \\
x(T) &= -1.7 \\
x(2T) &= 1.45 \\
x(3T) &= -1.025 \\
x(4T) &= 0.6625
\end{aligned}
$$

(c) Since

$$X(z) = \frac{z^{-1}\left(1+0.2z^{-1}\right)^2}{\left(1-0.4z^{-2}\right)^2} = \frac{z^{-1}+0.4z^{-2}+0.04z^{-3}}{1-0.8z^{-2}+0.16z^{-4}}$$

The long division is (only enough to give the required values are shown)

$$
\begin{array}{r|l}
 & z^{-1}+0.40z^{-2}+0.84z^{-3}+0.32z^{-4} \\
\hline
1-0.80z^{-2}+0.16z^{-4} & z^{-1}+0.40z^{-2}+0.04z^{-3} \\
 & z^{-1}-0.80z^{-3} \\
\cline{2-2}
 & 0.40z^{-2}+0.84z^{-3} \\
 & 0.40z^{-2}-0.32z^{-4} \\
\cline{2-2}
 & 0.84z^{-3}+0.32z^{-4} \\
 & 0.84z^{-3} \\
\cline{2-2}
 & 0.32z^{-4}
\end{array}
$$

Thus

$$
\begin{aligned}
x(0) &= 0 \\
x(T) &= 1 \\
x(2T) &= 0.4 \\
x(3T) &= 0.84 \\
x(4T) &= 0.32
\end{aligned}
$$

Problem 8-30

Since

$$X(z)=\frac{\left(1-z^{-1}\right)^{4}}{1-0.3z^{-1}+0.3z^{-2}-0.5z^{-3}}$$

Or, equivalently

$$X(z)=\frac{1-4z^{-1}+6z^{-2}-4z^{-3}+z^{-4}}{1-0.3z^{-1}+0.3z^{-2}-0.5z^{-3}}$$

The long division is

$$\begin{array}{r|l}
 & 1-3.70z^{-1}+4.59z^{-2}-1.013z^{-3}-2.5309z^{-4} \\
\hline
1-0.30z^{-1}+0.30z^{-2}-0.50z^{-3} & 1-4z^{-1}+6z^{-2}-4z^{-3}+z^{-4} \\
 & 1-0.30z^{-1}+0.30z^{-2}-0.50z^{-3} \\
\cline{2-2}
 & -3.70z^{-1}+5.70z^{-2}-3.50z^{-3}+z^{-4} \\
 & -3.70z^{-1}+1.11z^{-2}-1.11z^{-3}+1.85z^{-4} \\
\cline{2-2}
 & 4.59z^{-2}-2.39z^{-3}-0.85z^{-4} \\
 & 4.59z^{-2}-1.377z^{-3}+1.377z^{-4} \\
\cline{2-2}
 & -1.013z^{-3}-2.227z^{-4} \\
 & -1.013z^{-3}+0.3039z^{-4} \\
\cline{2-2}
 & -2.5309z^{-4}
\end{array}$$

Thus

$$\begin{aligned}
x(0)&=1\\
x(T)&=-3.7\\
x(2T)&=4.57\\
x(3T)&=-1.013\\
x(4T)&=-2.5309
\end{aligned}$$

Problem 8-31

(a) This problem is solved in exactly the same manner as Example 8-8. Note that the numerator must be expressed as [1 0 0 0] so that the polynomials have the correct degree. This is necessary since MATLAB expresses polynomials in terms of positive powers.

```
» num = [1 0 0 0];
» den = [1 -0.3 0.3 -0.5];
» x = dimpulse(num,den,5);
» x

x =

    1.0000
    0.3000
   -0.2100
    0.3470
    0.3171
```

The column vector represents $x(0), x(T),\ldots,x(4T)$.

(b) This part of the problem is solved exactly as the previous part of the problem. Note the use of the convolution operation to square the denominator. Note also that the denominator is written out so that we can be confident of constructing the numerators and denominators with the correct orders.

```
» den = conv([1 0.5],[1 0.5]);
» den

den =

    1.0000    1.0000    0.2500

» num = [1 -0.7 0];
» x = dimpulse(num,den,5);
» x

x =

    1.0000
   -1.7000
    1.4500
   -1.0250
    0.6625
```

(c) In this part of the problem the convolution operation is used to form both the numerator and the denominator. Note the manner in which the numerator polynomial is formed.

```
» den = conv([1 0 -0.4],[1 0 -0.4]);
» den

den =

    1.0000         0   -0.8000         0    0.1600

» num1 = conv([1 0.2],[1 0.2]);
» num1

num1 =

    1.0000    0.4000    0.0400

» num = [0,num1,0];
» x = dimpulse(num,den,5);
» x

x =

         0
    1.0000
    0.4000
    0.8400
    0.3200
```

(d) Problem 8-30 is worked below. Note the use of the convolution operation to form the numerator. Also note that a trailing "0" must be added to the denominatior to give it the correct degree.

```
» a = [1 -1];
» b = conv(a,a);
» num = conv(b,b);
» num

num =

     1    -4     6    -4     1

» den = [1 -0.3 0.3 -0.5 0];
» x = dimpulse(num,den,5);
» x

x =

    1.0000
   -3.7000
    4.5900
   -1.0130
   -2.5309
```

Problem 8-32

(a) Since

$$X(z)=\frac{2}{\left(1-z^{-1}\right)\left(1+0.2z^{-1}\right)}=\frac{2z^2}{(z-1)(z+0.2)}$$

The partial fraction expansion is

$$\frac{X(z)}{z}=\frac{2z}{(z-1)(z+0.2)}=\frac{A}{z-1}+\frac{B}{z+0.2}$$

The value of A is

$$A=\frac{2}{1.2}=\frac{20}{12}=\frac{5}{3}$$

and the value of B is

$$B=\frac{2(-0.2)}{-1.2}=\frac{1}{3}$$

Thus

$$X(z)=\frac{5}{3}\frac{1}{1-z^{-1}}+\frac{1}{3}\frac{1}{1+0.2z^{-1}}$$

so that

$$x(nT)=\left[\frac{5}{3}+\frac{1}{3}(-0.2)^n\right]u(n)$$

(b) Since

$$X(z)=\frac{2}{\left(1-z^{-1}\right)\left(1+z^{-1}\right)}=\frac{2z^2}{(z-1)(z+1)}$$

The partial fraction expansion is

$$\frac{X(z)}{z}=\frac{2z}{(z-1)(z+1)}=\frac{A}{z-1}+\frac{B}{z+1}$$

where

$$A=1,\quad B=\frac{-2}{-2}=1$$

Thus

$$X(z) = \frac{1}{1 - z^{-1}} + \frac{1}{1 + z^{-1}}$$

and

$$x(nT) = \left[1 + (-1)^n\right]u(n)$$

(c) Since

$$X(z) = \frac{1 + 0.3z^{-1}}{(1 + 0.2z^{-1})(1 - 0.4z^{-1})} = \frac{z(z + 0.3)}{(z + 0.2)(z - 0.4)}$$

Thus

$$\frac{X(z)}{z} = \frac{z + 0.3}{(z + 0.2)(z - 0.4)} = \frac{A}{z + 0.2} + \frac{B}{z - 0.4}$$

where

$$A = \frac{0.1}{-0.6} = -\frac{1}{6}$$

and

$$B = \frac{0.7}{0.6} = \frac{7}{6}$$

Therefore $X(z)$ is given by

$$X(z) = -\frac{1}{6}\frac{1}{1 + 0.2z^{-1}} + \frac{7}{6}\frac{1}{1 - 0.4z^{-1}}$$

and $x(nT)$ is given by

$$x(nT) = \left[-\frac{1}{6}(-0.2)^n + \frac{7}{6}(0.4)^n\right]u(n)$$

(d) Since

$$X(z) = \frac{1 + 0.3z^{-1}}{(1 + 0.2z^{-1})(2 - 0.4z^{-1})} = \frac{z(0.5z + 0.15)}{(z + 0.2)(z - 0.2)}$$

we can write the partial fraction expansion as

$$\frac{X(z)}{z} = \frac{0.5z + 0.15}{(z+0.2)(z-0.2)} = \frac{A}{z+0.2} + \frac{B}{z-0.2}$$

The value of A is

$$A = \frac{0.5(-0.2)+0.15}{-0.4} = -\frac{0.05}{0.40} = -\frac{1}{8}$$

and the value of B is

$$B = \frac{0.5(0.2)+0.15}{0.4} = \frac{25}{40} = \frac{5}{8}$$

This gives

$$X(z) = -\frac{1}{8}\frac{1}{1+0.2z^{-1}} + \frac{5}{8}\frac{1}{1-0.2z^{-1}}$$

Thus $x(nT)$ is given by

$$x(nT) = \left[-\frac{1}{8}(-0.2)^n + \frac{5}{8}(0.2)^n\right]u(n)$$

Problem 8-33

(a) Since

$$X(z) = \frac{1}{(1-z^{-1})(1+0.5z^{-1})(1-0.2z^{-1})}$$

we can write

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)(z+0.5)(z-0.2)} = \frac{A}{z-1} + \frac{B}{z+0.5} + \frac{C}{z-0.2}$$

The value of A is

$$A = \frac{1}{1.5(0.8)} = \frac{2}{3}\frac{5}{4} = \frac{5}{6}$$

The value of B is

$$B=\frac{(-0.5)^2}{(-1.5)(-0.7)}=\frac{1}{3}\frac{5}{7}=\frac{5}{21}$$

The value of C is

$$C=\frac{(0.2)^2}{(-0.8)(0.7)}=-\frac{1}{4}\frac{2}{7}=-\frac{1}{14}$$

Thus

$$X(z)=\frac{5}{6}\frac{1}{1-z^{-1}}+\frac{5}{21}\frac{1}{1+0.5z^{-1}}-\frac{1}{14}\frac{1}{1-0.2z^{-1}}$$

so that

$$x(nT)=\left[\frac{5}{6}+\frac{5}{21}(-0.5)^n-\frac{1}{14}(0.2)^n\right]u(n)$$

(b) For

$$X(z)=\frac{1-0.5z^{-1}}{(1-z^{-1})(1+0.5z^{-1})(1-0.2z^{-1})}$$

the partial fraction expansion is

$$\frac{X(z)}{z}=\frac{z(z-0.5)}{(z-1)(z+0.5)(z-0.2)}=\frac{A}{z-1}+\frac{B}{z+0.5}+\frac{C}{z-0.2}$$

The value of A is

$$A=\frac{0.5}{(1.5)(0.8)}=\frac{1}{3}\frac{5}{4}=\frac{5}{12}$$

The value of B is

$$B=\frac{-0.5(-1)}{(-1.5)(-0.7)}=\frac{1}{3}\frac{10}{7}=\frac{10}{21}$$

The value of C is

$$C = \frac{0.2(-0.3)}{(-0.8)(0.7)} = \frac{1}{4}\frac{3}{7} = \frac{3}{28}$$

Thus

$$X(z) = \frac{5}{12}\frac{1}{1-z^{-1}} + \frac{10}{21}\frac{1}{1+0.5z^{-1}} + \frac{3}{28}\frac{1}{1-0.2z^{-1}}$$

which yields

$$x(nT) = \left[\frac{5}{12} + \frac{10}{21}(-0.5)^n + \frac{3}{28}(0.2)^n\right]u(n)$$

Problem 8-34

The MATLAB solutions to both parts (a) and (b) are illustrated below.

```
syms n z
xza = z^3/((z-1)*(z+0.5)*(z-0.2));                  % Part (a)
xzb = ((z^2)*(z-0.5))/((z-1)*(z+0.5)*(z-0.2));      % Part (b)
xna = iztrans(xza,z,n);                             % Solution to part (a)
xnb = iztrans(xzb,z,n);                             % Solution to part (b)
xna                                                 % Show part (a) solution

xna =

5/6+5/21*(-1/2)^n-1/14*(1/5)^n

xnb                                                 % Show part (b) solution

xnb =

5/12+10/21*(-1/2)^n+3/28*(1/5)^n
```

Problem 8-35

(a) Since

$$X(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})^2} = \frac{z^3}{(z-1)(z-0.5)^2}$$

the partial fraction expansion can be written

$$\frac{X(z)}{z}=\frac{z^2}{(z-1)(z-0.5)^2}=\frac{A}{z-1}+\frac{B}{z-0.5}+\frac{C}{(z-0.5)^2}$$

The value of A is

$$A=\frac{1}{(0.5)^2}=4$$

and the value of C is

$$C=\frac{(0.5)^2}{-0.5}=-0.5$$

The value of B is given by

$$\begin{aligned}B&=\frac{d}{dz}\left\{\frac{z}{z-1}\right\}\Bigg|_{z=0.5}\\&=\frac{(z-1)-z}{(z-1)^2}\Bigg|_{z=0.5}\\&=\frac{-1}{(-0.5)^2}=-4\end{aligned}$$

Thus

$$X(z)=\frac{1}{1-z^{-1}}-\frac{4}{1-0.5z^{-1}}-\frac{0.5z^{-1}}{(1-0.5z^{-1})^2}$$

which gives

$$x(nT)=\left[1-4(0.5)^n-n(0.5)^n\right]u(n)$$

(b) $X(z)$ can be written

$$X(z)=\frac{1}{(1-0.81z^{-1})^2}=\frac{z^2}{(z-0.81)^2}$$

The partial fraction expansion is

$$\frac{X(z)}{z}=\frac{z}{(z-0.81)^2}=\frac{A}{z-0.81}+\frac{B}{(z-0.81)^2}$$

The value of B is

$$B = 0.81$$

and A is

$$A = \frac{d}{dz}(z) = 1$$

Thus

$$X(z) = \frac{0.81}{\left(1-0.81z^{-1}\right)^2} + \frac{1}{\left(1-0.81z^{-1}\right)}$$

which gives, for $n \geq 0$,

$$x(nT) = (0.81)^n n + (0.81)^n$$

Problem 8-36

Since $X(z)$ can be written

$$X(z) = \frac{1}{1+0.81z^{-2}} = \frac{z^2}{z^2+0.81}$$

Thus

$$\frac{X(z)}{z} = \frac{z}{(z+j0.9)(z-j0.9)} = \frac{A}{z+j0.9} + \frac{B}{z-j0.9}$$

The value of A is given by

$$A = \frac{-j0.9}{2(-j0.9)} = \frac{1}{2}$$

and B is given by

$$B = \frac{j0.9}{2(j0.9)} = \frac{1}{2}$$

Thus

$$X(z) = \frac{1}{2}\frac{1}{1+j0.9z^{-1}} + \frac{1}{2}\frac{1}{1-j0.9z^{-1}}$$

which yields

$$x(nT) = \frac{1}{2}e^{-j0.9n} + \frac{1}{2}e^{j0.9n} = \sin(0.9n)$$

Problem 8-37

The MATLAB solution is illustrated below.

```
syms n z                              % Declare symbolic
xz35a = z^2/((z-1)*(z-0.5)^2);        % X(z) for P-35(a)
xn35a = iztrans(xz35a,z,n);           % Compute x(n) for P-35(a)
xn35a                                 % Display result for P-35(a)

xn35a =

4-4*(1/2)^n-2*(1/2)^n*n


xz35b = z^2/((z-0.81)^2);             % X(z) for P-36(a)
xn35b = iztrans(xz35b,z,n);           % Compute x(n) for P-36(b)
xn35b                                 % Display result for P-35(b)

xn35b =

(81/100)^n+(81/100)^n*n


xz36 = z^2/(z^2+0.81);                % Given X(z) for P8-36
xn36 = iztrans(xz36,z,n);             % Compute x(n) for P8-36
xn36                                  % Display result for P8-36

xn36 =

1/2*(9/10)^n*((-1)^n)^(1/2)+1/2*(-9/10)^n*((-1)^n)^(1/2)
```

The preceding result is written in a rather strange manner but it is correct and in agreement with the previously derived result.

Problem 8-38

The inverse z-transform of

$$H(z) = \frac{z^2 + z}{(z+0.5)(z-0.5)}$$

is determined by first writing

$$\frac{H(z)}{z} = \frac{z+1}{(z+0.5)(z-0.5)} = \frac{A}{z+\frac{1}{2}} + \frac{B}{z-\frac{1}{4}}$$

The residues are

$$A = \frac{-\frac{1}{2}+1}{-\frac{1}{2}-\frac{1}{4}} = \frac{\frac{1}{2}}{-\frac{5}{4}} = -\frac{2}{3}$$

and

$$B = \frac{\frac{1}{4}+1}{\frac{1}{4}-+\frac{1}{2}} = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3}$$

Thus

$$h(nT) = \left[\frac{5}{3}\left(\frac{1}{4}\right)^n - \frac{2}{3}\left(-\frac{1}{2}\right)^n\right] u(n)$$

Problem 8-39

Since

$$X(z)=\frac{5}{\left(1-z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}=\frac{5z^{3}}{(z-1)\left(z-\frac{1}{8}\right)\left(z+\frac{1}{4}\right)}$$

The partial fraction expansion is

$$\frac{X(z)}{z}=\frac{5z^{2}}{(z-1)\left(z-\frac{1}{8}\right)\left(z+\frac{1}{4}\right)}=\frac{A}{z-1}+\frac{B}{z-\frac{1}{8}}+\frac{C}{z+\frac{1}{4}}$$

The value of A is

$$A=\frac{5}{\left(\frac{7}{8}\right)\left(\frac{5}{4}\right)}=\frac{32}{7}$$

The value of B is

$$B=\frac{5\left(\frac{1}{8}\right)^{2}}{\left(-\frac{7}{8}\right)\left(\frac{3}{8}\right)}=-\frac{5}{21}$$

The value of C is

$$C=\frac{5\left(-\frac{1}{4}\right)^{2}}{\left(-\frac{5}{4}\right)\left(-\frac{3}{8}\right)}=\frac{2}{3}$$

This gives

$$X(nT)=\left[\frac{32}{7}-\frac{5}{21}\left(\frac{1}{8}\right)^{n}+\frac{2}{3}\left(-\frac{1}{4}\right)^{n}\right]u(n)$$

In order to perform the necessary long division we write $X(z)$ as

$$X(z) = \frac{5}{1 - \frac{7}{8}z^{-1} - \frac{5}{32}z^{-2} + \frac{1}{32}z^{-3}}$$

Thus

$$\begin{array}{r|l}
 & 5 + \frac{35}{8}z^{-1} + \frac{290}{64}z^{-2} \\
\hline
1 - \frac{7}{8}z^{-1} - \frac{5}{32}z^{-2} + \frac{1}{32}z^{-3} & 5 \\
 & 5 - \frac{35}{8}z^{-1} - \frac{25}{32}z^{-2} + \frac{5}{32}z^{-3} \\
\hline
 & \frac{35}{8}z^{-1} + \frac{25}{32}z^{-2} - \frac{5}{32}z^{-3} \\
 & \frac{35}{8}z^{-1} - \frac{245}{64}z^{-2} - \frac{175}{256}z^{-3} \\
\hline
 & \frac{290}{64}z^{-2}
\end{array}$$

Thus

$$x(0) = 5, \quad x(T) = \frac{35}{8}, \quad x(2T) = \frac{290}{64}$$

Evaluating $x(nT)$ at $n = 0$, 1, 2 shows that we have agreement. The final value, $x(\infty)$, is

$$x(\infty) = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

Thus

$$x(\infty) = \frac{5}{\left(1 - \frac{1}{8}\right)\left(1 + \frac{1}{4}\right)} = \frac{32}{7}$$

which is in agreement with the general expression for $x(nT)$.

Problem 8-40

For

$$H(z) = \frac{2}{\left(1-z^{-1}\right)^2\left(1-\frac{1}{2}z^{-1}\right)}$$

we expand

$$\frac{H(z)}{z} = \frac{2z^2}{(z-1)^2\left(z-\frac{1}{2}\right)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-\frac{1}{2}}$$

The values of A and C are

$$A = \frac{2}{\frac{1}{2}} = 4, \qquad C = \frac{2\left(\frac{1}{4}\right)}{\frac{1}{4}} = 2$$

The value of B is

$$B = \frac{d}{dz}\left[\frac{2z^2}{z-\frac{1}{2}}\right]_{z=1}$$

$$= \left.\frac{4z\left(z-\frac{1}{2}\right)-2z^2}{\left(z-\frac{1}{2}\right)^2}\right|_{z=1} = 0$$

Thus

$$H(z) = \frac{4z^{-1}}{\left(1-z^{-1}\right)^2} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

From which

$$h(nT) = \left[4n + 2\left(\frac{1}{2}\right)^n\right] u(n)$$

This gives

$$h(0) = 2$$
$$h(T) = 4 + 1 = 5$$
$$h(2T) = 8 + \frac{1}{2} = \frac{17}{2}$$
$$h(\infty) = \infty$$

The value of $h(\infty)$ is checked using the final-value theorem

$$\lim_{z \to 1} \left(1 - z^{-1}\right) H(z) = \lim_{z \to 1} \frac{2}{\left(1 - z^{-1}\right)\left(1 - \frac{1}{2} z^{-1}\right)} = \infty$$

The values of $h(0)$, $h(T)$ and $h(2T)$ are checked using long division. First we write

$$H(z) = \frac{2}{1 - \frac{5}{2} z^{-1} + 2z^{-2} - \frac{1}{2} z^{-3}}$$

In performing the long division we only need retain the terms through z^{-2}.

$$\begin{array}{r|l} & 2 + 5z^{-1} + \frac{17}{2} z^{-2} \\ \hline 1 - \frac{5}{2} z^{-1} + 2z^{-2} - \frac{1}{2} z^{-3} & 2 \\ & \underline{2 - 5z^{-1} + 4z^{-2}} \\ & 5z^{-1} - 4z^{-2} \\ & \underline{5z^{-1} - \frac{25}{2} z^{-2}} \\ & \frac{17}{2} z^{-2} \end{array}$$

Thus

$$h(0) = 2$$
$$h(T) = 5$$
$$h(2T) = \frac{17}{2}$$

These values agree with those obtained using the general expression for $h(nT)$.

Problem 8-41

The inverse z-transform of

$$H(z)=\frac{1}{\left(1-z^{-1}\right)\left(1+0.8z^{-1}\right)\left(1-0.5z^{-1}\right)}$$

is found by first writing

$$\frac{H(z)}{z}=\frac{z^2}{(z-1)\left(z+\frac{4}{5}\right)\left(z-\frac{1}{2}\right)}=\frac{A}{z-1}+\frac{B}{z+\frac{4}{5}}+\frac{C}{z-\frac{1}{2}}$$

The residues are

$$A=\frac{1}{\frac{9}{5}\left(\frac{1}{2}\right)}=\frac{10}{9},\qquad B=\frac{\frac{16}{25}}{-\frac{9}{5}\left(-\frac{13}{10}\right)}=\frac{32}{117},\qquad C=\frac{\frac{1}{4}}{-\frac{1}{2}\left(\frac{13}{10}\right)}=-\frac{5}{13}$$

Thus

$$H(z)=\frac{10}{9}\,\frac{1}{1-z^{-1}}+\frac{32}{117}\,\frac{1}{1+\frac{4}{5}z^{-1}}-\frac{5}{13}\,\frac{1}{1-\frac{1}{2}z^{-1}}$$

This gives

$$h(nT)=\left[\frac{10}{9}-\frac{32}{117}\left(-\frac{4}{5}\right)^n-\frac{5}{13}\left(\frac{1}{2}\right)^n\right]u(n)$$

Problem 8-42

$X(z)$ can be written in the form

$$\frac{X(z)}{z}=\frac{1}{(z-1)(z+1)(z-0.5)(z-0.2)}=\frac{A}{z-1}+\frac{B}{z+1}+\frac{C}{z-0.5}+\frac{D}{z-0.2}$$

The value of A is given by

$$A=\frac{1}{(2)(0.5)(0.8)}=\frac{5}{4}$$

The value of B is

$$B=\frac{1}{(-1.5)(-2)(-1.2)}=-\frac{1}{3.6}=-\frac{5}{18}$$

The value of C is

$$C=\frac{1}{(-0.5)(1.5)(0.3)}=-\frac{40}{9}$$

Finally, the value of D is

$$D=\frac{1}{(-0.8)(1.2)(-0.3)}=\frac{125}{36}$$

Thus

$$X(z)=\frac{5}{4}\frac{1}{1-z^{-1}}-\frac{5}{18}\frac{1}{1+z^{-1}}-\frac{40}{9}\frac{1}{1-0.5z^{-1}}+\frac{125}{36}\frac{1}{1-0.2z^{-1}}$$

This gives

$$x(nT)=\left[\frac{5}{4}-\frac{5}{18}(-1)^n-\frac{40}{9}(0.5)^n+\frac{125}{36}(0.2)^n\right]u(n)$$

Problem 8-43

$X(z)$ can be written

$$X(z) = \frac{z^{-1}}{(1-z^{-1})(1+0.2z^{-1})^2} = \frac{z^2}{(z-1)(z+0.2)^2}$$

The partial fraction expansion is written

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z+0.2)^2} = \frac{A}{z-1} + \frac{B}{z+0.2} + \frac{C}{(z+0.2)^2}$$

The value of A is

$$A = \frac{1}{(1.2)^2} = \frac{25}{36}$$

and the value of C is

$$C = \frac{-0.2}{-1.2} = \frac{1}{6}$$

The value of B is

$$B = \frac{d}{dz}\left(\frac{z}{z-1}\right)\Bigg|_{z=-0.2} = \frac{-1}{(1.2)^2} = -\frac{25}{36}$$

Thus

$$X(z) = \frac{25}{36}\left[\frac{1}{1-z^{-1}} - \frac{1}{1+0.2z^{-1}}\right] + \frac{1}{6(0.2)}\frac{0.2z^{-1}}{(1+0.2z^{-1})^2}$$

which gives

$$x(nT) = \frac{25}{36}\left[1-(-0.2)^n\right]u(n) + \frac{5}{6}(0.2)^n u(n)$$

Problem 8-44

Since

$$X(z) = \frac{1}{(1-z^{-1})(1+0.2z^{-1})} = \frac{z^2}{(z-1)(z+0.2)}$$

The partial fraction expansion is

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z+0.2)} = \frac{A}{z-1} + \frac{B}{z+0.2}$$

The value of A is

$$A = \frac{1}{1.2} = \frac{5}{6}$$

and the value of B is

$$B = \frac{-0.2}{-1.2} = \frac{1}{6}$$

Thus

$$X(z) = \frac{5}{6}\frac{1}{1-z^{-1}} + \frac{1}{6}\frac{1}{1+0.2z^{-1}}$$

This gives

$$x(nT) = \left[\frac{5}{6} + \frac{1}{6}(-0.2)^n\right]u(n)$$

Problem 8-45

(a) With

$$y(nT) - y(nT-T) = x(nT)$$

the pulse transfer function is

$$H(z) = \frac{1}{1-z^{-1}}$$

and the unit pulse response is

$$h(nT) = u(n)$$

(b) With

$$y(nT) - 2y(nT-T) + y(nT-2T) = x(nT) + 3x(nT-3T)$$

the pulse transfer function is

$$H(z) = \frac{1+3z^{-3}}{1-2z^{-1}+z^{-2}} = \frac{z^3+3}{z(z-1)^2}$$

The partial-fraction expansion has the form

$$\frac{H(z)}{z} = \frac{z^3+3}{z^2(z-1)^2} = \frac{A}{z^2}+\frac{B}{z}+\frac{C}{(z-1)^2}+\frac{D}{z-1}$$

The values of A and C are

$$A = 3, \qquad C = \frac{1+3}{1} = 4$$

The value of B is

$$B = \frac{d}{dz}\left[\frac{z^3+3}{(z-1)^2}\right]_{z=0} = \frac{-3(2)(-1)}{1} = 6$$

and the value of D is

$$D = \frac{d}{dz}\left[\frac{z^3+3}{z^2}\right]_{z=1} = \frac{1(3)-(4)(2)}{1} = -5$$

Thus $H(z)$ can be written

$$H(z) = 3z^{-1}+6+4\frac{z^{-1}}{\left(1-z^{-1}\right)^2}-5\frac{1}{1-z^{-1}}$$

Taking the inverse z-transform gives

$$h(nT) = 3\delta(n-1)+6\delta(n)+(4n-5)\ u(n)$$

(c) The difference equation

$$y(nT)-y(nT-T)+0.16y(nT-2T) = x(nT)$$

yields the transfer function

$$H(z) = \frac{1}{1-z^{-1}+0.16z^{-2}} = \frac{z^2}{z^2-z+\frac{4}{25}}$$

from which

$$\frac{H(z)}{z} = \frac{z}{\left(z-\frac{1}{5}\right)\left(z-\frac{4}{5}\right)} = \frac{A}{z-\frac{1}{5}}+\frac{B}{z-\frac{4}{5}}$$

The residues are

$$A = \frac{\frac{1}{5}}{-\frac{3}{5}} == \frac{1}{3}, \qquad\qquad B = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{5}$$

This yields

$$H(z) = -\frac{1}{3}\ \frac{1}{1-\frac{1}{5}z^{-1}} + \frac{4}{5}\ \frac{1}{1-\frac{4}{5}z^{-1}}$$

Taking the z-transform gives

$$h(nT) = \left[\frac{4}{5}\left(\frac{4}{5}\right)^n - \frac{1}{3}\left(\frac{1}{5}\right)^n\right] u(n)$$

(d) The difference equation

$$y(nT) + 0.6y(nT-T) - 0.16y(nT-2T) = x(nT) + 4x(nT-T)$$

gives the pulse transfer function

$$H(z) = \frac{1+4z^{-1}}{1+0.6z^{-1}-0.16z^{-2}} = \frac{z(z+4)}{\left(z-\frac{1}{5}\right)\left(z+\frac{4}{5}\right)}$$

To find $h(nT)$ we expand

$$\frac{H(z)}{z} = \frac{z+4}{\left(z-\frac{1}{5}\right)\left(z+\frac{4}{5}\right)} = \frac{A}{z-\frac{1}{3}} + \frac{B}{z+\frac{4}{5}}$$

The residues are

$$A = \frac{4+\frac{1}{5}}{1} = \frac{21}{5}, \qquad\qquad B = \frac{4-\frac{4}{5}}{-1} = -\frac{16}{5}$$

Thus

$$H(z) = \frac{21}{5}\ \frac{1}{1-\frac{1}{5}z^{-1}} - \frac{16}{5}\ \frac{1}{1+\frac{4}{5}z^{-1}}$$

from which

$$h(nT) = \left[\frac{21}{5}\left(\frac{1}{5}\right)^n - \frac{16}{5}\left(-\frac{4}{5}\right)^n\right] u(n)$$

(e) The difference equation

$$y(nT) - 0.707y(nT - T) + 0.25y(nT - 2T) = x(nT) + x(nT - 2T)$$

gives the pulse transfer function

$$H(z) = \frac{1 + z^{-2}}{1 - 0.707z^{-1} + 0.25z^{-2}} = \frac{z^2 + 1}{z^2 - 0.707z + 0.25}$$

We therefore expand

$$\frac{H(z)}{z} = \frac{z^2 + 1}{z\left(z^2 - 0.707z + 0.25\right)} = \frac{A}{z} + \frac{Bz + C}{z^2 - 0.707z + 0.25}$$

By inpsection $A = 4$. The values of B and C are found by solving

$$z^2 + 1 = 4\left(z^2 - 0.707z + 0.25\right) + z(Bz + C)$$

Equating coefficients of like powers of z gives

$$1 = 4 + B, \qquad B = -3$$
$$0 = -4(0.707) + C, \qquad C = 2.828$$

Thus

$$H(z) = 4 + \frac{-3 + 2.828z^{-1}}{1 - 0.707z^{-1} + 0.25z^{-2}}$$

Consulting Table 8-1 we see that we want

$$e^{-aT} = \frac{1}{2}$$
$$e^{-2aT} = \frac{1}{4}$$
$$2e^{-aT}\cos(bT) = 0.707, \qquad bT = \frac{\pi}{4}$$
$$\sin(bT) = 0.707$$

To put $H(z)$ in the form required for taking the inverse z-transform we first write

$$H(z) = 4 - 3\frac{1 - \frac{4}{3}(0.707)z^{-1}}{1 - 0.707z^{-1} + 0.25z^{-2}}$$

which can be written

$$H(z) = 4 - 3\frac{1 - \frac{1}{2}(0.707)z^{-1} - \frac{5}{6}(0.707)z^{-1}}{1 - 0.707z^{-1} + 0.25z^{-2}}$$

or

$$H(z) = 4 - 3\frac{1 - \frac{1}{2}(0.707)z^{-1}}{1 - 0.707z^{-1} + 0.25z^{-2}} + 5\frac{\frac{1}{2}(0.707)z^{-1}}{1 - .0707z^{-1} + 0.25z^{-2}}$$

Taking the inverse z-transform yields

$$h(nT) = 4\delta(n) - 3\left(\frac{1}{2}\right)^n \cos\left(\frac{n\pi}{4}\right) u(n) + 5\left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)$$

Problem 8-46

For the system to be stable, the poles must lie within the unit circle (the system is assumed causal).

(a) For

$$y(nT) - Ky(nT - T) + K^2 y(nT - 2T) = x(nT)$$

the pulse transfer function is

$$H(z) = \frac{z^2}{z^2 - Kz + K^2}$$

The pole locations are

$$z_i = \frac{1}{2}\left[K \pm \sqrt{K^2 - 4K^2}\ \right]$$

$$= \frac{K}{2}\left[1 \pm j\sqrt{3}\ \right], \qquad i = 1,2$$

Thus

$$|z_i| = \frac{|K|}{2}\left[\left(1 - j\sqrt{3}\right)\left(1 + j\sqrt{3}\right)\right]^{1/2} = |K|$$

For the system to be stable

$$|K| < 1$$

or

$$-1 < K < 1$$

(b) For

$$y(nT) - 2Ky(nT - T) + K^2 y(nT - 2T) = x(nT)$$

the pulse transfer function is

$$H(z) = \frac{z^2}{z^2 - 2Kz + K^2} = \frac{z^2}{(z-K)^2}$$

which yields a second-order pole at $z = K$. Thus $|K| < 1$ or

$$-1 < K < 1$$

(c) For

$$y(nT) - K^2 y(nT - 2T) = x(nT)$$

the pulse transfer function can be written

$$H(z) = \frac{z^2}{z^2 - K^2} = \frac{z^2}{(z-K)(z+K)}$$

We have poles at $z = \pm K$. Thus, once again, stability requires that $|K| < 1$ or

$$-1 < K < 1$$

Problem 8-47

(a) The transformation is defined by

$$\mathcal{H}[x(nT)] = a x(nT) + b$$

Therefore

$$\mathcal{H}[x_1(nT)] = a x_1(nT) + b$$

$$\mathcal{H}[x_2(nT)] = a x_2(nT) + b$$

so that

$$\mathcal{H}[x_1(nT)] + \mathcal{H}[x_2(nT)] = a[x_1(nT) + x_2(nT)] + 2b$$

By definition of the transformation

$$\mathcal{H}[x_1(nT) + x_2(nT)] = a[x_1(nT) + x_2(nT)] + b$$

Since

$$\mathcal{H}[x_1(nT) + x_2(nT)] \neq \mathcal{H}\ [x_1(nT)] + \mathcal{H}[+x_2(nT)]$$

(unless $b = 0$) the system is nonlinear. It is clear from observation of $\mathcal{H}[x(nT)]$ that if

$$x_1(nT) = x_2(nT)\ , \qquad n \leq n_o$$

then

$$\mathcal{H}[x_1(nT)] = \mathcal{H}[x_2(nT)], \qquad n \le n_o$$

Thus the system is causal (note that this system is instantaneous). It also follows that the system is shift-invariant since

$$y(nT) = a\,x(nT) + b$$

$$y(nT - n_oT) = a\,x(nT - n_oT) + b$$

(b) The transformation is defined by

$$\mathcal{H}[x(nT)] = x(nT - n_oT)$$

Let

$$\mathcal{H}[x_1(nT)] = x_1(nT - n_oT)$$

$$\mathcal{H}[x_2(nT)] = x_2(nT - n_oT)$$

Since

$$\begin{aligned}\mathcal{H}[\alpha_1 x_2(nT) + \alpha_2 x_2(nT)] &= \alpha_1 x_1[nT - n_oT] + \alpha_2 x_2[nT - n_oT] \\ &= \alpha_1 \mathcal{H}[x_1(nT)] + \alpha_2 \mathcal{H}[x_2(nT)]\end{aligned}$$

it follows that the system is linear.

Observation of $\mathcal{H}[x(nT)]$ shows that if $n_o < 0$, the output depends on future values of the input. Thus, the sytem is non-causal if $n_o < 0$.

Noting that

$$\mathcal{H}[x(nT - n_kT)] = x(nT - n_kT - n_oT)$$

shows that the system is shift invariant.

(c) The transformation is defined by

$$\mathcal{H}[x(nT)] = x(nT + T) + x(nT - T)$$

Let

$$\mathcal{H}[x_1(nT)] = x_1(nT + T) + x_1(nT - T)$$

$$\mathcal{H}[x_2(nT)] = x_2(nT + T) + x_2(nT - T)$$

It follows that

$$\begin{aligned}\mathcal{H}[\alpha_1 x_1(nT) + \alpha_2 x_2(nT)] &= \alpha_1 x_1(nT + T) + \alpha_1 x_1(nT - T) + \alpha_2 x_2(nT + T) + \alpha_2 x_2(nT - T) \\ &= \alpha_1[x_1(nT + T) + x_1(nT - T)] + \alpha_2[x_2(nT + T) + x_2(nT - T)] \\ &= \alpha_1 \mathcal{H}[x_1(nT)] + \alpha_2 \mathcal{H}[x_2(nT)]\end{aligned}$$

and the system is therefore linear.

The system is not causal because of the presence of the $x(nT + T)$ term which makes $\mathcal{H}[x(nT)]$ dependent upon future values of the input.

Noting that

$$\mathcal{H}\left[x(nT - n_k T)\right] = x(nT - n_k T + T) + x(nT - n_k T - T)$$

shows that the system is shift invariant.

Problem 8-48

Let

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + x(3T)z^{-3} + \cdots$$

$$H(z) = h(0) + h(T)z^{-1} + h(2T)z^{-2} + h(3T)z^{-3} + \cdots$$

and

$$Y(z) = y(0) + y(T)z^{-1} + y(2T)z^{-2} + y(3T)z^{-3} + \cdots$$

Forming the product $H(z)X(z)$ yields

$$\begin{aligned} H(z)X(z) = &\left[h(0) + h(T)z^{-1} + h(2T)z^{-2} + h(3T)z^{-3} + \cdots\right] \\ &\cdot\left[x(0) + x(T)z^{-1} + x(2T)z^{-2} + x(3T)z^{-3} + \cdots\right] \end{aligned}$$

which is

$$\begin{aligned} H(z)X(z) = h(0)x(0) &+ \left[h(0)x(T) + h(T)x(0)\right]z^{-1} \\ &+ \left[h(0)x(2T) + h(T)x(T) + h(2T)x(0)\right]z^{-2} \\ &+ \left[h(0)x(3T) + h(T)x(2T) + h(2T)x(T) + h(3T)x(0)\right]z^{-3} \\ &+ \cdots \end{aligned}$$

Equating $H(z)X(z)$ to $Y(z)$ and equating like powers of z shows that

$$\begin{aligned} y(0) &= h(0)x(0) \\ y(T) &= h(0)x(T) + h(T)x(T) \\ y(2T) &= h(0)x(2T) + h(T)x(T) + h(2T)x(0) \\ y(3T) &= h(0)x(3T) + h(T)x(2T) + h(2T)x(T) + h(3T)x(0) \end{aligned}$$

In general

$$y(nT) = \sum_{k=0}^{n} h(kT)x(nT - kT)$$

Problem 8-49

The MATLAB command **conv** is easily applied to this problem. For the sample values given the result is

```
h =

     0     0     0     0     2     2     2     2     2     0     0

x =

     0     0     2     2     2     2     0     0     0     0     0

y =

  Columns 1 through 12

     0     0     0     0     0     0     4     8    12    16    16    12

  Columns 13 through 21

     8     4     0     0     0     0     0     0     0
```

See Problem 8-51 for the figure.

Problem 8-50

As with the previous problem, the MATLAB command **conv** is easily applied to this problem. For the sample values given the result is

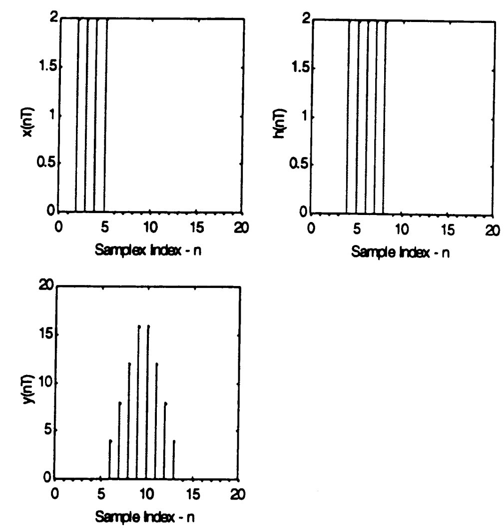
```
h =

     0     1     2     2     1     0    -1    -1    -1     0     0

x =

     1     1     2     2     1     0    -1    -1     0     0     0

y =

  Columns 1 through 12

     0     1     3     6     9    10     7     1    -6    -9    -8    -4

  Columns 13 through 21

     0     2     2     1     0     0     0     0     0
```

Once again, see Problem 8-51 for the figure.

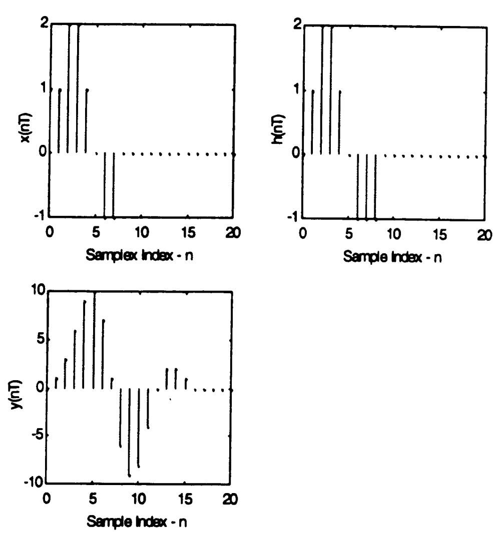
Problem 8-51

This problem is solved using MATLAB by simply modifying the computer code used in Example 8-12 to specify the correct sample values for h(nT) and x(nT). We should also set nz equal to 0 since the vector lengths for h(nT) and x(nT) are equal.

The result for the sample sequences given in Problem 8-49 is



and the result for the sample sequences given in Problem 8-50 is



Problem 8-52

As in Problems 8-49 and 8-50 we can use MATLAB. The result is

```
h =

     2     2     1     1     0     0     1     2     2     1     1

x =

    -1     0     1     2     2     2     1     0    -1     0     0

y =

  Columns 1 through 12

    -2    -2     1     5     9    11     9     4     0     2     6    10

  Columns 13 through 21

    12    10     5     1    -1    -1    -1     0     0
```

Problem 8-53

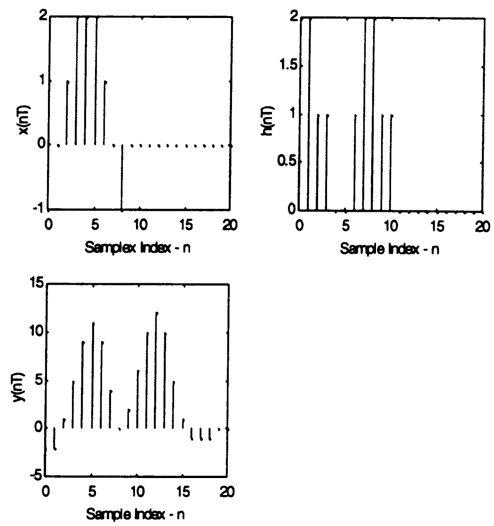
Using MATLAB the result is

```
h =

     0     1     2     1     0     0     0     0     0     0     0


x =

     0     0     2     0     0     0     0     0     0     1     0

» y

y =

  Columns 1 through 12

     0     0     0     2     4     2     0     0     0     0     1     2

  Columns 13 through 21

     1     0     0     0     0     0     0     0     0
```

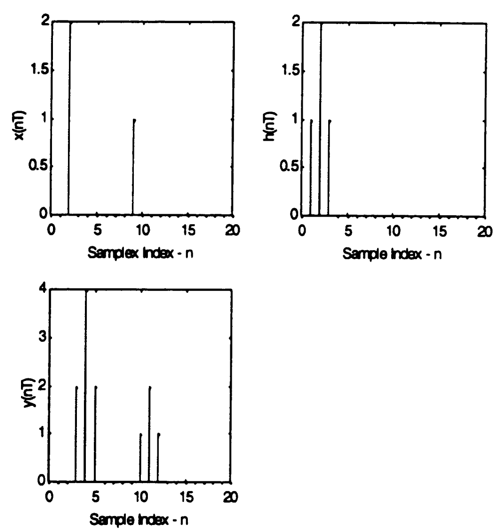
Problem 8-54

As in Problem 8-51, MATLAB can be used to solve the problem and plot the results by entering the proper sample values into the script file for Example 8-12.

The result for Problem 8-52 is



and the result for Problem 8-53 is



Note that since x(nT) consists of two impulses, y(nT) consists of two impulse responses. The responses are shifted to correspond to the points at which the impulses occur and are multiplied by the impulse weights (2 and 1).

Problem 8-55

With

$$x(nT) = \left(\frac{1}{4}\right)^n u(n), \qquad h(nT) = \left(\frac{1}{3}\right)^n u(n)$$

the output becomes

$$y(nT) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k u(k)\left(\frac{1}{3}\right)^{n-k} u(n-k)$$

which is, for $n \geq 0$,

$$y(nT) = \left(\frac{1}{3}\right)^n \sum_{k=0}^{n}\left(\frac{3}{4}\right)^k$$

Performing the summation with the aid of (8-140) yields

$$y(nT) = \left(\frac{1}{3}\right)^n \frac{1-\left(\frac{3}{4}\right)^{n+1}}{1-\frac{3}{4}}, \qquad n \geq 0$$

Since $y(nT) = 0$ for $n < 0$ we can write

$$y(nT) = 4\left[\left(\frac{1}{3}\right)^n - \frac{3}{4}\left(\frac{1}{4}\right)^n\right] u[n]$$

This result is easily verified using z-transform techniques.

Problem 8-56

With the input and unit pulse response defined by

$$x(nT) = \left(\frac{1}{4}\right)^{n-3} u(n-3), \qquad h(nT) = \left(\frac{1}{2}\right)^{n-5} u(n-5)$$

the output becomes

$$y(nT) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^{k-3} u(k-3)\left(\frac{1}{2}\right)^{n-k-5} u(n-k-5)$$

We see that $y(nT) = 0$ unless $n \geq 3 + 5 = 8$. For $n \geq 8$ the sum becomes

$$y(nT) = \sum_{k=3}^{n-5}\left(\frac{1}{4}\right)^{k-3} u(k-3)\left(\frac{1}{2}\right)^{n-k-5}$$

This yields

$$y(nT) = \left(\frac{1}{2}\right)^n (4)^3 (2)^5 \sum_{k=3}^{n-5} \left(\frac{1}{2}\right)^k$$

with the change of index $m = k - 3$ the preceding expression can be written

$$y(nT) = \left(\frac{1}{2}\right)^n (2)^{11} \sum_{m=0}^{n-8} \left(\frac{1}{2}\right)^{m+3}$$

or

$$y(nT) = \left(\frac{1}{2}\right)^n (2)^8 \frac{1-\left(\frac{1}{2}\right)^{n-7}}{1-\frac{1}{2}}$$

which is

$$y(nT) = \left(\frac{1}{2}\right)^n (2)^9 \left[1-\left(\frac{1}{2}\right)^{n-7}\right]$$

Thus, the complete expression for $y(nT)$ is

$$y(nT) = \left(\frac{1}{2}\right)^n (2)^9 \left[1-\left(\frac{1}{2}\right)^{n-7}\right] u(n-8)$$

Problem 8-57

The input and unit pulse response are defined by

$$x(nT) = \left(\frac{1}{4}\right)^n u(n-3), \qquad h(nT) = \left(\frac{1}{2}\right)^n u(n-5)$$

The output can be written

$$y(nT) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k \left(\frac{1}{2}\right)^{n-k} u(k-3)\ u(n-k-5)$$

The product $u(k - 3)\ u(n-k-5) = 0$ unless $3 \le k \le n-5$ which shows that $n-5 \ge 3$ or $n \ge 8$. Thus $y(nT) = 0$ for $n < 8$. For $n > 8$

$$y(nT) = \sum_{k=3}^{n-5} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^k$$

With the change of index $m = k - 3$ we have

$$y(nT) = \left(\frac{1}{2}\right)^n \left(\frac{1}{8}\right) \sum_{m=0}^{n-8} \left(\frac{1}{2}\right)^m$$

or

$$y(nT) = \frac{1}{8}\left(\frac{1}{2}\right)^n \frac{1-\left(\frac{1}{2}\right)^{n-7}}{1-\frac{1}{2}}$$

which is

$$y(nT) = \frac{1}{4}\left(\frac{1}{2}\right)^n\left(1-\left(\frac{1}{2}\right)^{n-7}\right)$$

Thus

$$y(nT) = \begin{cases} 0, & n < 8 \\ \frac{1}{4}\left(\frac{1}{2}\right)^n\left(1-\left(\frac{1}{2}\right)^{n-7}\right), & n \geq 8 \end{cases}$$

<u>Problem 8-58</u>

Let

$$h(nT) = h_1(nT) - h_2(nT)$$

where

$$h_1(nT) = \left(\frac{1}{3}\right)^n u(n-1)$$

and

$$h_2(nT) = \left(\frac{1}{4}\right)^n u(n-3)$$

Then

$$y(nT) = y_1(nT) - y_2(nT)$$

where

$$y_1(nT) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \left(\frac{1}{3}\right)^{n-k} u(n-k-1)$$

and

$$y_2(nT) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \left(\frac{1}{4}\right)^{n-k} u(n-k-3)$$

First let's evaluate $y_1(nT)$. Clearly $y_1(nT) = 0$ for $n < 1$ since the product of the two unit steps is zero unless $k \geq 0$ and $n \geq k + 1$. For $n \geq 1$ we have

$$y_1(nT) = \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-k}$$

which can be written

$$y_1(nT) = \left(\frac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\frac{3}{2}\right)^k$$

Performing the summation yields

$$y_1(nT) = \left(\frac{1}{3}\right)^n \frac{1-\left(\frac{3}{2}\right)^n}{1-\frac{3}{2}}$$

$$= 2\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n\right] \qquad n \geq 1$$

Thus

$$y_1(nT) = 2\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n\right] u(n-1)$$

We now evaluate $y_2(nT)$. First observe that $y_2(nT) = 0$ for $n < 3$. For $n \geq 3$ we have

$$y_2(nT) = \sum_{k=0}^{n-3} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k}$$

which is

$$y_2(nT) = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n-3} (2)^k$$

Performing the summation yields

$$y_2(nT) = -\left(\frac{1}{4}\right)^n \left[1-(2)^{n-2}\right], \qquad n \geq 3$$

Thus

$$y_2(nT) = \left[\frac{1}{4}\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u(n-3)$$

Combining $y_1(nT)$ and $y_2(nT)$ yields

$$y(nT) = 2\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n\right] u(n-1) - \left[\frac{1}{4}\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u(n-3)$$

Problem 8-59

If the unit pulse response of a filter is $h(kT)$, the unit step response is

$$y(nT) = \sum_{n=-\infty}^{\infty} h(kT)\, u(n-k)$$

Since $u(n-k) = 0$ for $k > n$ this becomes

$$y(nT) = \sum_{n=-\infty}^{n} h(kT)$$

This yields

$$y(nT) = \sum_{n=-\infty}^{n} \left\{\left(\frac{1}{3}\right)^k u(k-1) - \left(\frac{1}{4}\right)^k u(k-3)\right\} = A - B$$

where

$$A = \left(\sum_{k=1}^{n}\left(\frac{1}{3}\right)^k\right) u(n-1)$$

and

$$B = \left(\sum_{k=3}^{n}\left(\frac{1}{4}\right)^k\right) u(n-3)$$

(Note: The unit steps $u(n-1)$ and $u(n-3)$ result by examining the regions of overlap.)

Now

$$\sum_{k=1}^{n}\left(\frac{1}{3}\right)^k = \sum_{m=0}^{n-1}\left(\frac{1}{3}\right)^{m+1} = \left(\frac{1}{3}\right) \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}$$

$$= \frac{1}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$$

The second sum is

$$\sum_{k=3}^{n}\left(\frac{1}{4}\right)^{k}=\sum_{m=0}^{n-3}\left(\frac{1}{4}\right)^{m+3}=\left(\frac{1}{4}\right)^{3}\frac{1-\left(\frac{1}{4}\right)^{n-2}}{1-\frac{1}{4}}$$

$$=\frac{1}{48}\left(1-16\left(\frac{1}{4}\right)^{n}\right)$$

Thus

$$y(nT)=\frac{1}{2}\left(1-\left(\frac{1}{3}\right)^{n}\right)u(n-1)-\frac{1}{48}\left(1-16\left(\frac{1}{4}\right)^{n}\right)u(n-3)$$

Problem 8-60

With

$$x(nT)=h(nT)=\left(\frac{1}{4}\right)^{n}u(n-4)$$

the expression for $y(nT)$ becomes

$$y(nT)=\sum_{k=-\infty}^{\infty}\left(\frac{1}{4}\right)^{k}u(k-4)\left(\frac{1}{4}\right)^{n-k}u(n-k-4)$$

It follows that

$$y(nT)=0,\qquad n<8$$

For $n\geq 8$ we have

$$y(nT)=\sum_{k=4}^{n-4}\left(\frac{1}{4}\right)^{n}=\left(\frac{1}{4}\right)^{n}\sum_{k=0}^{n-8}(1)$$

which is

$$y(nT)=(n-7)\left(\frac{1}{4}\right)^{n}$$

Thus

$$y(nT)=\begin{cases}(n-7)\left(\frac{1}{4}\right)^{n}, & n\geq 8\\ 0, & n<8\end{cases}$$

Problem 8-61

For the given difference equation

$$y(nT) = x(nT) - x(nT - 10T)$$

the pulse transfer function is

$$H(z) = 1 - z^{-10} = \left(z^{5} - z^{-5}\right)z^{-5}$$

The sinusoidal steady-state frequency response is therefore given by

$$H\left(e^{j2\pi r}\right) = \left(e^{j10\pi r} - e^{-j10\pi r}\right)e^{-j10\pi r}$$

$$= 2j\sin(10\pi r)e^{-j10\pi r}$$

The amplitude response

$$A(r) = 2\left|\sin(10\pi r)\right|$$

and the phase response is, for $0 \le r \le \frac{1}{2}$,

$$\phi(r) = \begin{cases} \frac{\pi}{2} - 10\pi r & \text{when } \sin 10\pi r \ge 0 \\ \frac{\pi}{2} - 10\pi r \pm \pi & \text{when } \sin 10\pi r < 0 \end{cases}$$

The amplitude and phase responses are generated using Math CAD.

Define j: $j := \sqrt{-1}$

Define z(r): $z(r) := \exp(j \cdot 2 \cdot \pi \cdot r)$

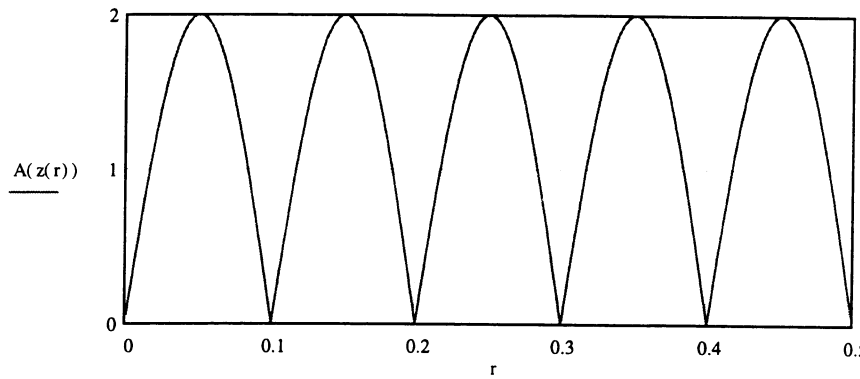
Define the transfer function: $H(z) := 1 - z^{-10}$

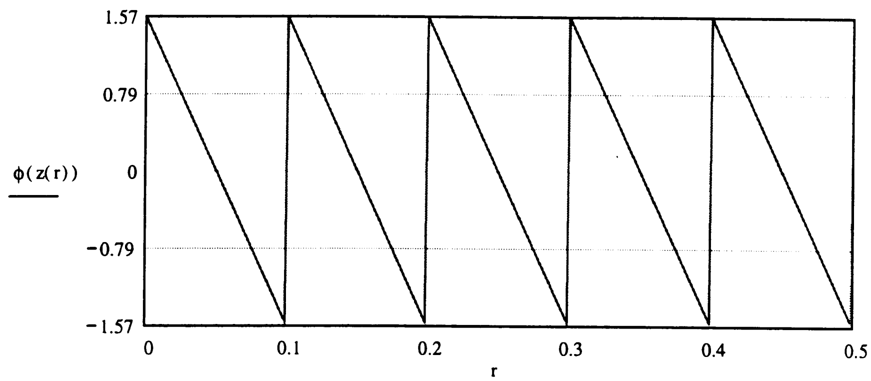
Define the amplitude response: $A(z) := |H(z)|$

Define the phase response: $\phi(z) := \arg(H(z))$

Define the range for the plot: $r := 0.001, 0.002 .. 0.5$

The amplitude and phase responses are shown below.





Problem 8-62

For the difference equation

$$y(nT) = 2x(nT) + 4x(nT - T) + 2x(nT - 2T)$$

gives the pulse transfer function

$$\begin{aligned} H(z) &= 2 + 4z^{-1} + 2z^{-2} \\ &= 2\left(2 + \left(z + z^{-1}\right)\right)z^{-1} \end{aligned}$$

The sinusoidal steady-state response is therefore

$$\begin{aligned} H\left(e^{-j2\pi r}\right) &= 2\left(2 + \left(e^{j2\pi r} + e^{-j2\pi r}\right)\right)e^{-j2\pi r} \\ &= 2(2 + 2\cos 2\pi r)e^{-j2\pi r} \\ &= 4(1 + \cos 2\pi r)e^{-j2\pi r} \end{aligned}$$

It follows that the amplitude response is, for $0 \le r \le 0.5$,

$$A(r) = 4(1 + \cos 2\pi r)$$

and the phase response is

$$\phi(r) = -2\pi r$$

These are determined below using MathCAD.

Define j: $j := \sqrt{-1}$

Define z(r): $z(r) := \exp(j \cdot 2 \cdot \pi \cdot r)$

Define the transfer function: $H(z) := 2 + 4 \cdot z^{-1} + 2 \cdot z^{-2}$

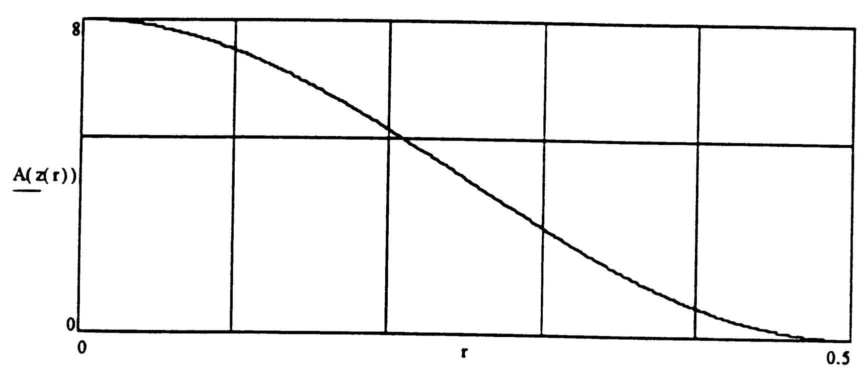
Define the amplitude response: $A(z) := |H(z)|$

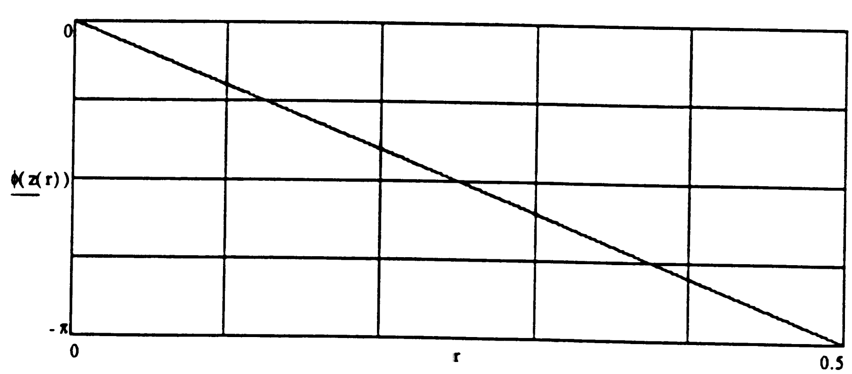
Define the phase response: $\phi(z) := \arg(H(z))$

Define the range: $r := 0.001, 0.002 .. 0.4999$

Note that we bound r away from 0.5 so that an error does not result when we calculate the phase angle.

The amplitude and phase responses are as shown below.





Problem 8-63

We determine the solution of this problem using MathCAD.

First we define j as

$$j := \sqrt{-1}$$

and then define z, which for sinusoidal steady-state frequency response is a function of r. Thus

$$z(r) := \exp(j \cdot 2 \cdot \pi \cdot r)$$

The difference equation y(nT)+0.25y(nT-T)=x(nT-T) has the pulse transfer function

$$H(z) := \frac{z^{-1}}{1 + 0.25 \cdot z^{-1}}$$

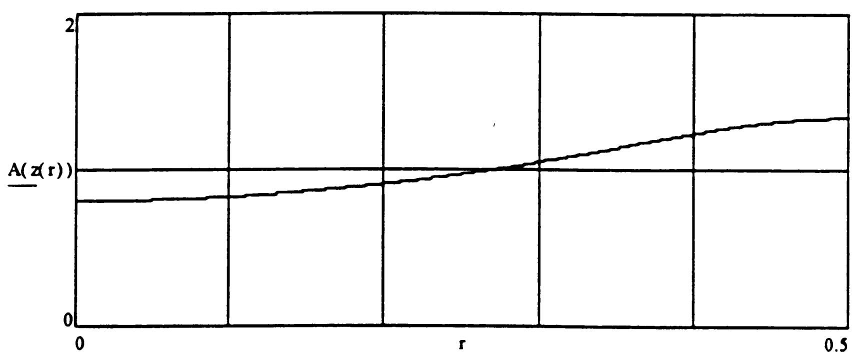
The amplitude and phase responses corresponding to the difference equation are

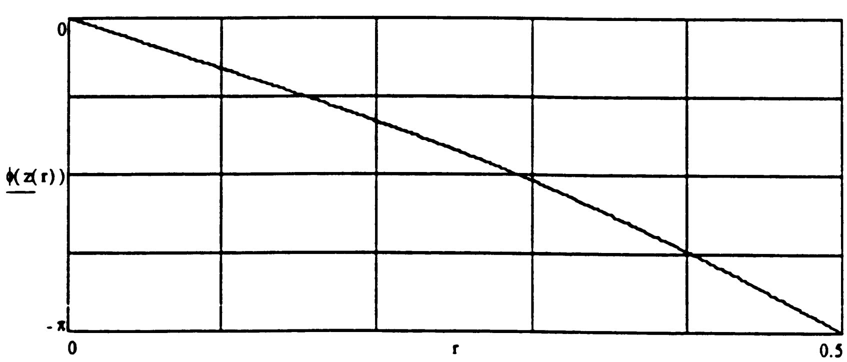
$$A(z) := |H(z)| \qquad \phi(z) := \arg(H(z))$$

The range variable for the plot is defined as

$$r := 0, 0.001 .. 0.5$$

This yields the amplitude and phase responses shown below.





Problem 8-64

The difference equation

$$y(nT) = \frac{1}{T}\{x(nT) - x(nT - T)\}$$

which defines an approximation to a differentiator having the transfer function

$$H(z) = \frac{1}{T}\left(1 - z^{-1}\right)$$

The sinusoidal steady-state frequency response is defined by

$$H\left(e^{j2\pi r}\right) = \frac{1}{T}\left(e^{j\pi r} - e^{-j\pi r}\right)e^{-j\pi r}$$

$$= j\frac{2}{T}\sin(\pi r)e^{-j\pi r}$$

For $0 \leq r < 0.5$, the amplitude and phase responses are given by

$$A(r) = \frac{2}{T}\sin \pi r$$

and

$$\phi(r) = \frac{\pi}{2} - \pi r$$

In order to compare the digital differentiator with an ideal analog differentiator defined by

$$H_a(s) = \frac{1}{s}$$

we write the sinusoidal steady-state response of the digital differentiator in the form

$$H\left(e^{j\omega T}\right) = j\frac{2}{T}\sin\left(\frac{\omega T}{2}\right)e^{-j\omega T/2}$$

For $\omega T << 1$ the $\sin\left(\frac{\omega T}{2}\right) \approx \frac{\omega T}{2}$ and the amplitude response is

$$\left|H\left(e^{j\omega T}\right)\right| = \frac{2}{T}\left(\frac{\omega T}{2}\right) = \omega$$

and the phase response is

$$\angle H\left(e^{j\omega T}\right) = \frac{\pi}{2} - \frac{T}{2}\omega$$

The analog filter has the steady-state response

$$H_a(j\omega) = \omega \angle \frac{\pi}{2}$$

Thus the amplitude response of the digital differentiator closely approximates the amplitude response of the ideal analog integrator for small input frequencies. The phase responses of the digital differentiator and the ideal analog integrator are identical except for a $T/2$ second group delay exhibited by the digital differentiator. The plots for the amplitude and phase responses are shown in Figure 9-9 and need not be repeated here.

Problem 8-65

The unit-pulse response is defined as

$$h(nT) = 4[u(n) - u(n-12)] = \begin{cases} 4, & 0 \le n \le 11 \\ 0, & otherwise \end{cases}$$

The frequency response is

$$H(e^{j\omega T}) = \sum_{n=0}^{11} 4e^{-jn\omega T} = 4\frac{1-e^{-j12\omega T}}{1-e^{-j\omega T}}$$

The preceding expression can be placed in the form

$$H(e^{j\omega T}) = 4\frac{\left(e^{j6\omega T} - e^{-j6\omega T}\right)e^{-j6\omega T}}{\left(e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}\right)e^{-j\omega\frac{T}{2}}}$$

or

$$H(e^{j\omega T}) = 4\frac{\sin(6\omega T)}{\sin\left(\omega\frac{T}{2}\right)}e^{-j11\omega\frac{T}{2}}$$

Thus $A(\omega) = 6\omega T$, $B(\omega) = \omega\frac{T}{2}$, $C(\omega) = -11\omega\frac{T}{2}$ and $k = 4$.

In terms of normalized frequency, r, the frequency response is

$$H(e^{j2\pi r}) = 4\frac{\sin(12\pi r)}{\sin(\pi r)}e^{-j11\pi r}$$

with $f_s = 1000$ Hz, $f = 0$, 25 and 50 Hz corresponds to $r = 0$, 0.025 and 0.05, respectively.

Thus the following table can be made

f	r	$A(r)$	$\phi(r) - rad$
0	0	48	0
25	0.025	41.25	-0.275π
50	0.05	24.32	-0.550π

Problem 8-66

(a) For $y(nT) = x(nT) + \frac{1}{2}y(nT-T)$ the pulse transfer function is

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

The dc response is

$$H(1) = \frac{1}{1-\frac{1}{2}} = 2$$

and the response at $f = \frac{1}{2}f_s$

$$H(-1) = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

(b) For $y(nT) = x(nT) + \frac{1}{2}x(nT-T) + \frac{1}{2}y(nT-T)$ the pulse transfer function is

$$H(z) = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

The dc response is

$$H(1) = \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$$

and the response at $f = \frac{1}{2}f_s$ is

$$H(-1) = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$$

(c) For $y(nT) = x(nT) - y(nT-T) + \frac{1}{2}y(nT-2T)$

the pulse transfer function is

$$H(1) = \frac{1}{1+z^{-1}-\frac{1}{2}z^{-2}}$$

The dc response is

$$H(1) = \frac{1}{1+1-\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

and the response at $f = \frac{1}{2}f_s$ is

$$H(-1) = \frac{1}{1-1-\frac{1}{2}} = -2$$

(d) For $y(nT) = x(nT) - \frac{3}{2}y(nT-T) + \frac{1}{2}y(nT-2T)$

the pulse transfer function is

$$H(z) = \frac{1}{1+\frac{3}{2}z^{-1}-\frac{1}{2}z^{-2}}$$

The dc response is

$$H(1) = \frac{1}{1+\frac{3}{2}-\frac{1}{2}} = \frac{1}{2}$$

and the response at $f = \frac{1}{2}f_s$ is

$$H(-1) = \frac{1}{1-\frac{3}{2}-\frac{1}{2}} = -1$$

<u>**Problem 8-67**</u>

For $f = 0.25 f_s$

$$e^{j\omega T} = e^{j2\pi(0.25)} = e^{j\pi/2} = j1$$

If $y(nT) = x(nT-T) + 0.5y(nT) - 0.2y(nT-T)$ the pulse transfer function is

$$H(z) = \frac{z^{-1}}{1-0.5z^{-1}+0.2z^{-2}}$$

For f_s = 1600 Hz and f = 400 Hz, we compute

$$H(j1) = \frac{-j}{1 + j0.5 - 0.2} = -\frac{-j}{0.8 + j0.5}$$

which is

$$H(j1) = \frac{1\angle -90^\circ}{0.943\angle 32^\circ} = 1.06\angle -122^\circ$$

Problem 8-68

For the difference equation

$$y(nT) = x(nT) + 0.2y(nT - T) - 0.5y(nT - 2T)$$

the pulse transfer function is given by

$$H(z) = \frac{1}{1 - 0.2z^{-1} + 0.5z^{-2}}$$

The dc $(f = 0)$ response is given by $H(1)$, which is

$$H(1) = \frac{1}{1 - 0.2 + 0.5} = \frac{10}{13} = 0.7692$$

The frequency $f = 500$ corresponds to $\frac{1}{4}f_s$. We therefore evaluate $H(z)$ at $z = j1$.

This gives

$$H(j1) = \frac{1}{1 + j0.2 - 0.5} = \frac{1}{0.5 + j0.2} = 1.8570\angle -21.8^\circ$$

The frequency $f = 1000$ corresponds to $\frac{1}{2}f_s$. We therefore evaluate $H(z)$ at $z = -1$. This gives

$$H(-1) = \frac{1}{1 + 0.2 + 0.5} = \frac{10}{17} = 0.5882$$

Problem 8-69

For

$$y(nT) = x(nT) + 0.3y(nT - T) - 0.1y(nT - 2T)$$

the pulse transfer function is

$$H(z) = \frac{1}{1 - 0.3z^{-1} + 0.1z^{-2}}$$

The dc $f = 0$) response is found by letting $z = 1$. Thus

$$H(1) = \frac{1}{1 - 0.3 + 0.1} = \frac{1}{0.8} = 1.25$$

For $f = 250$, which is $\frac{1}{4}f_s$, the response is found by letting $z = j1$. This gives

$$H(j1) = \frac{1}{1 + j0.3 - 0.1} = \frac{1}{0.9 + j0.3} = 1.0541 \underline{/-18.435^\circ}$$

For $f = 500$, which is $\frac{1}{2}f_s$, the response is found by letting $z = -1$. This gives

$$H(-j1) = \frac{1}{1 + 0.3 + 0.1} = \frac{1}{1.4} = 0.7143$$

Problem 8-70

For

$$y(nT) = \frac{8}{3}x(nT) + \frac{3}{4}y(nT - T) - \frac{1}{8}y(nT - T)$$

The pulse transfer function is

$$H(z) = \frac{\frac{8}{3}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

The dc ($f = 0$) response is found by letting $z = 1$. This gives

$$H(1) = \frac{\frac{8}{3}}{1 - \frac{3}{4} + \frac{1}{8}} = \frac{\frac{8}{3}}{\frac{3}{8}} = \frac{64}{9} = 7.1111$$

For $f = 250$, which is $\frac{1}{4}f_s$, the response is found by letting $z = j1$. This gives

$$H(j1) = \frac{\frac{8}{3}}{1 + j\frac{3}{4} + \frac{1}{8}} = \frac{\frac{8}{3}}{\frac{7}{8} + j\frac{3}{4}} = \frac{\frac{64}{3}}{7 + j6} = 2.3139 \underline{/-40.601^\circ}$$

For $f = 500$, which is $\frac{1}{2}f_s$, the response is found by letting $z = -1$. This gives

$$H(-1) = \frac{\frac{8}{3}}{1+\frac{3}{4}+\frac{1}{8}} = \frac{\frac{8}{3}}{\frac{15}{8}} = \frac{64}{45} = 1.4222$$

Problem 8-71

For

$$y(nT) = x(nT) - a_1 y(nT-T) - a_2 y(nT-2T)$$

the pulse transfer function is

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Since the dc response is $H(1)$

$$H(1) = \frac{1}{1 + a_1 + a_2} = 1$$

Since the response at $\left(f = \frac{1}{2}f_s\right)$ is $H(-1)$

$$H(-1) = \frac{1}{1 - a_1 + a_2} = 0.1$$

Thus, we have the system of equations

$$1 + a_1 + a_2 = 1$$
$$0.1(1 - a_1 + a_2) = 1$$

which can be written

$$a_1 + a_2 = 0$$
$$-a_1 + a_2 = 9$$

Thus

$$2a_2 = 9, \qquad a_2 = 4.5$$
$$a_1 = -4.5$$

Problem 8-72

Since

$$y(nT) = \frac{3}{4}y(nT-T) + 3x(nT)$$

The pulse transfer function is

$$H(z) = \frac{3}{1 - \frac{3}{4}z^{-1}}$$

The input is of the form

$$x(nT) = 5\cos(\omega nT)$$

where $\omega T = \frac{\pi}{2}$ (note that the frequency of the input is 1/4 of the sampling frequency). Thus $e^{j\omega T} = j$ and the pulse transfer function, evaluated at the input frequency, is

$$H(j) = \frac{3}{1 + j\frac{3}{4}} = 2.4 \angle{-36.87^\circ}$$

The steady-state output is

$$y_{ss}(nT) = 2.4\cos\left(\frac{\pi}{2}n - 36.87^\circ\right)$$

Problem 8-73

We solve this problem using MathCAD.

Define j: $j := \sqrt{-1}$

Define x(r): $x(r) := \exp(-j \cdot 2 \cdot \pi \cdot r)$

In order for the dc (f=0) gain to be unity k+α must equal one. Thus if α=-3/4, k=1-k or 7/4. Thus

$$k := \frac{7}{4} \qquad \alpha := -\frac{3}{4}$$

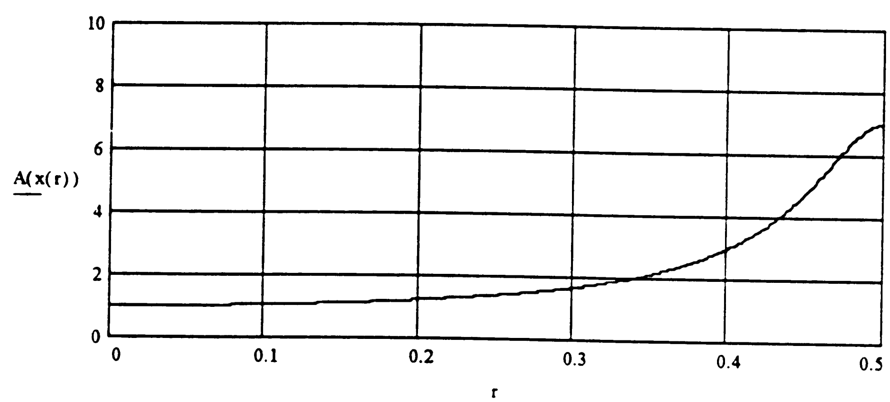
Define transfer function: $H(x) := \dfrac{k}{1 - \alpha \cdot x}$

Define amplitude response: $A(x) := |H(x)|$

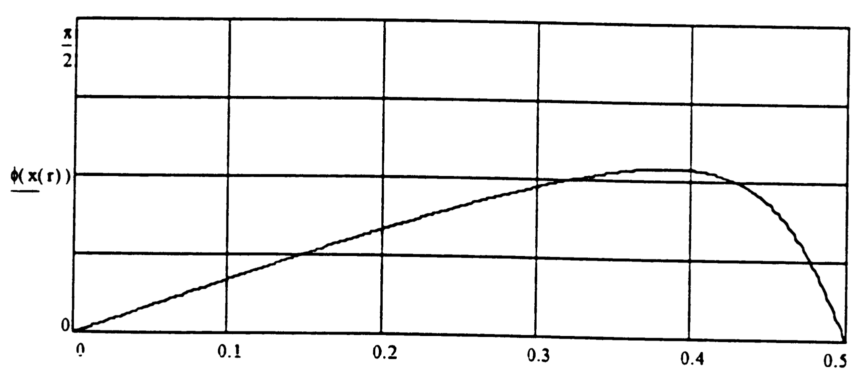
Define phase response: $\phi(x) := \arg(H(x))$

Define range: $r := 0.001, 0.002 .. 0.5$

On a linear scale the amplitude response is shown below.



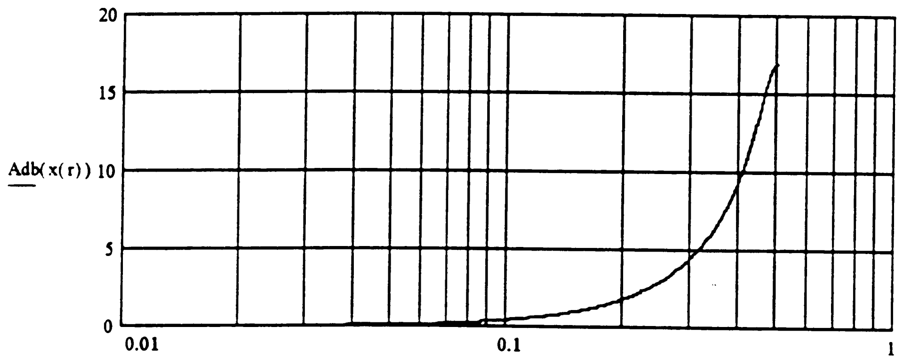
The phase response is shown below.



It is interesting to look at the amplitude in dB on a log frquuency scale. Thus we define

$$Adb(x) := 20 \cdot \log(A(x))$$

This results in the plot shown below



Problem 8-74

We solve this problem using MathCAD.

Define j: $j := \sqrt{-1}$

Define x(r): $x(r) := \exp(-j \cdot 2 \cdot \pi \cdot r)$

In order for the dc (f=0) gain to be unity k+α must equal one. Thus if α=-3/4, k=1-k or 7/4. Thus

$$k := \frac{31}{16} \qquad \alpha := -\frac{15}{16}$$

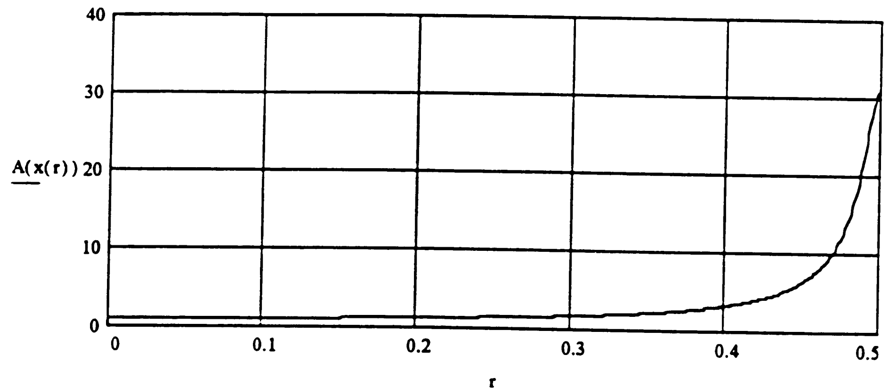
Define transfer function: $H(x) := \dfrac{k}{1 - \alpha \cdot x}$

Define amplitude response: $A(x) := |H(x)|$

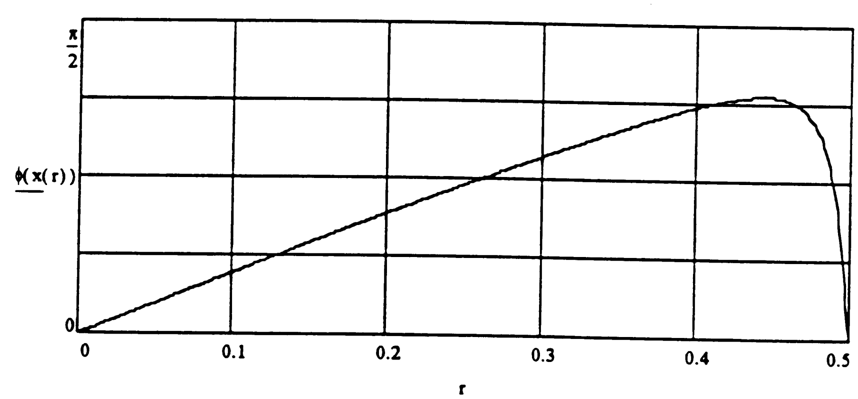
Define phase response: $\phi(x) := \arg(H(x))$

Define range: $r := 0.001, 0.002 .. 0.5$

On a linear scale the amplitude response is shown below.



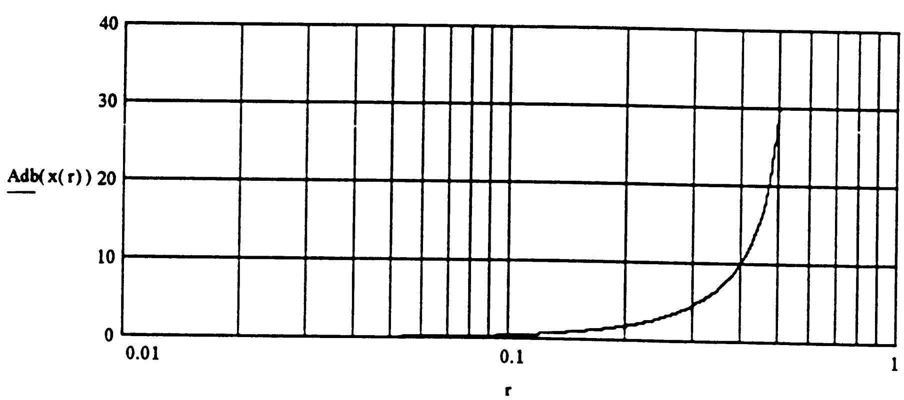
The phase response is shown below.



It is interesting to look at the amplitude in dB on a log frquuency scale. Thus we define

$$\mathrm{Adb}(x) := 20 \cdot \log(A(x))$$

This results in the plot shown below



Problem 8-75

Since

$$X(z) = \frac{1}{\left(1 - z^{-1}\right)\left(1 - 0.2z^{-1}\right)} = \frac{z^2}{(z-1)(z-0.2)}$$

the inverse z-transform can be written

$$x(nT) = \frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz$$

$$= \frac{1}{2\pi j}\oint_c \frac{z^{n+1}dz}{(z-1)(z-0.2)}$$

We take the contour, c, to lie in the region of convergence $|z| > 1$. For $n \geq -1$ we have two poles in the contour, $z = 1$ and $z = 0.2$. Thus

$$x(nT) = \frac{(1)^{n+1}}{1-0.2} + \frac{(0.2)^{n+1}}{0.2-1}, \qquad n \geq -1$$

or

$$x(nT) = \frac{5}{4} - \frac{1}{4}(0.2)^n, \qquad n \geq -1$$

Note that $x(-T) = 0$. Thus

$$x(nT) = \left[\frac{5}{4} - \frac{1}{4}(0.2)^n\right] u(n)$$

which is in agreement with Example 8-9. Other results are possible by choosing different regions of integration. We chose c to yield a causal filter.

Problem 8-76

Since

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{10}z^{-1}\right)} = \frac{z^2}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{10}\right)}$$

the inverse z-transform can be written

$$x(nT) = \frac{1}{2\pi j}\oint_c \frac{z^{n+1}dz}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{10}\right)}$$

If we choose the region of convergence to be $|z| > \frac{1}{4}$, we obtain

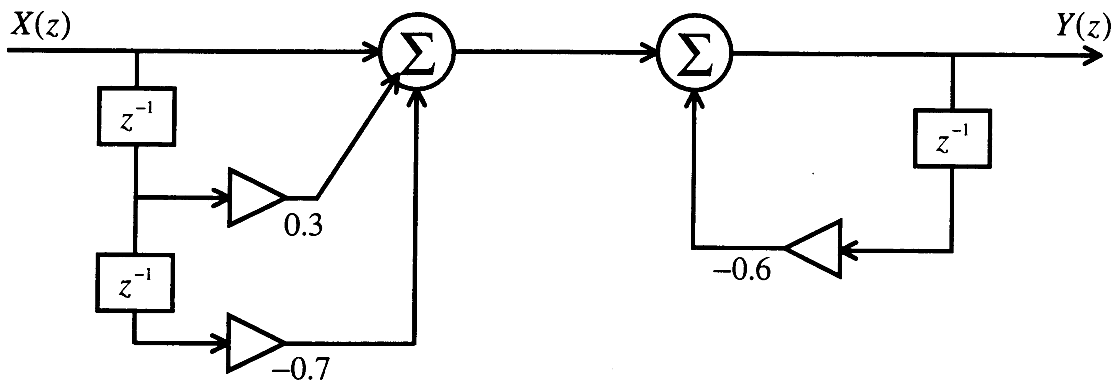
CHAPTER 9

Problem 9-1

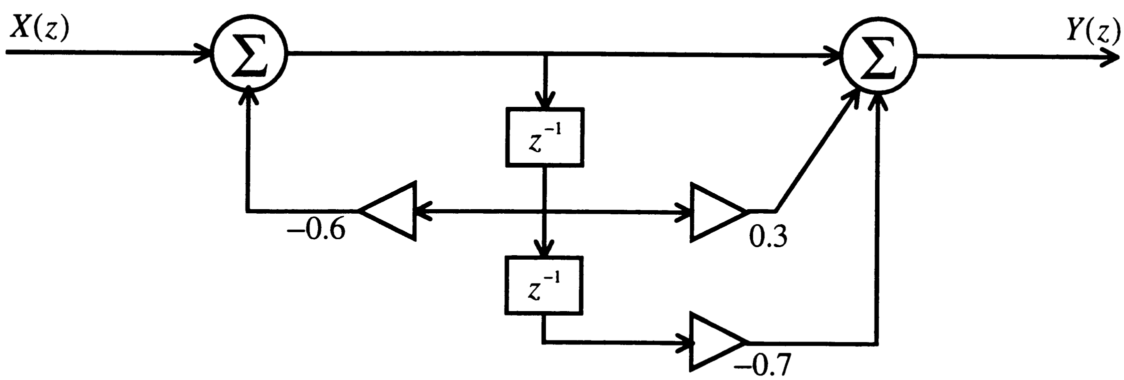
(a) For

$$H(z) = \frac{1 + 0.3z^{-1} - 0.7z^{-2}}{1 + 0.6z^{-1}}$$

the Direct Form I implementation is



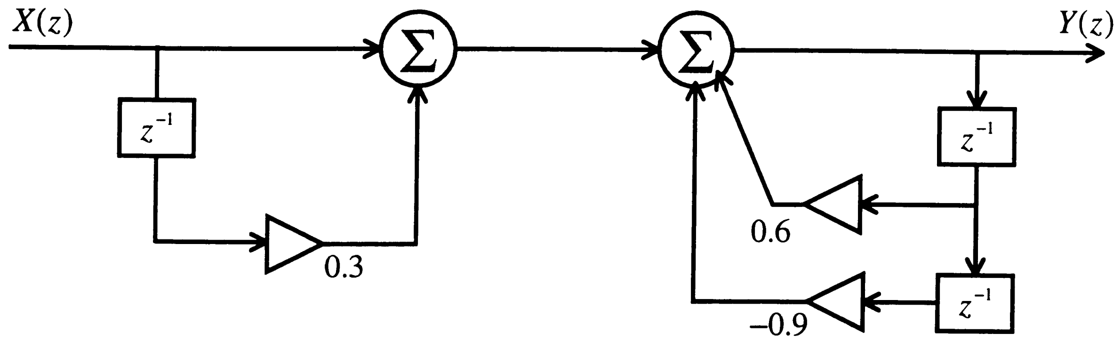
and the Direct Form II implementation is



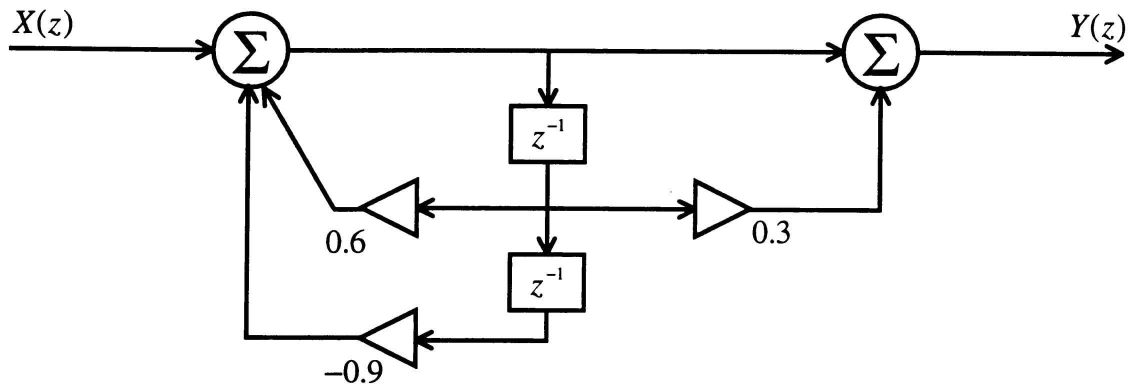
(b) For

$$H(z) = \frac{1 + 0.3z^{-1}}{1 - 0.6z^{-1} + 0.9z^{-2}}$$

the Direct Form I implementation is



and the Direct Form II realization is



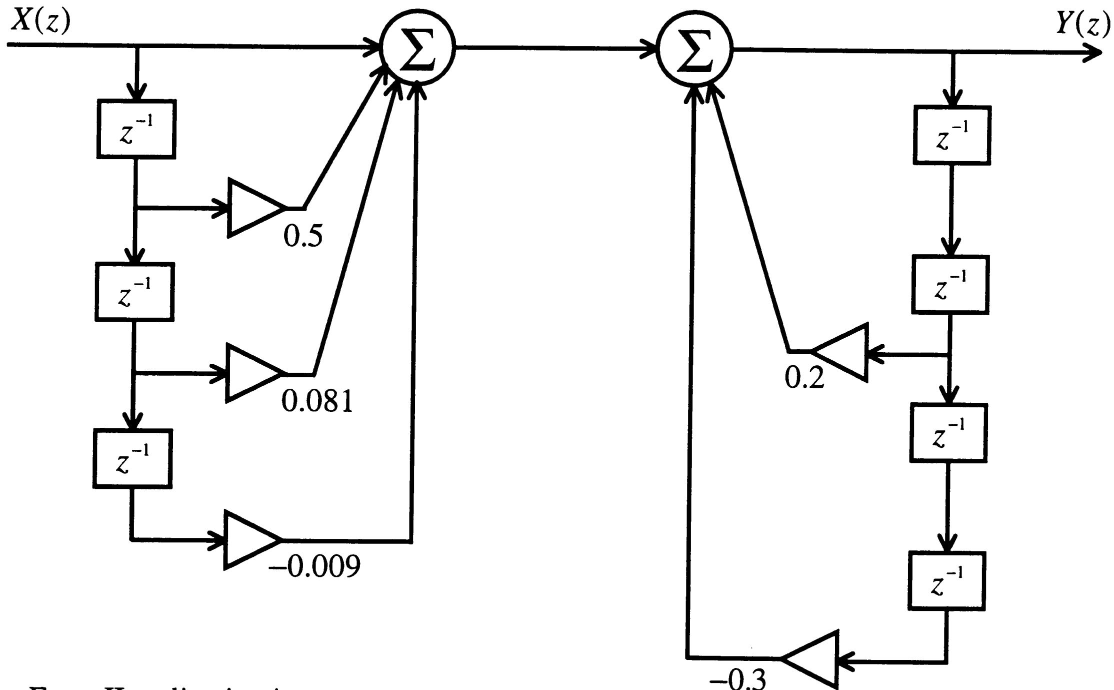
(c) Since

$$\left(1+0.3z^{-1}\right)^{2}\left(1-0.1z^{-1}\right)=\left(1+0.6z^{-1}+0.09z^{-2}\right)\left(1-0.1z^{-1}\right)$$
$$=1+0.5z^{-1}+0.081z^{-2}-0.009z^{-3}$$

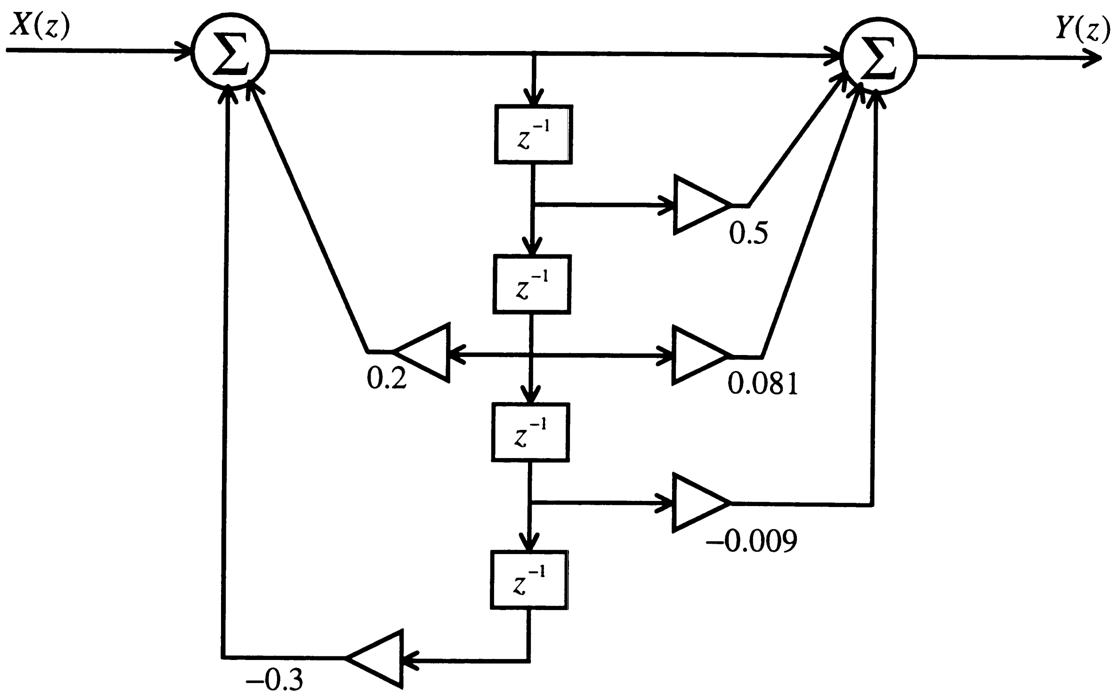
the given $H(z)$ can be written

$$H(z)=\frac{1+0.5z^{-1}+0.081z^{-2}-0.009z^{-3}}{1-0.2z^{-2}+0.3z^{-4}}$$

The Direct Form I realization is



The Direct Form II realization is

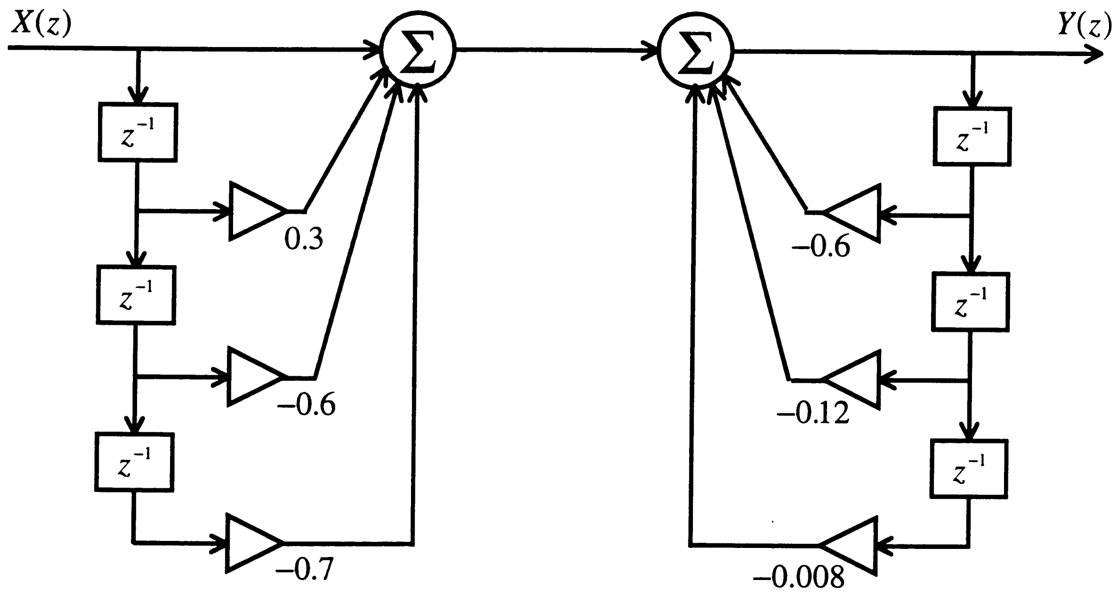


(d) For

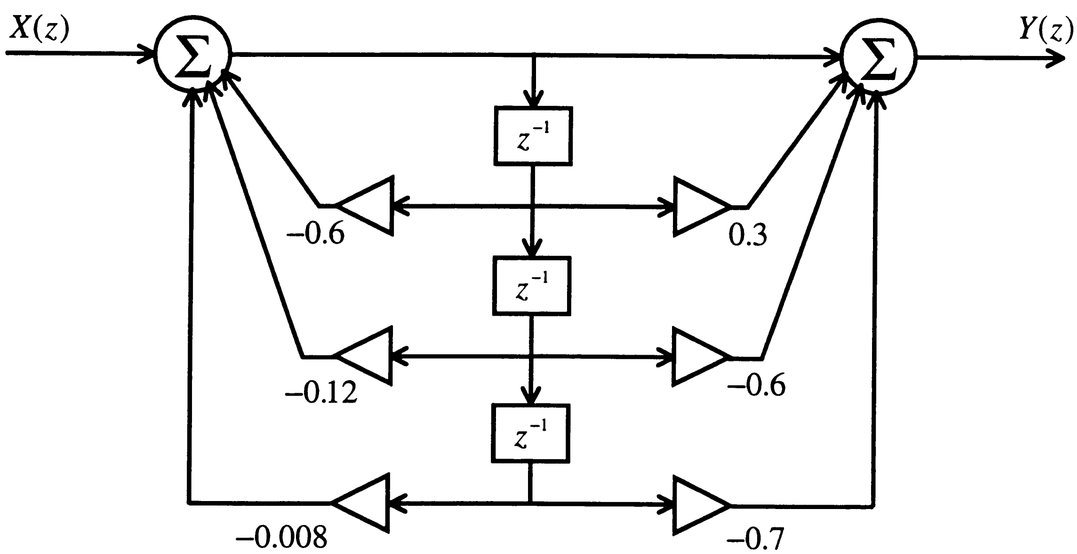
$$H(z) = \frac{1+0.3z^{-1}-0.6z^{-2}-0.7z^{-3}}{\left(1+0.2z^{-1}\right)^3}$$

$$= \frac{1+0.3z^{-1}-0.6z^{-2}-0.7z^{-3}}{1+0.6z^{-1}+0.12z^{-2}+0.008z^{-3}}$$

the Direct Form I realization is



and the Direct Form II realization is

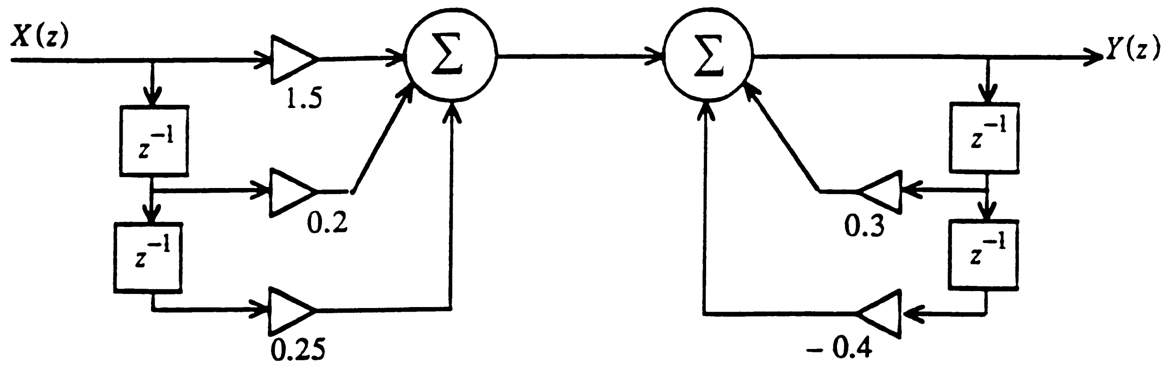


Problem 9-2

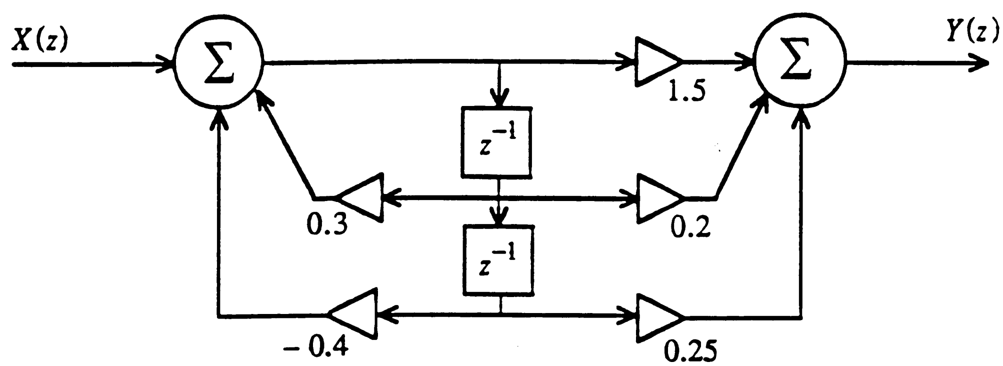
(a) For

$$H(z) = \frac{3 + 0.4z^{-1} + 0.5z^{-2}}{2 - 0.6z^{-1} + 0.8z^{-2}} = \frac{1.5 + 0.2z^{-1} + 0.25z^{-2}}{1 - 0.3z^{-1} + 0.4z^{-2}}$$

the Direct Form I realization is



and the Direct Form II realization is



(b) The pulse transfer function

$$H(z) = \frac{1-0.2z^{-2}}{4-0.4z^{-1}} + \frac{1}{1-0.2z^{-2}}$$

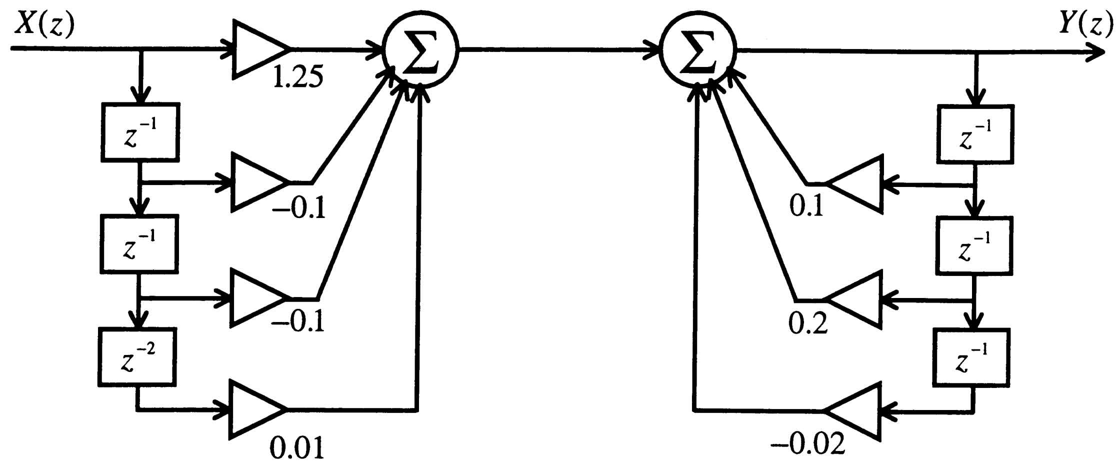
can be written

$$H(z) = \frac{5-0.4z^{-1}-0.4z^{-2}+0.04z^{-4}}{4-0.4z^{-1}-0.8z^{-2}+0.08z^{-3}}$$

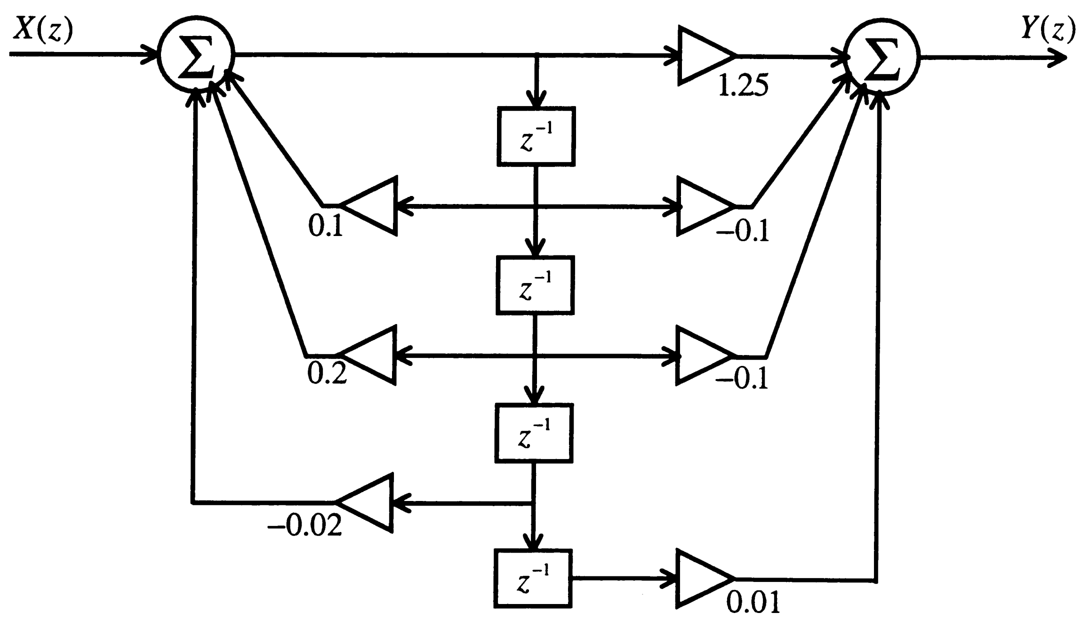
which, in standard form, is

$$H(z) = \frac{1.25-0.1z^{-1}-0.1z^{-2}+0.01z^{-4}}{1-0.1z^{-1}-0.2z^{-2}+0.02z^{-3}}$$

The Direct Form I realization is



and the Direct Form II realization is



(c) Since

$$\left(1-z^{-1}\right)^3\left(1-0.3z^{-2}\right)=\left(1-3z^{-1}+3z^{-2}-z^{-3}\right)\left(1-0.3z^{-2}\right)$$
$$=1-3z^{-1}+2.7z^{-2}-0.1z^{-3}-0.9z^{-4}+0.9z^{-5}$$

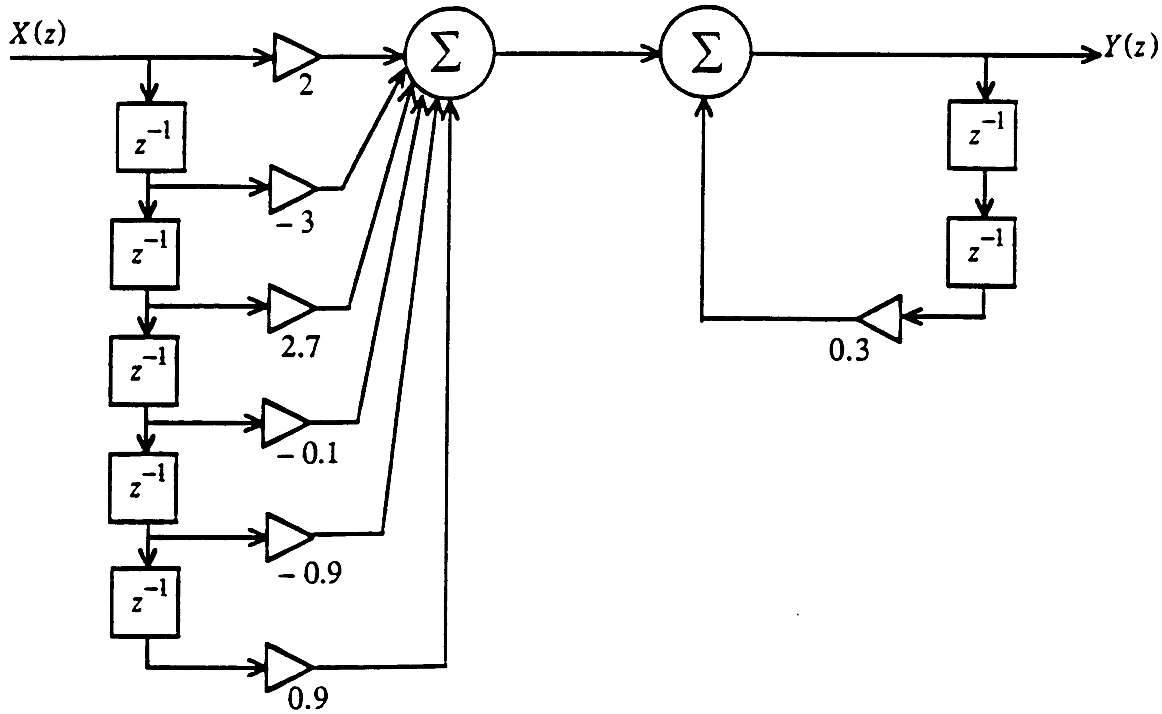
The pulse transfer function

$$H(z)=\left(1-z^{-1}\right)^3+\frac{1}{1-0.3z^{-2}}$$

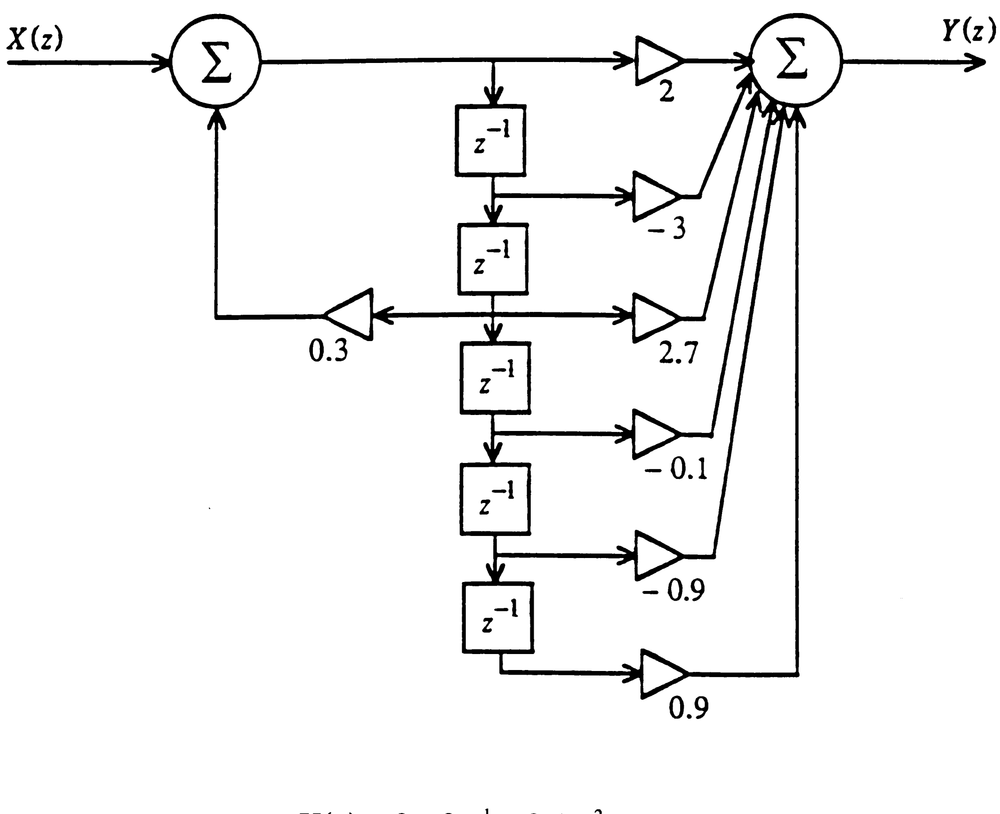
can be written

$$H(z)=\frac{2-3z^{-1}+2.7z^{-2}-0.1z^{-3}-0.9z^{-4}+0.9z^{-5}}{1-0.3z^{-2}}$$

The Direct From I realization is

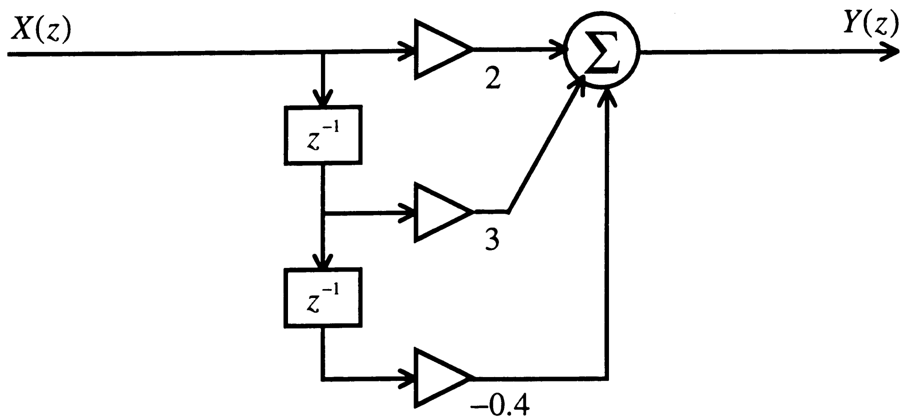


and the Direct Form II realization is



(d) For

$$H(z) = 2 + 3z^{-1} - 0.4z^{-2}$$



Since $H(z)$ has no poles, the Direct Form II realization is identical to the Direct Form I realization.

Problem 9-3

(a) The given $H(z)$

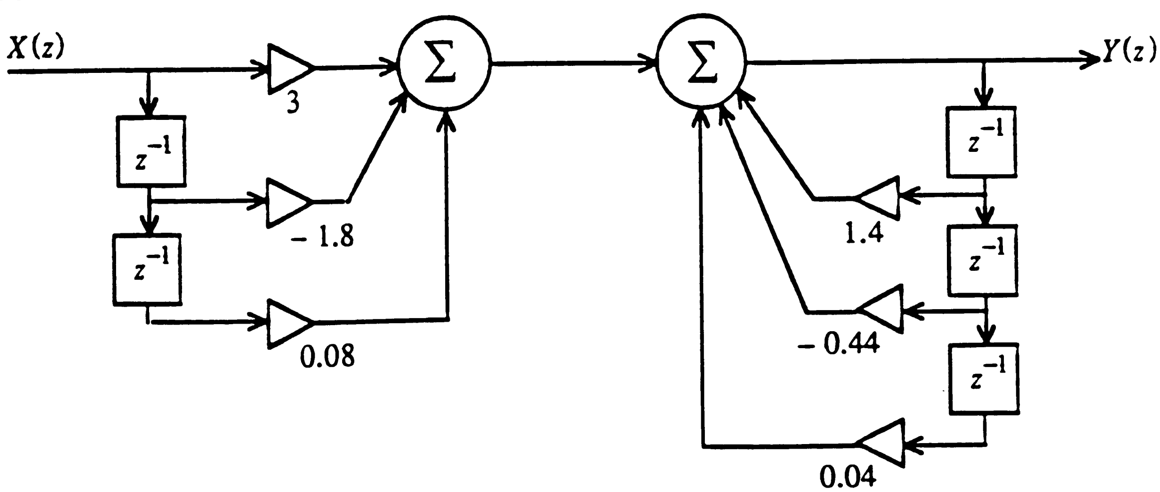
$$H(z)=\frac{2}{1-z^{-1}}+\frac{1}{\left(1-0.2z^{-1}\right)^2}$$

must first be written as a ratio of polynomial in z^{-1}. This yields

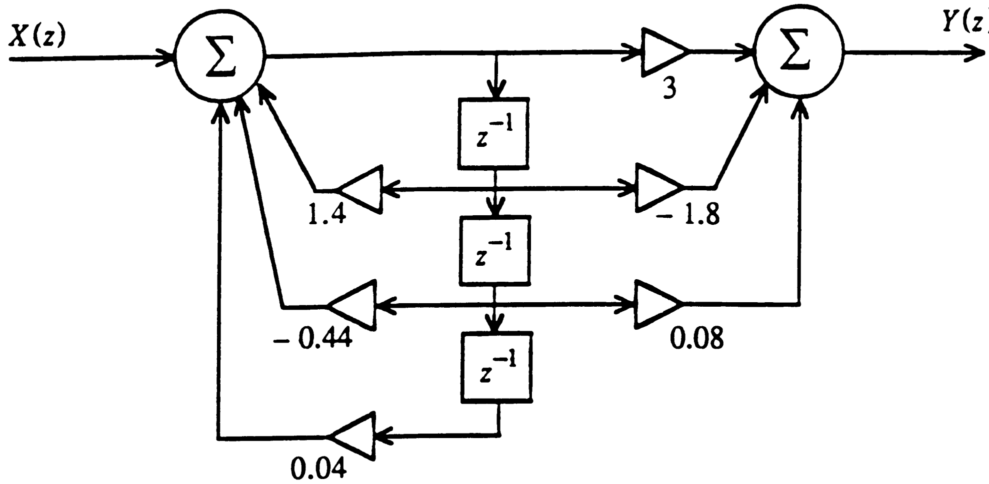
$$H(z)=\frac{2\left(1-0.2z^{-1}\right)^2+\left(1-z^{-1}\right)}{\left(1-z^{-1}\right)\left(1-0.2z^{-1}\right)^2}$$

$$=\frac{3-1.8z^{-1}+0.08z^{-2}}{1-1.4z^{-1}+0.44z^{-2}-0.04z^{-3}}$$

The Direct Form I realization is



and the Direct Form II realization is



(b) The given $H(z)$

$$H(z)=\left(1-z^{-1}\right)+\frac{3}{\left(1-0.3z^{-1}\right)^{2}}$$

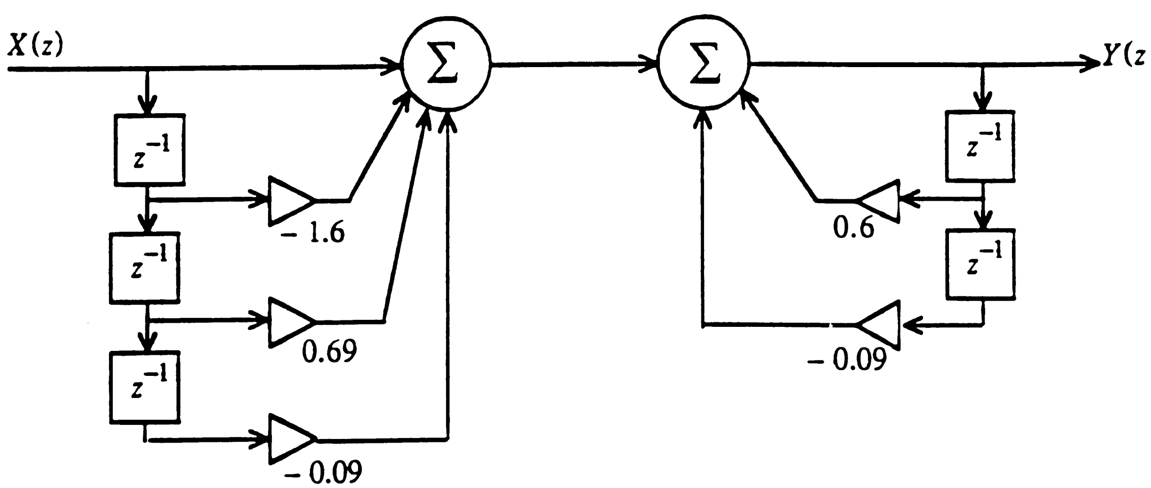
must first be written

$$H(z)=\frac{3+\left(1-z^{-1}\right)\left(1-0.3z^{-1}\right)^{2}}{\left(1-0.3z^{-1}\right)^{2}}$$

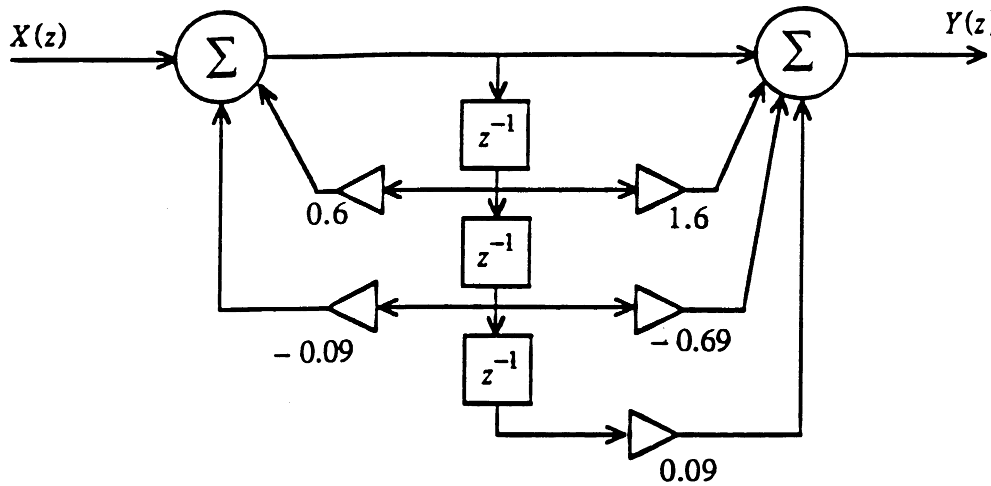
which is

$$H(z)=\frac{1-1.6z^{-1}+0.69z^{-2}-0.09z^{-3}}{1-0.6z^{-1}+0.09z^{-2}}$$

The Direct Form I realization is



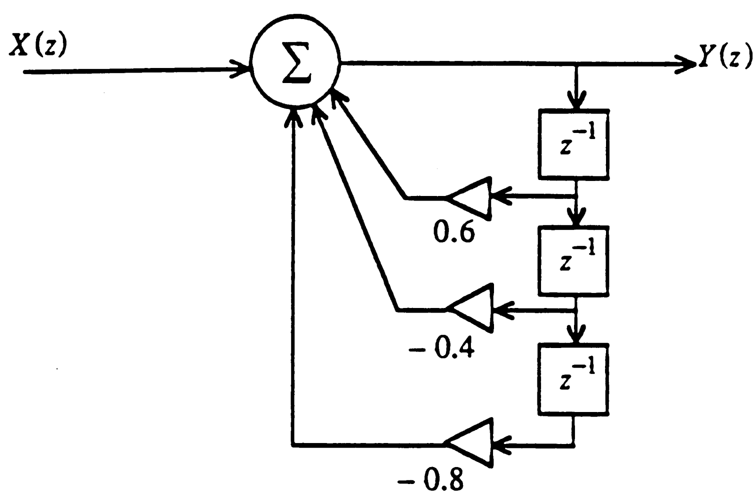
and the Direct Form II realization is



(c) For

$$H(z) = \frac{1}{1 - 0.6z^{-1} + 0.4z^{-2} + 0.8z^{-3}}$$

The Direct Form I realization is



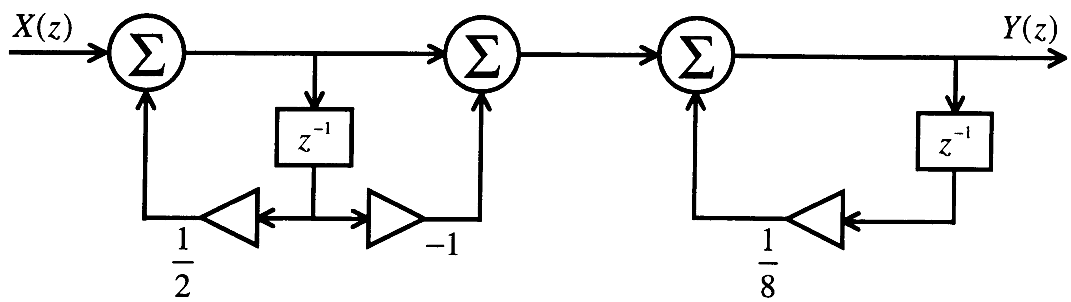
Since the system has no zeros, the Direct Form II realiztion is identical to the Direct Form I realization.

Problem 9-4

The cascade realization is obtained by writing $H(z)$ in the form

$$H(z) = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{8}z^{-1}\right)}$$

Thus, a possible cascade realization is as follows:



The parallel realization is obtained by writing

$$\frac{H(z)}{z}=\frac{z-1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{8}\right)}=\frac{A}{z-\frac{1}{2}}+\frac{B}{z-\frac{1}{8}}$$

The residues are

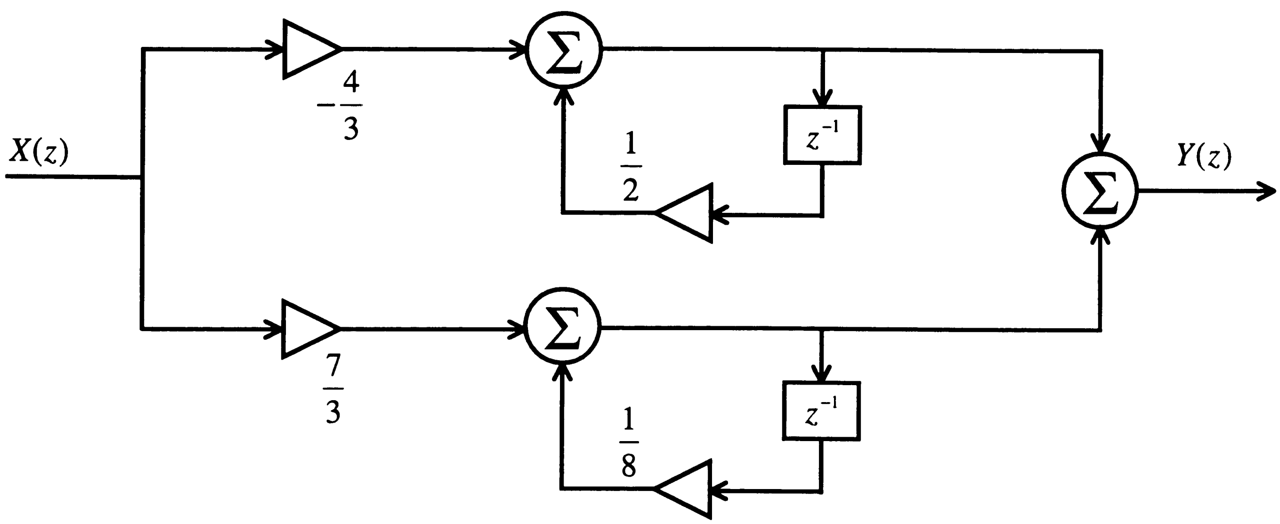
$$A=\frac{-\frac{1}{2}}{\frac{3}{8}}=-\frac{4}{3}$$

$$B=\frac{-\frac{7}{8}}{-\frac{3}{8}}=\frac{7}{3}$$

We can therefore write

$$H(z)=-\frac{4}{3}\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{7}{3}\frac{1}{1-\frac{1}{8}z^{-1}}$$

The parallel realization corresponding to the given $H(z)$ is



Problem 9-5

First we write $H(z)$ in the form

$$\frac{H(z)}{z} = \frac{z^2}{(z-0.1)\left(z^2-0.9z+0.81\right)}$$

Since the roots of $z^2 - 0.9z + 0.81 = 0$ are complex we expand as follows:

$$\frac{H(z)}{z} = \frac{A}{z-0.1} + \frac{B+Cz}{z^2-0.9z+0.81}$$

The value of A is

$$A = \frac{0.01}{0.01-0.9(0.1)+0.81} = \frac{1}{73} = 0.0137$$

The value of B can be found by letting $z = 0$. This gives

$$0 = -10A + \frac{B}{0.81}$$

which is

$$B = 0.81(10A) = 0.11096$$

with $z = 1$

$$\frac{1}{(0.9)(0.91)} = \frac{A}{0.9} + \frac{B+C}{0.91}$$

Thus

$$C = \frac{1}{0.9} - \frac{0.91}{0.9}A - B = 0.98630$$

Note that by multiplying $\frac{H(z)}{z}$ by z and taking the limit as $z \to \infty$ shows that

$$1 = A + C$$

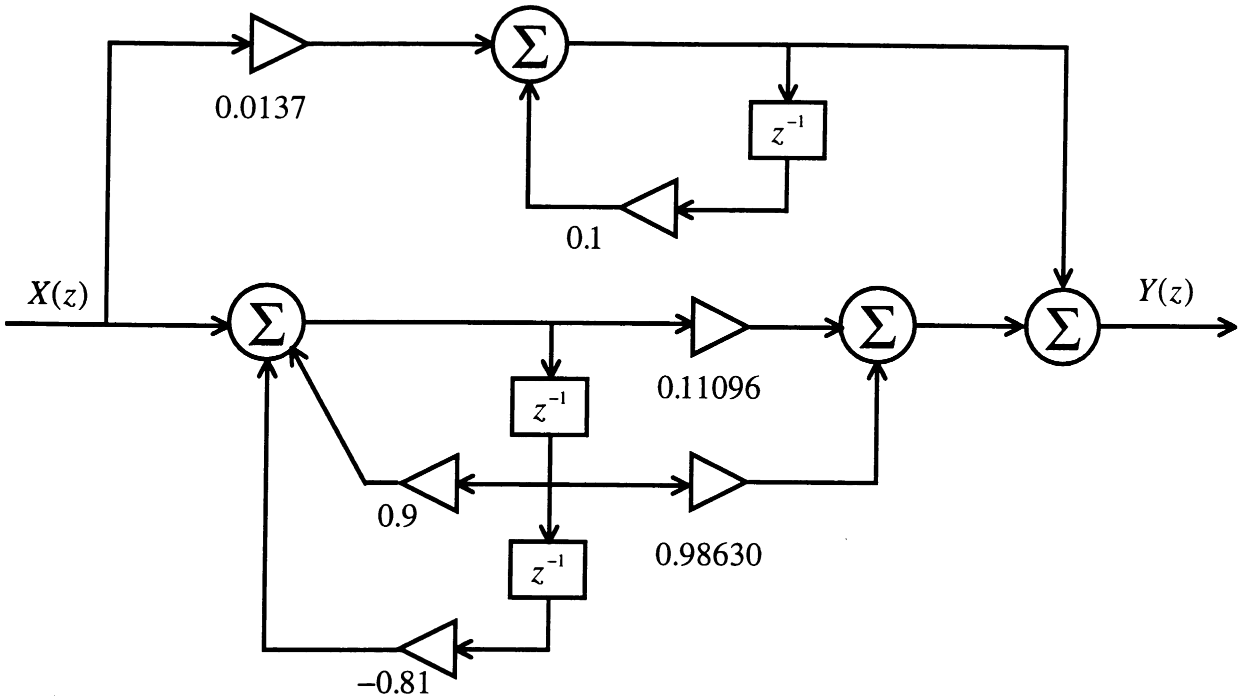
or

$$C = 1 - A$$

This provides a nice check on the results. The expression for $H(z)$ is

$$H(z) = 0.0137\frac{1}{1-0.1z^{-1}} + \frac{0.11096+0.98630z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

The parallel realization is



Problem 9-6

$$H(z) = \frac{z^{-3}}{\left(1-0.1z^{-1}\right)\left(1-0.5z^{-1}\right)^2} = \frac{1}{(z-0.1)(z-0.5)^2}$$

Thus

$$\frac{H(z)}{z} = \frac{1}{z(z-0.1)(z-0.5)^2}$$
$$= \frac{A}{z} + \frac{B}{z-0.1} + \frac{C}{(z-0.5)^2} + \frac{D}{z-0.5}$$

The values of A, B, C, and D are given by

$$A = \frac{-10}{0.25} = -40$$

$$B = \frac{10}{0.16} = 62.5$$

$$C = \frac{2}{0.4} = 5$$

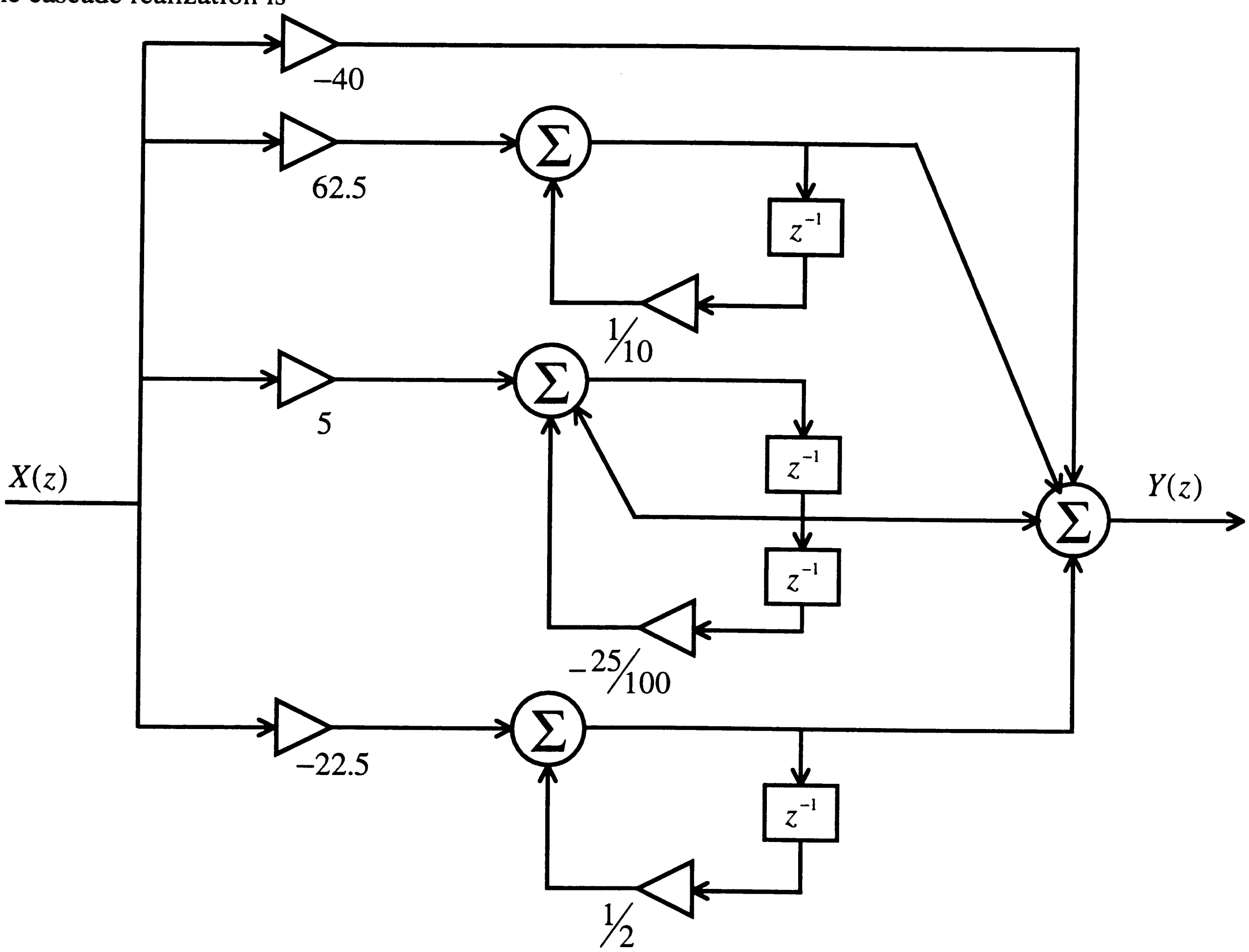
$$D = \frac{d}{dz}\left[\frac{1}{z(z-0.1)}\right]\Bigg|_{z=0.5}$$

$$= \frac{-[2(0.5)-0.1]}{[0.5(0.4)]^2} = -22.5$$

$H(z)$ can therefore be written

$$H(z) = -40 + 62.5\frac{1}{1-0.1z^{-1}} + 5\frac{z^{-1}}{1-z^{-1}+0.25z^{-2}} - 22.5\frac{1}{1-0.5z^{-1}}$$

The cascade realization is



Problem 9-7

The given $H(z)$

$$H(z) = \frac{z^{-3}}{(1-0.1z^{-1})(1-0.4z^{-1})^3}$$

can be written in the form

$$\frac{H(z)}{z} = \frac{1}{(z-0.1)(z-0.4)^3}$$

The partial fraction expansion is

$$\frac{H(z)}{z} = \frac{A}{z-0.1} + \frac{B}{(z-0.4)^3} + \frac{C}{(z-0.4)^2} + \frac{D}{z-0.4}$$

The values of A and B are given by

$$A = \frac{1}{(-0.3)^3} = \frac{-1}{0.027} = 37.037$$

$$B = \frac{1}{0.3} = 3.333$$

The value of C is

$$C = \frac{d}{dz}\left[\frac{1}{z-0.1}\right]\Bigg|_{z=0.4}$$

$$= \frac{-1}{0.09} = -11.111$$

Thus

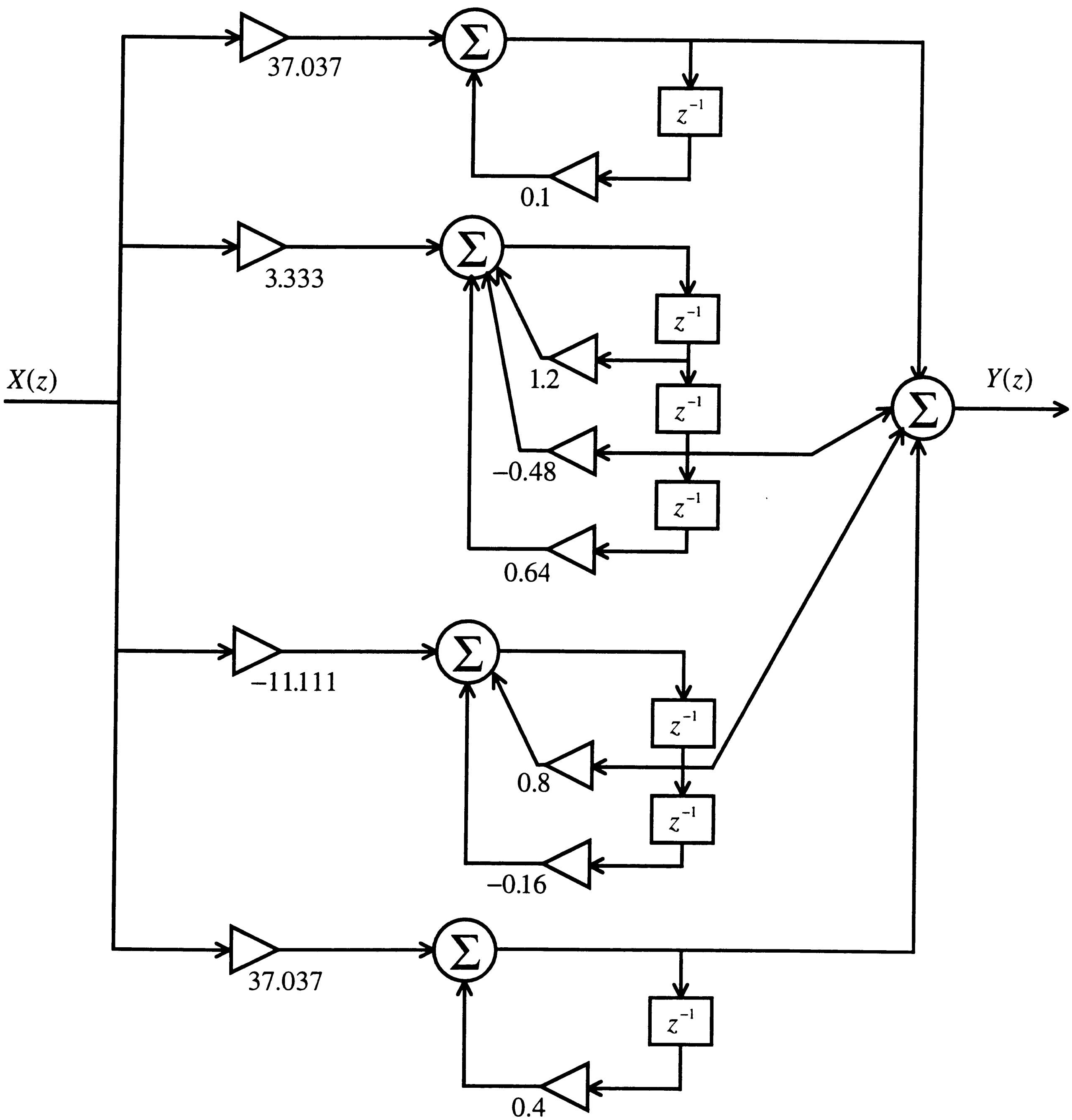
$$H(z) = 37.307\frac{1}{1-0.1z^{-1}} + 3.333\frac{z^{-2}}{1-1.2z^{-1}+0.48z^{-2}-0.064} - 1.111\frac{z^{-1}}{1-0.8z^{-1}+0.16z^{-2}} + D\frac{1}{1-0.4z^{-1}}$$

The value of D is given by

$$2D = \frac{d^2}{dz^2}\left\{\frac{1}{z-0.1}\right\}\Bigg|_{z=0.4}$$

or

$$D = 37.037$$



Problem 9-8

$$H(z)=\frac{z^{-3}}{(1-z^{-1})(2-z^{-1})(4+z^{-1})}=\frac{1}{8}\frac{1}{(z-1)\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$$

Thus

$$\frac{H(z)}{z}=\frac{1}{8}\frac{1}{z(z-1)\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$$
$$=\frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-\frac{1}{2}}+\frac{D}{z+\frac{1}{4}}$$

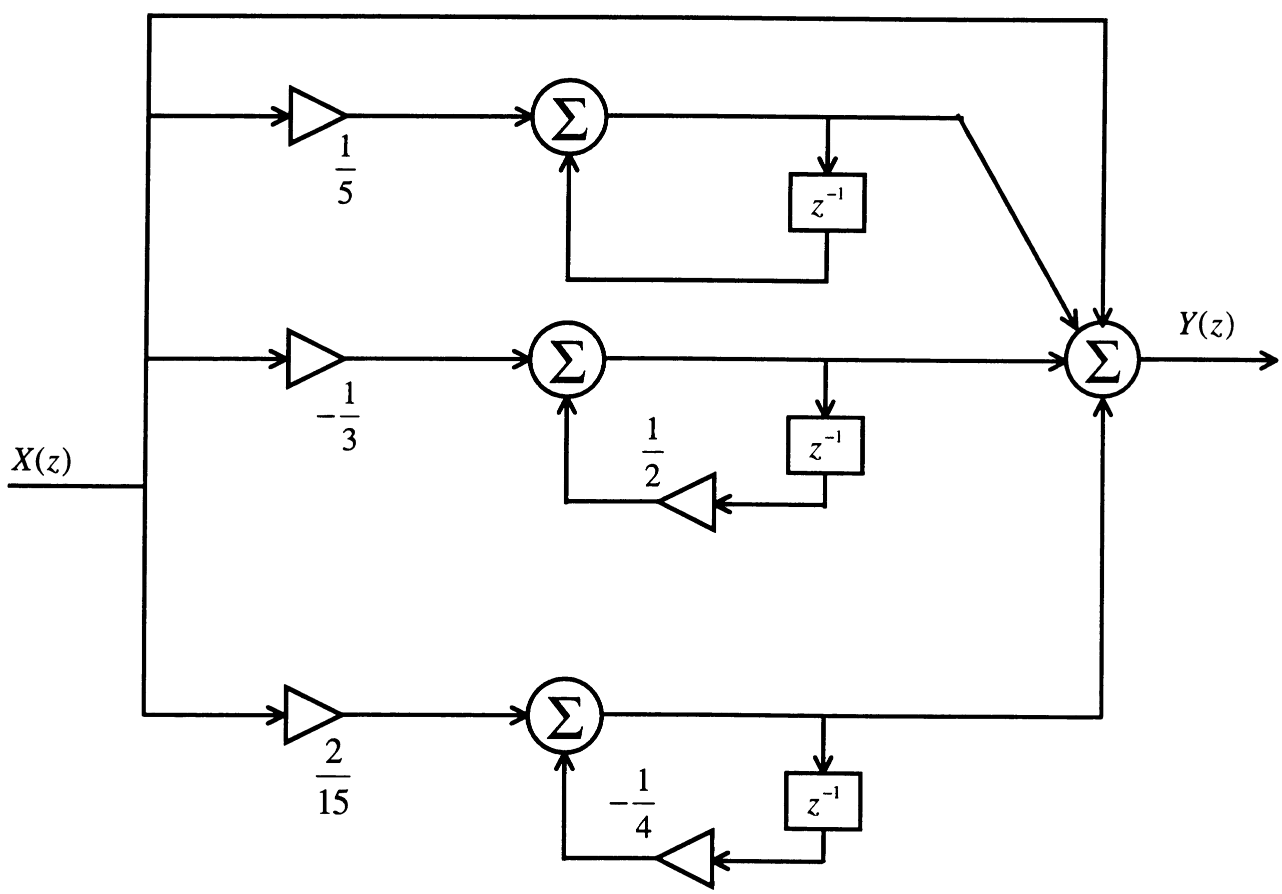
The values of A, B, C and D are

$$A=1,\ B=\frac{1}{5},\ C=-\frac{1}{3},\ D=\frac{2}{15}$$

Thus

$$H(z)=1+\frac{1}{5}\frac{1}{1-z^{-1}}-\frac{1}{3}\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{2}{15}\frac{1}{1+\frac{1}{4}z^{-1}}$$

The parallel realization is



Problem 9-9

$$H(z)=\frac{z^{-2}}{\left(1-0.3z^{-1}\right)^{2}}=\frac{1}{(z-0.3)^{2}}$$

Writing $\frac{H(z)}{z}$ yields

$$\frac{H(z)}{z}=\frac{1}{z(z-0.3)^{2}}=\frac{A}{z}+\frac{B}{(z-0.3)^{2}}+\frac{C}{z-0.3}$$

the values of A and B are

$$A=\frac{1}{0.09}=11.111,\ B=\frac{1}{0.3}=3.333$$

Since

$$\frac{1}{z}=\frac{A}{z}(z-0.3)^{2}+B+C(z-0.3)$$

The value of C is given by

$$C=\frac{d}{dz}\left(\frac{1}{z}\right)\bigg|_{z=0.3}$$
$$=\frac{-1}{z^{2}}\bigg|_{z=0.3}=-\frac{1}{0.09}=-11.111$$

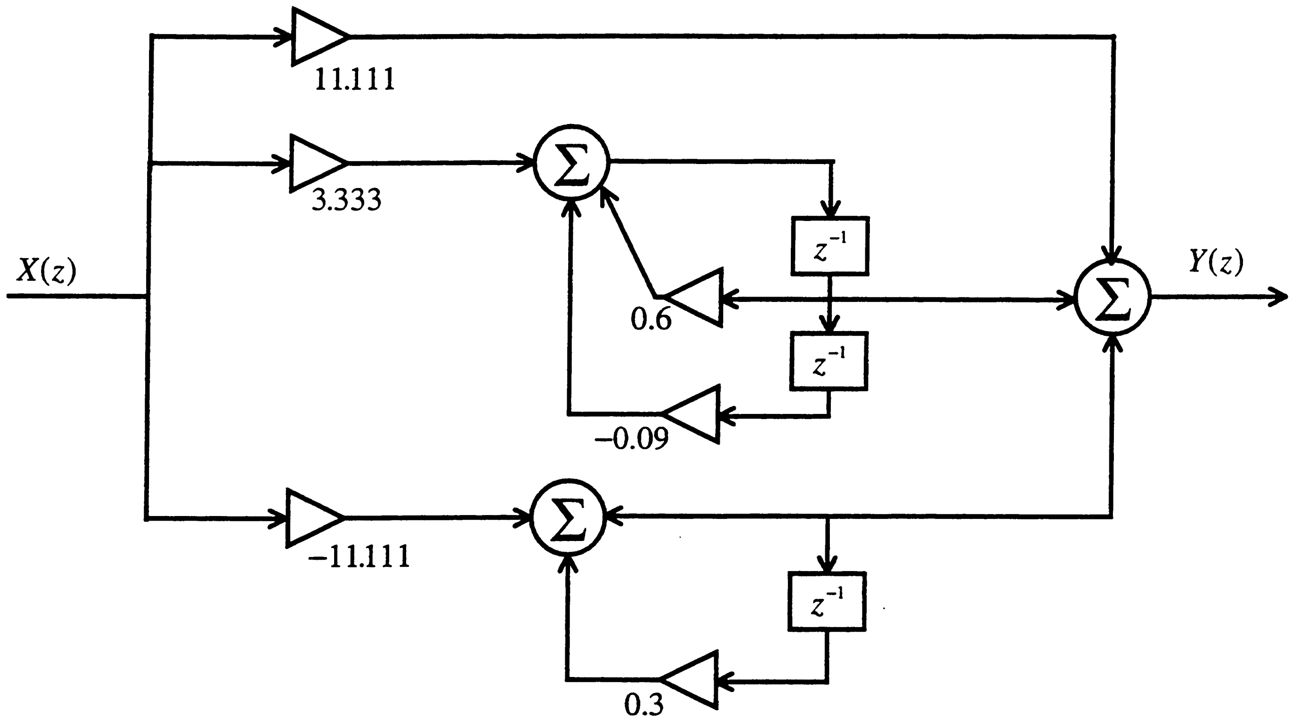
Thus

$$H(z)=11.111+3.333\frac{z^{-1}}{\left(1-0.3z^{-1}\right)^{2}}-11.111\frac{1}{1-0.3z^{-1}}$$

or

$$H(z)=11.111+3.333\frac{z^{-1}}{1-0.6z^{-1}+0.09z^{-2}}-11.111\frac{1}{1-0.3z^{-1}}$$

The parallel realization is



Problem 9-10

The pulse transfer function is defined as

$$H(z) = \frac{z^{-4}}{\left(1-z^{-1}\right)\left(1-z^{-2}\right)} = \frac{z^{-1}}{(z-1)\left(z^2-1\right)}$$

Thus

$$\frac{H(z)}{z} = \frac{1}{z^2(z+1)(z-1)^2}$$

$$= \frac{A}{z^2} + \frac{B}{z} + \frac{C}{z+1} + \frac{D}{(z-1)^2} + \frac{E}{z-1}$$

The values of A, C and D are easy to determine. We have

$$A = 1, \quad C = \frac{1}{4}, \quad D = \frac{1}{2}$$

The value of B is given by

$$B = \frac{d}{dz}\left[\frac{1}{(z+1)(z-1)^2}\right]\Bigg|_{z=0}$$

$$= \frac{d}{dz}\left[\frac{1}{z^3 - z^2 - z + 1}\right]\Bigg|_{z=0}$$

$$= -(-1) = 1$$

The value of E is given by

$$E = \frac{d}{dz}\left[\frac{1}{z^2(z+1)}\right]\Bigg|_{z=1}$$

This yields

$$E = \frac{1}{4}\left[-\left(3z^2 + 2z\right)\right]\Big|_{z=1} = -\frac{5}{4}$$

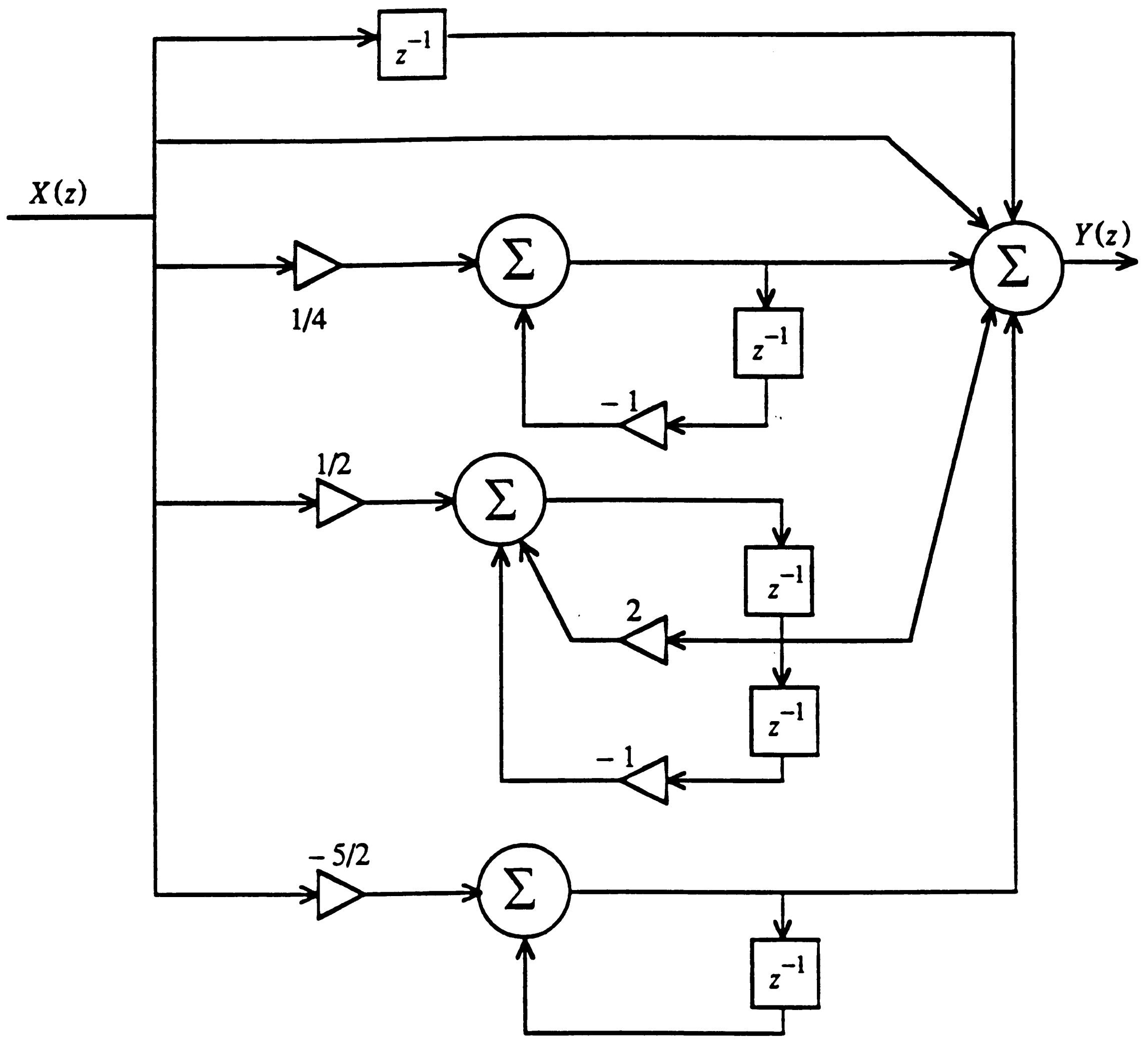
Thus

$$H(z) = \frac{1}{z} + 1 + \frac{1}{4}\frac{z}{z+1} + \frac{1}{2}\frac{z}{(z-1)^2} - \frac{5}{4}\frac{z}{z-1}$$

The final form desired is

$$H(z) = z^{-1} + 1 + \frac{1}{4}\frac{1}{1+z^{-1}} + \frac{1}{2}\frac{z^{-1}}{1-2z^{-1}+z^{-2}} - \frac{5}{4}\frac{1}{1-z^{-1}}$$

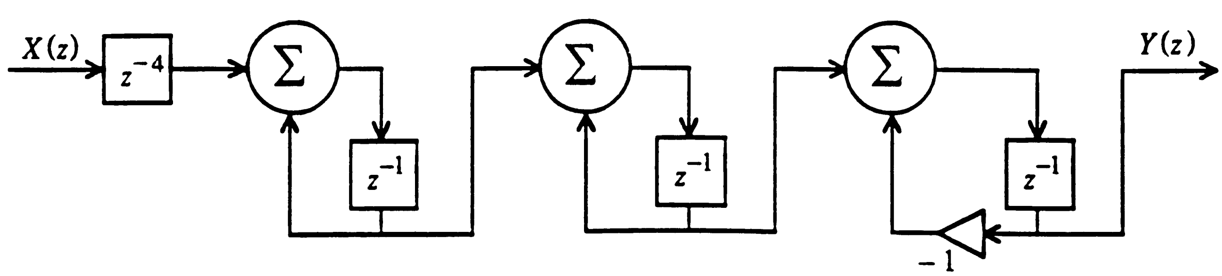
The parallel realization is



A cascade realization is determined by writing $H(z)$ as

$$H(z)=\frac{z^{-4}}{\left(1-z^{-1}\right)^2\left(1+z^{-1}\right)}$$

Thus



Problem 9-11

Denoting $E(z)$ as the signal at the output of the first summation

$$Y(z) = 4X(z) + 4E(z) + 4z^{-1}E(z)$$

$E(z)$ is given by

$$E(z) = X(z) - 0.3z^{-1}E(z)$$

$$E(z) = \frac{1}{1+0.3z^{-1}}X(z)$$

Thus

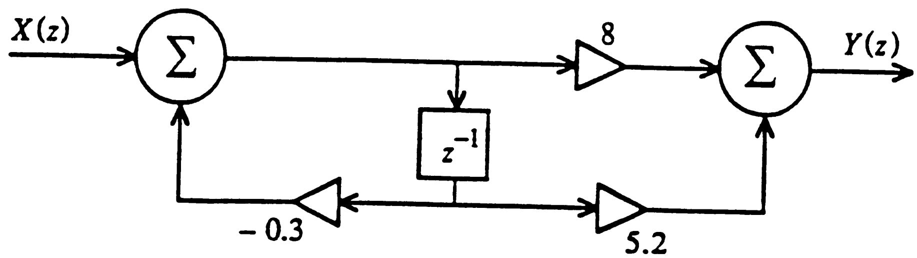
$$Y(z) = 4X(z) + 4\,\frac{1+z^{-1}}{1+0.3z^{-1}}\,X(z)$$

and

$$H(z) = \frac{4\left(1+0.3z^{-1}\right)+4\left(1+z^{-1}\right)}{1+0.3z^{-1}}$$

$$= \frac{8+5.2z^{-1}}{1+0.3z^{-1}}$$

The Direct Form II realization is



Problem 9-12

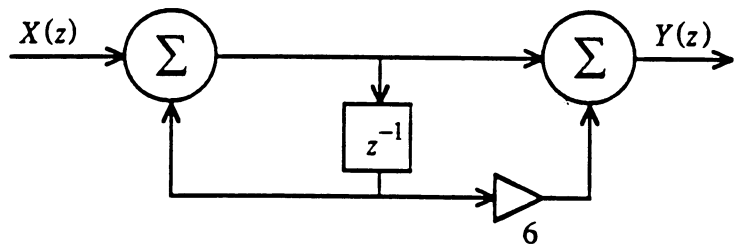
By inspection

$$Y(z)=\frac{1+3z^{-1}}{1-z^{-1}}X(z)+3z^{-1}\frac{1}{1-z^{-1}}X(z)$$

The transfer function is

$$H(z)=\frac{1+3z^{-1}+3z^{-1}}{1-z^{-1}}=\frac{1+6z^{-1}}{1-z^{-1}}$$

The system is realized, using a single z^{-1} element, with a Direct Form II realiztion. This gives



Problem 9-13

The transfer function can be written by inspection

$$H(z)=1+\frac{1+bz^{-1}}{1-az^{-1}}+\frac{1+dz^{-1}}{1-cz^{-1}}$$

This yields

$$H(z)=\frac{N(z)}{D(z)}$$

Where

$$\begin{aligned}D(z)&=\left(1-az^{-1}\right)\left(1-cz^{-1}\right)\\&=1-(a+c)z^{-1}+acz^{-2}\end{aligned}$$

and

$$\begin{aligned}N(z)&=\left(1-az^{-1}\right)\left(1-cz^{-1}\right)+\left(1+bz^{-1}\right)\left(1-cz^{-1}\right)+\left(1+dz^{-1}\right)\left(1-az^{-1}\right)\\&=3+(-a-c+b-c+d-a)z^{-1}+(ac-bc-ad)z^{-2}\\&=3+(b+d-2c-2a)z^{-1}+(ac-bc-ad)z^{-2}\end{aligned}$$

Thus

$$H(z)=\frac{3+(b+d-2c-2a)z^{-1}+(ac-bc-ad)z^{-2}}{1-(a+c)z^{-1}+acz^{-2}}$$

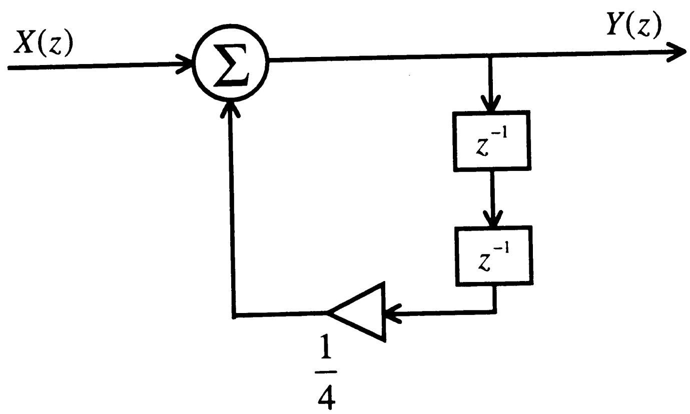
Problem 9-14

By inspection

$$H(z)=\frac{1}{1-\frac{1}{2}z^{-1}}\cdot\frac{1}{1+\frac{1}{2}z^{-1}}$$

$$=\frac{1}{1-\frac{1}{4}z^{-2}}$$

the Direct Form II realization is



Problem 9-15

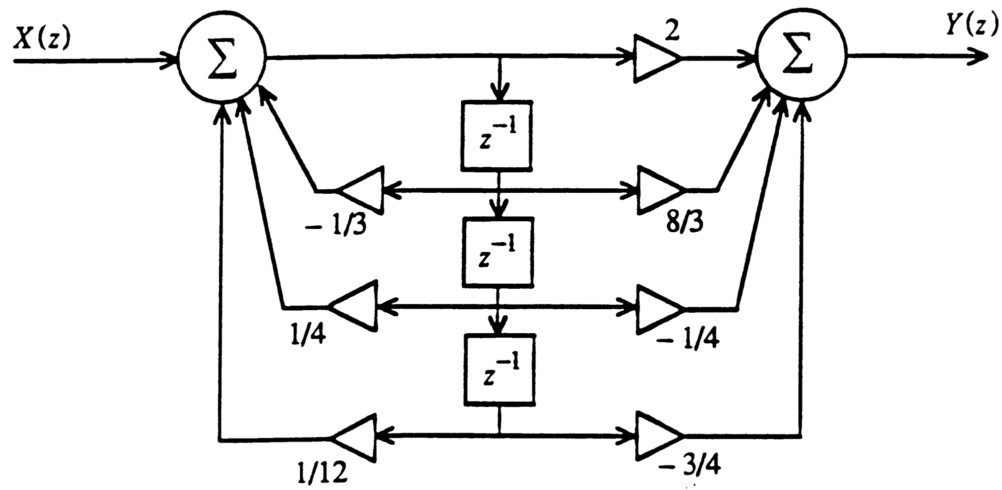
By inspection

$$H(z)=\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}+\frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}}$$

$$=\frac{\left(1+\frac{1}{3}z^{-1}\right)+\left(1+3z^{-1}\right)\left(1-\frac{1}{4}z^{-2}\right)}{\left(1-\frac{1}{4}z^{-2}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$

$$=\frac{1-\frac{1}{3}z^{-1}+1+3z^{-1}-\frac{1}{4}z^{-2}-\frac{3}{4}z^{-3}}{1+\frac{1}{3}z^{-1}-\frac{1}{4}z^{-2}-\frac{1}{12}z^{-3}}$$

The Direct Form II realiztion is $X(z)$



Problem 9-16

By inspection

$$H(z)=\frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}+\frac{1+\frac{1}{8}z^{-1}}{1+\frac{1}{2}z^{-1}}$$

which is

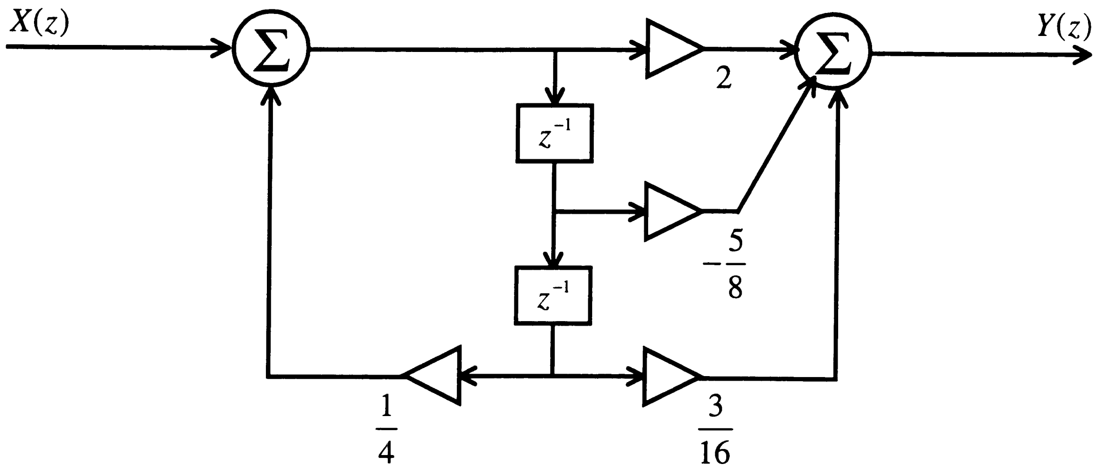
$$H(z)=\frac{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)+\left(1+\frac{1}{8}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}$$

$$=\frac{1+z^{-1}+\frac{1}{4}z^{-2}+1-\frac{3}{8}z^{-1}-\frac{1}{16}z^{-2}}{1-\frac{1}{4}z^{-2}}$$

Collecting terms

$$H(z)=\frac{2-\frac{5}{8}z^{-1}+\frac{3}{16}z^{-2}}{1-\frac{1}{4}z^{-2}}$$

The Direct Form II realization is



Problem 9-17

By inspection

$$H(z) = 2 + \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}} + \frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}}$$

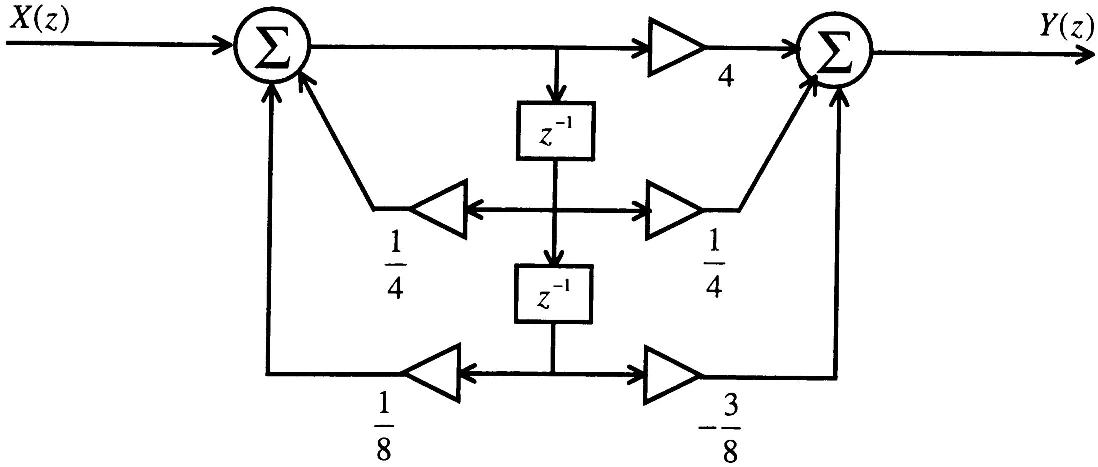
Combining terms yields

$$H(z) = \frac{2\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)+\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)+\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

Performing the indicated multiplications

$$\begin{aligned} H(z) &= \frac{2-\frac{1}{2}z^{-1}-\frac{1}{4}z^{-2}+1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}+1-\frac{1}{4}z^{-2}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} \\ &= \frac{4+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} \end{aligned}$$

The Direct Form II realization is



Problem 9-18

By inspection

$$H(z) = 1 + \frac{1+z^{-1}}{1-z^{-1}} + \frac{1+\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}{1+\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

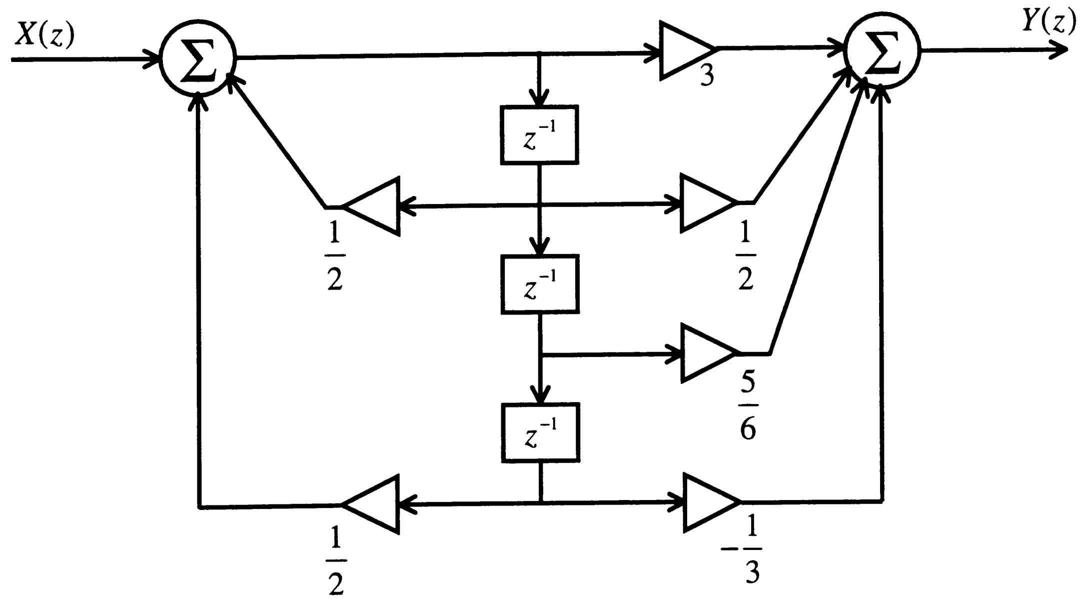
which can be written

$$H(z) = \frac{\left(1-z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}\right)+\left(1+z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}\right)+\left(1-z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}\right)}{\left(1-z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}\right)}$$

or

$$H(z) = \frac{3+\frac{1}{2}z^{-1}+\frac{5}{6}z^{-2}-\frac{1}{3}z^{-3}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-3}}$$

The Direct Form II realization is



Problem 9-19

The pulse transfer function is

$$H(z) = \frac{z^2}{\left(z - ae^{j\theta}\right)\left(z - ae^{-j\theta}\right)}$$

From which

$$\frac{H(z)}{z} = \frac{z}{\left(z - ae^{j\theta}\right)\left(z - ae^{-j\theta}\right)} = \frac{A}{z - ae^{j\theta}} + \frac{A^*}{z - ae^{-j\theta}}$$

Note that the residues are complex conjugates since the poles are complex conjugates. The values of A and A^* are

$$A = \frac{ae^{j\theta}}{ae^{j\theta} - ae^{-j\theta}} = \frac{e^{j\theta}}{2j\sin\theta}, \quad A^* = -\frac{e^{-j\theta}}{2j\sin\theta}$$

Thus, for $n \geq 0$,

$$h(nT) = \frac{1}{2j\sin\theta}\left[e^{j\theta}\left(ae^{j\theta}\right)^n - e^{-j\theta}\left(ae^{-j\theta}\right)^n\right]$$
$$= \frac{1}{2j\sin\theta}a^n\left[e^{j(n+1)\theta} - e^{-j(n+1)\theta}\right]$$

Therefore

$$h(nT) = \frac{a^n}{\sin\theta}\sin[(n+1)\theta]u[n]$$

Problem 9-20

Using the results of the preceding problem

$$H\left(e^{j\theta}\right)=\frac{e^{j2\theta}}{\left(e^{j\theta}-ae^{j\theta}\right)\left(e^{j\theta}-ae^{-j\theta}\right)}$$

$$=\frac{1}{(1-a)\left(1-ae^{-j2\theta}\right)}$$

The magnitude of $A = 1\text{-}ae^{-j2\theta}$ is

$$|A|=\sqrt{AA^*}=\sqrt{\left(1-ae^{-j2\theta}\right)\left(1-ae^{j2\theta}\right)}$$

$$=\sqrt{1+a^2-2a\cos 2\theta}$$

Therefore, since a is real,

$$\left|H\left(e^{j\theta}\right)\right|=\frac{1}{(1-a)\sqrt{1+a^2-2a\cos 2\theta}}$$

For $\theta=\frac{\pi}{2}$, $\cos 2\theta=-1$, and

$$\left|H\left(e^{j\theta}\right)\right|=\frac{1}{(1-a)\sqrt{1+2a+a^2}}$$

$$=\frac{1}{1-a^2}$$

Problem 9-21

$$y(nT) = 0.5[x(nT) + x(nT - T)]$$

Thus

$$Y(z) = 0.5[1 + z^{-1}]X(z)$$

This yields the pulse transfer function

$$H(z) = 0.5[1 + z^{-1}]$$

The steady-state response is

$$\begin{aligned} H(e^{j2\pi r}) &= 0.5[1 + e^{-j2\pi r}] \\ &= 0.5[e^{j\pi r} + e^{-j\pi r}]e^{-j\pi r} \\ &= \cos \pi r \, e^{-j\pi r} \end{aligned}$$

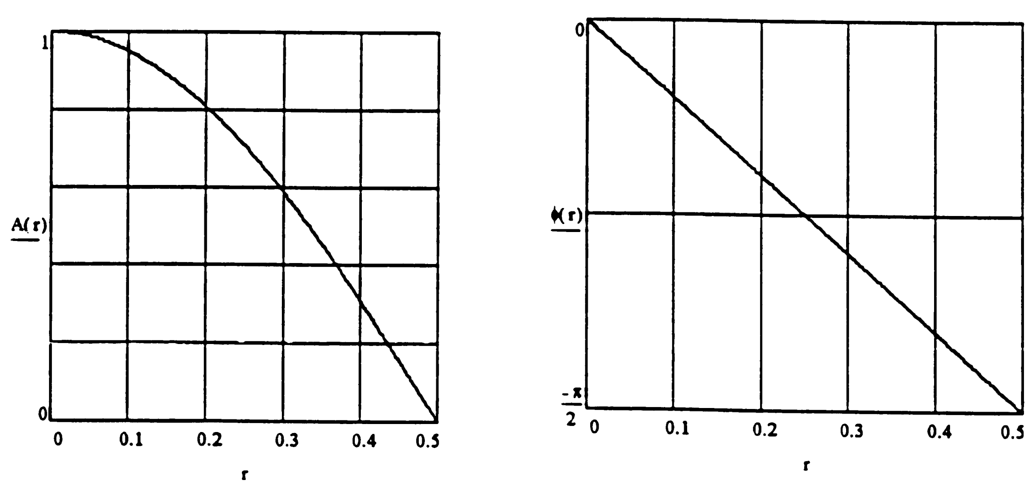
The amplitude response is

$$A(r) = \cos \pi r, \quad 0 \le r \le 0.5$$

and the phase response is

$$\phi(r) = -\pi r, \quad 0 \le r \le 0.5$$

These are shown below.



Problem 9-22

$$y(nT) = 0.25[x(nT) + x(nT - T) + x(nT - 2T) + x(nT - 3T)]$$

The pulse transfer function is

$$H(z) = 0.25\left[1 + z^{-1} + z^{-2} + z^{-3}\right]$$
$$= 0.25\left[z^{3/2} + z^{1/2} + z^{-1/2} + z^{-3/2}\right]z^{-3/2}$$

The yields the steady-state frequency response

$$H\left(e^{j2\pi r}\right) = 0.25\left[e^{j3\pi r} + e^{j\pi r} + e^{-j\pi r} + e^{-j3\pi r}\right] \cdot e^{-j3\pi r}$$

This should be grouped as shown

$$H\left(e^{j2\pi r}\right) = 0.25\left[\left(e^{j3\pi r} + e^{-j3\pi r}\right) + \left(e^{j\pi r} + e^{-j\pi r}\right)\right] \cdot e^{-j3\pi r}$$

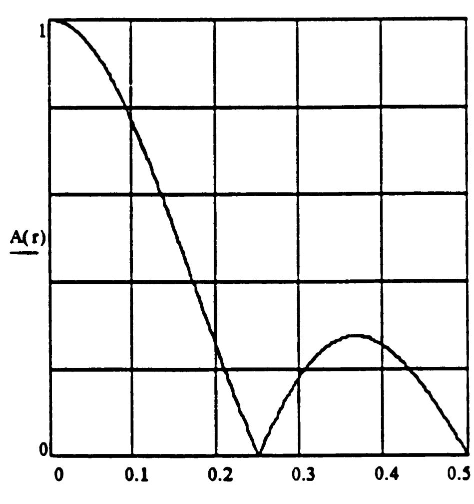
or

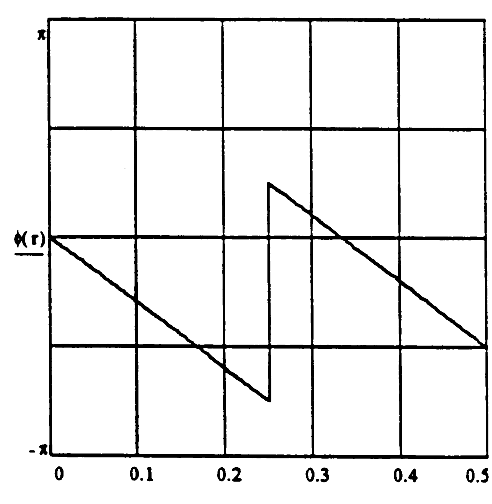
$$H\left(e^{j2\pi r}\right) = 0.5\,[\cos\pi r + \cos 3\pi r]\; e^{-j3\pi r}$$

Thus the amplitude response is

$$A(r) = 0.5\,|\cos\pi r + \cos 3\pi r|, \quad 0 \le r \le 0.5$$

Note that $\cos \pi r + \cos 3\pi r$ is zero at $r = 1/4$, and therefore changes sign at $r = 1/4$. This is reflected in a π radian phase shift in $\phi(r)$ at $r = 1/4$.





Problem 9-23

The sinusoidal steady-state frequency response of the given system is defined by

$$H\left(e^{j2\pi r}\right)=\frac{1}{N}\sum_{k=0}^{N-1}e^{-j2\pi kr}$$

Using the summation formula

$$\sum_{n=0}^{N-1}x^n=\frac{1-x^N}{1-x}$$

yields

$$H\left(e^{j2\pi r}\right)=\frac{1}{N}\frac{1-e^{-j2\pi Nr}}{1-e^{-j2\pi r}}$$

which can be written

$$H\left(e^{j2\pi r}\right)=\frac{1}{N}\frac{\left(e^{j\pi Nr}-e^{-j\pi Nr}\right)e^{-j\pi Nr}}{\left(e^{j\pi r}-e^{-j\pi r}\right)e^{-j\pi r}}$$

The amplitude response is therefore given by

$$A(r)=\frac{1}{N}\left|\frac{\sin \pi Nr}{\sin \pi r}\right|$$

Problem 9-24

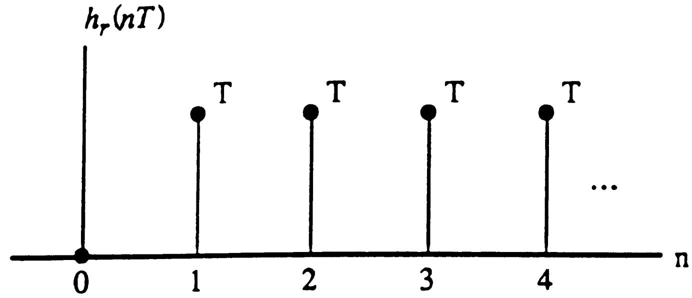
From (9-22) we see that the pulse transfer function of a rectangular integrator is

$$H_r(z)=\frac{T\,z^{-1}}{1-z^{-1}}$$

Thus the unit-pulse response of a rectangular integrator is

$$h_r(nT)=T\,u(n-1)$$

as shown at the top of the following page



The pulse transfer function of a trapezoidal integrator is

$$H_t(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

Thus

$$\frac{H_t(z)}{z} = \frac{T}{2}\frac{z+1}{z(z-1)} = \frac{A}{2} + \frac{B}{z-1}$$

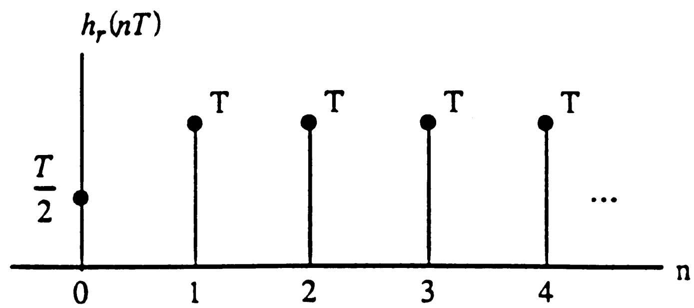
The residues are

$$A = -\frac{T}{2}, \quad B = T$$

so that the unit-pulse response is

$$h_t(nT) = -\frac{T}{2}\delta(n) + T\,u(n)$$

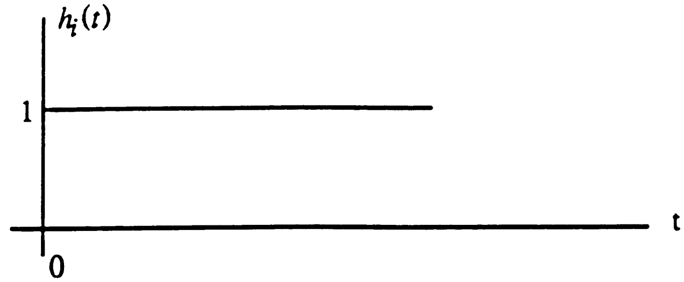
The unit-pulse response of the trapezoidal integrator is



The ideal analog integrator has the unit pulse response

$$h_i(t) = u(t)$$

as shown



We see that for $n > 0$, both the rectangular integrator and the trapezoidal integrator have a unit-pulse response corresponding to samples of the ideal unit step scaled by the sampling period T. The rectangular integrator clearly has a T second group delay as we saw in Section 9-3. The trapezoidal integrator has a non-zero response at $n = 0$, and this one sample is the only difference, in the time-domain, between rectangular and trapezoidal integration.

Problem 9-25

For an ideal analog integrator $H_a(s) = \frac{1}{s}$. The transform of the input is

$$X_a(s) = \frac{A}{s+\alpha}$$

so that the output of the ideal integrator is

$$Y_a(s) = H_a(s)X_a(s) = \frac{A}{s(s+\alpha)} = \frac{K_1}{s} + \frac{K_2}{s+\alpha}$$

The values of K_1 and K_2 are given by

$$K_1 = \frac{A}{\alpha}, \qquad K_2 = -\frac{A}{\alpha}$$

Thus, $y_a(t)$, the continuous-time output signal, is given by

$$y_a(t) = \frac{A}{\alpha}\left(1 - e^{-\alpha t}\right) u(t)$$

Note that the final value of the output is

$$y_a(\infty) = \frac{A}{\alpha}$$

We now consider the discrete-time integrators. The z-transform of the given sampled input signal is

$$X(z) = \frac{A}{1 - e^{-\alpha T} z^{-1}}$$

For the rectangular integrator

$$H_r(z) = \frac{T\, z^{-1}}{1 - z^{-1}}$$

so that the z-transform of the rectangular integrator output is

$$Y_r(z) = \frac{A}{1 - e^{-\alpha T} z^{-1}} \frac{T\, z^{-1}}{1 - z^{-1}} = \frac{A\, T\, z}{(z-1)\left(1 - e^{-\alpha T} z^{-1}\right)}$$

Thus

$$\frac{Y_r(z)}{z} = \frac{A\, T}{(z-1)\left(z - e^{-\alpha T}\right)} = \frac{K_1}{z-1} + \frac{K_2}{z - e^{-\alpha T}}$$

where

$$K_1 = \frac{A\, T}{1 - e^{-\alpha T}}, \qquad K_2 = -\frac{A\, T}{1 - e^{-\alpha T}}$$

This gives

$$Y_r(z) = \frac{A\, T}{1 - e^{-\alpha T}} \left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-\alpha T} z^{-1}}\right]$$

The sample values are therefore given by

$$y_r(nT) = \frac{A\, T}{1 - e^{-\alpha T}} \left[1 - e^{-\alpha n T}\right], \qquad n \geq 0$$

Note that the final value is

$$y_r(\infty) = \frac{A\, T}{1 - e^{-\alpha T}}$$

Comparing $y_a(t)$ and $y_r(nT)$ shows that, except for a scale change, $y_r(nT)$ is the sampled version of $y_a(t)$. Note that for T small (large sampling frequency) $\alpha T << 1$ and

$$1-e^{-\alpha T} \approx 1-(1-\alpha T)=\alpha T$$

so that

$$\frac{A\,T}{1-e^{-\alpha T}} \approx \frac{AT}{\alpha T}=\frac{A}{\alpha},\ \ \alpha T << 1$$

Thus the scale change becomes negligible as $T \to 0$ or $f_s \to \infty$.

For the trapezoidal integrator

$$H_t(z)=\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

and

$$Y_t(z)=H_t(z)\,X(z)=\frac{AT}{2}\frac{1}{1-e^{-\alpha T}z^{-1}}\frac{1+z^{-1}}{1-z^{-1}}$$

which gives

$$\frac{Y_t(z)}{z}=\frac{AT}{2}\frac{1}{z-e^{-\alpha T}}\frac{z+1}{z-1}=\frac{K_1}{z-e^{-\alpha T}}+\frac{K_2}{z-1}$$

The values of K_1 and K_2 are given by

$$K_1=-\frac{AT}{2}\frac{1+e^{-\alpha T}}{1-e^{-\alpha T}},\ K_2=\frac{AT}{2}\frac{2}{1-e^{-\alpha T}}$$

Thus

$$Y_t(z)=\frac{AT}{1-e^{-\alpha T}}\left[\frac{1}{1-z^{-1}}-\frac{1}{2}\left(1+e^{-\alpha T}\right)\frac{1}{1-e^{\alpha T}z^{-1}}\right]$$

The sample values are given by

$$y_t(nT)=\frac{AT}{1-e^{-\alpha T}}\left[1-\frac{1}{2}\left(1+e^{-\alpha T}\right)e^{-\alpha nT}\right],\ \ n\ge 0$$

Note that for $\alpha T << 1$

$$\frac{1}{2}\left(1+e^{-\alpha T}\right)\approx\frac{1}{2}(1+(1-\alpha T))=1-\frac{\alpha T}{2}\approx 1$$

Thus, for T small (f_s large) $y_t(nT) \approx y_r(nT)$ which is also $y_a(t)$.

We now plot the results using MathCAD.

First we define the values of A and α. For example consider the following values:

$$A := 5 \qquad \alpha := 2$$

The continuous x(t) is

$$x(t) := A\cdot \exp(-\alpha\cdot t)$$

and the continuous-time integrator output is

$$y(t) := \int_0^t x(t)\, dt$$

We now consider the discrete-time integrators. First a sampling period must be defined. For this example let

$$T := 0.1$$

It is convenient to let the discrete-time samples be represented using vector notation. We shall compute 101 samples. The samples at k=0 will be represented by initial conditions. Thus the iteration index k is defined as

$$k := 1, 2 .. 100$$

The sample values for the integrator input is defined by

$$x_k := A\cdot \exp(-\alpha\cdot T)^k$$

The initial conditions are defined by,

$$x_0 := A \qquad yr_0 := 0 \qquad yt_0 := 0.5\cdot T\cdot x_0$$

where yr represents the output of the rectangular integrator and yt represents the output of the trapezoidal integrator. The rectangular integrator is represented by the difference equation

$$yr_k := yr_{k-1} + T\cdot x_{k-1}$$

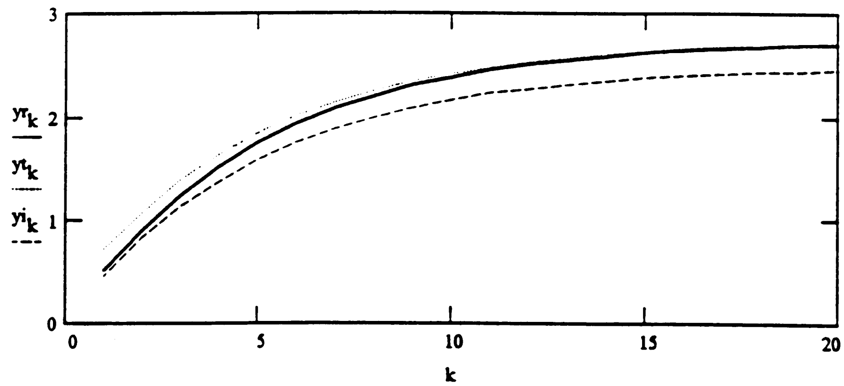
and the trapezoidal integrator is defined by the difference equation

$$yt_k := yt_{k-1} + T\cdot x_{k-1} + 0.5\cdot T\cdot \left(x_k - x_{k-1}\right)$$

We now determine the outputs of the three integrators (ideal, rectangular and trapezoidal). The vector representation of the ideal integrator output is defined by

$$yi_k := y(k\cdot T)$$

The outputs are shown below.



As can from the legend, the response of the trapezoidal integrator is the top curve, the response of the rectangular integrator is the middle curve and the ideal integrator output is the bottom curve. It is clear that the two discrete-time integrators approach a different (higher) asymptote than does the output of the ideal integrator. We have shown that this must be the case. The approximate values of the asymptotes are

$yi_{100} = 2.5$ $\qquad yr_{100} = 2.758$ $\qquad yt_{100} = 2.758$

We now show that these are the correct values. For the ideal integrator output the asymptote is

$$yai := \frac{A}{\alpha}$$

which gives

$$yai = 2.5$$

This is the correct value for the ideal integrator. The more interesting case is the discrete-time integrators. As we have shown, the asymptotes for both discrete-time integrators is

$$yad := \frac{A \cdot T}{1 - \exp(-\alpha \cdot T)}$$

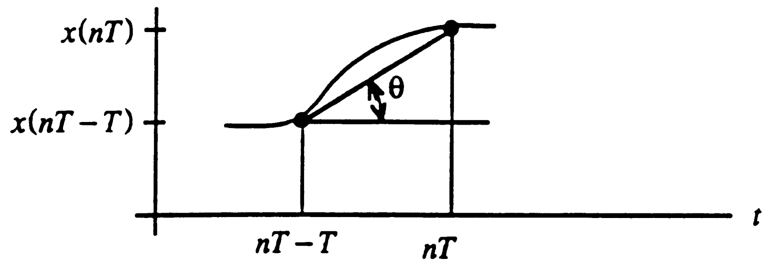
This evaluates to

$$yad = 2.758$$

which agrees with the plot.

Problem 9-26

(a) The differentiation rule is shown below



The slope is

$$\text{Slope} = \tan\theta = \frac{x(nT) - x(nT - T)}{T}$$

If the differentiator output is taken as the slope we have the difference equation

$$y(nT) = \frac{1}{T}\{x(nT) - x(nT - T)\}$$

(b) The pulse transfer function can be determined from the difference equation. This yields

$$Y(z) = \frac{1}{T}\left\{1 - z^{-1}\right\} X(z)$$

which gives the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{T}\left(1 - z^{-1}\right)$$

The sinusoidal steady-state frequency response is

$$H\left(e^{j2\pi r}\right) = \frac{1}{T}\left(1 - e^{-j2\pi r}\right) = \frac{1}{T}\left(e^{j\pi r} - e^{-j\pi r}\right)e^{-j\pi r}$$

which can be placed in the form

$$H\left(e^{j2\pi r}\right) = j\frac{2}{T}\sin\pi r \; e^{-j\pi r}$$

The amplitude response is

$$A(r) = \left|H\left(e^{j2\pi r}\right)\right| = \frac{2}{T}\sin\pi r, \quad 0 \le r \le \frac{1}{2}$$

and the phase response is

$$\phi(r) = \frac{\pi}{2} - \pi r,\ 0 \le r \le \frac{1}{2}$$

The ideal differentiator is defined by

$$H_i(j\omega) = j\omega$$

Since

$$2\pi r = 2\pi \frac{f}{f_s} = 2\pi f T = \omega T$$

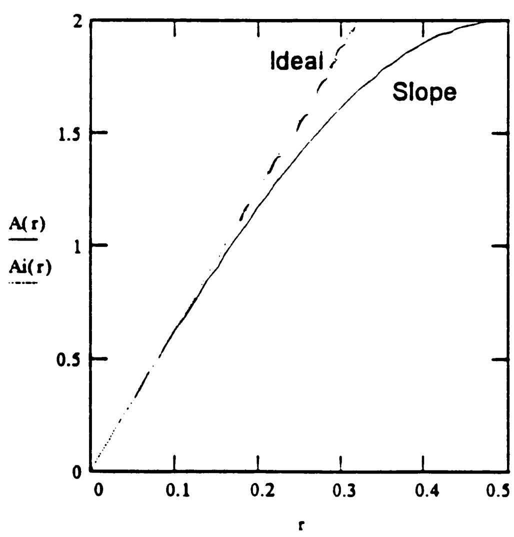
we can write

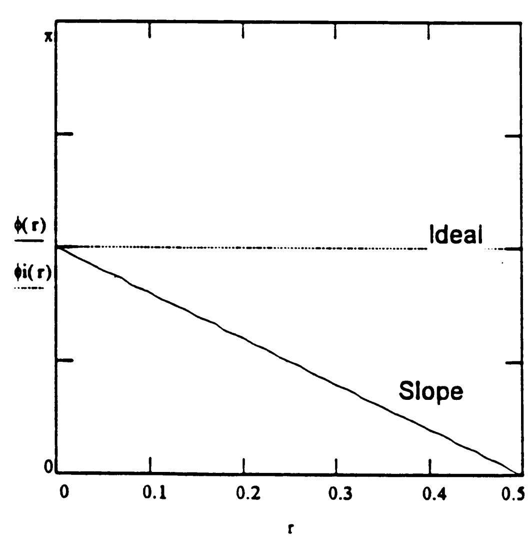
$$A_i(r) = \frac{2\pi r}{T}, \qquad r \ge 0$$

and the phase response of the ideal differentiator is given by

$$\phi_i(r) = \frac{\pi}{2}, \qquad r \ge 0$$

The slope differentiator and the ideal differentiator are compared below for $T = 1$.





We now examine $H\left(e^{j2\pi r}\right)$ for small r. Since

$$r = \frac{f}{f_s} = fT = \frac{\omega T}{2\pi}$$

we can write

$$H\left(e^{j2\pi r}\right) = j\frac{2}{T}\sin\frac{\omega T}{2}\ e^{-j\omega T/2}$$

Since sin $(x) \approx x$ for small x the preceding expression becomes

$$\begin{aligned} H\left(e^{j2\pi r}\right) &= j\frac{2}{T}\frac{\omega T}{2}\ e^{-j\omega T/2} \\ &= j\omega\left(e^{j\omega T/2}\right) \end{aligned}$$

Except for the delay of $\frac{T}{2}$, which results from the phase shift that is the linear function of frequency, we see that slope differentiator is equivalent to the ideal differentiator for small ω. Note that there is no phase distortion since the difference between the actual phase response and the ideal phase response is a linear function of frequency.

Problem 9-27

The transfer function for an ideal integrator is defined by

$$H_a(s) = \frac{1}{s}$$

Thus

$$h_a(t) = u(t)$$

and, by definition

$$h_a(nT) = u(n)$$

The pulse transfer function of the impulse-invariant integrator is therefore

$$H_i(z) = \frac{T}{1 - z^{-1}}$$

We see that this is equivalent to rectangular integration without the one sample delay.

The step-invariant integerator is determined from

$$Y_a(s) = \frac{1}{s} H_a(s) = \frac{1}{s^2}$$

Thus

$$y_a(t) = t\,u(t)$$

and

$$y_a(nT) = nT\,u(n)$$

From Table 8-1 we see that the pulse transfer function is

$$H_s(z) = \left(1 - z^{-1}\right)\left[\frac{T z^{-1}}{\left(1 - z^{-1}\right)^2}\right] = \frac{T z^{-1}}{1 - z^{-1}}$$

We now develop the amplitude and phase responses using MathCAD.

First we define j: $j := \sqrt{-1}$

and then define z: $z(r) := \exp(j \cdot 2 \cdot \pi \cdot r)$

The transfer function of the impulse invariant integrator (with T=1) is:

$$Hi(z) := \frac{1}{1 - z^{-1}}$$

The transfer function of the step invariant integrator (with T=1) is:

$$Hs(z) := \frac{z^{-1}}{1 - z^{-1}}$$

The amplitude response (in dB) and the phase responses are defined as:

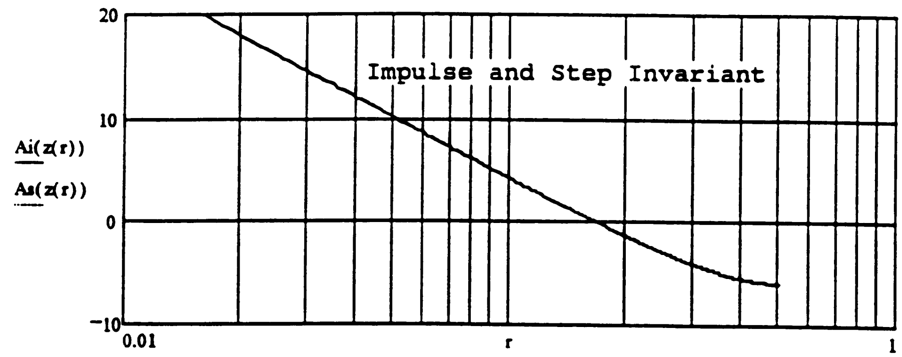
$Ai(z) := 20 \cdot \log(|Hi(z)|)$ $As(z) := 20 \cdot \log(|Hs(z)|)$

$\phi i(z) := \arg(Hi(z))$ $\phi s(z) := \arg(Hs(z))$

Define the range of r as (note that we bound ourselves away from f=0)

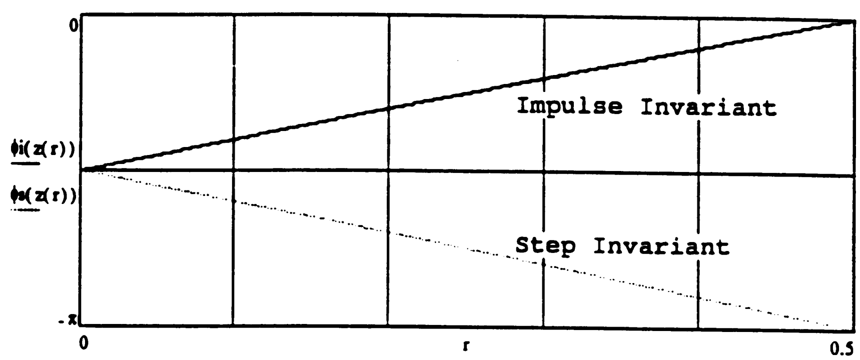
$r := 0.001, 0.002 .. 0.5$

The plot of the amplitude responses are:



Note that the two responses are identical. It should be remembered that the amplitude response of an integrator is infinite at f=0. We knew that the amplitude responses would be identical since the two responses differ only by phase term.

The phase response (on a linear scale) is:



The impulse invariant integrator has a phase response of 0 at r=0.5 while the step invariant integrator has a phase response of $-\pi$ at r=0.5. Note that both phase responses are linear.

Problem 9-28

The analog integrator is defined by

$$H_a(s) = \frac{1}{s}$$

The bilinear z-transform integrator is formed by substituting

$$C\frac{1-z^{-1}}{1+z^{-1}}$$

for s in $H_a(s)$ with $C = \frac{2}{T}$. Thus

$$H(z) = \frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

Comparison of this expression with (9-26) shows it to define trapezoidal integration.

Problem 9-29

For the ideal continuous-time differentiator

$$H_a(s) = s$$

Thus

$$Y_a(s) = \frac{1}{s}H_a(s) = 1$$

which has the inverse Laplace transform

$$y_a(t) = \delta(t)$$

The equivalent sampled sequence is $\delta(n)$ and has the z-transform 1. Thus the step-invariant differentiator has the transfer function

$$H(z) = 1 - z^{-1}$$

The sinusoidal steady-state frequency response is

$$H\left(e^{j2\pi r}\right) = \left(e^{j\pi r} - e^{-j\pi r}\right)e^{-j\pi r} = 2j\sin\pi r \ e^{-j\pi r}$$

Thus the amplitude response is

$$A(r) = 2\sin\pi r, \qquad 0 \le r \le 0.5$$

and the phase response is

$$\phi(r) = \frac{\pi}{2} - \pi r, \; 0 \le r \le 0.5$$

Except for the scaling by the sampling period T, the amplitude response is identical to the amplitude response of the step differentiator developed in Problem 9-26. The phase responses are also identical. Thus the amplitude and phase responses need not be redrawn here.

Problem 9-30

(a)

$$H_a(s) = \frac{8}{(s+2)(s+4)} = \frac{A}{s+2} = \frac{B}{s+4}$$

The residues are

$$A = 4, \qquad B = -4$$

Thus, the sampled unit pulse response is

$$h_a(nT) = \left[4e^{-2nT} - 4e^{-4nT}\right]u(n)$$

This gives the pulse impulse invariant transfer function

$$H(z) = \frac{4T}{1-e^{-2T}z^{-1}} - \frac{4T}{1-e^{-4T}z^{-1}}$$

(b)

$$H_a(s) = \frac{8}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

The residues are

$$A = 1, \quad B = \frac{8}{(-2)(2)} = -2, \quad C = \frac{8}{(-4)(-2)} = 1$$

Thus, the sampled unit pulse response is

$$h_a(nT) = \left[1 - 2e^{-2nT} + e^{-4nT}\right]u(n)$$

This gives the impulse invariant pulse transfer function

$$H(z) = \frac{T}{1-z^{-1}} - 2\frac{T}{1-e^{-2T}z^{-1}} + \frac{T}{1-e^{-4T}z^{-1}}$$

(c)

$$H_a(s) = \frac{s+1}{(s+0.5)(s+4)} = \frac{A}{s+0.5} + \frac{B}{s+4}$$

The residues are

$$A = \frac{\frac{1}{2}}{\frac{7}{2}} = \frac{1}{7}, \quad B = \frac{-3}{-\frac{7}{2}} = \frac{6}{7}$$

Thus, the sampled unit impulse response is

$$h_a(nT) = \left[\frac{1}{7}e^{-\frac{1}{2}nT} + \frac{6}{7}e^{-4nT}\right]u(n)$$

This gives the impulse invariant transfer function

$$H(z) = \frac{1}{7}\frac{T}{1-e^{-\frac{1}{2}T}z^{-1}} + \frac{6}{7}\frac{T}{1-e^{-4T}z^{-1}}$$

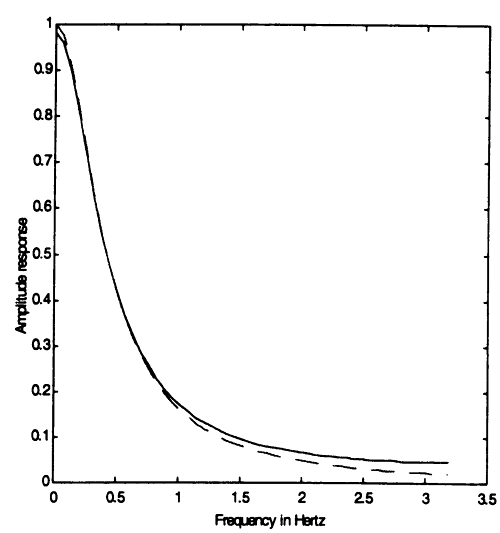
Problem 9-31

(a) For the first part of the problem we first develop the appropriate MATLAB m-file. The m-file shown below should be understandable with out additional comment.

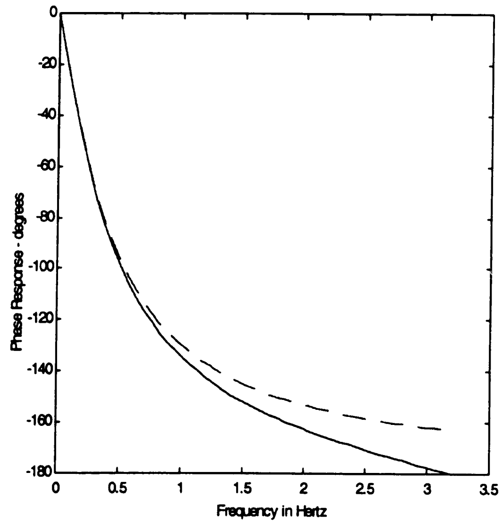
```
Ws = 40;                                % Sampling frequency in rad/s
fs = Ws/(2*pi);                         % Sampling frequency in Hertz
T = 1/fs;                               % Sampling period in seconds

r = 0:0.01:0.5;                         % Normalized frequency vector

% The following lines of code specify the filter
A = 4*T;
a = exp(-2*T);
B = -4*T;
b = exp(-4*T);
```

```
Hr1 = A./(1-a*exp(-j*2*pi*r));          % First term
Hr2 = B./(1-b*exp(-j*2*pi*r));          % Second term
Hd = Hr1+Hr2;                           % Filter response


b = [8];                                % Numerator vector
a = [1 6 8];                            % Denominator vector
Wa = Ws*r;                              % Frequency vector in rad/s
Ha = freqs(b,a,Wa);                     % Compute analog filter response

ampd = abs(Hd);                         % Digital amplitude response
ampa = abs(Ha);                         % Analog amplitude response
phid = (180/pi)*angle(Hd);              % Digital filter phase response
phia = (180/pi)*angle(Ha);              % Analog filter phase response
f = Wa/(2*pi);                          % Frequency vector in Hertz
plot(f,ampd,f,ampa,'--')                % Plot amplitude response
xlabel('Frequency in Hertz')            % Label x axis
ylabel('Amplitude response')            % Label y axis
pause                                   % Pause to look
plot(f,phid,f,phia,'--')                % Plot phase response
xlabel('Frequency in Hertz')            % Label x axis
ylabel('Phase Response - degrees')      % Label y axis
```

Executing the plot gives the following results for the amplitude response. The solid curve represents the digital filter and the dashed curve represents the analog filter.



The phase response is shown at the top of the following page.

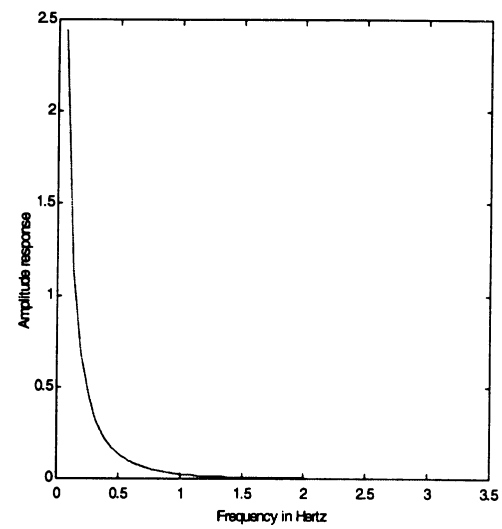


(b) For the (b) part of the problem we write the following MATLAB script, which is identical to the previous part of the problem except for the specification of the filter.

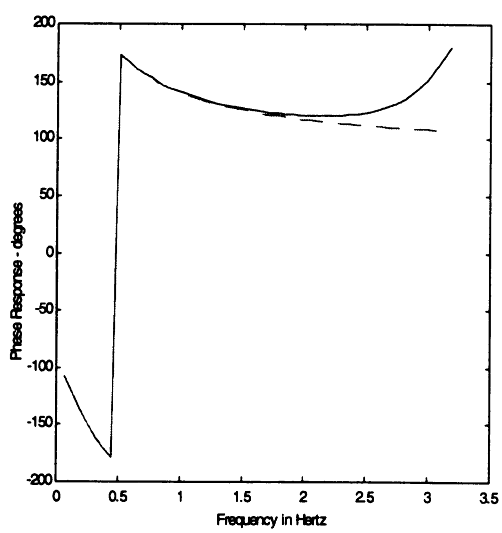
```
Ws = 40;                                % Sampling frequency in rad/s
fs = Ws/(2*pi);                         % Sampling frequency in Hertz
T = 1/fs;                               % Sampling period in seconds

r = 0:0.01:0.5;                         % Normalized frequency vector

% The following lines of code specify the filter
A = T;
a = 1;
B = -2*T;
b = exp(-2*T);
C = T;
c = exp(-4*T);

Hr1 = A./(1-a*exp(-j*2*pi*r));          % First term
Hr2 = B./(1-b*exp(-j*2*pi*r));          % Second term
Hr3 = C./(1-c*exp(-j*2*pi*r));          % Third term
Hd = Hr1+Hr2+Hr3;                       % Filter response


b = [8];                                % Numerator vector
a = [1 6 8 0];                          % Denominator vector
Wa = Ws*r;                              % Frequency vector in rad/s
Ha = freqs(b,a,Wa);                     % Compute analog filter response

ampd = abs(Hd);                         % Digital amplitude response
ampa = abs(Ha);                         % Analog amplitude response
phid = (180/pi)*angle(Hd);              % Digital filter phase response
phia = (180/pi)*angle(Ha);              % Analog filter phase response
f = Wa/(2*pi);                          % Frequency vector in Hertz
plot(f,ampd,f,ampa,'--')                % Plot amplitude response
```

```
xlabel('Frequency in Hertz')              % Label x axis
ylabel('Amplitude response')              % Label y axis
pause                                     % Pause to look
plot(f,phid,f,phia,'--')                  % Plot phase response
xlabel('Frequency in Hertz')              % Label x axis
ylabel('Phase Response - degrees')        % Label y axis
```

Executing the program yields the amplitude response



and the phase response

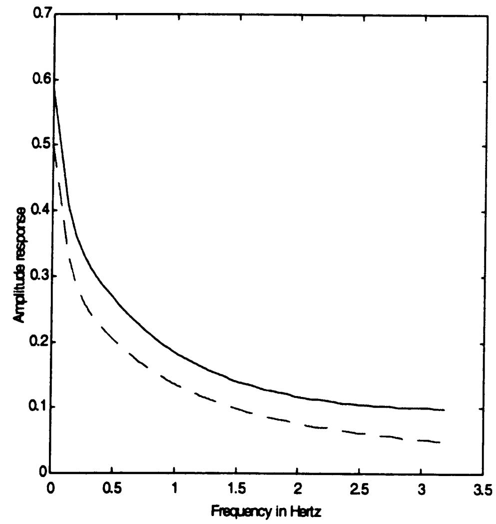


Note that once again, the solid curve represents the digital filter and the dashed curve represents the analog filter. Note also that agreement is so close that the difference between the amplitude responses of the analog and digital filters cannot be distinguished.

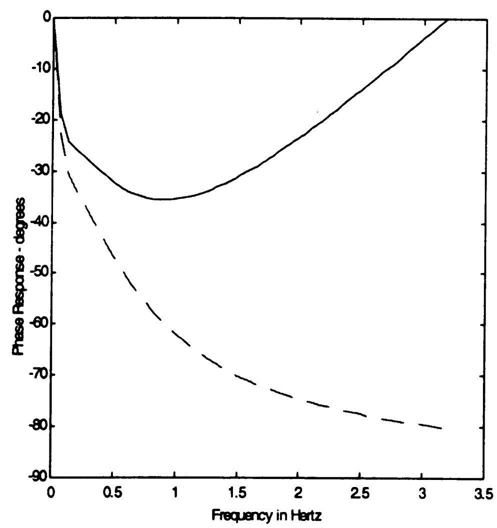
(c) For the third part of the problem, the following MATLAB script is written.

```
Ws = 40;                                % Sampling frequency in rad/s
fs = Ws/(2*pi);                         % Sampling frequency in Hertz
T = 1/fs;                               % Sampling period in seconds

r = 0:0.01:0.5;                         % Normalized frequency vector

% The following lines of code specify the filter
A = (1/7)*T;
a = exp(-(1/2)*T);
B = (6/7)*T;
b = exp(-4*T);

Hr1 = A./(1-a*exp(-j*2*pi*r));          % First term
Hr2 = B./(1-b*exp(-j*2*pi*r));          % Second term
Hd = Hr1+Hr2;                           % Filter response


b = [1 1];                              % Numerator vector
a = [1 4.5 2];                          % Denominator vector
Wa = Ws*r;                              % Frequency vector in rad/s
Ha = freqs(b,a,Wa);                     % Analog filter response

ampd = abs(Hd);                         % Digital amplitude response
ampa = abs(Ha);                         % Analog amplitude response
phid = (180/pi)*angle(Hd);              % Digital filter phase response
phia = (180/pi)*angle(Ha);              % Analog filter phase response
f = Wa/(2*pi);                          % Establish frequency vector
plot(f,ampd,f,ampa,'--')                % Plot amplitude response
xlabel('Frequency in Hertz')            % Label x axis
ylabel('Amplitude response')            % Label y axis
pause                                   % Pause to look
plot(f,phid,f,phia,'--')                % Plot phase response
xlabel('Frequency in Hertz')            % Label x axis
ylabel('Phase Response - degrees')      % Label y axis
```

We execute the preceding code to obtain the amplitude responses and the phase responses. Once again, the solid curve represents the digital filter and the dashed curve represents the analog filter. The amplitude responses are illustrated at the top of the following page.



The phase responses are shown below.



Problem 9-32

(a) First we form

$$\frac{H_a(s)}{s}=\frac{8}{s(s+2)(s+4)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+4}$$

where

$$A=1,\quad B=\frac{8}{-2(2)}=-2,\quad C=\frac{8}{-4(-2)}=1$$

Inverse transforming and sampling yields

$$1-2e^{-2nT}+e^{-4nT}$$

for the right-hand side of $\frac{H_a(s)}{s}$. Thus

$$H(z)=\left(1-z^{-1}\right)\left[\frac{1}{1-z^{-1}}-2\frac{1}{1-e^{-2T}z^{-1}}+\frac{1}{1-e^{-4T}z^{-1}}\right]$$

(b) First we form

$$\frac{H_a(s)}{s}=\frac{8}{s^2(s+2)(s+4)}=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s+2}+\frac{D}{s+4}$$

where

$$A=1,\quad C=\frac{8}{4(2)}=1,\quad D=\frac{8}{16(-2)}=-\frac{1}{4}$$

The value of B is given by

$$B=\frac{d}{ds}\left[\frac{8}{s^2+6s+8}\right]s=0$$

$$B=\frac{-8(6)}{64}=-\frac{3}{4}$$

Inverse transforming and sampling yields

$$nT-\frac{3}{4}+e^{-2nT}-\frac{1}{4}e^{-4nT}$$

for the right-hand side of $\frac{H_a(s)}{s}$. Thus

$$H(z) = (1 - z^{-1})\left[\frac{Tz^{-1}}{(1 - z^{-1})^2} - \frac{3}{4}\frac{1}{1 - z^{-1}} + \frac{1}{1 - e^{-2T}z^{-1}} - \frac{1}{4}\frac{1}{1 - e^{-4T}z^{-1}}\right]$$

(c) First we form

$$\frac{H_a(s)}{s} = \frac{s+1}{s(s+0.5)(s+4)} = \frac{A}{s} + \frac{B}{s+0.5} + \frac{C}{s+4}$$

where

$$A = \frac{1}{2},\quad B = \frac{0.5}{-0.5(3.5)} = -\frac{2}{7},\quad C = \frac{-3}{-4(-3.5)} = -\frac{3}{14}$$

Inverse transforming and sampling yields

$$\frac{1}{2} - \frac{2}{7}e^{-0.5nT} - \frac{3}{14}e^{-4nT}$$

for the right-hand side of $\frac{H_a(s)}{s}$. Thus

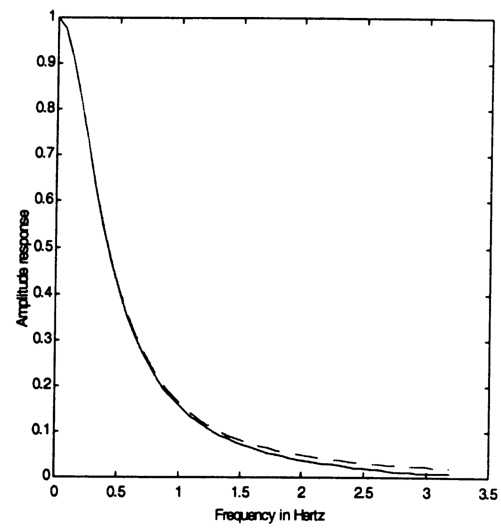
$$H(z) = (1 - z^{-1})\left[\frac{1}{2}\frac{1}{1 - z^{-1}} - \frac{2}{7}\frac{1}{1 - e^{-0.5T}z^{-1}} - \frac{3}{14}\frac{1}{1 - e^{-4T}z^{-1}}\right]$$

The amplitude and phase responses may be plotted using the same MATLAB programs used for Problem 9-31. Only the specification of the digital filter changes. Thus the MATLAB code given in the following sections may simply be substituted for the corresponding lines of code in the solutions for Problem 9-31.

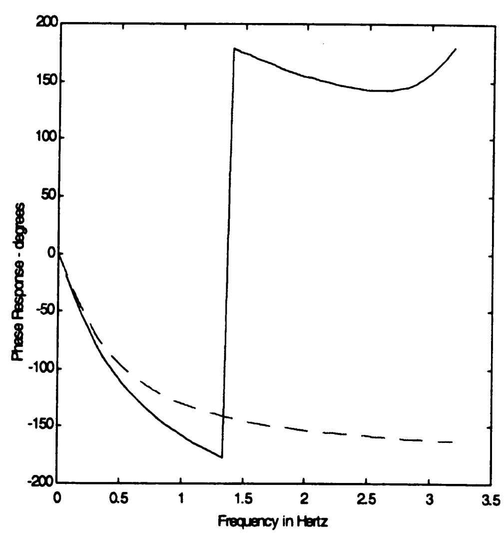
(a) For the first part of the problem the filter specification is as follows:

```
% The following lines of code specify the filter
A = 1;
a = 1;
B = -2;
b = exp(-2*T);
C = 1;
c = exp(-4*T);

Hr1 = A./(1-a*exp(-j*2*pi*r));          % First term
Hr2 = B./(1-b*exp(-j*2*pi*r));          % Second term
Hr3 = C./(1-c*exp(-j*2*pi*r));          % Third term
Hd = (1-exp(-j*2*pi*r)).*(Hr1+Hr2+Hr3); % Filter response
```

Executing the resulting MATLAB program yields the amplitude response and the phase response. The amplitude response is given below. As in the previous problem the solid curve represents the digital filter and the dashed curve represents the analog filter.



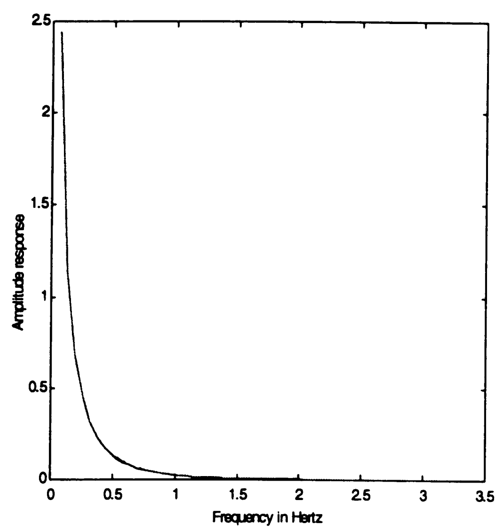
The phase response is as follows.



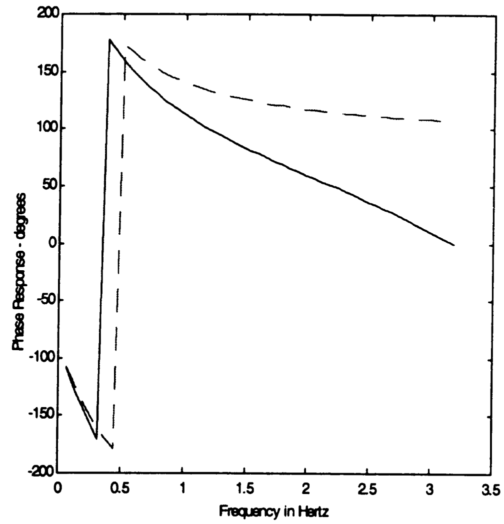
(b) For the second part of the problem we use the following MATLAB code to specify the filter.

```
% The following lines of code specify the filter
A = 1*T;
a = 1;
B = -3/4;
b = 1;
C = 1;
c = exp(-2*T);
D = -1/4;
d = exp(-4*T);

Hr1n = (A*exp(-j*2*pi*r));              % Numerator of first term
Hr1d = (1-a*exp(-j*2*pi*r)).^2;         % Denominator of first term
Hr1 = Hr1n./Hr1d;                       % First term
Hr2 = B./(1-b*exp(-j*2*pi*r));          % Second term
Hr3 = C./(1-c*exp(-j*2*pi*r));          % Third term
Hr4 = D./(1-d*exp(-j*2*pi*r));          % Fourth term
Hd1 = (1-exp(-j*2*pi*r));               % Multiplier
Hd = Hd1.*(Hr1+Hr2+Hr3+Hr4);            % Filter response
```

Executing the MATLAB problem yields the results shown below for the amplitude response. Note that the difference between the amplitude response of the analog filter and the amplitude response of the digital filter cannot be observed given the scaling of the plot.



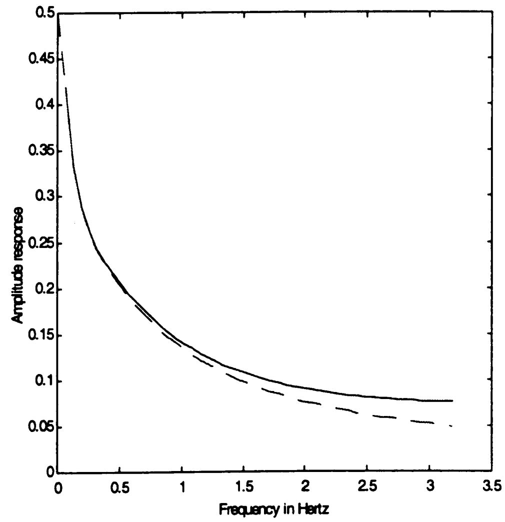
The phase response is shown at the top of the following page.



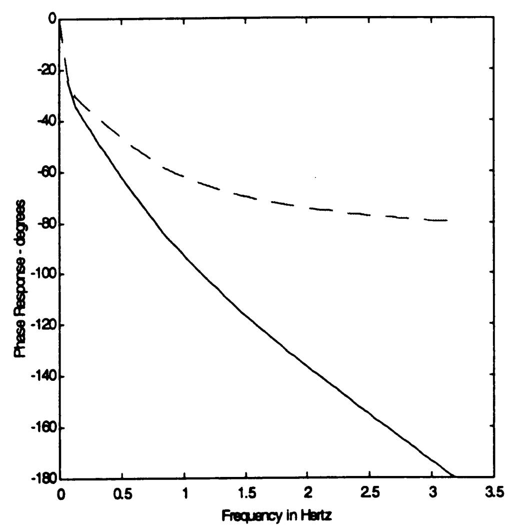
(c) For the last part of the problem, the following MATLAB code is used for the filter specification.

```
 % The following lines of code specify the filter
A = 1/2;
a = 1;
B = -2/7;
b = exp(-0.5*T);
C = -3/14;
c = exp(-4*T);

Hr1 = A./(1-a*exp(-j*2*pi*r));          % First term
Hr2 = B./(1-b*exp(-j*2*pi*r));          % Second term
Hr3 = C./(1-c*exp(-j*2*pi*r));          % Third term
Hd = (1-exp(-j*2*pi*r)).*(Hr1+Hr2+Hr3); % Filter response
```

Executing the resulting code yields the amplitude responses shown on the following page.



Finally, the phase responses are illustrated below.



Problem 9-33

The analog transfer function is first expanded as

$$H_a(s) = \frac{20}{(s+0.5)(s+2)(s+20)} = \frac{A}{s+0.5} + \frac{B}{s+2} + \frac{C}{s+20}$$

The residues are

$$A = \frac{20(2)(2)}{3(39)} = \frac{80}{117}$$

$$B = -\frac{20(2)}{3(18)} = -\frac{20}{27}$$

and

$$C = \frac{20(2)}{(39)(18)} = \frac{20}{351}$$

Thus

$$H_a(s) = \frac{80}{117}\frac{1}{s+0.5} - \frac{20}{27}\frac{1}{s+2} + \frac{20}{351}\frac{1}{s+20}$$

and the sampled unit pulse response is

$$h_a(nT) = \left[\frac{80}{117}e^{-0.5nT} - \frac{20}{27}e^{-2nT} + \frac{20}{351}e^{-20nT}\right]u(n)$$

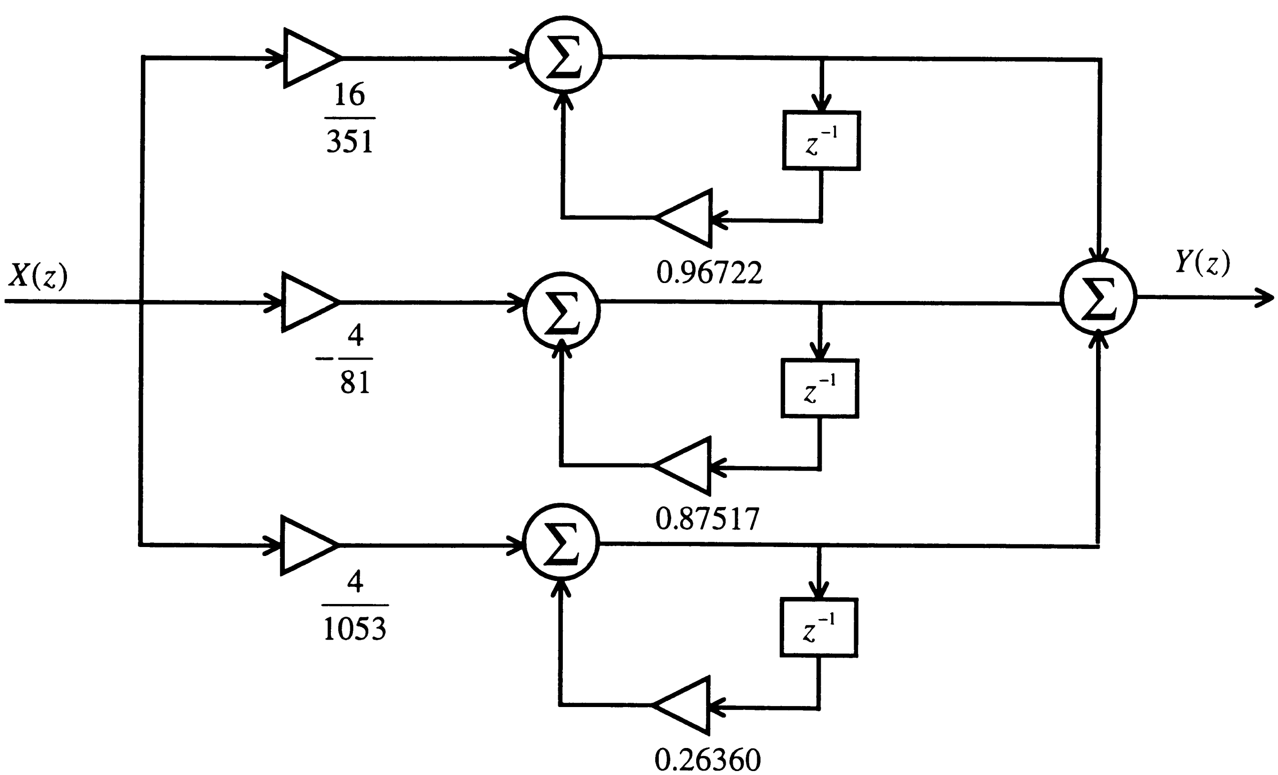
With $f_s = 15$, $T = \frac{1}{15}$ and

$$e^{-0.5T} = 0.96722,\ e^{-2T} = 0.87517,\ e^{-20T} = 0.26360$$

The pulse transfer function is therefore given by, after multiplying by $T = \frac{1}{15}$

$$H(z) = \frac{16}{351}\left(\frac{1}{1-0.96722z^{-1}}\right) - \frac{4}{81}\left(\frac{1}{1-0.87517z^{-1}}\right) + \frac{4}{1053}\left(\frac{1}{1-0.26360z^{-1}}\right)$$

The parallel realization of the filter is

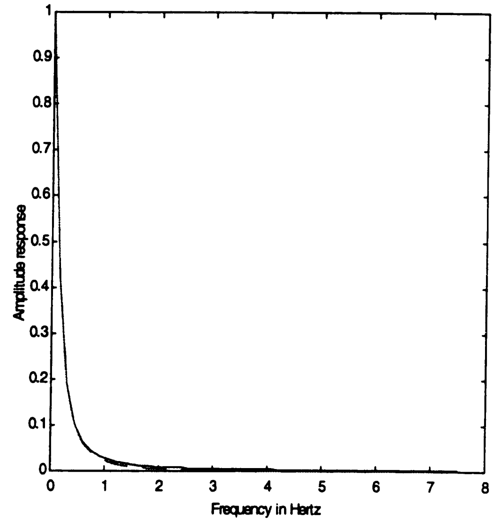


Problem 9-34

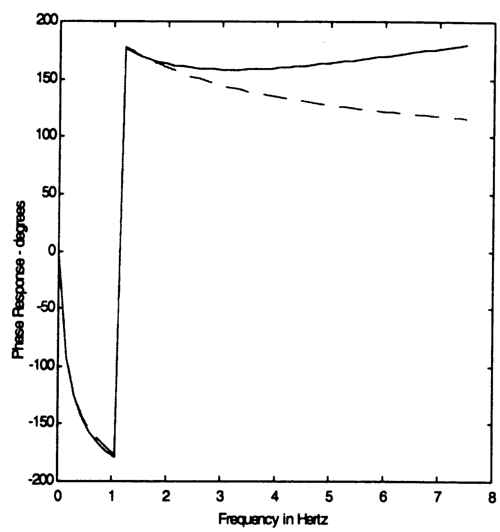
The MATLAB program used to plot the amplitude and phase responses of the analog and digital filters is essentially the same program used in Problems 9-31 and 9-32. Only the filter specification is changed. The script file to plot the amplitude response and the phase response of the digital filter found in Problem 9-34 is shown below.

```
fs = 15;                                % Sampling frequency in Hertz
T = 1/fs;                               % Sampling period in seconds
Ws = 2*pi*fs;                           % Sampling frequency in rad/s
r = 0:0.01:0.5;                         % Normalized frequency vector

% The following lines of code specify the filter
A = (80/117)*T;
a = 0.96722;
B = (-20/27)*T;
b = 0.87517;
C = (20/351)*T;
c = 0.26360;

Hr1 = A./(1-a*exp(-j*2*pi*r));          % First term
Hr2 = B./(1-b*exp(-j*2*pi*r));          % Second term
Hr3 = C./(1-c*exp(-j*2*pi*r));          % Third term
Hd = Hr1+Hr2;                           % Filter response


b = [20];                               % Numerator vector
a = [1 22.5 51 20];                            % Denominator vector
Wa = Ws*r;                              % Frequency vector in rad/s
Ha = freqs(b,a,Wa);                     % Compute analog freq. response

ampd = abs(Hd);                         % Digital filter amp. response
ampa = abs(Ha);                         % Analog filter amp. response
phid = (180/pi)*angle(Hd);              % Digital filter phase response
phia = (180/pi)*angle(Ha);              % Analog filter phase response
f = Wa/(2*pi);                          % Frequency vector in Hertz
plot(f,ampd,f,ampa,'--')                % Plot amplitude response
xlabel('Frequency in Hertz')            % Label x axis
ylabel('Amplitude response')            % Label y axis
pause                                   % Pause to look
plot(f,phid,f,phia,'--')                % Plot phase response
xlabel('Frequency in Hertz')            % Label x axis
ylabel('Phase Response - degrees')      % Label y axis
```

Executing the program generates the amplitude response and the phase response. The amplitude response is shown at the top of the next page.



The digital filter is designated by the solid curve and the analog filter is designated by the dashed curve. For the amplitude response, the difference between the digital and the analog filters cannot be distinguished. The phase responses are shown below.



Note that the results are in very close agreement over most of the frequency range.

Problem 9-35

First we write

$$\frac{H_a(s)}{s} = \frac{20}{s(s+0.5)(s+2)(s+20)} = \frac{A}{s} + \frac{B}{s+0.5} + \frac{C}{s+2} + \frac{D}{s+20}$$

where

$$A = 1, \quad B = \frac{20}{-0.5\left(\frac{3}{2}\right)\left(\frac{39}{2}\right)} = -\frac{160}{117}, \quad C = \frac{20}{-2\left(-\frac{3}{2}\right)(18)} = \frac{10}{27}$$

and

$$D = \frac{20}{-20\left(-\frac{39}{2}\right)(-18)} = -\frac{1}{351}$$

Inverse tranforming and sampling yields

$$1 - \frac{160}{117}e^{-0.5nT} + \frac{10}{27}e^{-2nT} - \frac{1}{351}e^{-20nT}$$

It therefore follows that the step-invariant digital filter is defined by

$$H(z) = \left(1 - z^{-1}\right)\left[\frac{1}{1-z^{-1}} - \frac{160}{117}\frac{1}{1-e^{-0.5T}z^{-1}} + \frac{10}{27}\frac{1}{1-e^{-2T}z^{-1}} - \frac{1}{351}\frac{1}{1-e^{-20T}z^{-1}}\right]$$

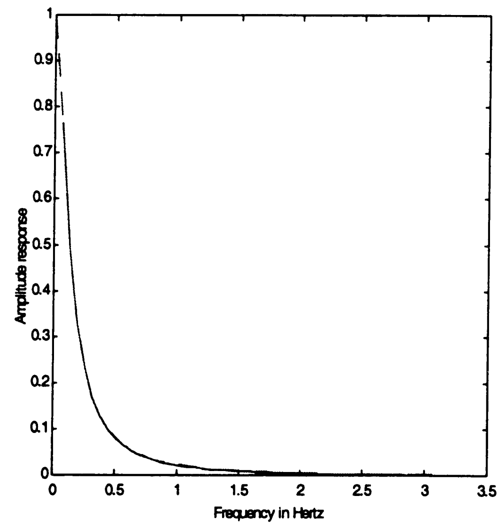
With $T = \frac{1}{15}$ we have

$$H(z) = \left(1 - z^{-1}\right)\left[\frac{1}{1-z^{-1}} - \frac{160}{117}\frac{1}{1-0.96722z^{-1}} + \frac{10}{27}\frac{1}{1-0.87517z^{-1}} - \frac{1}{351}\frac{1}{1-0.26360z^{-1}}\right]$$

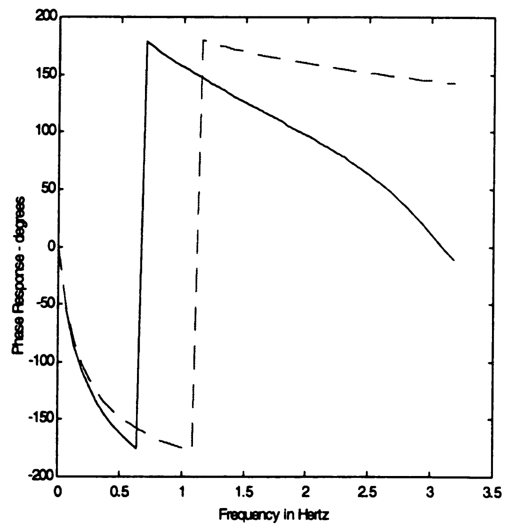
The MATLAB script for solving this problem is shown below.

```
Ws = 40;                              % Sampling frequency in rad/s
fs = Ws/(2*pi);                       % Sampling frequency in Hertz
T = 1/fs;                             % Sampling period in seconds

r = 0:0.01:0.5;                       % Normalized frequency vector

% The following lines of code specify the filter
A = 1;
a = 1;
B = -160/117;
b = exp(-0.5*T);
C = 10/27;
c = exp(-2*T);
D = -1/351;
d = exp(-20*T);

Hr1 = A./(1-a*exp(-j*2*pi*r));             % First term
Hr2 = B./(1-b*exp(-j*2*pi*r));             % Second term
Hr3 = C./(1-c*exp(-j*2*pi*r));             % Third term
Hr4 = D./(1-d*exp(-j*3*pi*r));             % Fourth term
Hdm = (1-exp(-j*2*pi*r));                  % Multiplier
Hd = Hdm.*(Hr1+Hr2+Hr3+Hr4);               % Filter response


b = [20];                             % Numerator vector
a = [1 22.5 51 20];                   % Denominator vector
Wa = Ws*r;                            % Frequency vector in rad/s
Ha = freqs(b,a,Wa);                   % Analog frequency response

ampd = abs(Hd);                       % Digital amplitude response
ampa = abs(Ha);                       % Analog amplitude response
phid = (180/pi)*angle(Hd);            % Digital filter phase response
phia = (180/pi)*angle(Ha);            % Analog filter phase response
f = Wa/(2*pi);                        % Frequency vector in Hertz
plot(f,ampd,f,ampa,'--')              % Plot amplitude response
xlabel('Frequency in Hertz')          % Label x axis
ylabel('Amplitude response')          % Label y axis
pause                                 % Pause to look
plot(f,phid,f,phia,'--')              % Plot phase response
xlabel('Frequency in Hertz')          % Label x axis
ylabel('Phase Response - degrees')    % Label y axis
```

Executing the code gives the amplitude response shown below.



After the pause, the phase response is computed and displayed as shown below.



Problem 9-36

For the impulse-invariant filter we write

$$H_a(s) = \frac{10(s+1)}{(s+2)(s+6)} = \frac{A}{s+2} + \frac{B}{s+6}$$

where

$$A = \frac{10(-1)}{4} = -\frac{5}{2}, \quad B = \frac{10(-5)}{-4} = \frac{25}{2}$$

Thus

$$H_a(s) = -\frac{5}{2}\frac{1}{s+2} + \frac{25}{2}\frac{1}{s+6}$$

The z-domain transfer function is therefore given by

$$H(z) = -\frac{5}{2}\frac{T}{1-e^{-2T}z^{-1}} + \frac{25}{2}\frac{T}{1-e^{-6T}z^{-1}}$$

For the step-invariant filter we write

$$\frac{H_a(s)}{s} = \frac{10(s+1)}{s(s+2)(s+6)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+6}$$

where

$$A = \frac{5}{6}, \quad B = \frac{10(-1)}{(-2)(4)} = \frac{5}{4}, \quad C = \frac{10(-5)}{(-6)(-4)} = -\frac{25}{12}$$

Thus

$$\frac{H_a(s)}{s} = \frac{5}{6} + \frac{5}{4}\frac{1}{s+2} - \frac{25}{12}\frac{1}{s+6}$$

The pulse transfer function of the step-invariant filter is therefore given by

$$H(z) = \left(1-z^{-1}\right)\left[\frac{5}{6}\frac{1}{1-z^{-1}} + \frac{5}{4}\frac{1}{1-e^{-2T}z^{-1}} - \frac{25}{12}\frac{1}{1-e^{-6T}z^{-1}}\right]$$

Problem 9-37

(a) For

$$H_a(s) = \frac{5}{s+5}$$

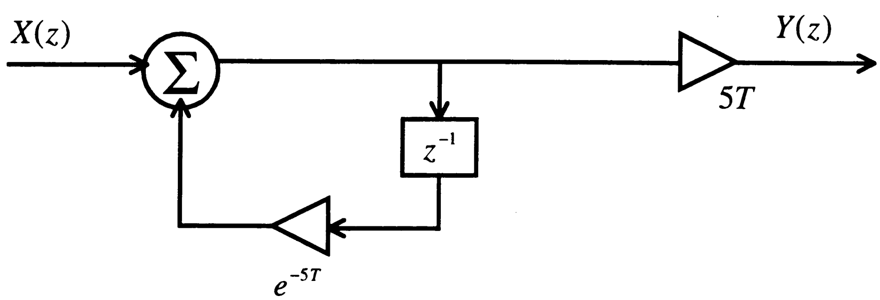
The sample unit impulse response is

$$h_a(nT) = 5e^{-5nT}u(n)$$

The corresponding pulse transfer function is

$$H(z) = \frac{5T}{1 - e^{-5T}z^{-1}}$$

The Direct Form II realization of the impulse-invariant filter is



The step response of the given filter is

$$Y_a(s) = \frac{1}{s}H_a(s) = \frac{5}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

The values of A and B are

$$A = 1, \quad B = -1$$

So that $y_a(nT)$ is given by

$$y_a(nT) = \left(1 - e^{-5nT}\right)u(n)$$

which has the z-transform

$$Y(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-5T}z^{-1}}$$

The step-invariant digital filter is then defined by

$$H(z) = \left(1 - z^{-1}\right)Y(z) = 1 - \frac{1 - z^{-1}}{1 - e^{-5T}z^{-1}}$$

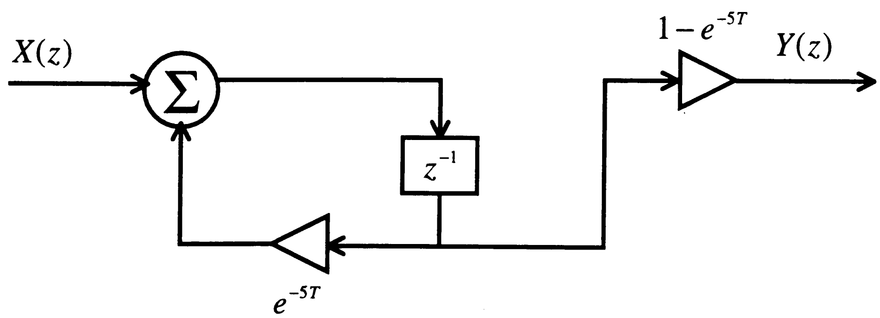
In order to show the Direct Form II realization $H(z)$ must be expressed as a ratio of polynomials in z^{-1}. Thus

$$H(z)=\frac{1-e^{-5T}z^{-1}-1+z^{-1}}{1-e^{-5T}z^{-1}}$$

which is

$$H(z)=\frac{\left(1-e^{-5T}\right)z^{-1}}{1-e^{-5T}z^{-1}}$$

The Direct Form II realization is



(b) From the problem statement

$$H_a(s)=\frac{5}{s(s+5)}=\frac{A}{s}+\frac{B}{s+5}$$

The residues are

$$A=1,\quad B=-1$$

Thus

$$h_a(nT)=\left(1-e^{-5nT}\right)u(n)$$

and the pulse transfer function of the impulse-invariant digital filter is

$$H(z)=\frac{T}{1-z^{-1}}-\frac{T}{1-e^{-5T}z^{-1}}$$

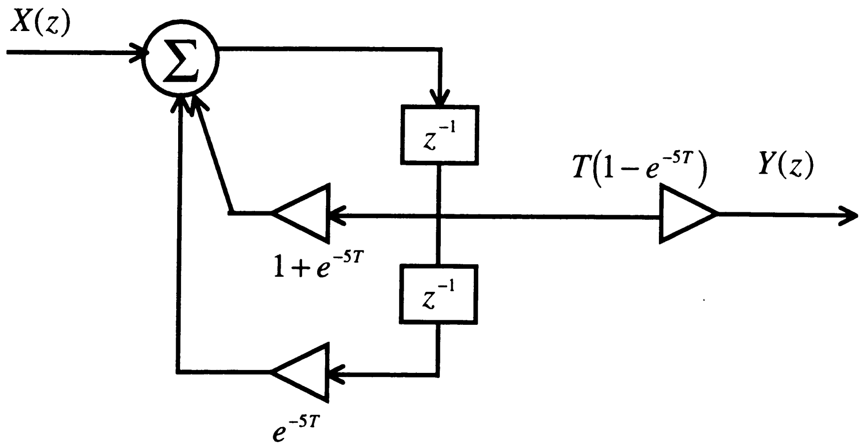
In polynomial form this becomes

$$H(z)=\frac{T\left(1-e^{-5T}z^{-1}\right)-T\left(1-z^{-1}\right)}{\left(1-z^{-1}\right)\left(1-e^{-5T}z^{-1}\right)}$$

or

$$H(z)=\frac{T\left(1-e^{-5T}\right)z^{-1}}{1-\left(1+e^{-5T}\right)z^{-1}+e^{-5T}z^{-2}}$$

The Direct Form II implementation is



For the step-invariant filter we write

$$\begin{aligned}Y_a(s)&=\frac{1}{s}H_a(s)=\frac{5}{s^2(s+5)}\\&=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s+5}\end{aligned}$$

The values of A and C are

$$A=1,\quad C=\frac{1}{5}$$

The value of B is

$$B=\frac{d}{ds}\left[\frac{5}{s+5}\right]\Bigg|_{s=0}=-\frac{1}{5}$$

Therefore

$$Y_a(s)=\frac{1}{s^2}-\frac{1}{5}\frac{1}{s}+\frac{1}{5}\frac{1}{s+5}$$

$$H(z) = \left(1 - z^{-1}\right)\left[\frac{Tz^{-1}}{\left(1 - z^{-1}\right)^2} - \frac{1}{5}\frac{1}{1 - z^{-1}} + \frac{1}{5}\frac{1}{1 - e^{-5T}z^{-1}}\right] = \frac{1}{5} + \frac{Tz^{-1}}{1 - z^{-1}} + \frac{1}{5}\frac{1 - z^{-1}}{1 - e^{-5T}z^{-1}}$$

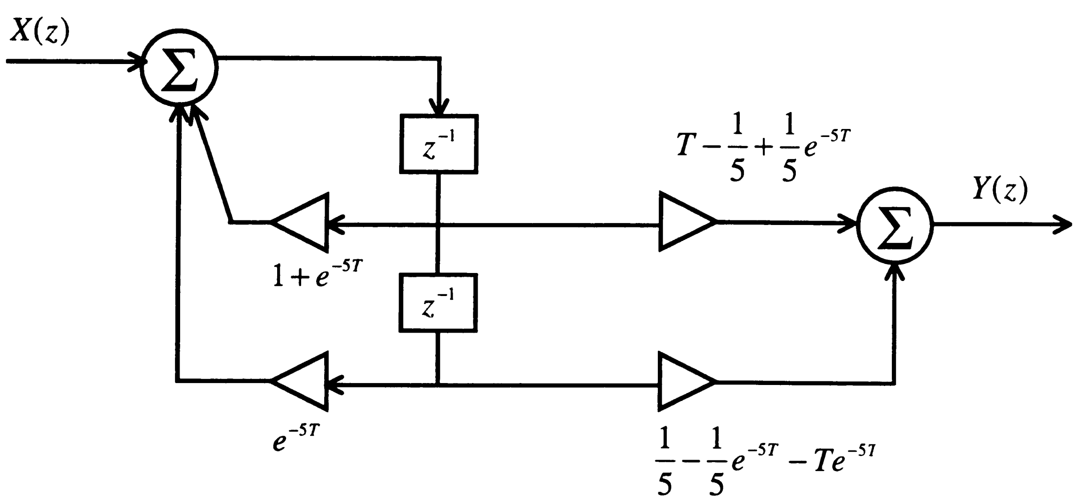
We must now place $H(z)$ in polynominal form. This gives

$$H(z) = \frac{Tz^{-1}\left(1 - e^{-5T}z^{-1}\right) - \frac{1}{5}\left(1 - z^{-1}\right)\left(1 - e^{-5T}z^{-1}\right) + \frac{1}{5}\left(1 - z^{-1}\right)^2}{\left(1 - z^{-1}\right)\left(1 - e^{-5T}z^{-1}\right)}$$

or

$$H(z) = \frac{\left(T - \frac{1}{5} + \frac{1}{5}e^{-5T}\right)z^{-1} + \left(\frac{1}{5} - \frac{1}{5}e^{-5T} - Te^{-5T}\right)z^{-2}}{1 - \left(1 + e^{-5T}\right)z^{-1} + e^{-5T}z^{-2}}$$

The Direct Form II realization is



Problem 9-38

For the impulse invariant filter we inverse transform

$$H_a(s) = \frac{100}{(s+10)^2}$$

which yields

$$h_a(t) = 100t\, e^{-10t}\, u(t)$$

The sample sequence corresponding to $h_a(t)$ is

$$h_a(nT) = 100nT\, e^{-10nT}\, u(n)$$

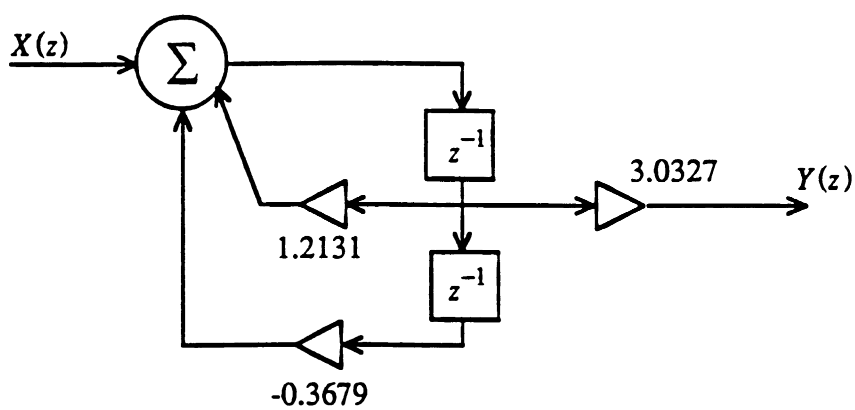
which has the z-transform

$$H(z) = \frac{100T\, e^{-10T} z^{-1}}{\left(1 - e^{-10T} z^{-1}\right)^2}$$

For a sampling frequency of 20 Hz, $T = \frac{1}{20}$ and $H(z)$ becomes

$$H(z) = \frac{5(0.6065)z^{-1}}{\left(1 - 0.6065z^{-1}\right)^2} = \frac{3.0327z^{-1}}{1 - 1.2131z^{-1} + 0.3679z^{-2}}$$

The Direct Form II realization is shown below



For the step invariant filter we write

$$\frac{H_a(s)}{s} = \frac{100}{s(s+10)^2} = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{(s+10)^2}$$

where

$$A=1, \quad C=-10, \quad B=100\frac{d}{ds}\left(\frac{1}{s}\right)\Big|_{s=-10}=-1$$

Thus

$$\frac{H_a(s)}{s}=\frac{1}{s}-\frac{1}{s+10}-\frac{10}{(s+10)^2}$$

The transfer function of the step-invariant filter is

$$H(z)=1-\frac{1-z^{-1}}{1-e^{-10T}z}-\frac{10Te^{-10T}\left(1-z^{-1}\right)z^{-1}}{\left(1-e^{-10T}z^{-1}\right)^2}$$

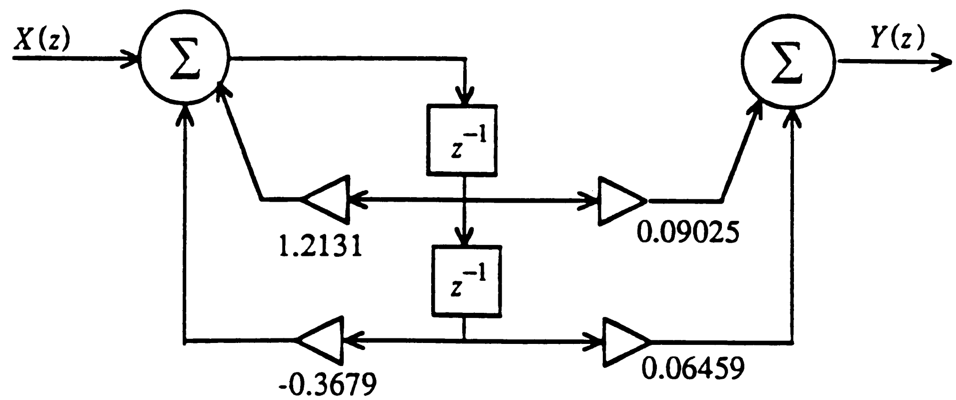
with $T=\frac{1}{20}$ this yields

$$H(z)=1-\frac{1-z^{-1}}{1-0.6065z^{-1}}-\frac{0.5(0.6065)\left(1-z^{-1}\right)z^{-1}}{\left(1-0.6065z^{-1}\right)^2}$$

which can be placed in the form

$$H(z)=\frac{0.09025z^{-1}+0.06459z^{-2}}{1-1.2131z^{-1}+0.3679z^{-2}}$$

This yields the Direct Form II realization shown below



Problem 9-39

For the impulse invariant filter we write

$$H_a(s) = \frac{100}{(s+1)(s+10)^2} = \frac{A}{s+1} + \frac{B}{s+10} + \frac{C}{(s+10)^2}$$

where

$$A = \frac{100}{(9)^2} = \frac{100}{81}$$

$$C = \frac{100}{-9} = -\frac{100}{9}$$

and

$$B = \frac{d}{ds}\left[\frac{100}{s+1}\right]_{s=-10} = -\frac{100}{81}$$

Thus

$$H_a(s) = \frac{100}{81}\frac{1}{s+1} - \frac{100}{81}\frac{1}{s+10} - \frac{100}{9}\frac{1}{(s+10)^2}$$

and

$$h_a(nT) = \left[\frac{100}{81}e^{-nT} - \frac{100}{81}e^{-10nT} - \frac{100}{9}(nT)e^{-10nT}\right], \qquad n \geq 0$$

With $T = \frac{1}{20}$ we have

$$e^{-T} = 0.9512, \qquad e^{-10T} = 0.6065$$

Thus

$$h_a(nT) = \left[\frac{100}{81}(0.9512)^n - \frac{100}{81}(0.6065)^n - \frac{5}{9}n(0.6065)^n\right], \; n \geq 0$$

The z-transform of $h_a(nT)$ yields the transfer function

$$H(z) = \frac{100}{81}\frac{1}{1-0.9512z^{-1}} - \frac{100}{81}\frac{1}{1-0.6065z^{-1}} - \frac{5}{9}\frac{0.6065z^{-1}}{\left(1-0.6065z^{-1}\right)^2}$$

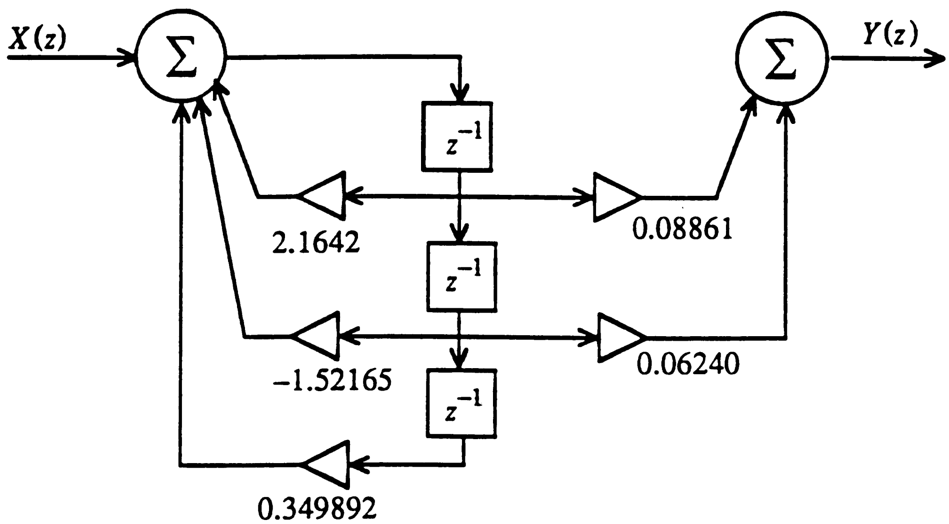
Combining terms yields

$$H(z)=\frac{0.08861z^{-1}+0.06240z^{-2}}{\left(1-0.9512z^{-1}\right)\left(1-0.6065z^{-1}\right)^2}$$

which yields the final form

$$H(z)=\frac{0.08861z^{-1}+0.06240z^{-2}}{1-2.1642z^{-1}+1.52165z^{-2}-0.349892z^{-3}}$$

The Direct Form II realization is shown below



For the step invariant filter we first inverse transform

$$\frac{H_a(s)}{s}=\frac{100}{s(s+1)(s+10)^2}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+10}+\frac{D}{(s+10)^2}$$

where

$$A=1,\qquad B=\frac{100}{(-1)(9)^2}=-\frac{100}{81},\qquad D=\frac{100}{-10(-9)}=\frac{10}{9}$$

and

$$C=\frac{d}{ds}\left[\frac{100}{s^2+s}\right]_{s=-10}=\frac{-100(-19)}{(90)^2}=\frac{19}{81}$$

Thus

$$\frac{H_a(s)}{s}=\frac{1}{s}-\frac{100}{81}\frac{1}{s+1}+\frac{19}{81}\frac{1}{s+10}+\frac{10}{9}\frac{1}{(s+10)^2}$$

This yields the z-domain transfer function

$$H(z)=\left(1-z^{-1}\right)\left[\frac{1}{1-z^{-1}}-\frac{100}{81}\frac{1}{1-e^{-T}z^{-1}}+\frac{19}{81}\frac{1}{1-e^{-10T}z^{-1}}+\frac{10}{9}\frac{Te^{-10T}z^{-1}}{\left(1-e^{-10T}z^{-1}\right)^2}\right]$$

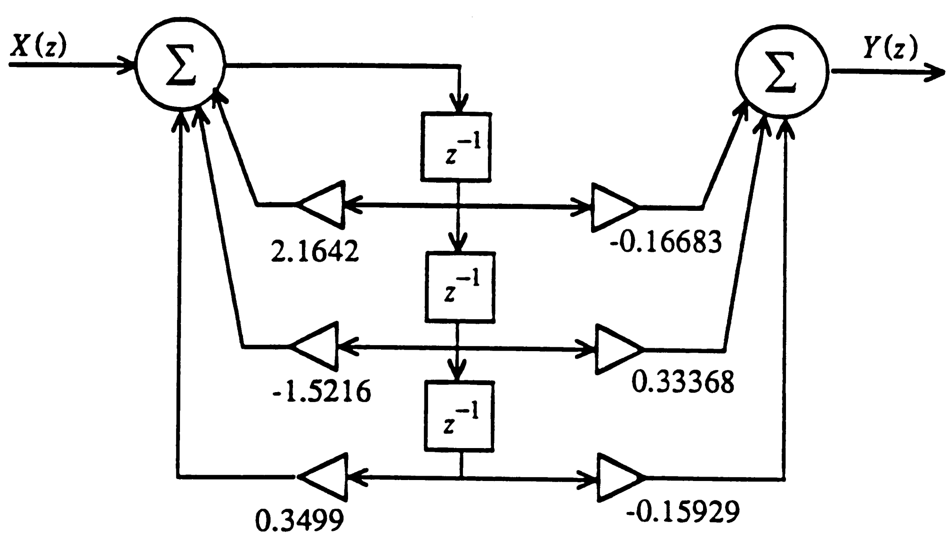
As for the impulse invariant filter we let $T=\frac{1}{20}$. This gives

$$H(z)=1-\frac{100}{81}\frac{1-z^{-1}}{1-0.9512z^{-1}}+\frac{19}{81}\frac{1-z^{-1}}{1-0.6065z^{-1}}+\frac{2}{9}\frac{0.6065\left(1-z^{-1}\right)z^{-1}}{\left(1-0.6065z^{-1}\right)^2}$$

Collecting terms yields

$$H(z)=\frac{-0.16683z^{-1}+0.33368z^{-2}-0.15929z^{-3}}{1-2.1642z^{-1}+1.5216z^{-2}-0.3499z^{-3}}$$

The Direct Form II realization is shown below



Problem 9-40

The exponential invariant digital filter is given by (9-52) with

$$H_a(s) = \frac{5}{s+5}$$

and

$$X_a(s) = \frac{1}{s+3}$$

This gives

$$Y_a(s) = H_a(s)X_a(s) = \frac{5}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

The residues are

$$A = \frac{5}{2}, \quad B = -\frac{5}{2}$$

This gives

$$y_a(t) = \frac{5}{2}\left[e^{-3t} - e^{-5t}\right] u(t)$$

The sampled $y_a(nT)$ is

$$y_a(nT) = \frac{5}{2}\left[e^{-3nT} - e^{-5nT}\right] u(n)$$

With a sampling frequency of 20 hz, $T = 0.05$ and

$$y_a(nT) = \frac{5}{2}\left[e^{-0.15n} - e^{-0.25n}\right] u(n)$$
$$= \frac{5}{2}\left[(0.8607)^n - (0.7788)^n\right] u(n)$$

Thus

$$Y(z) = \frac{5}{2}\left[\frac{1}{1-0.8607z^{-1}} - \frac{1}{1-0.7788z^{-1}}\right]$$
$$= \frac{5}{2}\,\frac{0.0819z^{-1}}{\left(1-0.8607z^{-1}\right)\left(1-0.7788z^{-1}\right)}$$

It follows from the definition of $x(t)$ that $x(nT) = e^{-3nT}u(n)$. Since $T = 0.05$ we have

$$X(s) = \frac{1}{1-e^{-0.15}z^{-1}} = \frac{1}{1-0.8607z^{-1}}$$

Thus

$$H(z) = \frac{G}{X(z)} \; Y(z) = \frac{5}{2} G \frac{0.0819 z^{-1}}{1 - 0.7788 z^{-1}}$$

The dc response is

$$H(1) = \frac{5}{2} G \frac{0.0819}{1 - 0.7788} = \frac{5}{2} G(0.3721)$$

Since $H(1) = 1$

$$G = \frac{2}{5} \frac{1}{0.3721} = \frac{2}{5}(2.6874)$$

Thus

$$H(z) = 2.6874 \frac{0.0819 z^{-1}}{1 - 0.7788 z^{-1}}$$

$$= \frac{0.2201 z^{-1}}{1 - 0.7788 z^{-1}}$$

Problem 9-41

An ideal continuous-time differentiator is defined by

$$H_a(s) = s$$

Choosing the bilinear z-transform filter to closely approximate the response of the ideal differentiator $(\omega_r = 0)$ yeilds

$$H(z) = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

The steady-state frequency response of the bilinear z-transform differentiator is

$$H\left(e^{j2\pi r}\right) = \frac{2}{T} \frac{1 - e^{-j2\pi r}}{1 + e^{-j2\pi r}} = \frac{2}{T} \frac{e^{j\pi r} - e^{-j\pi r}}{e^{j\pi r} - e^{-j\pi r}}$$

which can be written

$$H\left(e^{j2\pi r}\right) = j \frac{2}{T} \frac{\sin \pi r}{\cos \pi r} = j \frac{2}{T} \tan \pi r$$

For small input frequencies ($r << 1$) $\tan \pi r \approx \pi r$ so that

$$H\left(e^{j2\pi r}\right) = j\frac{2}{T}\,\pi r, \qquad r << 1$$

Since

$$r = \frac{\omega}{\omega_s} = \frac{\omega}{2\pi f_s} = \frac{\omega}{2\pi}T$$

we have

$$\frac{1}{T} = \frac{\omega}{2\pi r}$$

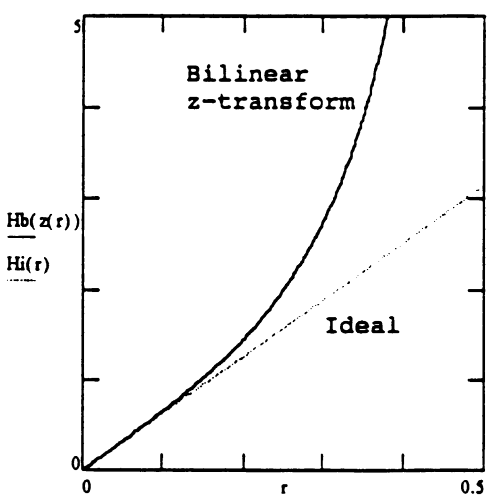
Thus

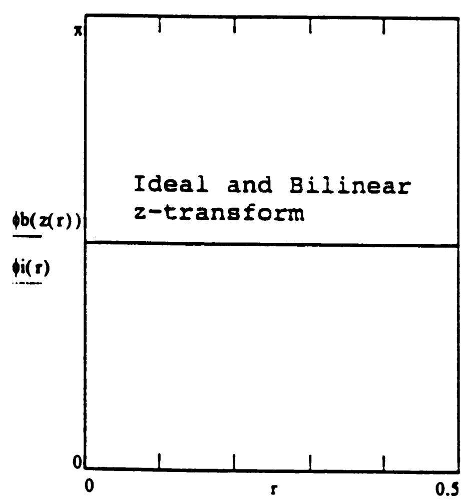
$$H\left(e^{j2\pi r}\right) = j2\left(\frac{\omega}{2\pi r}\right)\pi r = j\omega, \qquad r << 1$$

so that we do indeed see that the bilinear z-transform differentiator closely approximates the ideal differentiator for low frequencies. Note that both the ideal differentiator and the bilinear z-transform differentiator have the phase response

$$\phi(r) = \frac{\pi}{2}, \qquad 0 \le r \le \frac{1}{2}$$

The plots of the amplitude and phase responses are shown below. The bilinear z-transform differentiator is not an effective design because the amplitude response is infinite at $r = 0.5$. Thus it is an unstable system.





Problem 9-42

$$H_a(z) = \frac{5}{(s+1)(s+5)}$$

First we determine a general solution. This gives

$$H(z) = \frac{5}{\left[C\frac{1-z^{-1}}{1+z^{-1}}+1\right]\left[C\frac{1-z^{-1}}{1+z^{-1}}+5\right]}$$

$$= \frac{5\left(1+z^{-1}\right)^2}{\left[C\left(1-z^{-1}\right)+\left(1+z^{-1}\right)\right]\left[C\left(1-z^{-1}\right)+5\left(1+z^{-1}\right)\right]}$$

$$= \frac{5+10z^{-1}+5z^{-2}}{\left[(1+C)+(1-C)z^{-1}\right]\left[(5+C)+(5-C)z^{-1}\right]}$$

(a) If the response of the digital filter is to match the response of the analog filter at low frequencies

$$C = \frac{2}{T} = 2f_s = 24$$

Thus

$$H(z) = \frac{5+10z^{-1}+5z^{-2}}{\left(25-23z^{-1}\right)\left(29-19z^{-1}\right)}$$

$$= \frac{5+10z^{-1}+5z^{-2}}{725-1142z^{-1}+437z^{-2}}$$

$$= \frac{0.00690+0.01379z^{-1}+0.00690z^{-2}}{1-1.57517z^{-1}+0.60276z^{-2}}$$

(b) If the responses are to match at $f = 2\text{Hz}$

$$C = 2\pi(2)\ \cot\frac{2\pi(2)}{2(12)}$$

$$= \frac{4\pi}{\tan(\pi/6)} = 21.76559$$

This yields

$$H(z) = \frac{5 + 10z^{-1} + 5z^{-2}}{\left(22.76559 - 20.76559z^{-1}\right)\left(26.76559 - 16.76559z^{-1}\right)}$$

$$= \frac{5 + 10z^{-1} + 5z^{-2}}{609.33457 - 937.48202z^{-1} + 348.14746}$$

$$= \frac{0.00821 + 0.01642z^{-1} + 0.00821z^{-2}}{1 - 1.53853z^{-1} + 0.57136z^{-2}}$$

Problem 9-43

(a) For the case in which the digital filter and the digital filter are to have the same response for very low frequencies, no prewarping is applied. For this case the MATLAB code is that which follows:

```
Fs = 12;                                  % Sampling frequency in Hertz
nums = 5;                                 % Define numeratior in s domain
dens = [1 6 5];                           % Define denominator in s
domain
[numd,dend] = bilinear(nums,dens,Fs);     % Determine digital filter
numd                                      % Display numerator

numd =

    0.0069    0.0138    0.0069

dend                                      % Display denominator

dend =

    1.0         -1.5752    0.6028
```

(b) If prewarping is applied so that the analog and digital filters have the same response at 2 Hertz, the following MATLAB code solves the problem:

```
Fp = 2;                                   % Prewarping frequency in Hertz
Fs = 12;                                  % Sampling frequency in Hertz
nums = 5;                                 % Define numerator in s domain
dens = [1 6 5];                           % Define denominator in s
domain
[numd,dend] = bilinear(nums,dens,Fs,Fp);  % Determine digital filter
numd                                      % Display numerator

numd =

    0.0082    0.0164    0.0082

dend                                      % Display denominator

dend =

    1.0000   -1.5385    0.5714
```

We see that both results are agree (to the precision printed) with the previous problem.

__Problem 9-44__

The analog transfer function

$$H_a(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

has the bilinear z-transform

$$H(z) = \frac{1}{\left[C\frac{1-z^{-1}}{1+z^{-1}}\right]^3 + 2\left[C\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + 2C\frac{1-z^{-1}}{1+z^{-1}} + 1}$$

which can be written

$$H(z) = \frac{\left(1+z^{-1}\right)^3}{C^3\left(1-z^{-1}\right)^3 + 2C^2\left(1+z^{-1}\right)\left(1-z^{-1}\right)^2 + 2C\left(1+z^{-1}\right)^2\left(1-z^{-1}\right) + \left(1+z^{-1}\right)^3}$$

or

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{A + Bz^{-1} + Dz^{-2} + Ez^{-3}}$$

The values of A, B, C and D are given by

$$A = C^3 + 2C^2 + 2C + 1$$
$$B = -3C^3 - 2C^2 + 2C + 3$$
$$D = 3C^3 - 2C^2 - 2C + 3$$
$$E = -C^3 + 2C^2 - 2C + 1$$

With $\omega_r = 1$ and $T = 1/6$, the value of C is given by

$$C = (1)\ \cot\left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = 11.97221$$

The values of A, B, D and E are given by

$$A = 2027.63423$$
$$B = -5407.78984$$
$$D = 4840.45465$$
$$E = 1452.29905$$

Thus

$$H(z)=\frac{0.00049+0.00148z^{-1}+0.00148z^{-2}+0.00049z^{-3}}{1-2.66707z^{-1}+2.38724z^{-2}-0.71625z^{-3}}$$

Problem 9-45

The previous problem is worked using the following MATLAB code:

```
Fp = 1/(2*pi);                          % Prewarping frequency in Hertz
Fs = 6;                                 % Sampling frequency in Hertz
[nums,dens] = butter(3,1,'s');          % Define analog prototype
[numd,dend] = bilinear(nums,dens,Fs,Fp); % Determine digital filter
numd                                    % Display numerator

numd =

  4.9319e-004  1.4796e-003  1.4796e-003  4.9319e-004

dend                                    % Display denominator

dend =

  1.0000e+000 -2.6670e+000  2.3872e+000 -7.1625e-001
```

Note that the "short e" format was used so that the precision necessary to compare the results with those of the previous problem could be obtained. The results agree to the precision used to calculate the results obtained in the previous problem. Note that the results given here are more accurate than the results obtained in the previous problem.

Problem 9-46

This problem is identical to the preceding problem except that we let

$$C=\frac{2}{T}=2f_s=12$$

The values of A, B, D and E, as defined in the preceding problem, are

$$A=(12)^3+2(12)^2+2(12)+1=2041$$

$$B=-3(12)^3-2(12)^2+2(12)+3=-5445$$

$$D=3(12)^3-2(12)^2-2(12)+3=4875$$

$$E=-(12)^3+2(12)^2-2(12)+1=-1463$$

This yields

$$H(z)=\frac{0.00049+0.00147z^{-1}+0.00147z^{-2}+0.00049z^{-3}}{1-2.66781z^{-1}+2.38854z^{-2}-0.71681z^{-3}}$$

Problem 9-47

(a) The analog transfer function is given by

$$H_a(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{RCs + 1} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

With $R = 10\ \text{k}\Omega$ and $C = 20\ \mu f$ we have $RC = 0.2$. Thus the analog transfer function is

$$H_a(s) = \frac{5}{s+5}$$

For the impulse-invariant digital filter we have

$$H_i(z) = \frac{5T}{1 - e^{-5T} z^{-1}}$$

which, for $T = \frac{1}{15}$, yields

$$H_i(z) = \frac{0.3333}{1 - 0.7165 z^{-1}}$$

For the step-invariant filter we can write

$$\frac{H_a(s)}{s} = \frac{5}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

where

$$A = 1, \qquad B = -1$$

Thus the step-invariant pulse transfer function can be written

$$H_s(z) = \left(1 - z^{-1}\right)\left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-5T} z^{-1}}\right]$$

which, for $T = \frac{1}{15}$, yields

$$H_s(z) = 1 - \frac{1 - z^{-1}}{1 - 0.7165 z^{-1}}$$

This can be written in polynominal form

$$H_s(z) = \frac{0.2835 z^{-1}}{1 - 0.7165 z^{-1}}$$

For the bilinear z-transform filter we write

$$H_b(z) = \frac{5}{C\dfrac{1-z^{-1}}{1+z^{-1}}+5} = \frac{5+5z^{-1}}{(5+C)+(5-C)z^{-1}}$$

where C is given by

$$C = \omega_r \cot\left(\frac{\omega_r T}{2}\right)$$

With a reference freqneucy of 5 rad/sec and $T = \dfrac{1}{15}$ we have

$$C = 5\cot\left(\frac{5}{2(15)}\right) = 5\cot\left(\frac{1}{6}\right) = 29.7217$$

This gives the pulse transfer function

$$H_b(z) = \frac{5+5z^{-1}}{34.7217-24.7217z^{-1}} = \frac{0.1440+0.1440z^{-1}}{1-0.7120z^{-1}}$$

For the analog filter we have

$$r = \frac{\omega}{\omega_s}$$

So that

$$\omega = r\omega_s = r(2\pi(15)) = 30\pi r$$

Thus

$$H_a(j\omega) = \frac{5}{5+j30\pi r} = \frac{1}{1+j6\pi r}$$

(b) and (c) The amplitude and phase responses are easily determined using MathCAD as shown on the following pages..

(b) First we define j:

$$j := \sqrt{-1}$$

and z:

$$z(r) := \exp(j \cdot 2 \cdot \pi \cdot r)$$

The impulse-invariant filter is defined by

$$Hi(z) := \frac{0.3333}{1 - 0.7165 \cdot z^{-1}}$$

The step-invariant filter is defined by

$$Hs(z) := \frac{0.2835 \cdot z^{-1}}{1 - 0.7165 \cdot z^{-1}}$$

The bilinear z-transform filter is defined by

$$Hb(z) := \frac{0.1440 + 0.1440 \cdot z^{-1}}{1 - 0.7120 \cdot z^{-1}}$$

The response of the analog filter, in terms of the normalized frequency r, is

$$Ha(r) := \frac{1}{1 + j \cdot 6 \cdot \pi \cdot r}$$

The corresponding amplitude responses in dB are

$Ai(z) := 20 \cdot \log(|Hi(z)|)$ Impulse-invariant filter.

$As(z) := 20 \cdot \log(|Hs(z)|)$ Step-invariant filter.

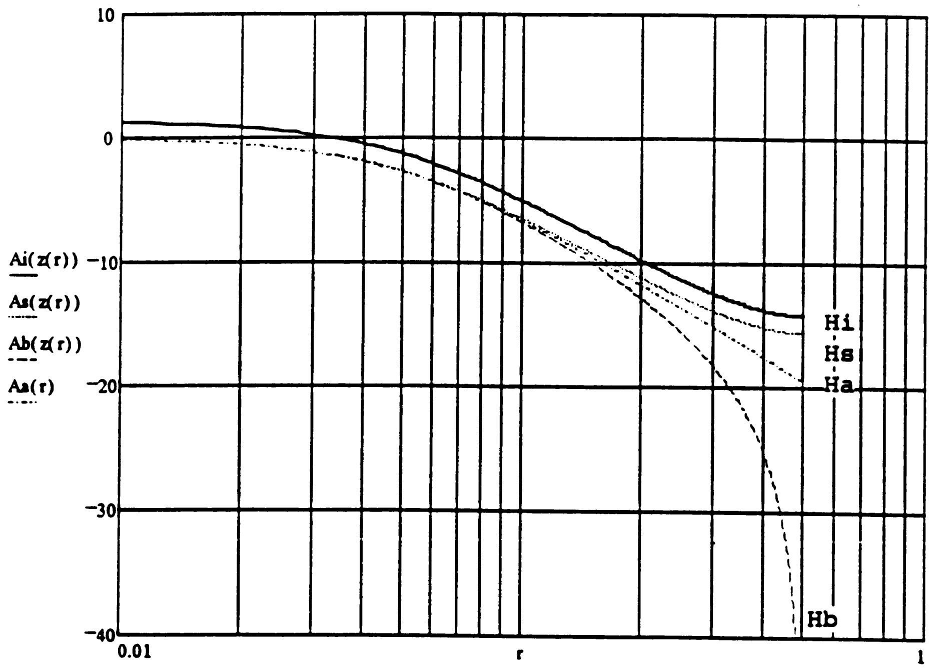
$Ab(z) := 20 \cdot \log(|Hb(z)|)$ Bilinear z-transform filter.

$Aa(r) := 20 \cdot \log(|Ha(r)|)$ Analog prototype.

The range of r is

$$r := 0.01, 0.011 .. 0.5$$

The amplitude response plots (in dB) are shown on the following page.



As indicated by the notations on the plot that identify the various filters, the plot of the amplitude response, the impulse-invariant response (Hi) lies above all of the others and has a dc value > 1 (0 dB). The step-invariant response (Hs) lies imediately below the impulse-invariant response. The next curve (Ha) is for the analog filter. The bilinear z-transform response (Hb) goes to $-\infty$ at r=0.5.

(c) The phase responses are defined as

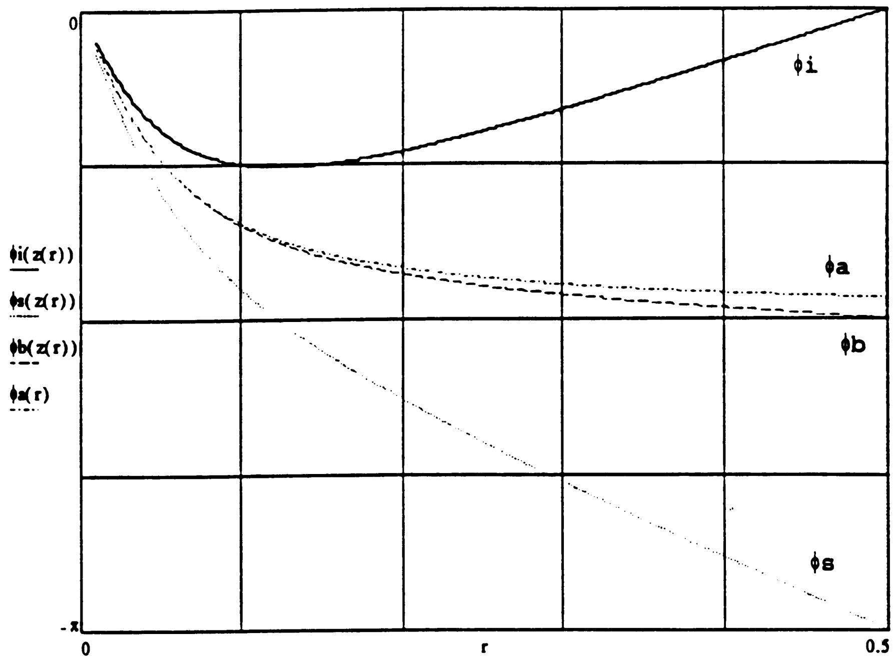
$\phi i(z) := \arg(Hi(z))$ Impulse-invariant filter.

$\phi s(z) := \arg(Hs(z))$ Step-invariant filter.

$\phi b(z) := \arg(Hb(z))$ Bilinear z-transform filter.

$\phi a(r) := \arg(Ha(r))$ Analog prototype.

These definitions give the plot shown at the top of the following page.



As before, the order of the curves is indicated by the notations on the curves.

Problem 9-48

The analog transfer function

$$H_a(s) = \frac{16}{s^2 + 12s + 16}$$

has the bilinear z-transform pulse transfer function

$$H(z) = \frac{16}{\left[C\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + 12C\frac{1-z^{-1}}{1+z^{-1}} + 16}$$

Multiplying the numerator and denominator by $\left(1-z^{-1}\right)^2$ gives

$$H(z) = \frac{16\left(1+2z^{-1}+z^{-2}\right)}{C^2\left(1-2z^{-1}+z^{-2}\right)+12C\left(1-z^{-2}\right)+16\left(1+2z^{-1}+z^{-2}\right)}$$

collecting like terms allows $H(z)$ to be placed in the form

$$H(z) = \frac{16+32z^{-1}+16z^{-2}}{\left(C^2+12C+16\right)+\left(32-2C^2\right)z^{-1}+\left(C^2-12C+16\right)z^{-2}}$$

The value of C is given by, since $\omega_r = 2\pi(4)$ and $f_s = 50$

$$C = 8\pi\cot\left(\frac{4\pi}{50}\right) = 97.8856$$

The coefficients are

$$C^2 + 12C + 16 = 10772.21$$

$$32 - 2C^2 = -19131.17$$

$$C^2 - 12C + 16 = 8422.96$$

Thus, after dividing numerator and denominator by 10722.21

$$H(z) = \frac{0.00149 + 0.00297z^{-1} + 0.00149z^{-2}}{1 - 1.77597z^{-1} + 0.78192z^{-2}}$$

Problem 9-49

The analog transfer function

$$H_a(s)=\frac{2(s+4)}{(s+1)(s+8)}=\frac{2s+8}{s^2+9s+8}$$

has the bilinear z-transform pulse transfer function

$$H(z)=\frac{2C\frac{1-z^{-1}}{1+z^{-1}}+8}{\left[C\frac{1-z^{-1}}{1+z^{-1}}\right]^2+9C\frac{1-z^{-1}}{1+z^{-1}}+8}$$

Multiplying numerator and denominator by $\left(1+z^{-1}\right)^2$ yields

$$H(z)=\frac{2C\left(1-z^{-2}\right)+8\left(1+2z^{-1}+z^{-2}\right)}{C^2\left(1-2z^{-1}+z^{-2}\right)+9C\left(1-z^{-2}\right)+8\left(1+2z^{-1}+z^{-2}\right)}$$

Collecting like terms gives $H(z)$ in the form

$$H(z)=\frac{(2C+8)+16z^{-1}+(8-2C)z^{-2}}{\left(C^2+9C+8\right)+\left(16-2C^2\right)z^{-1}+\left(C^2-9C+8\right)z^{-2}}$$

The reference frequency is

$$\omega_r=\frac{4}{\pi}(2\pi)=8$$

and the sampling frequency is

$$f_s=14$$

Thus, the constant C is

$$C=8\cot\left(\frac{4}{14}\right)=27.2339$$

The filter coefficients are

$$2C+8=62.4678$$
$$8-2C=-46.4678$$
$$C^2+9C+8=994.7914$$
$$16-2C^2=-1467.3724$$
$$C^2-9C+8=504.5810$$

This yields the pulse transfer function

$$H(z)=\frac{62.4678+16z^{-1}-46.4678z^{-2}}{994.7914-1467.3724z^{-1}+504.5810z^{-2}}$$

Thus

$$H(z) = \frac{0.06279 + 0.01608z^{-1} - 0.04671z^{-2}}{1 - 1.47506z^{-1} + 0.50722z^{-2}}$$

Problem 9-50

The following MATLAB script solves Problem 9-48.

```
Fs = 50;                                % Sampling frequency
Fr = 4;                                 % Reference frequency
num = 16;                               % Analog filter numerator
den = [1 12 16];                        % Analog filter denominator
[numd,dend] = bilinear(num,den,Fs,Fr);  % Compute filter
numd                                    % Display numerator
dend                                    % Display denominator
```

Execution of the program yields the following output.

```
numd =

    0.0015    0.0030    0.0015


dend =

    1.0000   -1.7760    0.7819
```

Comparison with the results of Problem 9-48 so that we have agreement to the percision printed.

Problem 9-51

The bilinear z-transform of

$$H_a(s) = \frac{2(s+3)}{3(s+1)(s+2)} = \frac{2s+6}{3s^2+9s+6}$$

is

$$H(z) = \frac{2C\dfrac{1-z^{-1}}{1+z^{-1}}+6}{3C^2\left(\dfrac{1-z^{-1}}{1-z^{-1}}\right)^2+9C\dfrac{1-z^{-1}}{1+z^{-1}}+6}$$

which can be placed in the form

$$H(z) = \frac{2C\left(1-z^{-1}\right)\left(1+z^{-1}\right)+6\left(1+z^{-1}\right)^2}{3C^2\left(1-z^{-1}\right)^2+9C\left(1-z^{-1}\right)\left(1+z^{-1}\right)+6\left(1+z^{-1}\right)^2}$$

The value of C is given by

$$C = 3\cot\left(\frac{3\pi}{20}\right) = 5.88783$$

The transfer function $h(z)$ can be placed in the form

$$H(z) = \frac{L_0+L_1z^{-1}+L_2z^{-2}}{K_0+K_1z^{-1}+K_2z^{-2}}$$

where

$$L_0 = 2C+6 = 17.77566$$
$$L_1 = 12$$
$$L_2 = -2C+6 = -5.77566$$
$$K_0 = 3C^2+9C+6 = 162.99016$$
$$K_1 = -6C^2+12 = -195.99936$$
$$K_2 = 3C^2-9C+6 = 57.00920$$

Dividing numerator and denominator by K_0 gives $H(z)$ in the standard form

$$H(z)=\frac{0.10906+0.07362z^{-1}-0.03544z^{-2}}{1-1.20252z^{-1}+0.34977z^{-2}}$$

See the following problem for the amplitude and phase plots.

Problem 9-52

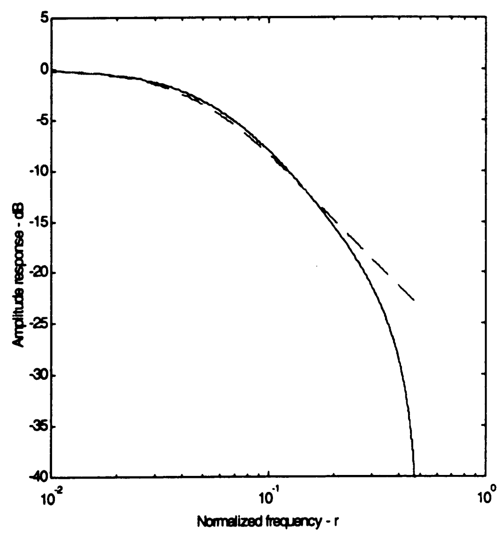
The amplitude and phase responses for the previous problem are developed using the following MATLAB script

```
Fs = 10/pi;                                 % Sampling fequency
Fr = 3/(2*pi);                              % Reference frequency
num = [2 6];                                % Analog numerator
den = [3 9 6];                              % Analog denominator
[numd,dend] = bilinear(num,den,Fs,Fr);      % Determine digital filter

numd                                        % Display numerator
dend                                        % Display denominaror

[Hd,W] = freqz(numd,dend);                  % Response of digital filter
Wa = W*Fs;                                  % Analog frequency in rad/s
Ha = freqs(num,den,Wa);                     % Analog response
r = W/(2*pi);                               % Normalized frequency vector
ampddB = 20*log10(abs(Hd));                 % Digital fil. response in dB
ampadB = 20*log10(abs(Ha));                 % Analog fil. response in dB
semilogx(r,ampddB,r,ampadB,'--')            % Plot amplitude responses
axis([0.01 1 -40 5]);                       % Adjust axis
xlabel('Normalized frequency - r')          % Label x axis
ylabel('Amplitude response - dB')           % Label y axis
pause                                       % Pause to look

phid = (180/pi)*angle(Hd);                  % Digital filter phase
phia = (180/pi)*angle(Ha);                  % Analog filter phase
plot(r,phid,r,phia,'--')                    % Plot phase responses
xlabel('Normalized frequency - r')          % Label x axis
ylabel('Phase response - degrees')          % Label y axis
```

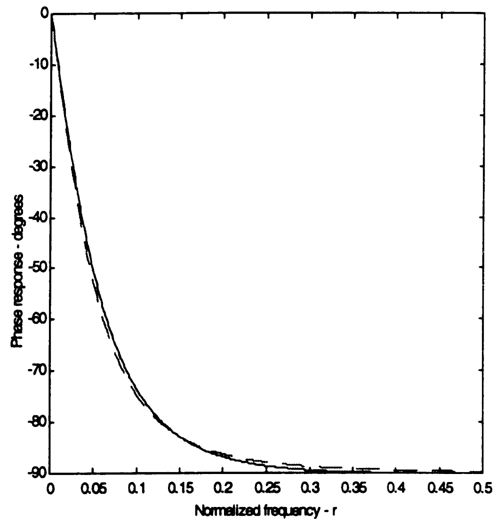
Executing the program first gives the numerator and denominator coefficients of the digital filter. These are

```
numd =

    0.1091    0.0736   -0.0354


dend =

    1.0000   -1.2025    0.3498
```

Comparison of these with the results of the previous problem show agreement to the precision printed. The next step is to generate the amplitude responses of the analog and digital filters. These are shown below. The digital filter is represented by the solid curve and the analog filter is represented by the dashed curve.



Next we plot the phase response. As before the digital filter is represented by the solid curve and the analog filter is represented by the dashed curve.



Problem 9-53

The analog transfer function

$$H_a(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

yields the bilinear z-transform filter

$$H(z) = \frac{1}{C^3\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 2C^2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 2C\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1}$$

which can be written in the form

$$H(z) = \frac{\left(1+z^{-1}\right)^3}{C^3\left(1-z^{-1}\right)^3 + 2C^2\left(1+z^{-1}\right)\left(1-z^{-1}\right)^2 + 2C\left(1+z^{-1}\right)^2\left(1-z^{-1}\right) + \left(1+z^{-1}\right)^3}$$

Performing the indicated multiplications gives the transfer function in the form

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{K_0 + K_1 z^{-1} + K_2 z^{-2} + K_3 z^{-3}}$$

where

$$K_0 = C^3 + 2C^2 + 2C + 1$$
$$K_1 = -3C^3 - 2C^2 + 2C + 3$$

$$K_2 = 3C^3 - 2C^2 - 2C + 3$$
$$K_3 = -C^3 + 2C^2 - 2C + 1$$

The value of C is given by

$$C = (1)\cot\left(\frac{1}{20}\right) = 19.9833$$

This gives

$$K_0 = 8819.647$$
$$K_1 = -24695.74$$
$$K_2 = 23104.41$$
$$K_3 = -7220.313$$

Dividing the numerator and denominator of $H(z)$ by K_0 yields $H(z)$ in the standard form

$$H(z) = \frac{1.1338(10^{-4}) + 3.4015(10^{-4})z^{-1} + 3.4015(10^{-4})z^{-2} + 1.1338(10^{-4})z^{-3}}{1 - 2.8001z^{-1} + 2.6197z^{-2} - 0.8187z^{-3}}$$

Problem 9-54

Problems 9-45 and 9-54, like Problems 9-44 and 9-53, are identical except for the change in the sampling frequency. This allow the student to observe the change in filter coefficients as the sampling frequency is changed. The MATLAB code for Problem 9-54 follows:

```
Fp = 1/(2*pi);                            % Prewarping frequency in Hertz
Fs = 10;                                  % Sampling frequency in Hertz
[nums,dens] = butter(3,1,'s');            % Define analog prototype
[numd,dend] = bilinear(nums,dens,Fs,Fp); % Determine digital filter
numd                                      % Display numerator

numd =

  1.1338e-004  3.4015e-004  3.4015e-004  1.1338e-004

dend                                      % Display denominator

dend =

  1.0000e+000 -2.8001e+000  2.6197e+000 -8.1866e-001
```

Note once again that the "short e" format was used to print the results so that the precision necessary to compare the results with those of the previous problem could be obtained. The results are consistent with those of the previous problem.

Problem 9-55

A second-order Butterworth prototype is defined by

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The transfer function of the digital filter is therefore given by

$$H(z) = \frac{1}{A^2\left(\dfrac{1 - Bz^{-1} + z^{-2}}{1 - z^{-2}}\right)^2 + \sqrt{2}A\left(\dfrac{1 - Bz^{-1} + z^{-2}}{1 - z^{-2}}\right) + 1}$$

which is

$$H(z) = \frac{\left(1 - z^{-2}\right)^2}{A^2\left(1 - Bz^{-1} + z^{-2}\right)^2 + \sqrt{2}A\left(1 - z^{-2}\right)\left(1 - Bz^{-1} + z^{-2}\right) + \left(1 - z^{-2}\right)^2}$$

Performing the indicated multiplications yields the transfer function in the form

$$H(z) = \frac{1 - 2z^{-2} + z^{-4}}{K_0 + K_1 z^{-1} + K_2 z^{-2} + K_3 z^{-3} + K_4 z^{-4}}$$

where

$$K_0 = A^2 + \sqrt{2}A + 1$$

$$K_1 = -2A^2B - \sqrt{2}AB$$

$$K_2 = A^2B^2 - 2A^2$$

$$K_3 = -2A^2B + \sqrt{2}AB$$

$$K_4 = A^2 - \sqrt{2}A + 1$$

With $f_s = 5000$, $f_u = 1200$ and $f_l = 800$, the values of A and B become

$$A = \cot\left[\pi\left(\frac{1200 - 800}{5000}\right)\right] = 3.894743$$

$$B = 2\frac{\cos\left(\pi\left(\dfrac{1200 + 800}{5000}\right)\right)}{\cos\left(\pi\left(\dfrac{1200 - 800}{5000}\right)\right)} = 0.638080$$

with these values the filter coefficients become

$$K_0 = 21.6770$$
$$K_1 = -22.8726$$
$$K_2 = 34.5141$$
$$K_3 = -15.8436$$
$$K_4 = 10.6610$$

Dividing the numerator and denominator by K_0 yields $H(z)$ in the standard form

$$H(z) = \frac{0.04613 - 0.92264z^{-2} + 0.04613z^{-4}}{1 - 1.05516z^{-1} + 1.59220z^{-2} - 0.73089z^{-3} + 0.49181z^{-4}}$$

Problem 9-56

We wish to solve for the reference frequency ω_r at which, using (9-90) with $\omega_r = \omega = \omega_1$,

$$\omega_r = 1.5 f_s \tan\frac{\omega_r T}{2}$$

or

$$\omega_r T = 1.5 \tan\frac{\omega_r T}{2}$$

Letting $x = \omega_r T$ yields the expression

$$x = 1.5 \tan\left(\frac{x}{2}\right)$$

This can be solved by trial and error or by using an equation solver program such as MathCAD. The MathCAD program follows and shows that $x = 1.689$. In order to better understand this number consider Figure 9-18 on page 439. The value of the abcissa corresponding to the normalized reference frequency is

$$\frac{\omega_r}{0.5\omega_s} = \frac{\omega_r}{\pi}T = \frac{x}{\pi} = \frac{1.689}{\pi} = 0.538$$

Observation of Figure 9-18 shows this to be a reasonable value. Consider the following MathCAD program:

Guess value $x := 3$

Given

$x = 1.5 \cdot \tan\left(\frac{x}{2}\right)$ $a := \text{Find}(x)$ $a = 1.689$

Problem 9-57

As shown in Appendix E, Equation (E-84), a bandreject filter is derived from a lowpass prototype by the transformation

$$H_{br}(s) = H\left(\frac{s\omega_b}{s^2 + \omega_c^2}\right)$$

For the bilinear z-transform we have

$$\frac{s\omega_b}{s^2 + \omega_C^2} \Rightarrow \frac{C\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\omega_b}{\left[C\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + \omega_c^2}$$

$$= \frac{C\left(1-z^{-2}\right)\omega_b}{C^2\left(1-z^{-1}\right)^2 + \omega_c^2\left(1+z^{-1}\right)^2}$$

The preceding can be placed in the form

$$\frac{s\omega_b}{s^2 + \omega_c^2} \Rightarrow \frac{C\omega_b}{C^2 + \omega_c^2} \frac{1-z^{-2}}{1 - 2\left[\frac{C^2 - \omega_c^2}{C^2 + \omega_c^2}\right]z^{-1} + z^{-2}}$$

Equating this to

$$D\frac{1-z^{-2}}{1 - Ez^{-1} + z^{-2}}$$

yields

$$D = \frac{C\omega_b}{C^2 + \omega_c^2}$$

and

$$E = 2\frac{C^2 - \omega_c^2}{C^2 + \omega_c^2}$$

As in (9-131) ω_c^2 is the geometric mean of the upper and lower critical frequencies, ω_u and ω_ℓ, respectively. Thus, under the bilinear z-transformation,

$$\omega_c^2 \Rightarrow C^2 \tan\frac{\omega_u T}{2} \tan\frac{\omega_\ell T}{2}$$

In terms of normalized frequency

$$\frac{\omega_u T}{2} = \frac{\pi f_u}{f_s} = \pi r_u$$

and

$$\frac{\omega_\ell T}{2} = \frac{\pi f_\ell}{f_s} = \pi r_\ell$$

where r_u and r_ℓ are the normalized values of ω_u and ω_ℓ. The normalized bandwidth is

$$\omega_b \Rightarrow C \ \tan \pi r_u - C \ \tan \pi r_\ell$$

Thus D becomes, after cancelling the C^2 terms,

$$D = \frac{\tan \pi r_u - \tan \pi r_\ell}{1 + \tan \pi r_u \ \tan r_\ell}$$

Using the trigonometric identity in (9-136) yields

$$D = \tan \pi(r_u - r_\ell)$$

The value of E becomes, after again cancelling the C^2 terms

$$E = 2\frac{1 - \tan \pi r_u \ \tan \pi r_\ell}{1 + \tan \pi r_n \ \tan r_\ell}$$

Using the trigonmetric identity of (9-140) yields

$$E = 2\frac{\cos \pi(r_u + r_\ell)}{\cos \pi(r_u - r_\ell)}$$

Problem 9-58

Since a first-order Butterworth prototype is given by

$$H_a(s) = \frac{1}{s+1}$$

The desired $H(z)$ for the notch filter is

$$H(a) = \frac{1}{\dfrac{D(1 - z^{-2})}{1 - Ez^{-1} + z^{-2}} + 1}$$

which can be placed in the form

$$H(z) = \frac{1 - Ez^{-1} + z^{-2}}{(1+D) - Ez^{-1} + (1-D)z^{-2}}$$

with $f_s = 2000$ Hz, $f_u = 300$ Hz and $f_l = 200$ Hz, we get

$$r_u = \frac{300}{2000} = \frac{3}{20}$$

and

$$r_l = \frac{200}{2000} = \frac{1}{10}$$

With these values

$$D = \tan \pi\left(\frac{3}{20} - \frac{1}{10}\right) = 0.15838$$

and

$$E = 2\frac{\cos \pi\left(\frac{3}{20} + \frac{1}{10}\right)}{\cos \pi\left(\frac{3}{20} - \frac{1}{10}\right)} = 1.43184$$

Thus $H(z)$ becomes

$$H(z) = \frac{1 - 1.43184z^{-1} + z^{-2}}{1.15835 - 1.43184z^{-1} + 0.84162z^{-2}}$$

which is, in standard form

$$H(z) = \frac{0.86327 - 1.23610z^{-1} + 0.86327z^{-2}}{1 - 1.23610z^{-1} + 0.72654z^{-2}}$$

The amplitude and phase response are computed using MathCAD.

First we define j: $j := \sqrt{-1}$

and then define z: $z(r) := \exp(j \cdot 2 \cdot \pi \cdot r)$

The filter constants are:

L0 := 0.86327 L1 := -1.23610 L2 := 0.86327

K1 := -1.23610 K2 := 0.72654

The transfer function is defined as:

$$H(z) := \frac{L0 + L1 \cdot z^{-1} + L2 \cdot z^{-2}}{1 + K1 \cdot z^{-1} + K2 \cdot z^{-2}}$$

The amplitude response (in dB) and the phase response is defined as:

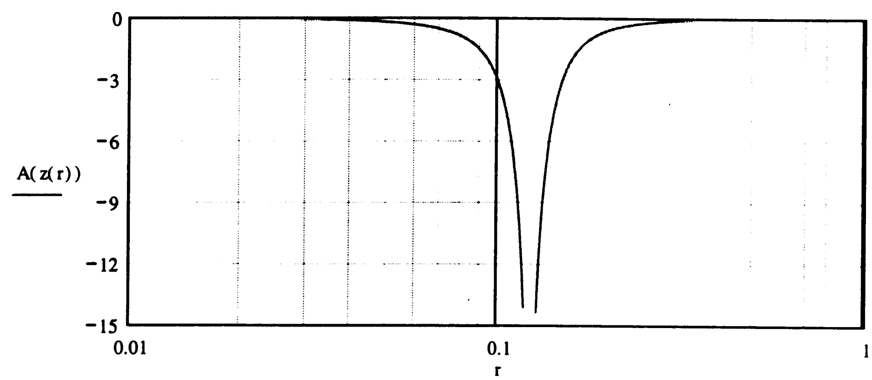
$$A(z) := 20 \cdot \log(|H(z)|)$$

$$\phi(z) := \arg(H(z))$$

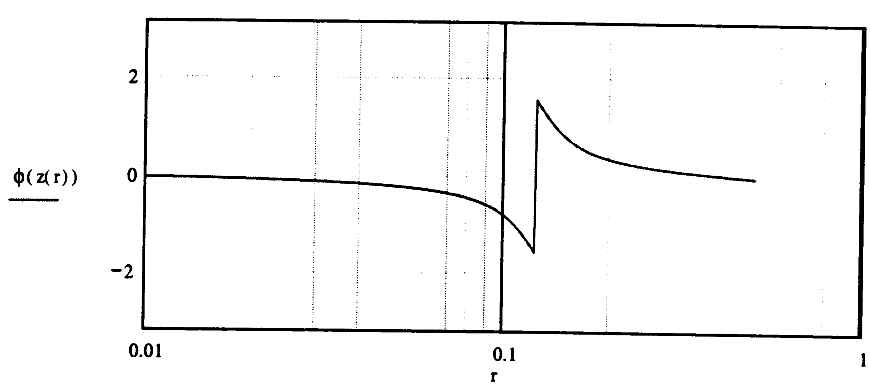
Define the range of r:

$$r := 0.001, 0.002 .. 0.5$$

The plot of the amplitude response is:



The phase response is:



Let's check the upper and lower 3-dB frequencies:

$A(z(0.10)) = -3.011$ Check!

$A(z(0.15)) = -3.01$ Check!

Problem 9-59

This problem is easily worked using MATLAB. We will, however, use MathCAD for variety. The amplitude response will be plotted for all three windows; rectangular, Hamming and triangular.

We shall plot the amplitude and the phase response of the filter using the unit-pulse response expressed as a vector. Since the starting point is a non-causal filter we must shift the origin. Thus

$$\text{ORIGIN} \equiv -8$$

The complex operator is first defined

$$i := \sqrt{-1}$$

The iteration variable

$$n := 1 .. 8$$

is next specified. The terms for h(n) can then be defined. This gives

$$h_n := \frac{1}{\pi \cdot n} \cdot \sin(0.3 \cdot \pi \cdot n)$$

The unit-pulse response is even and h(0) is 0.3. Thus

$$h_{-n} := h_n \qquad h_0 := 0.3$$

The iteration variable is now changed to

$$n := -8 .. 8$$

The Hamming window is defined as

$$wh_n := 0.54 + 0.46 \cdot \cos\left(\frac{\pi \cdot n}{8}\right)$$

The triangular window is defined as

$$wt_n := 1 - \frac{|n|}{9}$$

The windowed unit-pulse responses are next formed. For the rectangular window we have

$$hr_n := h_n$$

For the Hamming window we have

$$hh_n := h_n \cdot wh_n$$

Finally, for the triangular window we have

$$ht_n := h_n \cdot wt_n$$

The values of h(n), wh(n), wt(n), hh(n) and ht(n) for the non-causal filter are now fully determined. We therefore have

n	h_n	wh_n	wt_n	hh_n	ht_n
-8	0.03784	0.08	0.11111	0.003	0.004
-7	0.01405	0.115	0.22222	0.002	0.003
-6	-0.03118	0.215	0.33333	-0.007	-0.01
-5	-0.06366	0.364	0.44444	-0.023	-0.028
-4	-0.04677	0.54	0.55556	-0.025	-0.026
-3	0.03279	0.716	0.66667	0.023	0.022
-2	0.15137	0.865	0.77778	0.131	0.118
-1	0.25752	0.965	0.88889	0.249	0.229
0	0.3	1	1	0.3	0.3
1	0.25752	0.965	0.88889	0.249	0.229
2	0.15137	0.865	0.77778	0.131	0.118
3	0.03279	0.716	0.66667	0.023	0.022
4	-0.04677	0.54	0.55556	-0.025	-0.026
5	-0.06366	0.364	0.44444	-0.023	-0.028
6	-0.03118	0.215	0.33333	-0.007	-0.01
7	0.01405	0.115	0.22222	0.002	0.003
8	0.03784	0.08	0.11111	0.003	0.004

Now we make the filter causal by shifting the unit pulse response of the non-causal filter. For

$$k := 0..16$$

the unit pulse response of the causal filter with the rectangular window is

$$hrc_k := hr_{k-8}$$

The unit pulse response of thr causal filter with the Hamming window is

$$hhc_k := hh_{k-8}$$

The unit pulse response of thr causal filter with the triangular window is

$$htc_k := ht_{k-8}$$

With the unit-pulse response of the causal filters defined, the frequency responses of the causal filters can be defined. The results are

$$Hr(r) := \sum_k hrc_k \cdot \exp(-i \cdot 2 \cdot \pi \cdot k \cdot r) \qquad \text{(Rectangular window)}$$

$$Hh(r) := \sum_k hhc_k \cdot \exp(-i \cdot 2 \cdot \pi \cdot k \cdot r) \qquad \text{(Hamming window)}$$

and finally

$$Ht(r) := \sum_k htc_k \cdot \exp(-i \cdot 2 \cdot \pi \cdot k \cdot r) \qquad \text{(Triangular window)}$$

The amplitude and phase responses are defined as

$$Ar(r) := |Hr(r)| \qquad \text{(Rectangular window)}$$

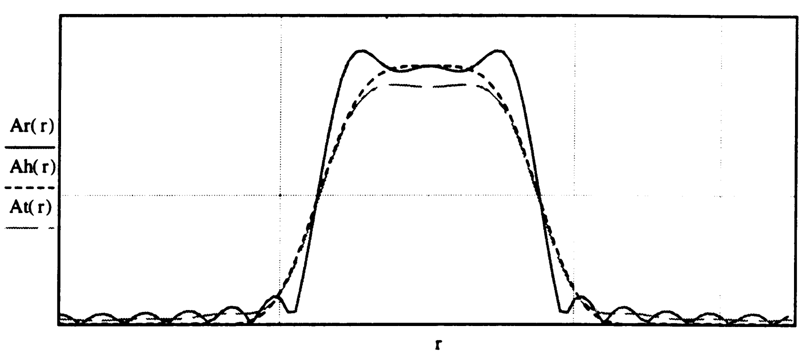
$$Ah(r) := |Hh(r)| \qquad \text{(Hamming window)}$$

$$At(r) := |Ht(r)| \qquad \text{(Triangular window)}$$

The range variable for the plot is defined as

$$r := -0.5, -0.49 .. 0.5$$

The amplitude responses are shown below.



Problem 9-60

The defining sum

$$W_r\left(e^{j2\pi r}\right) = \sum_{n=-M}^{M} e^{-j2\pi nr}$$

is first written

$$W_r\left(e^{j2\pi r}\right) = \sum_{k=0}^{2M} e^{-j2\pi(k-M)r}$$

This gives

$$W_r\left(e^{j2\pi r}\right) = e^{j2\pi Mr} \sum_{k=0}^{2M} e^{-j2\pi kr}$$

which is summed to give

$$W_r\left(e^{j2\pi r}\right) = e^{j2\pi Mr}\left(\frac{1-e^{-j2\pi(2M+1)r}}{1-e^{-j2\pi r}}\right)$$

This can be expressed

$$W_r\left(e^{j2\pi r}\right) = \frac{\left(e^{j\pi(2M+1)r} - e^{-j\pi(2M+1)r}\right)e^{-j\pi r}}{\left(e^{j\pi r} - e^{-j\pi r}\right)e^{-j\pi r}}$$

Thus

$$W_r\left(e^{j2\pi r}\right) = \frac{\sin \pi(2M+1)r}{\sin \pi r}$$

Problem 9-61

The unit pulse response is derived using (9-169). This gives

$$h_d(nT) = \int_{-1/2}^{0} 3e^{j2\pi nr}dr + \int_{0}^{1/2} 6e^{j2\pi nr}dr$$

which is

$$h_d(nT) = \frac{1.5}{j\pi n}\left(1-e^{-j\pi n}\right) + \frac{3}{j\pi n}\left(e^{j\pi n}-1\right)$$

Since

$$e^{-j\pi n} = e^{j\pi n} = \cos \pi n$$

We can write

$$h_d(nT) = j\frac{1.5}{\pi n} - j\frac{1.5}{\pi n}\cos \pi n,\ n \neq 0$$

The $n = 0$ term must be determined using

$$h_d(nT) = \int_{-1/2}^{0} 3dr + \int_{0}^{1/2} 6dr = 3\left(0 + \frac{1}{2}\right) + 6\left(\frac{1}{2} - 0\right) = \frac{9}{2}$$

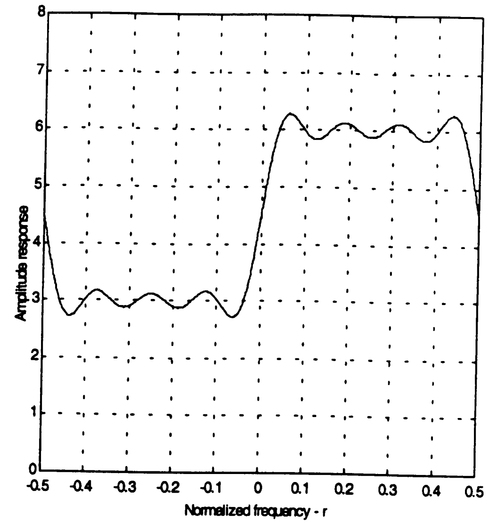
which could have been determined by inspection since the average value of the desired frequency response is clearly *4.5*.

Note that the unit pulse response, $h_d(nT)$, is complex. This was to be expected since the desired amplitude response is not an even function of frequency. It should be observed that if the average value of the desired frequency response is set equal to zero by subtracting 3 from $H_d(e^{j2\pi r})$, the desired frequency response is odd and, consequently, the unit pulse response becomes imaginary.

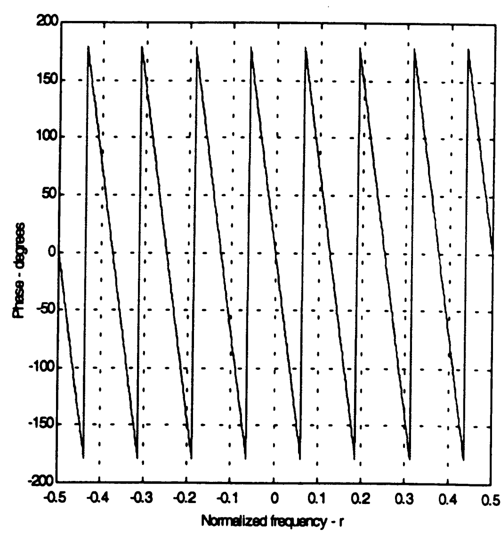
The MATLAB script for solving this problem is shown below.

```
n = -8:8;                                    % Vector n
hn = ((j*1.5/pi)./n).*(1-cos(n*pi));         % Weights
hn(9) = 4.5;                                 % Weight at n=0

r = -0.5:0.001:0.5;                          % Normalized frequency vector
re = exp(-j*2*pi*r);                         % Vector exponential

mx = zeros(17,1001);                         % Initialize matrix
for k=1:17                                   % Loop to fill matrix rows
mx(k,:) = hn(k)*(re(1,:) .^ (k-1));          % kth row of matrix
end                                          % End of loop

amp = abs(sum(mx));                          % Compute amplitude response
plot(r,amp)                                  % Plot amplitude response
axis([-0.5 0.5 0 8]);                        % Establish axis parameters
xlabel('Normalized frequency - r')           % Label x axis
ylabel('Amplitude response')                 % Label y axis
grid                                         % Place grid on plot
pause                                        % Pause to look
phase = (180/pi)*angle(sum(mx));             % Compute phase
plot(r,phase)                                % Plot phase response
xlabel('Normalized frequency - r')           % Label x axis
ylabel('Phase - degrees')                    % Label y axis
grid                                         % Place grid on plot
```

This yields the amplitude response



and the phase response



Problem 9-62

For the FIR filter shown

$$X(z)=\frac{1}{2}+z^{-2}+\frac{1}{2}z^{-4}$$

or

$$X(z)=\left[1+\frac{1}{2}\left(z^{2}+z^{-2}\right)\right]z^{-2}$$

In terms of the steady-state frequency variable, *r*, this is

$$X\left(e^{j2\pi r}\right)=\left[1+\frac{1}{2}\left(e^{j4\pi r}+e^{-j4\pi r}\right)\right]e^{-j4\pi r}$$

or

$$X\left(e^{j2\pi r}\right)=\left[1+\cos 4\pi r\right]e^{-j4\pi r}$$

The amplitude response is

$$A(r)=1+\cos 4\pi r$$

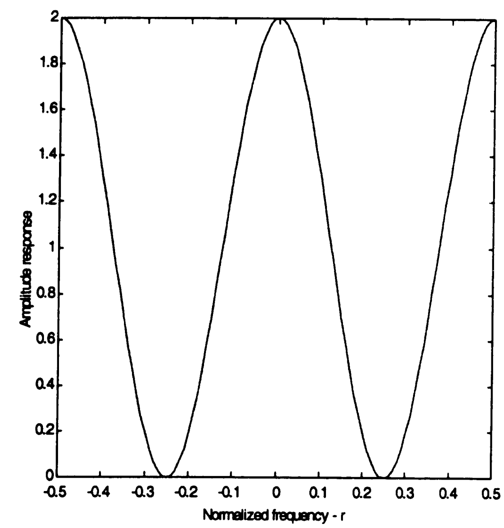
and the phase response is

$$\phi(r)=-4\pi r$$

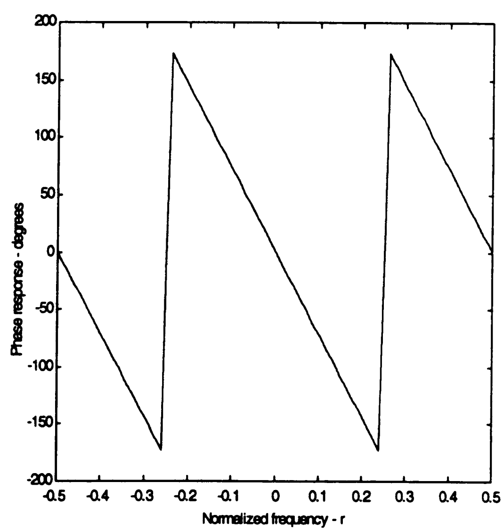
The MATLAB script for solving this problem is shown below.

```
r = -0.5:0.01:0.5;                            % Define frequency vector
z = exp(j*2*pi*r);                            % Steady-state response
hz = 0.5+(z.^(-2))+0.5*(z.^(-4));             % Transfer function
amp = abs(hz);                                % Amplitude response
plot(r,amp)                                   % Plot amplitude response
xlabel('Normalized frequency - r')            % Label x axis
ylabel('Amplitude response')                  % Label y axis
pause                                         % Pause to look
phase = (180/pi)*angle(hz);                   % Compute phase
plot(r,phase)                                 % Plot phase response
xlabel('Normalized frequency - r')            % Label x axis
ylabel('Phase response - degrees')            % Label y axis
```

Execution of this program gives the amplitude response



and the phase response



Problem 9-63

For the structure shown

$$X(z) = 5 + 5z^{-2} = 5(z + z^{-1})z^{-1}$$

In terms of the steady-state frequency variable, r, we have

$$X(e^{j2\pi r}) = 5(e^{j2\pi r} + e^{-j2\pi r})e^{-j2\pi r}$$

or

$$X(e^{j2\pi r}) = (10\cos 2\pi r)e^{-j2\pi r}$$

The amplitude response is

$$A(r) = |10\cos 2\pi r|$$

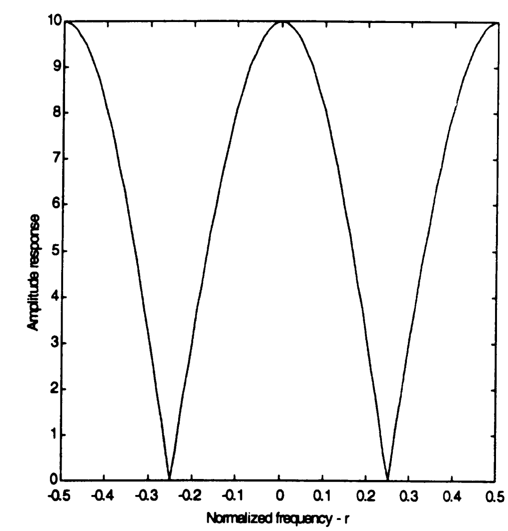
and the phase response is

$$\phi(r) = \begin{cases} -2\pi r, & \cos 2\pi r \geq 0 \\ -2\pi r \pm \pi, & \cos 2\pi r < 0 \end{cases}$$

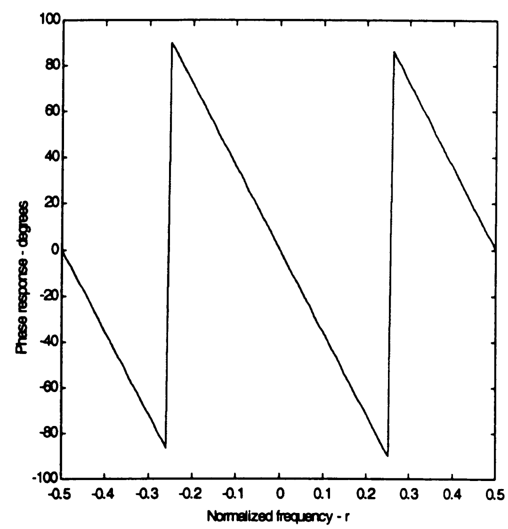
This problem is solved using the MATLAB script shown below.

```
r = -0.5:0.01:0.5;                          % Define frequency vector
z = exp(j*2*pi*r);                          % Steady-state response
hz = 5+5*(z.^(-2));                         % Transfer function
amp = abs(hz);                              % Amplitude response
plot(r,amp)                                 % Plot amplitude response
xlabel('Normalized frequency - r')          % Label x axis
ylabel('Amplitude response')                % Label y axis
pause                                       % Pause to look
phase = (180/pi)*angle(hz);                 % Compute phase
plot(r,phase)                               % Plot phase response
xlabel('Normalized frequency - r')          % Label x axis
ylabel('Phase response - degrees')          % Label y axis
```

Execution of this program yields the amplitude response



and the phase response



Problem 9-64

The unit pulse response coresponding to the desired frequency response is given by

$$h_d(nT) = \int_{-0.3}^{-0.1} 0.6e^{j2\pi nr}dr + \int_{-0.1}^{0.1} e^{j2\pi nr}dr + \int_{0.1}^{0.3} 0.6e^{j2\pi nr}dr$$

This yields

$$h_d(nT) = \frac{0.6}{j2\pi n}\left[e^{-j0.2\pi n} - e^{-j0.6\pi n}\right]$$
$$+\frac{1}{j2\pi n}\left[2^{j0.2\pi n} - e^{-j0.2\pi n}\right]$$
$$+\frac{0.6}{j2\pi n}\left[e^{j0.6\pi n} - e^{j0.2\pi n}\right]$$

Grouping terms together yields, for $n \neq 0$,

$$h_d(nT) = \frac{1}{\pi n}[\sin(0.2\pi n) + 0.6\sin(0.6\pi n) - 0.6\sin(0.2\pi n)]$$

The term for $n = 0$ is given by

$$h_d(nT) = \int_{-0.3}^{-0.1} 0.6e^{j2\pi nr}dr + \int_{-0.1}^{0.1} e^{j2\pi nr}dr + \int_{0.1}^{0.3} 0.6e^{j2\pi nr}dr$$

which is

$$h_d(0) = 0.6(-0.1 + 0.3) + (0.1 + 0.1) + 0.6(0.3 - 0.1) = 0.44$$

The filter weights are therefore given by

$$L_k = h_d(nT - 12T)\, w_h(n - 12)$$

where $w_h(n - 12)$ represents the delayed Hamming window function defined by

$$w_h(n-12) = \begin{cases} 0.54 + 0.46\cos\dfrac{\pi(n-12)}{12}, & 0 \le n \le 24 \\ 0, & \text{otherwise} \end{cases}$$

Problem 9-65

The amplitude and phase response of the filter developed in Problem 9-64 is easily determined using MathCAD.

We shall plot the amplitude and the phase response of the filter using the unit-pulse response expressed as a vector. Since the starting point is a non-causal filter we must shift the origin. Thus

$$\mathrm{ORIGIN} \equiv -12$$

The complex operator is first defined

$$i := \sqrt{-1}$$

The iteration variable

$$n := 1 .. 12$$

is next specified. The terms for h(n) can then be defined. This gives

$$h_n := \frac{1}{\pi \cdot n} \cdot (\sin(0.2 \cdot \pi \cdot n) + 0.6 \cdot \sin(0.6 \cdot \pi \cdot n) - 0.6 \cdot \sin(0.2 \cdot \pi \cdot n))$$

The unit-pulse response is even and h(0) is 0.44. Thus

$$h_{-n} := h_n \qquad h_0 := 0.44$$

The iteration variable is now changed to

$$n := -12 .. 12$$

The Hamming window is defined as

$$w_n := 0.54 + 0.46 \cdot \cos\left(\frac{\pi \cdot n}{12}\right)$$

The windowed unit-pulse response is next formed. This gives

$$hw_n := h_n \cdot w_n$$

The values of h(n), w(n) and hw(n) for the non-causal filter are now fully determined.

Thus we can form the following table.

n	h_n	w_n	hw_n
-12	0.00074	0.08	0.00006
-11	0.02332	0.096	0.00223
-10	0	0.142	0
-9	-0.0285	0.215	-0.00612
-8	-0.0011	0.31	-0.00034
-7	-0.00126	0.421	-0.00053
-6	-0.04275	0.54	-0.02308
-5	0	0.659	0
-4	0.06412	0.77	0.04937
-3	0.00294	0.865	0.00255
-2	0.00442	0.938	0.00414
-1	0.25648	0.984	0.25246
0	0.44	1	0.44
1	0.25648	0.984	0.25246
2	0.00442	0.938	0.00414
3	0.00294	0.865	0.00255
4	0.06412	0.77	0.04937
5	0	0.659	0
6	-0.04275	0.54	-0.02308
7	-0.00126	0.421	-0.00053
8	-0.0011	0.31	-0.00034
9	-0.0285	0.215	-0.00612
10	0	0.142	0
11	0.02332	0.096	0.00223
12	0.00074	0.08	0.00006

Now we make the filter causal by shifting the unit pulse of the non-causal filter. For

$$k := 0..24$$

the unit pulse response of the causal filter is

$$hc_k := hw_{k-12}$$

With the unit-pulse response of the causal filter defined, the frequency response of the causal filter can be defined. The result is

$$H(r) := \sum_k hc_k \cdot \exp(-i \cdot 2 \cdot \pi \cdot k \cdot r)$$

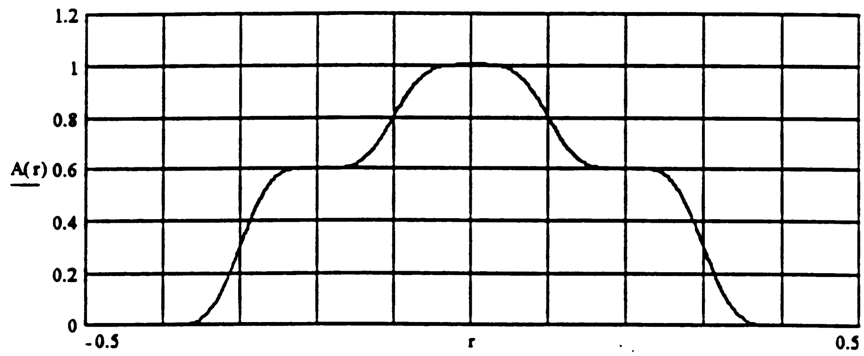
The amplitude and phase responses are defined as

$$A(r) := |H(r)| \qquad \phi(r) := \arg(H(r))$$

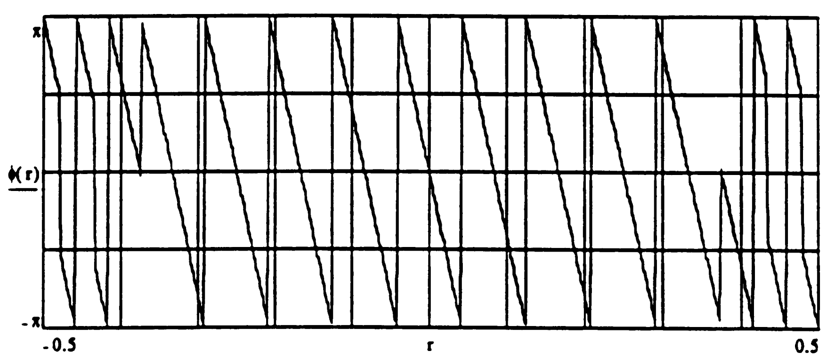
The range variable for the plot is defined as

$r := -0.5, -0.499 .. 0.5$

The amplitude response is shown below.



The phase response is shown below.



The student should experiment with various values of M and with various windows.

Problem 9-66

As a function of r, the desired frequency response can be written

$$H\left(e^{j2\pi r}\right)=\begin{cases}|1-5r|, & -\frac{1}{5}\le r\le\frac{1}{5}\\ 0, & \text{otherwise}\end{cases}$$

Thus

$$h(nT)=\int_{-1/5}^{0}(1+5r)e^{j2\pi nr}\,dr+\int_{0}^{1/5}(1-5r)e^{j2\pi nr}\,dr$$

which, by symmetry, can be written

$$h(nT)=2\int_{0}^{1/5}(1-5r)\cos(2\pi nr)dr$$

With the charge of variable $x = 2\pi nr$, we can write

$$\begin{aligned}h(nT)&=2\int_{0}^{0.4\pi n}\left(1-\frac{5x}{2\pi n}\right)\cos x\frac{dx}{2\pi n}\\ &=\frac{1}{\pi n}\sin(0.4\pi n)-\frac{5}{2(\pi n)^2}\int_{0}^{0.4\pi n}x\cos x\,dx\end{aligned}$$

Since

$$\int_{0}^{0.4\pi n}x\cos x\,dx=\frac{2}{5}(\pi n)\sin(0.4\pi n)+\cos(0.4\pi n)-1$$

we have

$$h(nT)=\frac{2.5}{(\pi n)^2}(1-\cos(0.4\pi n)),\qquad n\neq 0$$

Since $h(0)$ is the area under the desired frequency response

$$h(0)=2(1)\ (0.2)\left(\frac{1}{2}\right)=0.2$$

the filter weights of the causal filter are therefore given by

$$L_k=h(nT-12T)\ w_h(n-12)$$

where, $w_h(n-12)$ represents the delayed Hamming window function defined by

$$w_h(n-12)=\begin{cases}0.54+0.46\cos\dfrac{\pi(n-12)}{12}, & 0\le n\le 24\\ 0, & \text{otherwise}\end{cases}$$

Problem 9-67

The amplitude and phase response of the filter developed in Problem 9-66 is easily determined using MathCAD.

We shall plot the amplitude and the phase response of the filter using the unit-pulse response expressed as a vector. Since the starting point is a non-causal filter we must shift the origin. Thus

$$\mathrm{ORIGIN} \equiv -12$$

The complex operator is first defined

$$i := \sqrt{-1}$$

The iteration variable

$$n := 1..12$$

is next specified. The terms for h(n) can then be defined. This gives

$$h_n := \frac{2.5}{(\pi\cdot n)^2}\cdot(1-\cos(0.4\cdot\pi\cdot n))$$

The unit-pulse response is even and h(0) is 0.2. Thus

$$h_{-n} := h_n \qquad\qquad h_0 := 0.2$$

The iteration variable is now changed to

$$n := -12..12$$

The Hamming window is defined as

$$w_n := 0.54 + 0.46\cdot\cos\left(\frac{\pi\cdot n}{12}\right)$$

The windowed unit-pulse response is next formed. This gives

$$hw_n := h_n\cdot w_n$$

The values of h(n), w(n) and hw(n) for the non-causal filter are now fully determined.

We can therefore develop the following table of data:

n	h_n	w_n	hw_n
-12	0.00318	0.08	0.00025
-11	0.00145	0.096	0.00014
-10	0	0.142	0
-9	0.00216	0.215	0.00046
-8	0.00716	0.31	0.00222
-7	0.00935	0.421	0.00394
-6	0.00486	0.54	0.00263
-5	0	0.659	0
-4	0.01094	0.77	0.00842
-3	0.05091	0.865	0.04405
-2	0.11456	0.938	0.1075
-1	0.17503	0.984	0.17228
0	0.2	1	0.2
1	0.17503	0.984	0.17228
2	0.11456	0.938	0.1075
3	0.05091	0.865	0.04405
4	0.01094	0.77	0.00842
5	0	0.659	0
6	0.00486	0.54	0.00263
7	0.00935	0.421	0.00394
8	0.00716	0.31	0.00222
9	0.00216	0.215	0.00046
10	0	0.142	0
11	0.00145	0.096	0.00014
12	0.00318	0.08	0.00025

Now we make the filter causal by shifting the unit pulse of the non-causal filter. For

$$k := 0..24$$

the unit pulse response of the causal filter is

$$hc_k := hw_{k-12}$$

With the unit-pulse response of the causal filter defined, the frequency response of the causal filter can be defined. The result is

$$H(r) := \sum_k hc_k \cdot \exp(-i \cdot 2 \cdot \pi \cdot k \cdot r)$$

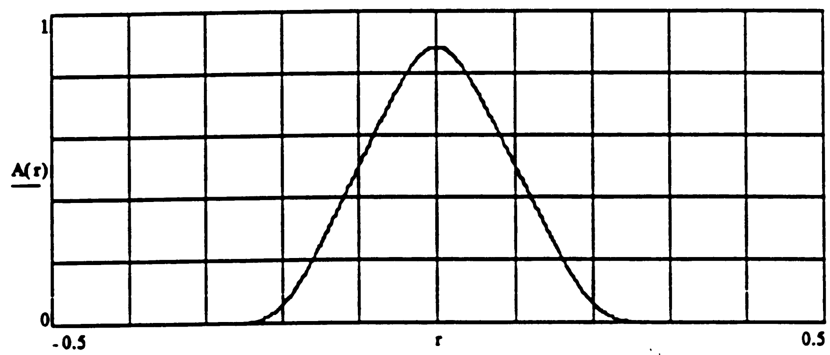
The amplitude and phase responses are defined as

$$A(r) := |H(r)| \qquad \phi(r) := \arg(H(r))$$

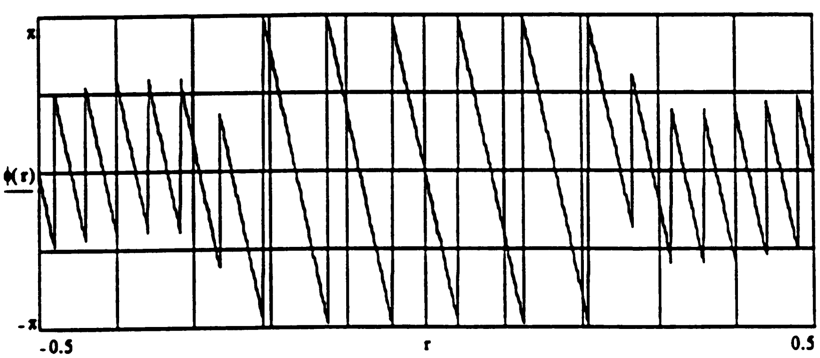
The range variable for the plot is defined as

$r := -0.5, -0.499 .. 0.5$

The amplitude response is shown below.



The phase response is shown below.



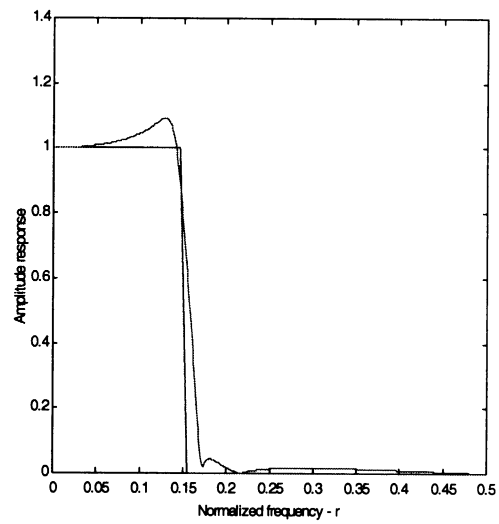
Note that the phase is linear, with a number of jumps when the response changes sign.

Problem 9-68

For the FIR filter we use the following script file:

```
f = [0 0.29 0.31 1];                    % Frequency vector
m = [1 1 0 0];                          % Magnitude vector
n = 10;                                 % Order of filter
[b,a] = yulewalk(n,f,m);                % Synthesize filter
[h,w] = freqz(b,a,512);                 % Determine frequency response
amp = abs(h);                           % Determine amplitude response
r = w/(2*pi);                           % Specify normalized frequency
plot(f/2,m,r,amp)                       % Execute plot
xlabel('Normalized frequency - r')      % Label x axis
ylabel('Amplitude response')            % Label y axis
```

This yields the amplitude response shown below.

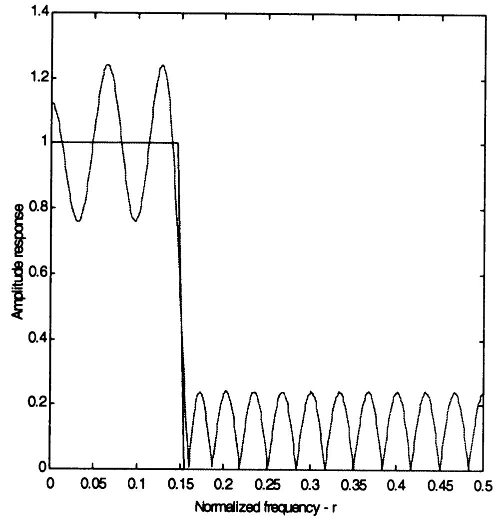


Note that both the desired and the actual response of the resulting IIR filter for order = 10 are shown. The interested student should investigate the effect of changing the filter order.

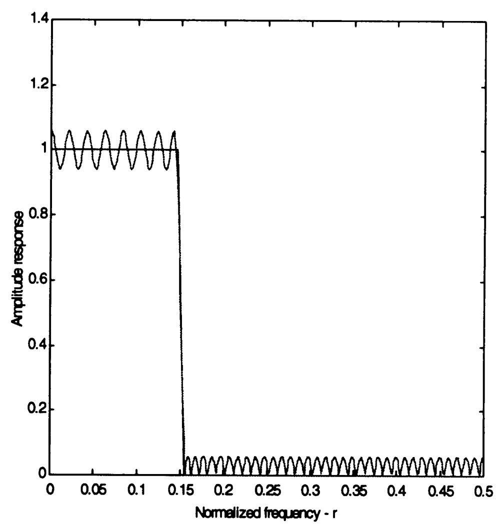
The FIR digital filter is designed using the following MATLAB script.

```
f = [0 0.29 0.31 1];                    % Frequency vector
m = [1 1 0 0];                          % Magnitude vector
n = 30;                                 % Order of filter
b = remez(n,f,m);                       % Synthesize filter
[h,w] = freqz(b,1,512);                 % Determine freuency response
amp = abs(h);                           % Determine amplitude response
r = w/(2*pi);                           % Specify normalized frequency
plot(f/2,m,r,amp)                       % Execute plot
xlabel('Normalized frequency - r')      % Label x axis
ylabel('Amplitude response')            % Label y axis
```

Executing this MATLAB program results in the amplitude response shown below.



As shown in the program, this result is for filter order = 30. For a filter order equal to 100, the amplitude response shown below results.



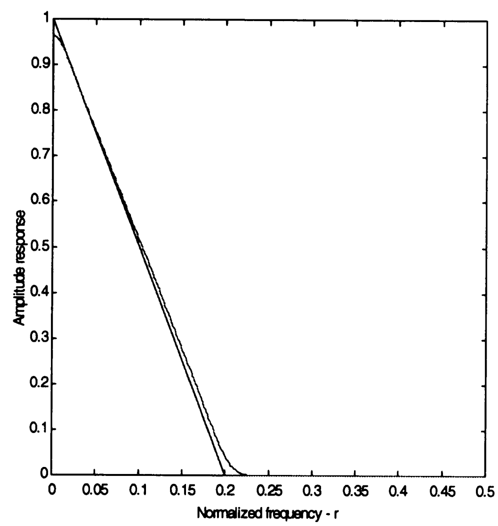
It can be seen that, as expected, increasing the filter order raises the frequency of the passband and stopband ripple and reduces the amplitude of the ripple.

Problem 9-69

First we design the IIR filter. The appropriate MATLAB script is shown below.

```
f = [0 0.1 0.2 0.3 0.4 1];                % Frequency vector
m = [1 0.75 0.5 0.25 0 0];                % Magnitude vector
n = 10;                                   % Order of filter
[b,a] = yulewalk(n,f,m);                  % Synthesize filter
[h,w] = freqz(b,a,512);                   % Determine frequency response
amp = abs(h);                             % Determine amplitude response
r = w/(2*pi);                             % Normalized frequency vector
plot(f/2,m,r,amp)                         % Execute plot
xlabel('Normalized frequency - r')        % Label x axis
ylabel('Amplitude response')              % Label y axis
```

Executing the program yields the amplitude response shown below for order = 10.

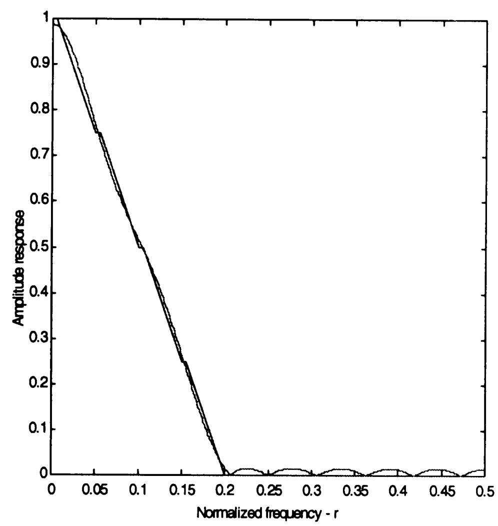


The results look reasonable. We now look at the FIR filter. This is more of a problem since the desired amplitude response must be entered in bands. In other words, the **remez** program assumes that the desired amplitude response is piecewise constant. Our desired amplitude is piecewise linear but not piecewise constant. There is a way around this problem for the desired amplitude response, as can be seen from the following MATLAB program.

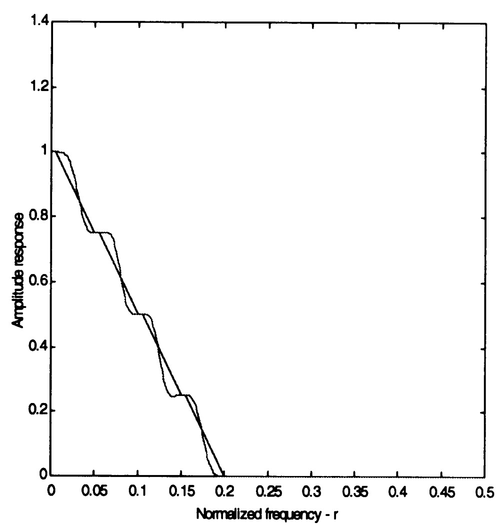
```
% The following two lines of code specify the frequency and
% magnitude vectors
f = [0 0.01 0.1 0.11 0.2 0.21 0.3 0.31 0.4 1];
m = [1 1 0.75 0.75 0.5 0.5 0.25 0.25 0 0];

n = 20;                                   % Order of filter
b = remez(n,f,m);                         % Synthesize filter
[h,w] = freqz(b,1,512);                   % Determine frequency response
amp = abs(h);                             % Determine amplitude response
```

```
r = w/(2*pi);                           % Normalized frequency vector
plot(f/2,m,r,amp)                       % Execute plot
xlabel('Normalized frequency - r')      % Label x axis
ylabel('Amplitude response')            % Label y axis
```

Execution of the program yields the following amplitude response.



Note that both the desired and the actual amplitude response are plotted. The results look reasonable for order = 20. Typically the result can be improved by increasing the order of the filter. Let's try order = 100. The result is shown below.



This result is clearly worse than that achieved for order = 20. This result occurs because the higher order filter results in an amplitude response that is a closer match to the desired filter in the designated frequency bands.

CHAPTER 10

Problem 10-1

From (10-1)

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} e^{-j2\pi nk/N}, \quad 0 \le k \le N-1$$

Use (4-136) to sum the series, giving the result

$$X_k = \begin{cases} N, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

That is, a sampled "dc" signal has a spectrum with a line at zero frequency and is zero elsewhere.

Problem 10-2

(a) In this case, the DFT sum is

$$X_k = \sum_{n=0}^{K} e^{-j2\pi nk/N} = \frac{1 - \left(e^{-j2\pi kn/N}\right)^{K+1}}{1 - e^{-j2\pi K/N}} = \frac{1 - e^{-j2\pi k(K+1)n/N}}{1 - e^{-j2\pi K/N}}, \quad 0 < k \le N-1$$

For $k = 0$ the sum must be considered as a special case; the result is $X_0 = K + 1$. The magnitude squared is

$$|X_k|^2 = \begin{cases} \dfrac{1 - \cos[2\pi k(K+1)/N]}{1 - \cos(2\pi K/N)}, & k = 1, 2, \ldots, N-1 \\ (K+1)^2, & k = 0 \end{cases}$$

(b) Since the given signal is a sampled version of a square pulse, we would like to have a $\sin(x)/x$ spectrum, but aliasing due to sampling provides a distorted version of this ideal shape.

Problem 10-3

(a) For the given signal,

$$\tilde{X}_k = \sum_{n=0}^{N-1} A e^{j(2\pi Ln/N + \theta_0)} e^{-j2\pi kn/N} = A e^{j\theta_0} \sum_{n=0}^{N-1} e^{j2\pi n(L-K)/N} = A e^{j\theta_0} \frac{1 - e^{j2\pi(L-k)}}{1 - e^{j2\pi(L-k)/N}}$$

which may be reduced to the result given in the problem (for $k = L$ we have to consider the sum as a special case).

(b) Case 1: The result is 8 for $k = 1$ and 0 otherwise. For case 1, we would like to have a large line near $k = 1.1$ and negligible ones otherwise. This is impossible, for the DFT can put out frequency estimates only at integer values, so the result is a large line at $k = 1$ and small, but significant, lines for other k-values due to leakage.

Problem 10-4

Write the given signal as

$$x_n = \frac{A}{2} e^{j(2\pi Ln/N + \theta_0)} + \frac{A}{2} e^{-j(2\pi Ln/N + \theta_0)}$$

Using superposition and the result of Problem 10-3, we obtain

$$X_k = \begin{cases} \dfrac{A}{2} \dfrac{\sin \pi(L-k)}{\sin \pi(L-k)/N} \cos[\pi(L-k)(1-1/N) + \theta_0], & k \neq L \\ \dfrac{NA}{2} \cos\theta_0, & k = L \end{cases}$$

Problem 10-5

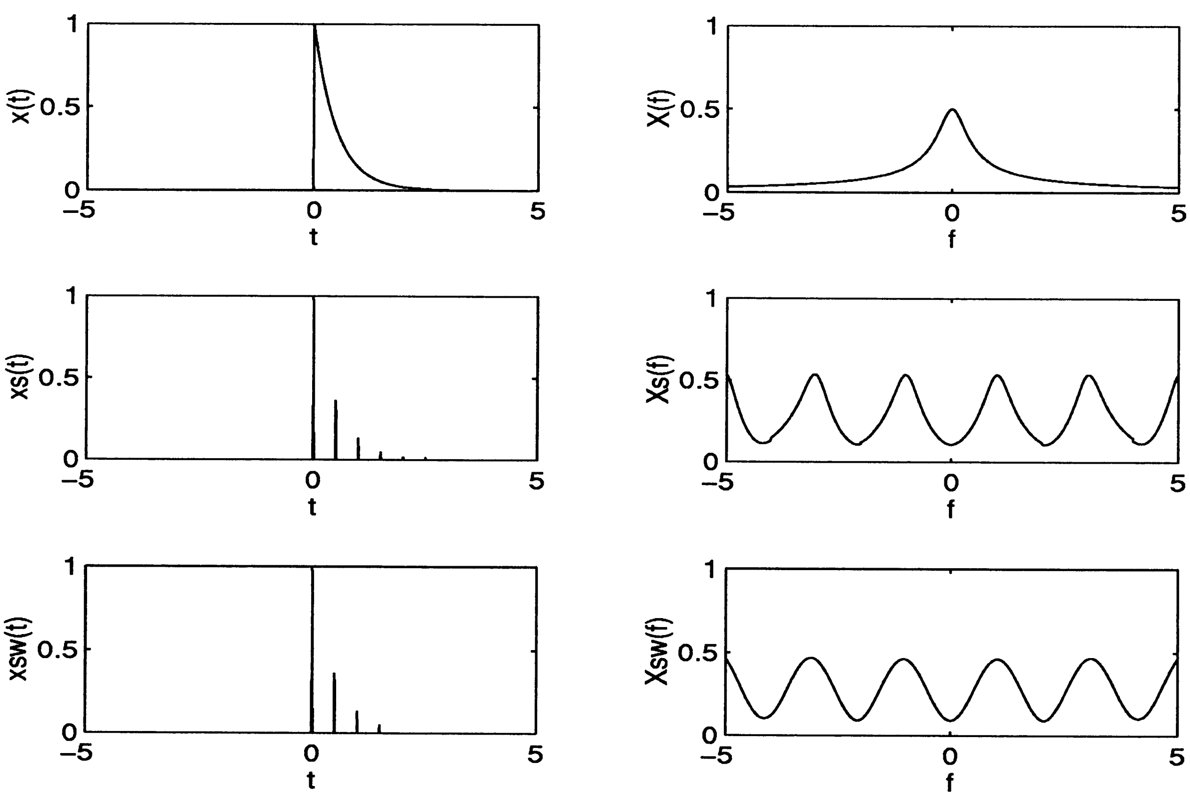
Only the first signal will be done. We have the transform pair

$$x(t) = Ae^{-\alpha t}u(t) \leftrightarrow X(f) = \frac{A}{\alpha + j2\pi f}$$

Plots for the various signals and spectra shown in Figure 10-1 are shown below for the parameter values $A = 1$, $\alpha = 2$, $f_s = 2$, and $T = 2$. A MATLAB program is given below for making the plots.

```
%       Plots for Problem 10-5
%
clg
del = 0.01;
fs = 2;
```

```
t = -3:del:3;
f = -7:del:7;
L = length(f);
fp = [2*f(1):del:2*f(L)];
LL = length(fp);
fpp = [fp(1)+f(1):del:f(L)+fp(LL)];
x = exp(-2*t).*stp_fn(t);
xs = x.*cmb_fn(t-.01,1/fs,6,.01)*.01;
xsw = xs.*pls_fn((t-1)/2);
X = 1./(2 + j*2*pi*f);
Xs = del*conv(X, cmb_fn(f,fs,10,.01));
Xw = 2*sinc(2*f);
Xsw = del*conv(Xs,Xw);
LL = length(Xsw)
subplot(3,2,1), plot(t,x,'-w'),xlabel('t'),ylabel('x(t)'),axis([-5 5 0 1])
subplot(3,2,2), plot(f,abs(X),'-w'),xlabel('f'),ylabel('X(f)'),axis([-5 5 0 1])
subplot(3,2,3), plot(t,xs,'-w'),xlabel('t'),ylabel('xs(t)'),axis([-5 5 0 1])
subplot(3,2,4), plot(fp,abs(Xs),'-w'),xlabel('f'),ylabel('Xs(f)'),axis([-5 5 0 1])
subplot(3,2,5), plot(t,xsw,'-w'),xlabel('t'),ylabel('xsw(t)'),axis([-5 5 0 1])
subplot(3,2,6), plot(fpp,abs(Xsw),'-w'),xlabel('f'),ylabel('Xsw(f)'),axis([-5 5 0 1])
```



The final spectrum is sampled each $1/T = 0.5$ Hz which repeats the time domain signal each 2 s.

Problem 10-6

(a) We first find the DFT of the signal

$$x_n = e^{-an}\cos(2\pi bn)$$

and then let $a = 1/10$ and $b = 1/2$. Note that this signal can be written as

$$x_n = 0.5e^{-an}e^{j2\pi bn} + 0.5e^{-an}e^{-j2\pi bn} = 0.5\tilde{x}_n + 0.5\tilde{x}_n^*$$

where the tilde denotes a complex quanitity. Also note that

$$\text{DFT}[\tilde{x}_n^*] = \tilde{X}_{-k}^*$$

Now,

$$\text{DFT}[\tilde{x}_n] = \tilde{X}_k = \sum_{n=0}^{N-1} e^{-an}e^{j2\pi bn}e^{-j2\pi nk/N} = \sum_{n=0}^{N-1}\left[e^{j2\pi b - j2\pi k/N - a}\right]^n$$
$$= \frac{1 - e^{j(2\pi b - j2\pi k/N - a)N}}{1 - e^{j2\pi b - j2\pi k/N - a}}$$

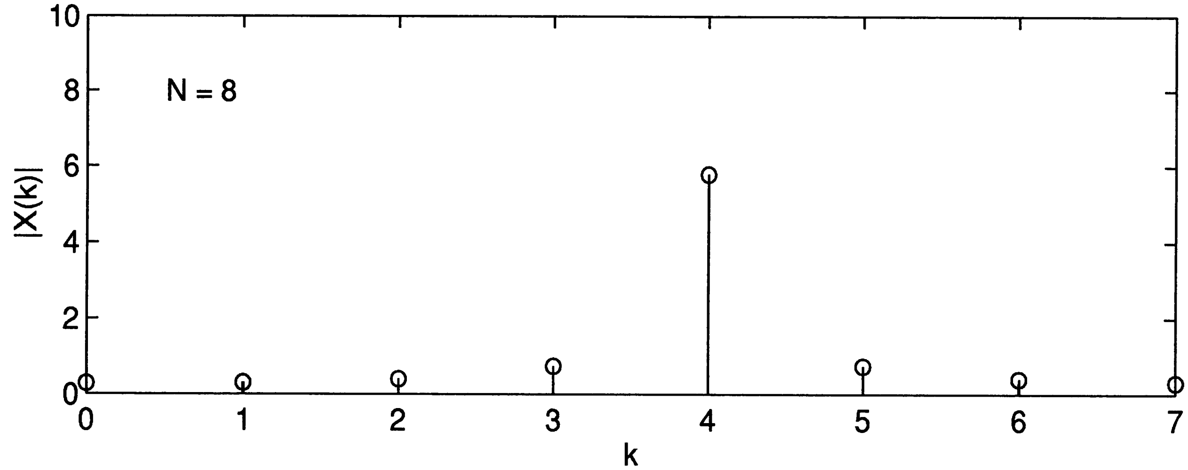
where (4-136) has been used. This may be simplified to

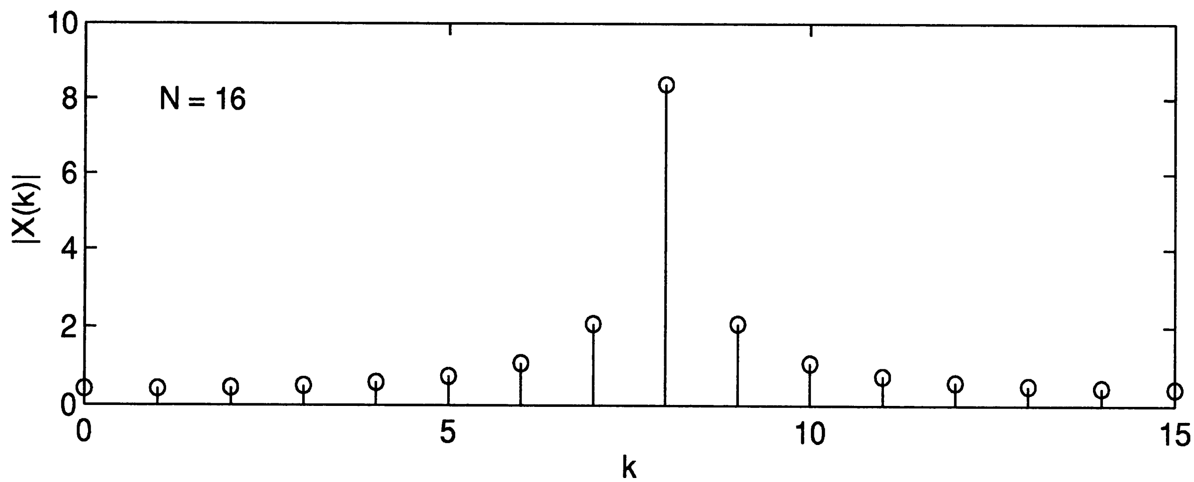
$$\tilde{X}_k = \frac{1 - e^{-an}e^{j2\pi bN}}{1 - e^{-a}e^{j2\pi(b - k/N)}}$$

where use has been made of the fact that $\exp(-j2\pi k) = 1$ for k an integer. Using linearity of the DFT, we obtain

$$\tilde{X}_k = \frac{1}{2}\frac{1 - e^{-an}e^{j2\pi bN}}{1 - e^{-a}e^{j2\pi(b - k/N)}} + \frac{1}{2}\frac{1 - e^{-an}e^{-j2\pi bN}}{1 - e^{-a}e^{-j2\pi(b + k/N)}}$$

(b) Plots are shown below for $N = 8$ and 16. The peaks are different due to aliasing as will be shown in part (c).





(c) Since

$$x(\Delta t) = e^{-2n\Delta t}\cos(20\pi n\Delta t) = e^{-a/10}\cos(\pi n)$$

We deduce that $\Delta t = 1/20$ s and $f_s = 20$ Hz. Thus, the right-most value of $k + 1$ for each graph corresponds to 20 Hz.

Using the s-shift theorem, we obtain the Laplace transform of the continuous signal as

$$X(s) = \frac{s+2}{(s+2)^2 + (20\pi)^2}$$

Let $s = j\omega$ to get the Fourier transform. It is

$$X(j\omega) = \frac{j\omega + 2}{(j\omega + 2)^2 + (20\pi)^2}$$

The magnitude of this Fourier transform peaks at about 20π rad/s or 10 Hz, which is where the discrete spectral plots shown above peak.

Problem 10-7

Write the input as

$$x_n = \cos^2(\pi n/4) = \frac{1}{2} + \frac{1}{2}\cos(\pi n/2) = \begin{cases} \frac{1}{2} + \frac{1}{2}\cos[2\pi(2n/8)], \; N = 8 \\ \frac{1}{2} + \frac{1}{2}\cos[2\pi(4n/16)], \; N = 16 \end{cases}$$

We may write $\cos[2\pi(\ell n/N)]$ as

$$\cos[2\pi(\ell n/N)] = \frac{1}{2}e^{j2\pi(\ell n/N)} + \frac{1}{2}e^{-j2\pi(\ell n/N)}$$

Following Example 10-2, we may use (10-18) to get

$$X_k = \frac{1}{2}(8)\delta_{k,0} + \frac{1}{4}(8)\delta_{k,2} + \frac{1}{4}(8)\delta_{k,-2}, \; N = 8$$

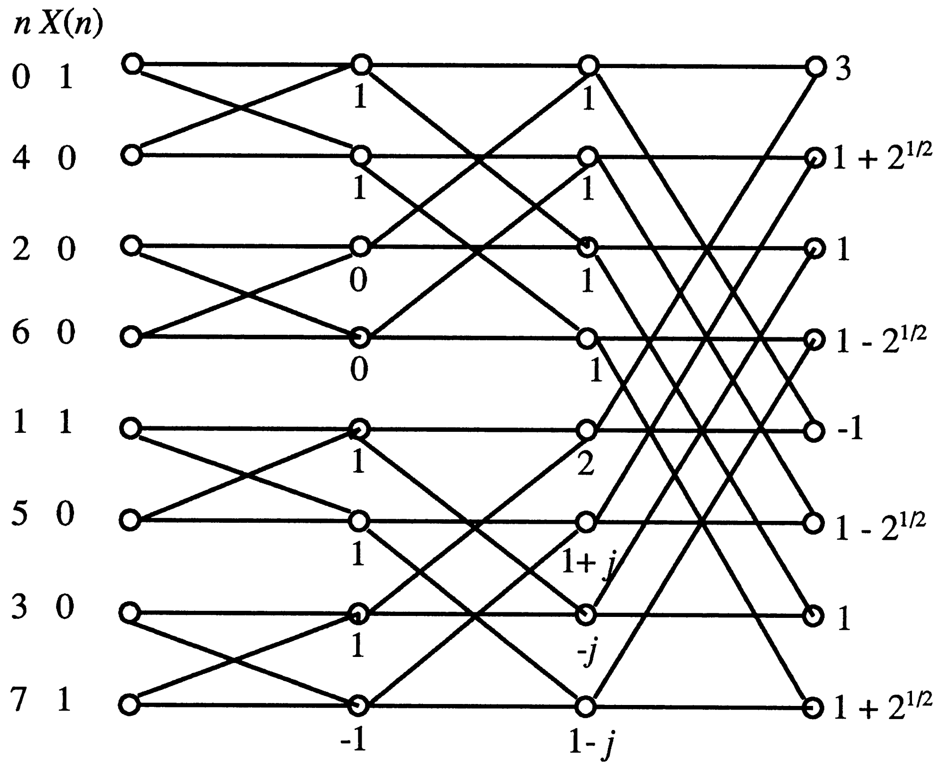
and

$$X_k = \frac{1}{2}(16)\delta_{k,0} + \frac{1}{4}(16)\delta_{k,4} + \frac{1}{4}(16)\delta_{k,-4}, \; N = 16$$

Thus, spectral lines are present at dc, and the discrete equivalent of 1/4 the sampling frequency.

Problem 10-8

(a) A flow graph for the FFT is shown below with corresponding numbers at the outputs of the recursions. For simplicity, powers of W_8 are not shown.



(b) For a square pulse signal, which this is the sampled version of, one would expect a $\sin(x)/x$ spectrum. However, aliasing distorts this desired ideal considerably.

(c) Consider this as an approximation for a sampled rectangular pulse train signal expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - 6T_s}{3T_s}\right)$$

Then the Fourier coeffients are

$$\overline{X_k} = \frac{3T_s}{8T_s}\text{sinc}(3T_s \times kf_0),\ f_0 = \frac{1}{8T_s}$$

where the overbar is used to distinguish these Fourier coefficients from the FFT values. We multiply through by 8 to normalize both sets of coefficients to be the same scale. Thus

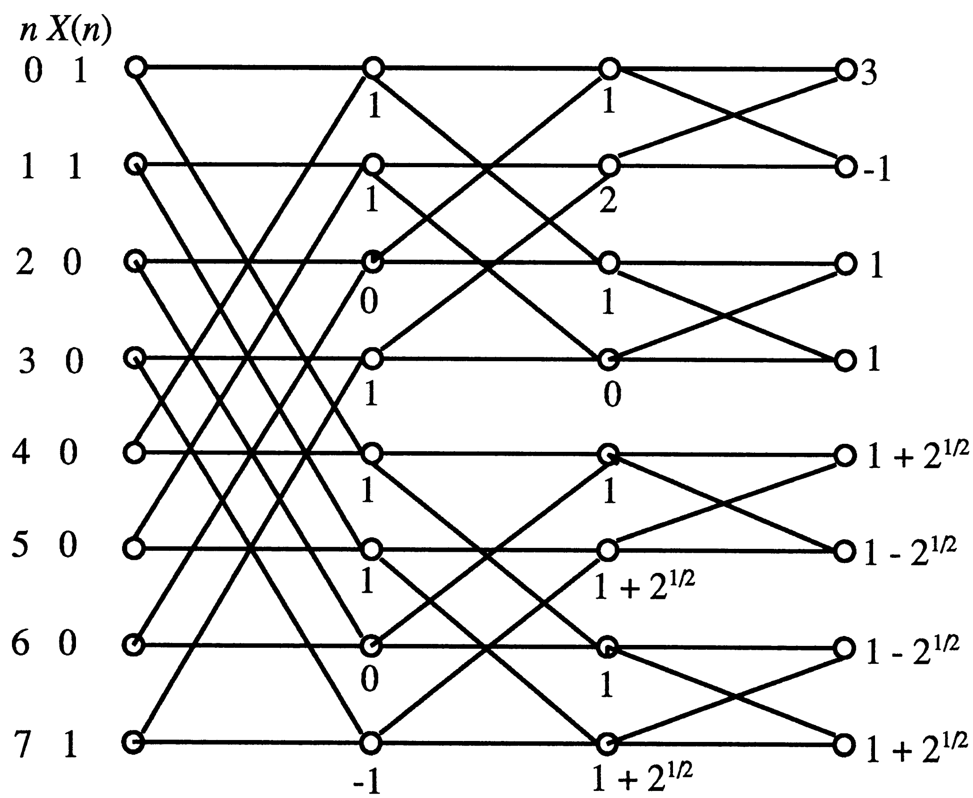
$$8\overline{X_k} = 3\,\mathrm{sinc}(3k/8)$$

The two sets of values are compared in the table below. The differences between the two sets of results are due to aliasing.

k	FFT coefficients	Fourier series coefficients
0	3	3
1	2.414	2.353
2	1	0.9
3	-0.414	-0.325
4	-1	-0.637

Problem 10-9

(a) The flow graph with numbers at the output of each recursion is shown below. The powers of W are not shown for simplicity. Taking into account the scrambling of the output points, this is the same result as obtained in Problem 10-8. (b) Use (4-142) to show that the original sequence results.



Problem 10-10

(a) The flow graph is similar to that of Problem 10-8. The result of the FFT operation on the given signal is

$$X_0 = 7;\ X_1 = 1;\ X_2 = -1;\ X_3 = 1;\ X_4 = -1;\ X_5 = 1;\ X_6 = -1;\ X_7 = 1$$

We may view the input samples as the result of sampling the periodic signal

$$x(t) = 1 - \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - 4T_s - nT_s}{T_s}\right)$$

The Fourier coefficients of this signal are

$$\overline{X_k} = \delta_{k,0} - \frac{T_s}{8T_s}\mathrm{sinc}(kf_0T_s) = \delta_{k,0} - \frac{1}{8}\mathrm{sinc}(k/8),\ f_0 = 1/8T_s$$

where δ_{ij} is 1 if $i = j$ and 0 otherwise. The difference between the above result (renormalized by 8) is considerable due to aliasing.

(b) Simply run the above output samples through the same algorithm again and get the original signal samples multiplied by 8.

Problem 10-11

First, from (10-46), note that each recursion is formed from butterflies of the form

$$X_{m+1}(p) = X_m(p) + W_N^s X_m(q)$$
$$X_{m+1}(q) = X_m(p) - W_N^s X_m(q)$$

where m is the recursion number. To get the power s, let

$$\mathrm{LN} = \log_2(N);\ \mathrm{LP} = \mathrm{LN} - m\ (m = \text{recursion number});\ \mathrm{LM} = 2^{\mathrm{LP}}$$

For the mth recursion, counting from the top,

$$s = \mathrm{LM}*I,\ I = 0, 1, \ldots, \frac{N}{2^{\mathrm{LP}} + 1} - 1$$

Problem 10-12

(a) Write the DFT sum as

$$X(k) = \sum_{m=0}^{N/4-1} x(4m)\,W_N^{4mk} + \sum_{m=0}^{N/4-1} x(4m+1)\,W_N^{(4m+1)k} + \sum_{m=0}^{N/4-1} x(4m+2)\,W_N^{(4m+2)k} + \sum_{m=0}^{N/4-1} x(4m+3)\,W_N^{(4m+3)k}$$

where it is assumed that N is a power of 4. Note that the above sums include all terms of the DFT sum. Now factor W_n^k from the second sum, W_N^{2k} from the third sum, and W_N^{2k} from the fourth sum. Also note that W_N^{4k} can be written as $W_{N/4}^{k}$. The resulting simplified sums are

$$X(k) = \sum_{m=0}^{N/4-1} x(4m)\,W_{N/4}^{mk} + W_N^k \sum_{m=0}^{N/4-1} x(4m+1)\,W_{N/4}^{mk} + W_N^{2k} \sum_{m=0}^{N/4-1} x(4m+2)\,W_{N/4}^{mk} + W_N^{3k} \sum_{m=0}^{N/4-1} x(4m+3)\,W_{N/4}^{mk}$$

The sums are DFTs of size $N/4$, and they are periodic with period $N/4$. Thus, write $X(k)$ as

$$X(k) = G(k) + W_N^k H(k) + W_N^{2k} I(k) + W_N^{3k} J(k)$$

For $N = 16$, the flow graph on the next page results. Only the first two and last two outputs are shown for clairity. For the 4-point DFT algorithm, see Example 10-4.

(b) Follow the pattern for simplifying the DFT equations developed in part (a) using the decimation-in-frequency approach. Another way is to flip the decimation-in-time diagram left-to-right.

Problem 10-13

The Butterfly equations are

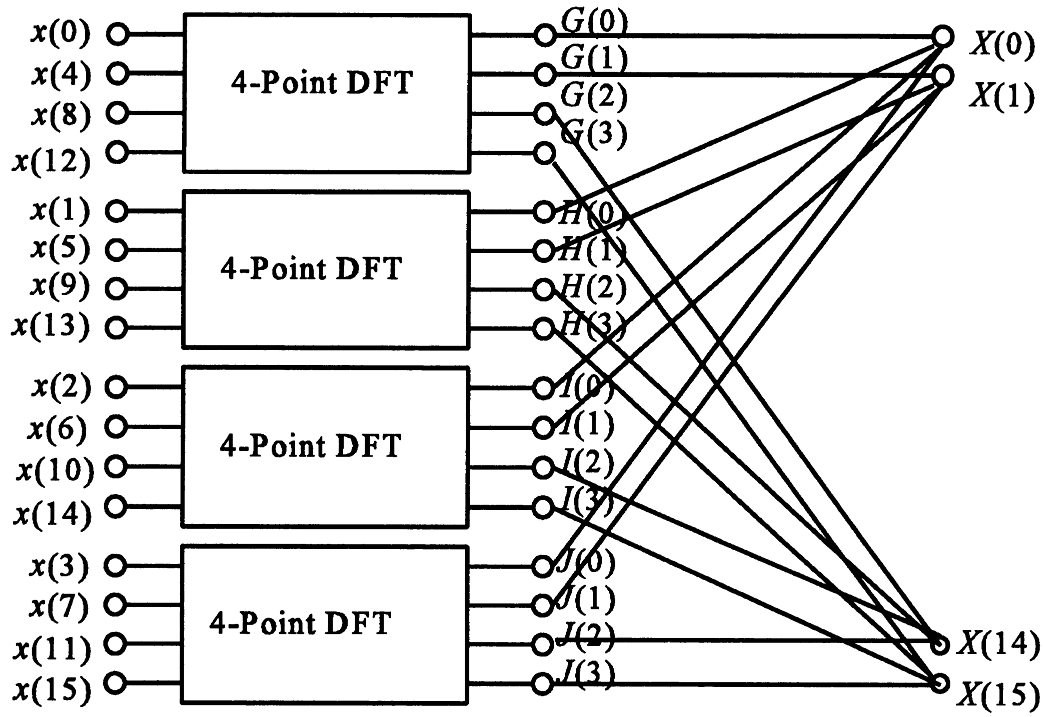
$$X_{m+1}(p) = X_m(p) + W_N^s X_m(q)$$
$$X_{m+1}(q) = X_m(p) - W_N^s X_m(q)$$

Let

$$X_m(p) = \text{Re}[X_m(p)] + j\text{Im}[X_m(p)] = RX_m(p) + jIX_m(p)$$

and similarly for the second equation. Also note that by Euler's theorem, the first equation becomes

$$RX_{m+1}(p) + jIX_{m+1}(p) = [RX_m(p) + RX_m(q)C + IX_m(q)S] + j[IX_m(p) + IX_m(q)C - RX_m(q)S]$$



Flow graph for Problem 10-12

Problem 10-13 - continued

where

$$W_N^s = e^{-j2\pi s/N} = \cos(2\pi s/N) - j\sin(2\pi s/N) = C - jS$$

The second equation becomes

$$RX_{m+1}(q) + jIX_{m+1}(q) = [RX_m(p) - RX_m(q)C - IX_m(q)S] + j[IX_m(p) - IX_m(q)C + RX_m(q)S]$$

Matching real and imaginary parts on both sides of the above two equations, we obtain

$$RX_{m+1}(p) = RX_m(p) + RX_m(q)C + IX_m(q)S$$
$$IX_{m+1}(p) = IX_m(p) + IX_m(q)C - RX_m(q)S$$
$$RX_{m+1}(q) = RX_m(p) - RX_m(q)C - IX_m(q)S$$
$$IX_{m+1}(q) = IX_m(p) - IX_m(q)C + RX_m(q)S$$

A block diagram can be constructed from these equations with the real and imaginary samples.

Problem 10-14

The butterfly expressions are

$$X_{m+1}(p) = X_m(p) + X_m(q)$$
$$X_{m+1}(q) = [X_m(p) - X_m(q)]\, W_N^s$$

From the first equation

$$RX_{m+1}(p) = RX_m(p) + RX_m(q) \text{ and } IX_{m+1}(p) = IX_m(p) + IX_m(q)$$

where the notation is the same as used in Problem 10-13. Using Euler's theorem to expand W_n^s in the second butterfly equation and equating real and imaginary parts on either side, we obtain

$$RX_{m+1}(q) = [RX_m(p) - RX_m(q)]\cos(2\pi s/N) - [IX_m(p) - IX_m(q)]\sin(2\pi s/N)$$
$$IX_{m+1}(q) = -[IX_m(p) - IX_m(q)]\cos(2\pi s/N) - [RX_m(p) - RX_m(q)]\sin(2\pi s/N)$$

This butterfly and the one considered in Problem 10-13 require exactly the same number of real multiplications and real additions.

Problem 10-15

Let

$$X_1(k) = \text{DFT}[x_1(n)] \text{ and } X_2(k) = \text{DFT}[x_2(n)]$$

Then

$$\begin{aligned}\text{DFT}[ax_1(n) + bx_2(n)] &= \sum_{n=0}^{N-1} [ax_1(n) + bx_2(n)]\, W_N^{kn} \\ &= \sum_{n=0}^{N-1} ax_1(n) + \sum_{n=0}^{N-1} bx_2(n)\, W_N^{kn} = aX_1(k) + bX_2(k)\end{aligned}$$

which proves the linearity property.

Problem 10-16

(a) Let

$$Y(k) = \text{DFT}[x(n-m)] = \sum_{n=0}^{N-1} x(n-m)W_N^{kn} = \sum_{\ell=-m}^{N--M} x(\ell)e^{-j2\pi(\ell+m)k/N}$$

where the substitution $\ell = n - m$ has been used. The periodicity of the inverse DFT implied by frequency sampling allows the shift of the sumto get

$$\left[\sum_{\ell=0}^{N-1} x(\ell)W_N^{\ell k}\right]e^{-j2\pi mk/N} = X(k)e^{-j2\pi mk/N}$$

(b) Frequency shift property:

$$\begin{aligned} X(k) &= \text{DFT}[x(n)e^{j2\pi nm/N}] \\ &= \sum_{n=0}^{N-1} x(n)e^{j2\pi nm/N}e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x(n)e^{-j2\pi(k-n)m/N} = X(k-m) \end{aligned}$$

Problem 10-17

Let

$$X(k) = \text{DFT}[x(n)] \text{ and } Y(k) = \text{DFT}[y(n)]$$

Using the inverse DFT, we have

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)e^{j2\pi kn/N} \text{ and } y(n-m) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)e^{j2\pi k(n-m)/N}$$

Therefore, the convolution sum can be written as

$$\begin{aligned} \sum_{m=0}^{N-1} x(m)x(n-m) &= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\left[\sum_{k=0}^{N-1} Y(k)e^{j2\pi k(n-m)/N}\right] \\ &= \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\left[\sum_{m=0}^{N-1} x(m)e^{-j2\pi km/N}\right]e^{j2\pi kn/N} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y(k)e^{j2\pi kn/N} = \text{DFT}^{-1}[X(k)Y(k)] \end{aligned}$$

Problem 10-18

Let

$$S = \sum_{n=0}^{N-1} x(n)x^*(n)$$

Use the inverse DFT to represent x(n):

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(k)W_N^{-nk} \text{ and } x^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} X^*(k)W_N^{nk}$$

Therefore

$$S = \sum_{n=0}^{N-1} x(n)\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)W_N^{nk} = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\sum_{n=0}^{N-1} x(n)W_N^{nk}$$

But the last sum is $X(k)$ by definition of the DFT. Hence

$$S = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)X(k) = \frac{1}{N}\sum_{k=0}^{n-1} |X)k)|^2$$

Problem 10-19

From the definition of the DFT,

$$Y(k-m) = \sum_{n=0}^{N-1} y(n)W_N^{(k-m)n}$$

Thus

$$\frac{1}{N}\sum_{m=0}^{N-1} X(m)Y(k-m) = \frac{1}{N}\sum_{m=0}^{N-1} X(m)\sum_{n=0}^{N-1} y(n)W_N^{kn}W_N^{-mn} = \sum_{n=0}^{N-1} y(n)\left[\frac{1}{N}\sum_{m=0}^{N-1} W_N^{-mn}\right]W_N^{kn}$$

But the sum inside the [ ] is $x(n)$, so

$$\frac{1}{N}\sum_{m=0}^{N-1} X(m)Y(k-m) = \sum_{n=0}^{N-1} x(n)y(n)W_N^{kn} = \text{DFT}[x(n)y(n)]$$

Problem 10-20

Let $\{w(n)\}$ be the given real sequence with $2N$ members. Let

$$x(n) = w(2n),\ n = 0, 1, \cdots, N-1 \text{ and } y(n) = w(2n+1),\ n = 0, 1, \cdots, N-1$$

Define

$$z(n) = x(n) + jy(n) = w(2p) + jw(2p+1),\ p = 0, 1, \cdots, N/2 - 1$$

Let

$$\text{DFT}[z(n)] = Z(k) = X(k) + jY(k)$$

where $X(k)$ is the DFT of x(n) and $Y(k)$ is the DFT of y(n). From property 10 of the DFT:

$$\begin{aligned} X(k) &= \text{even part of real part of } Z(k) + j[\text{odd part of imaginary part of } Z(k)] \\ Y(k) &= -[\text{odd part of real part of } Z(k)] + j[\text{even part of imaginary part of } Z(k)] \end{aligned}$$

So $X(k)$ and $Y(k)$ can be separated, and these are the DFTs of the even- and odd-indexed parts, respectively, of $w(n)$. Now $X(k)$ and $Y(k)$ must be combined to obtain the DFT of $\{w(n)\}$ which is denoted as $W(k)$. Consider the following development to accomplish this. From the definitions of $x(n)$ and $y(n)$, it follows that

$$X(k) = \sum_{n=0}^{N-1} w(2n) W_N^{nk} \text{ and } Y(k) = \sum_{n=0}^{N-1} w(2n+1) W_N^{nk}$$

Using these definitions, $W(k)$ can be obtained as follows:

$$\begin{aligned} W(k) &= \sum_{\ell=0}^{N-1} w(\ell) W_{2N}^{n\ell} = \sum_{n=0}^{N-1} w(2n) W_{2N}^{2kn} + \sum_{n=0}^{N-1} w(2n+1) W_{2N}^{2k(2n+1)} \\ &= \sum_{n=0}^{N-1} w(2n) W_{2N}^{2kn} + W_{2N}^{k} \sum_{n=0}^{N-1} w(2n) W_{2N}^{2kn} \end{aligned}$$

But

$$W_{2N}^{2kn} = e^{-j2\pi(2kn/2N)} = e^{-j2\pi kn/N} = W_N^{kn}$$

Therefore

$$W(k) = \sum_{n=0}^{N-1} w(2n) W_N^{kn} + W_N^{k/2} \sum_{n=0}^{N-1} w(2n+1) W_N^{k(2n+1)} = X(k) + W_N^{k/2} Y(k)$$

Problem 10-21

(a) Assuming that the samples are preceeded and followed by zeros, we pad with three additional zeros in order to use the FFT. Then one period of the circular convolution is

$$y_a(n) = \begin{cases} 0, & n = 0 \\ 0, & n = 1 \\ 1, & n = 2 \\ 3, & n = 3 \\ 2, & n = 4 \\ 0, & n = 5 \\ 0, & n = 6 \end{cases}$$

(b) Again, pad with three zeros to get

$$y_b(n) = \begin{cases} 0, & n = 0 \\ 0, & n = 1 \\ 2, & n = 2 \\ 1, & n = 3 \\ 0, & n = 4 \\ 0, & n = 5 \\ 0, & n = 6 \end{cases}$$

(c) One period of the circular convolution is

$$y_c(n) = \begin{cases} 0, & n = 0 \\ 0, & n = 1 \\ 1, & n = 2 \\ 3, & n = 3 \\ 3, & n = 4 \\ 1, & n = 5 \\ 0, & n = 6 \end{cases}$$

Problem 10-22

(a) To separate the FFTs due to the sine and cosine channels, use property 10 for DFTs. The last part of the solution to Problem 10-20 explains how to do this.
(b) Square the real and imaginary parts of each FFT output sample, add, and take the square root of the sum. A quick approximation to this operation is

$$|X(k)| = \max[RX(k), IX(k)] + \frac{1}{2}\min[RX(k), IX(k)]$$

where $RX(k)$ denotes the real part of $X(k)$ and $IX(k)$ denotes the imaginary part of $X(k)$.
(c) Find

$$\theta(k) = \tan^{-1}\left[\frac{IX(k)}{RX(k)}\right]$$

Problem 10-23

For the given $z(n) = x(n) + j\,y(n)$, we have

$$Z(k) = \begin{cases} 0, & k = 0 \\ 0, & k = 1 \\ 0, & k = 2 \\ j4, & k = 3 \end{cases}$$

From (10-65),

$$X_{\text{er}}(k) = \frac{1}{2}[\text{Re}Z(k) + \text{Re}Z(4-k)] = 0 \text{ and } Y_{\text{oi}}(k) = \frac{1}{2}[\text{Re}Z(k) - \text{Re}Z(4-k)] = 0$$

Also, from (10-66),

$$X_{\text{oi}}(k) = \frac{1}{2}[\text{Im}Z(k) - \text{Im}Z(4-k)] = \begin{cases} 0, & k = 0 \\ -2, & k = 1 \\ 0, & k = 2 \\ 2, & k = 3 \end{cases}$$

Again, from (10-66)

$$Y_{er}(k) = \frac{1}{2}[\text{Im}Z(k) + \text{Im}Z(4-k)] = \begin{cases} 0, & k=0 \\ 2, & k=1 \\ 0, & k=2 \\ 2, & k=3 \end{cases}$$

Finally, from (10-62),

$$X(k) = X_{er}(k) + jX_{oi}(k)] = \begin{cases} 0, & k=0 \\ -j2, & k=1 \\ 0, & k=2 \\ j2, & k=3 \end{cases}$$

and

$$Y(k) = Y_{er}(k) + jY_{oi}(k)] = \begin{cases} 0, & k=0 \\ 2, & k=1 \\ 0, & k=2 \\ 2, & k=3 \end{cases}$$

Problem 10-24

Given

$$Z(k) = \begin{cases} 1-j, & k=0 \\ 2+j, & k=1 \\ 2-j, & k=2 \\ 1+j, & k=3 \end{cases}$$

From (10-65),

$$X_{er}(k) = \frac{1}{2}[\text{Re}Z(k) + \text{Re}Z(4-k)] = \begin{cases} 1, & k=0 \\ 1.5, & k=1 \\ 2, & k=2 \\ 1.5, & k=3 \end{cases}$$

Also, from (10-65),

$$Y_{oi}(k) = \frac{1}{2}[\text{Re}Z(k) - \text{Re}Z(4-k)] = \begin{cases} 0, & k=0 \\ 0.5, & k=1 \\ 0, & k=2 \\ 0.5, & k=3 \end{cases}$$

From (10-66),

$$X_{oi}(k) = \frac{1}{2}[\text{Im}Z(k) - \text{Im}Z(4-k)] = 0$$

and also from (10-66),

$$Y_{er}(k) = \frac{1}{2}[\text{Im}Z(k) + \text{Im}Z(4-k)] = \begin{cases} -1, & k=0 \\ 1, & k=1 \\ -1, & k=2 \\ 1, & k=3 \end{cases}$$

Finally, from (10-62),

$$X(k) = X_{er}(k) + jX_{oi}(k)] = \begin{cases} 0, & k=0 \\ 1.5, & k=1 \\ 2, & k=2 \\ 1.5, & k=3 \end{cases}$$

and

$$Y(k) = Y_{er}(k) + jY_{oi}(k)] = \begin{cases} -1, & k=0 \\ 1+j0.5, & k=1 \\ -1, & k=2 \\ 1-j0.5, & k=3 \end{cases}$$

Problem 10-25

We have 50 points in $h(n)$. Therefore, we need $M - 1 = 49$ zeros to pad it. Thus, an FFT of size 49 + 50 = 99 is needed. To make the FFT size a power of 2, use $N = 128 = 2^7$. Therefore, we have 50 points for $h(n)$ and 128 - 50 = 78 zeros. 50 *new* points may be taken for each FFT from the long sequence. Thus, the requirement is 1000/50 = 20 FFTs (50 valid points per FFT).

Problem 10-26

For the given sequence, $R_x(m)$ is computed from (10-95 as

$$R_x(m) = \begin{cases} 0, & n \le -2 \\ 1/8, & n = -1 \\ 1/4, & n = 0 \\ 1/8, & n = 1 \\ 0, & n \ge 2 \end{cases}$$

FFT this using the flow diagram of Figure 10-9. This gives the spectrum estimate

$$S_x(k) = \begin{cases} 1/2, & n = 0 \\ (2 + \sqrt{2})/8, & n = 1 \\ 1/4, & n = 2 \\ (2 - \sqrt{2})/8, & n = 3 \\ 0, & n = 4 \\ (2 - \sqrt{2})/8, & n = 5 \\ 1/4, & n = 6 \\ (2 + \sqrt{2})/8, & n = 7 \end{cases}$$

(b) Now use

$$S_x(k) = X(k)X^*(k)/N = |X(k)|^2/N$$

From Problem 10-2, we obtain

$$X(k) = (1 - e^{-j\pi n/2})/(1 - e^{-j\pi n/4})$$

Use of this in the equation for $S_x(k)$ gives the same results as in part (a).

Problems 10-27 and 10-28

In general, let

$$w(n) = A - B\cos(2\pi n/M), \; n = 1, 1, \cdots, M-1$$

For the Hanning window, $A = B = 1/2$. For the Hamming window, $A = 0.54$ and $B = 0.46$. (See Table 10-4.) The DFT of the window is

$$\begin{aligned} W(k) &= \sum_{n=0}^{M-1} w(n) e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{M-1} A e^{-j2\pi kn/N} - \sum_{n=0}^{M-1} B \frac{e^{j2\pi n/M} + e^{-j2\pi n/M}}{2} e^{-j2\pi kn/N} \\ &= A \sum_{n=0}^{M-1} e^{-j2\pi kn/N} - \frac{B}{2}\sum_{n=0}^{M-1} e^{j2\pi n/M} e^{-j2\pi kn/N} - \frac{B}{2}\sum_{n=0}^{M-1} e^{-j2\pi n/M} e^{-j2\pi kn/N} \\ &= A \sum_{n=0}^{M-1} e^{-j2\pi kn/N} - \frac{B}{2}\sum_{n=0}^{M-1} e^{j2\pi n(1/M - k/N)} - \frac{B}{2}\sum_{n=0}^{M-1} e^{-j2\pi n(1/M + k/N)} \end{aligned}$$

To evaluate the sums, use

$$S_M = \sum_{n=0}^{M-1} x^n = \frac{1 - x^M}{1 - x}$$

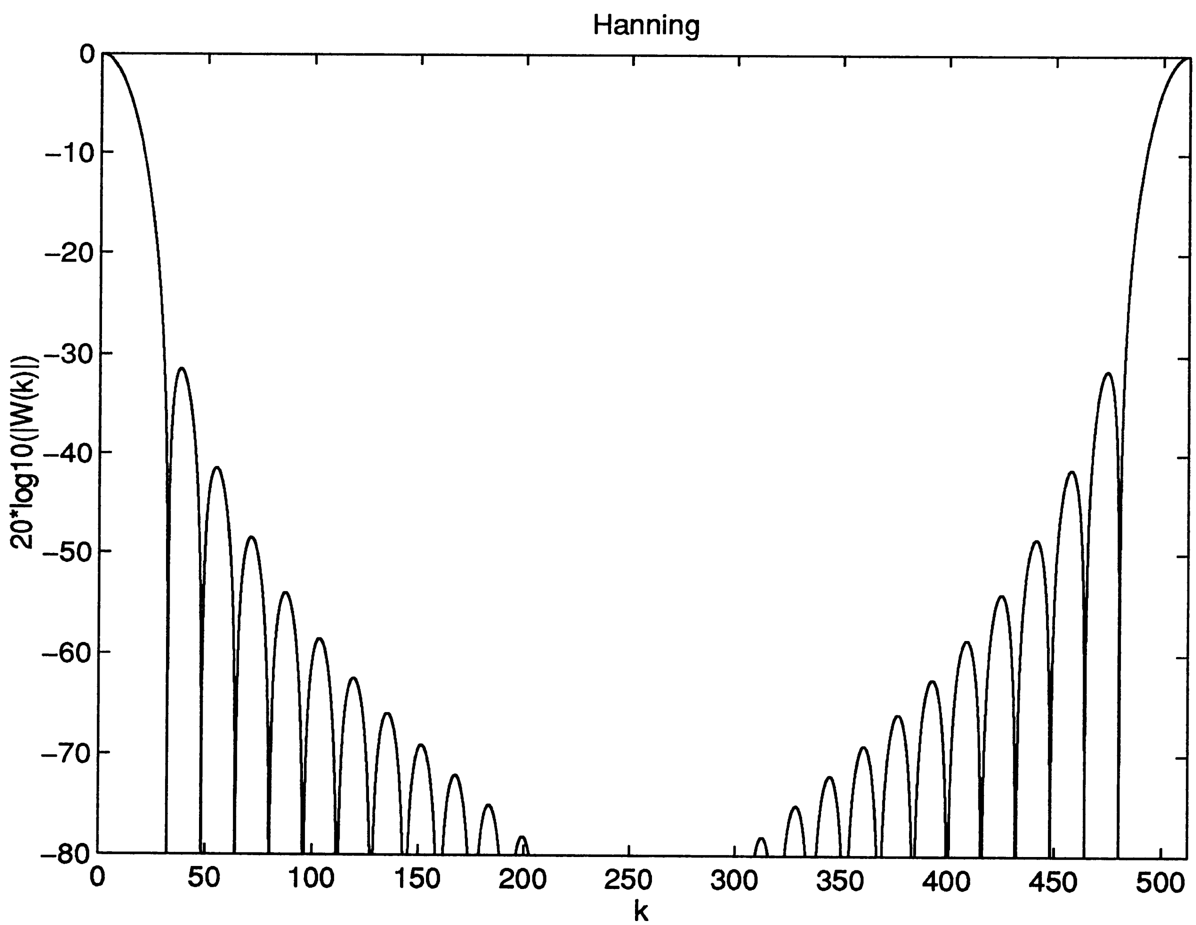
This gives

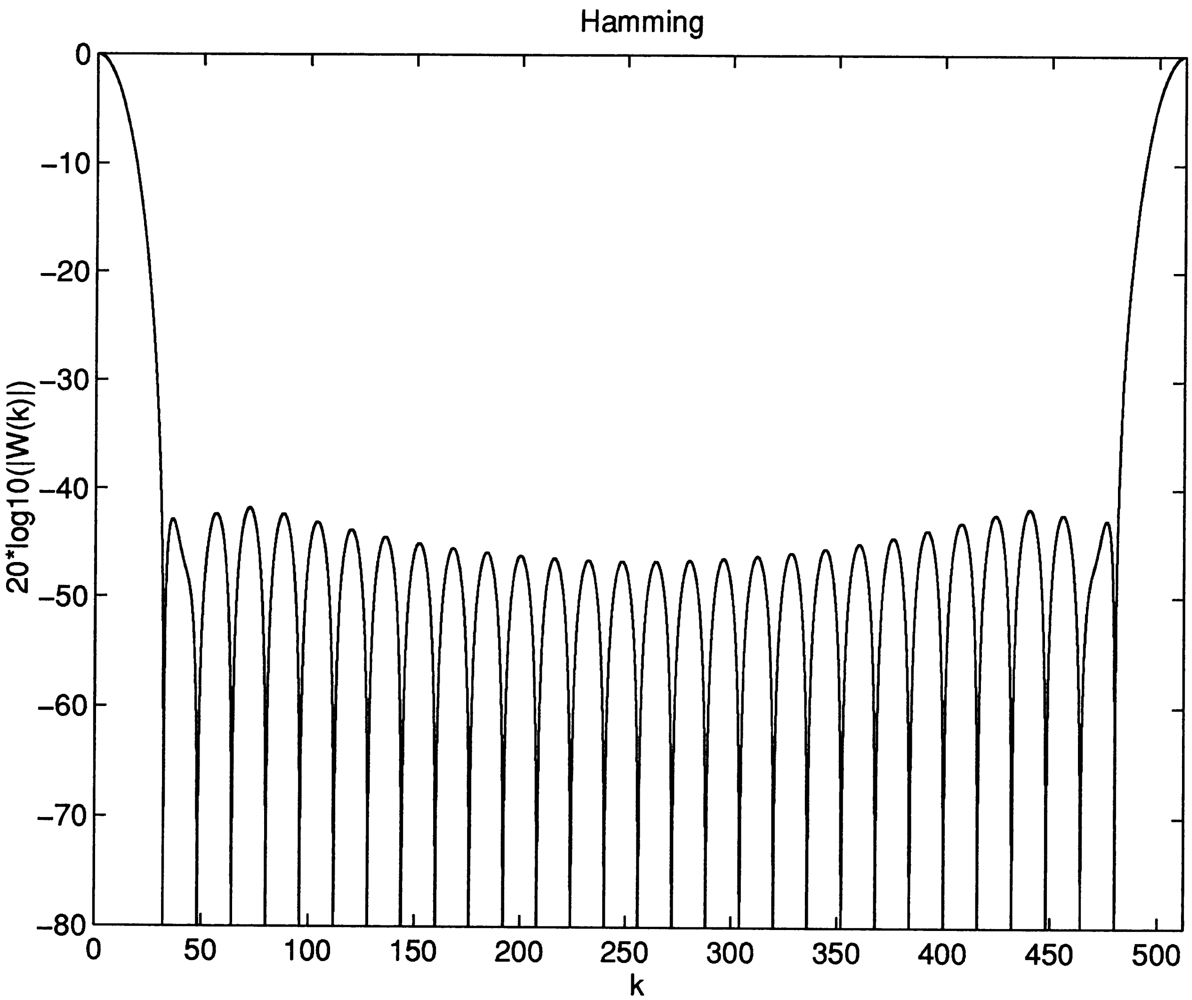
$$W(k) = A\frac{1 - e^{-j2\pi kM/N}}{1 - e^{-j2\pi k/N}} - \frac{B}{2}\frac{1 - e^{j2\pi M(1/M - k/N)}}{1 - e^{j2\pi(1/M - k/N)}} - \frac{B}{2}\frac{1 - e^{-j2\pi M(1/M - k/N)}}{1 - e^{-j2\pi(1/M - k/N)}}$$

The MATLAB program given below computes the DFTs of the Hanning and Hamming windows. The plots following use a fine step in k rather than a unit step to show the shape of the DFTs clearly.

```
%       Plots of Hamming and Hanning window functions; Problem 10-27,28
%
A = .5;
B = .5;
M = 32;
N = 512;
kk = 1.0001:0.01:N;
k = kk - 1;
T1 = A*(1 - exp(-j*2*pi*k*M/N))./(1 - exp(-j*2*pi*k/N));
```

```
T2 = 0.5*B*(1 - exp(j*2*pi*M*(1/M-k/N)))./(1 - exp(j*2*pi*(1/M-k/N)));
T3 = 0.5*B*(1 - exp(-j*2*pi*M*(1/M+k/N)))./(1 - exp(-j*2*pi*(1/M+k/N)));
W = T1-T2-T3;
plot(k,20*log10(abs(W)/max(abs(W))),'-w'),title('Hanning'),...
	axis([0 N -80 0]), xlabel('k'), ylabel('20*log10(|W(k)|)')
print c:\systems\pscr_fls\pr10_27 -dps
pause
A = .54;
B = .46;
kk = 1.0001:0.01:N;
k = kk - 1;
T1 = A*(1 - exp(-j*2*pi*k*M/N))./(1 - exp(-j*2*pi*k/N));
T2 = 0.5*B*(1 - exp(j*2*pi*M*(1/M-k/N)))./(1 - exp(j*2*pi*(1/M-k/N)));
T3 = 0.5*B*(1 - exp(-j*2*pi*M*(1/M+k/N)))./(1 - exp(-j*2*pi*(1/M+k/N)));
W = T1-T2-T3;
plot(k,20*log10(abs(W)/max(abs(W)))),title('Hamming'),...
	axis([0 N -80 0]), xlabel('k'), ylabel('20*log10(|W(k)|)')
```





Hamming
0
-10
-20
-30
-40
-50
-60
-70
-80
20*log10(|W(k)|)
0
50
100
150
200
250
300
350
400
450
500
k

Problem 10-29

The sampling rate is lower bounded by

$$f_s = N/T \geq 2W \text{ Hz} = 2(10 \text{ kHz}) = 20 \text{ kHz}$$

The spectral resolution requirement is

$$\Delta f = 100 \text{ Hz} = f_s/N = 1/T \text{ Hz}$$

Therefore, we need

$$T \geq 1/100 = 0.01 \text{ Hz and } N \geq f_s/\Delta f = 20000/100 = 2000 \text{ samples}$$

Thus, choose

$$N = 2{,}048 = 2^{11}$$

and

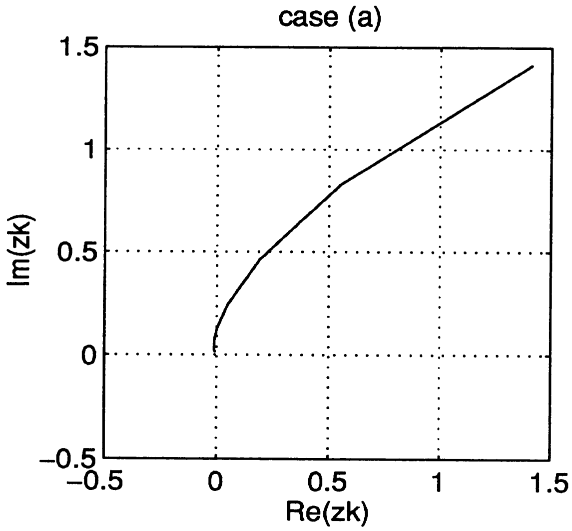
$$f_s = N/T = 2048/0.01 = 20{,}480 \text{ Hz}$$

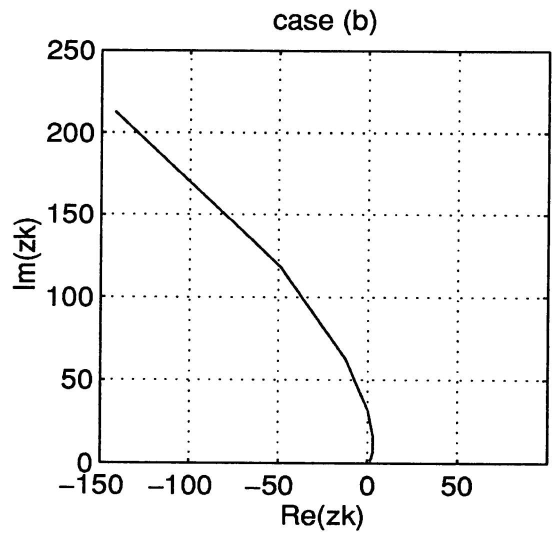
Problem 10-30

The appropriate equations are

$$z_k = AW^{-k},\ 0 \leq k \leq M-1, \text{ where } A = A_0 e^{j\theta_0} \text{ and } W = W_0 e^{-j\theta_0}$$

(a) Parameter values: $M = 8$; $W_0 = 2$; $\phi_0 = \pi/16$ radians; $A_0 = 2$; $\theta_0 = \pi/4$ radians. This case is shown below.
(b) Parameter values: $M = 8$; $W_0 = 1/2$; $\phi_0 = \pi/16$ radians; $A_0 = 2$; $\theta_0 = \pi/4$ radians. This case is shown below.





Problem 10-31

The operation to be performed is

$$X(z_k) = \frac{1}{h(k)} \sum_{n=0}^{N-1} g(n)h(k-n),\ ),\ 0 \le k \le M-1$$

where

$$h(k) = e^{jk^2\phi_0/2} = \cos(k^2\phi_0/2) + j\sin(k^2\phi_0/2)$$

and

$$h(k-n) = \cos[(k-n)^2\phi_0/2] + j\sin[(k-n)^2\phi_0/2]$$

Also

$$\begin{aligned} g(n) &= x(n)e^{-jn^2\phi_0/2} = [x_R(n) + jx_I(n)][\cos(n^2\phi_0/2) - j\sin(n^2\phi_0/2)] \\ &= [x_R(n)\cos(n^2\phi_0/2) + x_I(n)\sin(n^2\phi_0/2)] + j[x_I(n)\cos(n^2\phi_0/2) - x_R(n)\sin(n^2\phi_0/2)] \\ &= g_R(n) + jg_I(n) \end{aligned}$$

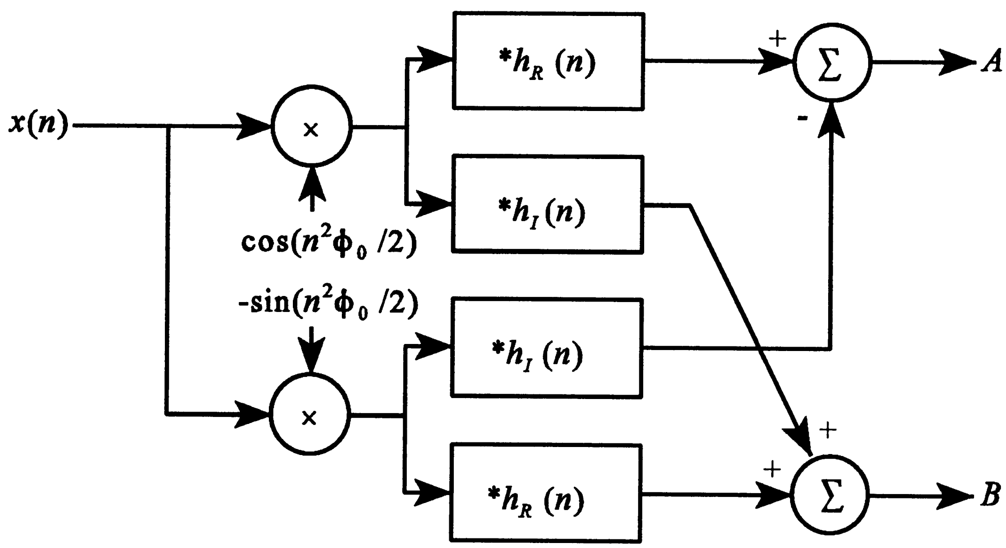
where x_R and x_I are the real and imaginary parts of $x(n)$, respectively, and the last equation defines the real and imaginary parts of $g(n)$. Therefore,

$$g(n)*h(n) = g_R*h_R(n) - g_I*h_I(n) + j[g_I*h_R(n) - g_R*h_I(n)]$$

where

$$h_R(n) = \cos(n^2\phi_0/2) \text{ and } h_I(n) = \sin(n^2\phi_0/2)$$

If $x(n)$ is lowpass, we can let it be real. The above equations result in the block diagram below.



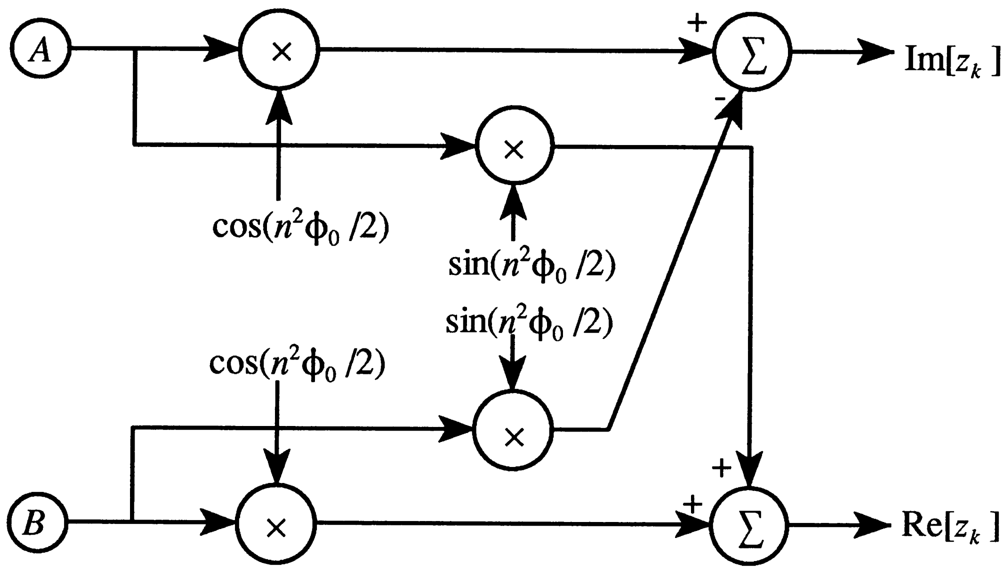
The A and B outputs are

$$A = \mathrm{Re}[g(n)*h(n)] \text{ and } B = \mathrm{Im}[g(n)*h(n)]$$

respectively. Finally,

$$\begin{aligned} X(z_k) &= e^{-jk^2\phi_0/2} g(n)*h(n) \\ &= \cos(k^2\phi_0/2)\mathrm{Re}[g(n)*h(n)] + \sin(k^2\phi_0/2)\mathrm{Im}[g(n)*h(n)] \\ &\quad + j\{\cos(k^2\phi_0/2)\mathrm{Im}[g(n)*h(n)] - \sin(k^2\phi_0/2)\mathrm{Re}[g(n)*h(n)]\} \end{aligned}$$

The diagram below accomplishes this operation:



Problem 10-32

The DFT and chirp-z transform are given by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \text{ and } X(z_k) = \sum_{n=0}^{N-1} x(n) z_k^{-n} \text{ where } z_k = e^{-jk\phi_0},\ 0 \le k \le M-1$$

respectively. Therefore

$$X(z_k) = \sum_{n=0}^{N-1} x(n) e^{-jkn\phi_0}$$

Resolution can be compared as follows:

1. FFT:

$$\frac{2\pi}{N} = \Delta f,\ 2\pi \sim f_s = \text{sampling interval}$$

Therefore

$$\frac{2\pi/N}{2\pi} = \frac{\Delta f}{f_s} \text{ or } \Delta f = \frac{f_s}{N}$$

2. Chirp-z:

$$\phi_0 \sim \Delta f,\ 2\pi \sim f_s$$

Therefore

$$\frac{\phi_0}{2\pi} = \frac{\Delta f}{f_s} \text{ or } \Delta f = \frac{\phi_0 f_s}{2\pi}$$

Comparison of the number of multiplications:

1. FFT: Each butterfly requires one complex multiplication. There are $\log_2 N$ recursions and $N/2$ butterflies per recursion. Thus, the total number of complex multiplications required is

$$N_{FFT} = \frac{N}{2}\log_2 N$$

We ignore the adds since they presumably take only a fraction of time required for a multiply.

2. Chirp-z transform: Implement it as

$$X(z_k) = \frac{1}{h(k)}\sum_{n=0}^{N-1} g(n)\,h(k-n),\ 0 \le k \le M-1$$

$$\text{where } h(k) = e^{j\phi_0 k^2/2} \text{ and } g(n) = x(n)\,e^{-j\phi_0 n^2/2}$$

The number of complex multiplications for the chirp-z transform can be deduced as follows: To form $g(n)$ requires one for each k value; to form $g(n)\ h(n-k)$ requires one each; there are N terms in each sum giving $2N$ multiplies to form each convolution. An additional multiply is required to premultiply by $1/h(n)$. There are M convolutions. Thus, the total number of complex multiplies required is

$$N_{\text{ch-z}} = (2N+2)M$$

We desire equal frequency resolutions for each approach. Thus, we require that

$$\frac{\phi_0 f_s}{2\pi} = \frac{f_s}{N} \text{ or } N = \frac{2\pi}{\phi_0} \text{ or } \phi_0 = \frac{2\pi}{N}$$

1. For the FFT , the total range of frequencies processed is $(0, f_s)$;
2. For the chirp-z, the total range of frequencies processed is $(0, M\phi_0)$, or $(0, f_x)$ where

$$f_x = \frac{M\phi_0}{2\pi}f_s = \frac{M(2\pi/N)f_s}{2\pi} = \frac{M}{N}f_s$$

The tables below illustrate some specific results. The last column is the total frequency range of the chirp-z divided by the total frequency range of the FFT (*M*/*N*, as derived above).

		Rng:	0.063	Rng:	0.031	Rng	0.016	Rng:	0.008	Rng:	0.004
N	*M*	N_{FFT}	$N_{ch\text{-}z}$	N_{FFT}	$N_{ch\text{-}z}$	N_{FFT}	$N_{ch\text{-}z}$	N_{FFT}	$N_{ch\text{-}z}$	N_{FFT}	$N_{ch\text{-}z}$
16	1	32	33								
32	2	80	130	"	65						
64	4	192	516	"	258	"	129				
128	8	448	2,056	"	1028	"	514	"	257		
256	16	1024	8,208	"	4104	"	2052	"	1026	"	513
512	32	2304	32800	"	16400	"	8200	"	4100	"	2050
1024	64	5120	131100	"	65568	"	32784	"	16392	"	8196

Only for small frequency ranges relative to the FFT and small *M* does the chirp-z come out ahead of the FFT. There does not appear to be a computationally more economical means for implementing the chirp-z using the FFT to do the convolution, since the number of output points will be the size of the FFT, which is the size of the input sequences or *N*.

APPENDIX B

Problem B-1

(a) For a translational damper,

$$f = B\dot{\Delta}$$

Since

$$f(t) = A\cos(\omega_0 t)$$

we have

$$\dot{\Delta} = \frac{A}{B}\cos(\omega_0 t)$$

and

$$P_{AV} = \frac{1}{T_0}\int_0^{T_0}(A\cos(\omega_0 t)\left[\frac{A}{B}\cos(\omega_0 t)\right]dt$$

$$= \frac{1}{T_0}\frac{A}{B}\int_0^{T_0}\cos^2(\omega_0 t)\,dt$$

$$= \frac{A^2}{2B}$$

(b) For the translational spring,

$$\dot{\Delta} = \frac{1}{K}\frac{df}{dt} = -\frac{A\omega_0}{K}\sin(\omega_0 t)$$

and

$$P_{AV} = \frac{1}{T_0}\int_0^{T_0} -\frac{A^2\omega_0}{K}\sin(\omega_0 t)\cos(\omega_0 t)\,dt = 0$$

For the translational mass element,

$$\dot{\Delta} = \frac{1}{M}\int^{t} f(\lambda)\,d\lambda = \frac{A}{M\omega_0}\sin(\omega_0 t)$$

and the average power is again zero as it was for the translational spring.

Problem B-2

(a) For the rotational damper,

$$\dot{\theta} = \frac{1}{B}T = \frac{A}{B}\cos(\omega_0 t)$$

and

$$P_{AV} = \frac{1}{T_0}\int_0^{T_0} T(t)\dot{\theta}(t)\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \frac{A^2}{B}\cos^2(\omega_0 t)\,dt$$

$$= \frac{A^2}{2B}\ \mathit{watts}$$

(b) For the rotational spring

$$\dot{\theta} = \frac{1}{K}\frac{dT(t)}{dt} = -\frac{A}{K}\omega_0\sin(\omega_0 t)$$

and

$$P_{AV} = \frac{1}{T_0}\int_0^{T_0} -\frac{A^2\omega_0}{K}\sin(\omega_0 t)\cos(\omega_0 t)\,dt = 0$$

For the rotational mass (inertance),

$$\dot{\theta} = \frac{1}{J}\int^{T_0} T(\lambda)d\lambda = \frac{A}{J\omega_0}\sin(\omega_0 t)$$

and

$$P_{AV} = \frac{1}{T_0}\int_0^{T_0}\frac{A^2}{J\omega_0}\sin(\omega_0 t)\cos(\omega_0 t)dt = 0$$

Problem B-3

Equations (B-19a) and (B-19b) are

$$R_1\frac{di_A}{dt} + \frac{1}{C}(i_A - i_B) = \frac{dv_s}{dt} \quad (a)$$

$$-\frac{1}{C}i_A + L\frac{d^2 i_B}{dt^2} + R_2\frac{di_B}{dt} + \frac{1}{C}i_B = 0 \quad (b)$$

Solve (a) for i_B:

$$i_B(t) = R_1 C\frac{di_A}{dt} + i_A - C\frac{dv_s}{dt}$$

Substitute into (b):

$$-\frac{1}{C}i_A + R_1 LC\frac{d^3 i_A}{dt^3} + \frac{d^2 i_A}{dt^2} - C\frac{d^3 v_s}{dt^3} + R_1 R_2 C\frac{d^2 i_A}{dt^2} + R_2\frac{di_A}{dt}$$

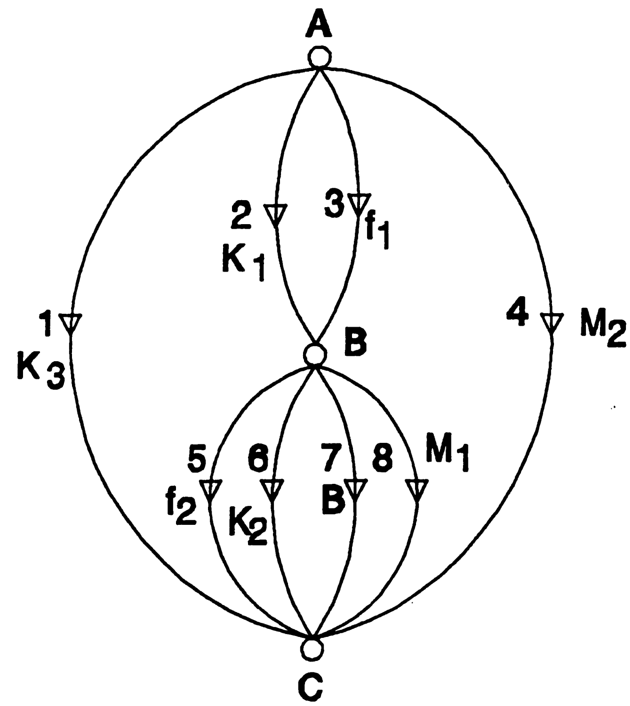
$$- R_2 C\frac{d^2 v_s}{dt^2} + R_1\frac{di^A}{dt} + \frac{1}{C}i_A - \frac{dv_s}{dt} = 0$$

Note that the terms i_A/C cancel. This allows one derivative to be removed. When rearranged, the result is

$$R_1 LC\frac{d^2 i_A}{dt^2} + (1 + R_1 R_2 C)\frac{di_A}{dt} + (R_1 + R_2)i_A = C\frac{d^2 v_s}{dt^2} + R_2 C\frac{dv_s}{dt} + v_s$$

Problem B-4

(a) *System graph:*



Node Postulate equations:

$$A: \quad f_1 + f_2 + f_3 + f_4 + \; = 0$$

$$B: \quad -f_2 - f_3 + f_5 + f_6 + f_7 + f_8 = 0$$

$$C: \quad -f_1 - f_5 - f_6 - f_7 - f_8 - f_4 = 0$$

Circuit Postulate equations (choose all circuit orientations clockwise):

$$1:\quad -\dot{\Delta}_1(t) + \dot{\Delta}_2(t) + \dot{\Delta}_5(t) = 0$$

$$2:\quad -\dot{\Delta}_2(t) + \dot{\Delta}_3(t) = 0$$

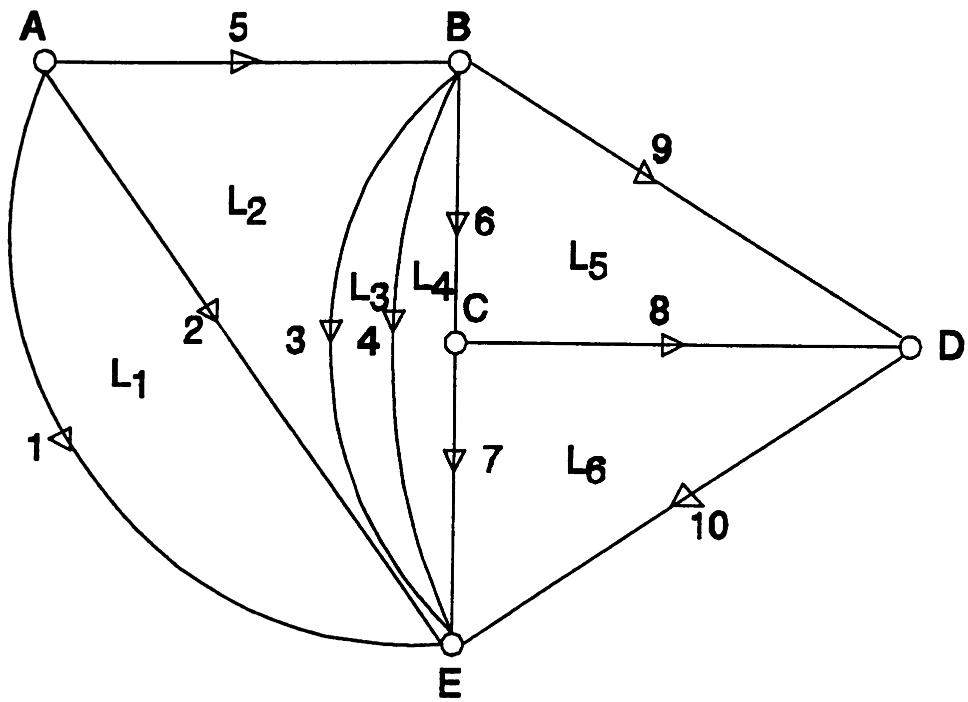
$$3:\quad -\dot{\Delta}_3(t) + \dot{\Delta}_4(t) - \dot{\Delta}_8(t) = 0$$

$$4:\quad -\dot{\Delta}_5(t) + \dot{\Delta}_6(t) = 0$$

$$5:\quad -\dot{\Delta}_6(t) + \dot{\Delta}_7(t) = 0$$

$$6:\quad -\dot{\Delta}_7(t) + \dot{\Delta}_8(t) = 0$$

(b) *System graph:*



Node Postulate equations:

$$A: \quad i_1 + i_2 + i_5 = 0$$

$$B: \quad -i_5 + i_3 + i_4 + i_6 = 0$$

$$C: \quad -i_6 + i_7 + i_8 = 0$$

$$D: \quad -i_9 - i_8 + i_{10} = 0$$

$$E: \quad -i_1 - i_2 - i_3 - i_4 - i_7 - i_{10} = 0$$

Circuit Postulate equations (assume all loop currents have a clockwise orientation):

$$L_1: \quad -v_1 + v_2 = 0$$

$$L_2: \quad -v_2 + v_5 + v_3 = 0$$

$$L_3: \quad -v_3 + v_4 = 0$$

$$L_4: \quad -v_4 + v_6 + v_7 = 0$$

$$L_5: \quad -v_5 + v_9 - v_8 = 0$$

$$L_6: \quad -v_7 + v_8 + v_{10} = 0$$

Problem B-5

See the diagram in Problem B-4a for the labeling of the nodes, branches and components.

Velocity relations:

$$\dot{\Delta}_1 = v_A = \dot{\Delta}_4$$

$$\dot{\Delta}_2 = v_A - v_B = \dot{\Delta}_3$$

$$\dot{\Delta}_5 = v_B = \dot{\Delta}_6 = \dot{\Delta}_7 = \dot{\Delta}_8$$

Terminal equations:

$$1: f_1(t) = K_3\int^t \dot{\Delta}_1(\lambda)d\lambda = K_3\int^t v_A(\lambda)d\lambda$$

$$2: f_2(t) = K_1\int^t \dot{\Delta}_2(\lambda)d\lambda = Kaub1\int^t (v_A - v_B)d\lambda$$

$$3: \tilde{f}_2(t) = f_3(t) \quad \textit{(the tilde denotes a source)}$$

$$4: f_4(t) = M_2\frac{dv_A}{dt}$$

$$5: f_5(t) = \tilde{f}_2(t)$$

$$6: f_6(t) = K_2\int^t v_B(\lambda)d\lambda$$

$$7: f_7(t) = Bv_B(t)$$

$$8: f_8(t) = M_1\frac{dv_B}{dt}$$

Node Postulate equations:

For node A:

$$f_1 + f_2 + f_3 + f_4 = 0$$

or

$$K_3\int^t v_A(\lambda)d\lambda + K_1\int^t (v_A - v_B)d\lambda + f_1(t) + M_2\frac{dv_A}{dt} = 0$$

For node B:

$$-f_2 - f_3 + f_5 + f_6 + f_7 + f_8 = 0$$

or

$$-K_1\int^t (v_A - v_B)d\lambda - \tilde{f}_1(t) + \tilde{f}_2(t) + K_2\int^t v_B d\lambda + Bv_B(t) + M_1\frac{dv_B}{dt} = 0$$

Problem B-6

Use the node and branch numbering shown in the solution of Problem B-4b.

Terminal equations:

$$v_1(t) = v(t)$$

$$v_2(t) = \frac{1}{C_1}\int^{t} i_2(\lambda)d\lambda$$

$$v_3(t) = \frac{1}{C_2}\int^{t} i_3(\lambda)d\lambda$$

$$v_4(t) = L_1\frac{di_4(t)}{dt}$$

$$v_5(t) = R_1 i_5(t)$$

$$v_6(t) = L_2\frac{di_6(t)}{dt}$$

$$v_7(t) = R_2 i_7(t)$$

$$v_8(t) = \frac{1}{C_3}\int^{t} i_8(\lambda)d\lambda$$

$$v_9(t) = \frac{1}{C_4}\int^{t} i_9(\lambda)d\lambda$$

$$v_{10}(t) = R_3 i_{10}(t)$$

Branch currents in terms of mesh (loop) currents (all mesh current references are taken as clockwise):

$$-i_1(t) = i_{L_1}(t) = i_2(t)$$

$$-i_2(t) = i_{L_2}(t) = i_3(t) = i_5(t)$$

$$-i_3(t) = i_{L_3}(t) = i_4(t)$$

$$-i_4(t) = i_{L_4}(t) = i_6(t) = i_7(t)$$

$$-i_6(t) = i_{L_1}(t) = i_9(t) = -i_8(t)$$

$$-i_7(t) = i_{L_1}(t) = i_8(t) = i_{10}(t)$$

From the Circuit Postulate equations of Problem B-4, we have six independent equations in six unknowns:

$$-v_1 + v_2 = 0 \rightarrow -v(t) + \frac{1}{C_1}\int^t i_{L_1}(\lambda)d\lambda = 0$$

$$-v_2 + v_5 + v_3 = 0 \rightarrow \frac{1}{C_1}\int^t i_{L_3}(\lambda)d\lambda + R_1 i_{L_3}(t) + \frac{1}{C_2}\int^t i_{L_3}(\lambda)d\lambda = 0$$

$$-v_3 + v_4 = 0 \rightarrow \frac{1}{C_2}\int^t i_{L_3}(\lambda)d\lambda + L_1\frac{di_{L_3}(t)}{dt} = 0$$

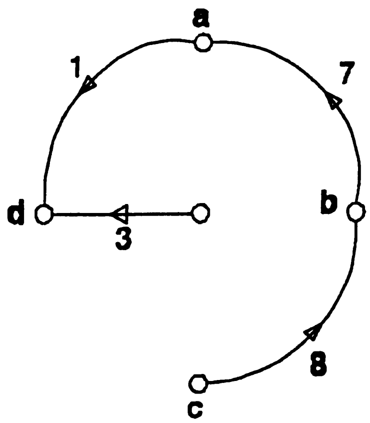
$$-v_4 + v_6 + v_7 = 0 \rightarrow L_1\frac{di_{L_4}(t)}{dt} + L_2\frac{di_{L_4}(t)}{dt} + R_2 i_{L_4}(t) = 0$$

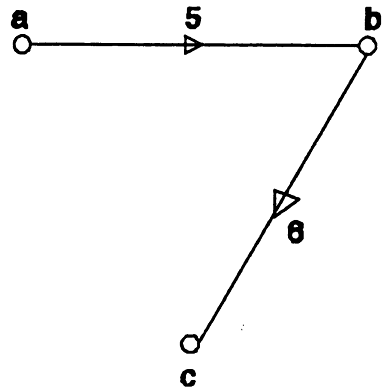
$$-v_5 + v_9 - v_8 = 0 \rightarrow R_1 i_{L_5}(t) + \frac{1}{C_4}\int^t i_{L_5}(\lambda)d\lambda + \frac{1}{C_3}\int^t i_{L_5}(\lambda)d\lambda = 0$$

$$-v_7 + v_8 + v_{10} = 0 \rightarrow R_2 i_{L_6}(t) + \frac{1}{C_3}\int^t i_{L_6}(\lambda)d\lambda + R_3 i_{L_6}(t) = 0$$

Problem B-7

(a) One possible tree for each graph is shown below:





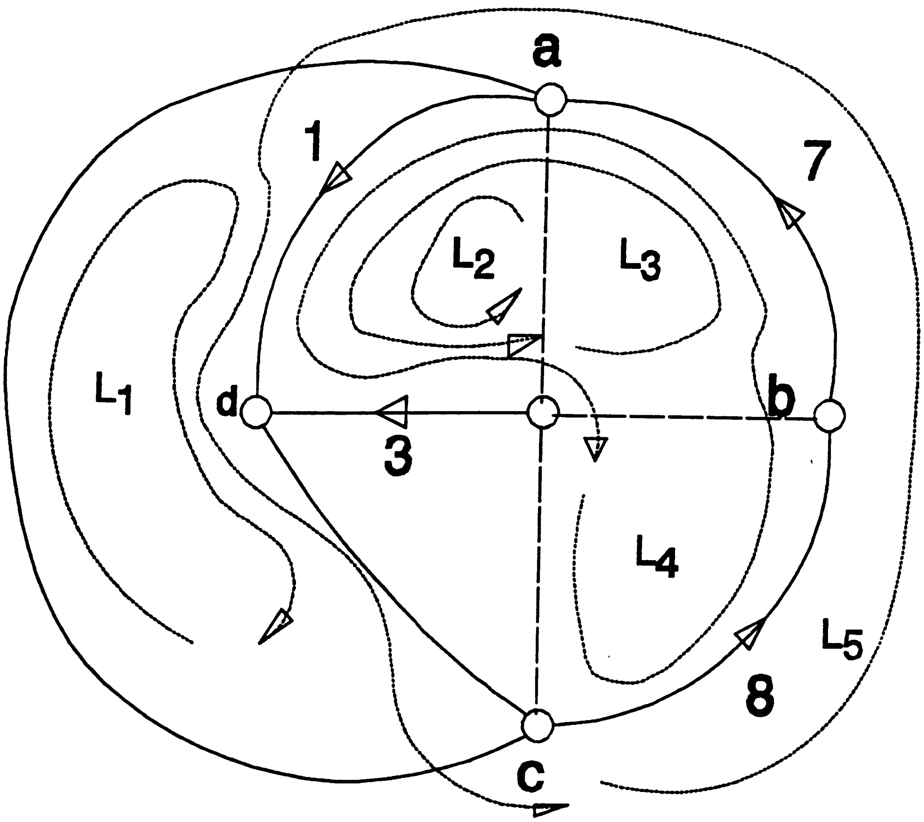
For the first circuit, chords or links are 4, 5, 6, and 9. For the second circuit, chords are 1, 2, 3, and 4.
(b) The number of independent Postulate I (KCL) equations is N_v - 1. The number of Postulate II (KVL) equations is

$$N_c = N_b - N_v + 1$$

The results for the above two graphs (circuits) are summarized in the table below.

Graph	N_v	N_b	N_v	Independent Postulate I Equations	Independent Postulate II Equations
LHS	5	9	5	4	5
RHS	3	6	4	2	4

(c) Loops for the first circuit are shown below:



The loop equations are:

$$i_{L_1} + (i_{L_{L_1}} + i_{L_5}) + v_s = 0$$

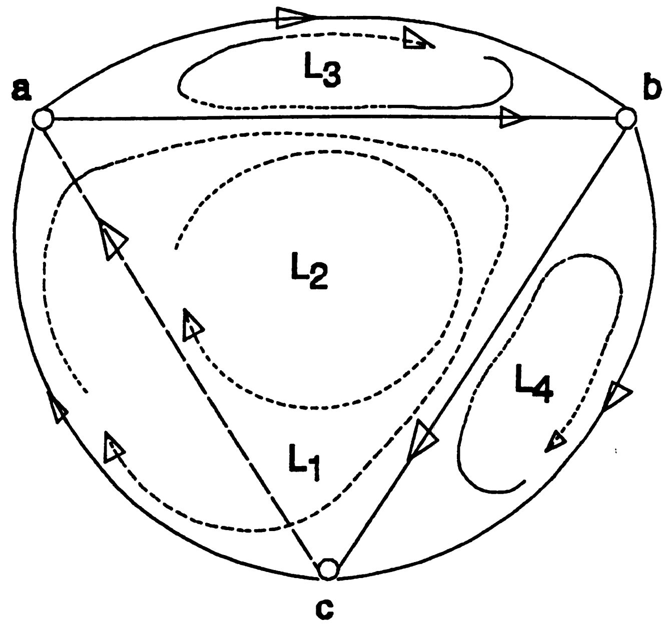
$$v_s + i_{L_2} + (i_{L_2} + i_{L_3} + i_{L_4}) = 0$$

$$v_s + (i_{L_2} + i_{L_3} + i_{L_4} + i_{L_5}) = 0$$

$$v_s + (i_{L_2} + i_{L_3} + i_{L_4}) + i_{L_4} + (i_{L_2} + i_{L_3} + i_{L_4} + i_{L_5}) = 0$$

$$v_s + (i_{L_2} + i_{L_3} + i_{L_4} + i_{L_5}) + (i_{L_4} + i_{L_5}) + i_{L_5} = 0$$

Loops for the second circuit are shown below. Note that the reference directions have been taken as clockwise - it makes no differenence as long as we are consistent.



The loop equations are:

$$v_s + (i_{L_1} - i_{L_3}) + (i_{L_1} - i_{L_4}) = 0$$

$$i_{L_2} + (i_{L_1} + i_{L_2} - i_{L_3}) + (i_{L_1} + i_{L_2} - i_{L_4}) = 0$$

$$i_{L_3} + (i_{L_3} - i_{L_1} - i_{L_3}) = 0$$

$$i_{L_4} + (i_{L_4} - i_{L_1} - i_{L_2}) = 0$$

(d) For the first circuit (assume the current source and branch references the same as before):

$$i_s + (v_a - v_c) + (v_a - v_b) + v_a = 0 \ (node\ a)$$

$$(v_b - v_a) + (v_b - b_c) + (v_b - b_e) = 0 \ (node\ b)$$

$$(v_d - v_a) + (v_d - v_e) + v_d = 0 \ (node\ d)$$

$$(v_e - v_a) + (v_e - v_b) + (v_e - v_d) + v_e = 0 \ (node\ e)$$

For the second circuit:

$$-i_s + 2(v_a - v_b) + v_a = 0$$

$$-2(v_b - v_a) + 2v_b = 0$$

APPENDIX E

Problem E-1

The recursion formula is

$$a_{k+1} = \frac{\cos(k\pi/2n)}{\sin[(k+1)(\pi/2n)]}\, a_k$$

with $a_o = 1$. For $n = 2$ we therefore have

$$a_0 = 1$$

$$a_1 = \frac{\cos(0)}{\sin(\pi/4)} = 1.41421$$

$$a_2 = \frac{\cos(\pi/4)}{\sin(\pi/2)}\, a_1 = 1$$

For $n = 3$ we have

$$a_0 = 1$$

$$a_1 = \frac{\cos(0)}{\sin(\pi/4)} = 1.41421$$

$$a_2 = \frac{\cos(\pi/4)}{\sin(\pi/2)}\, a_1 = 1$$

For $n = 4$ we have

$$a_0 = 1$$

$$a_1 = \frac{\cos(0)}{\sin(\pi/8)}\, a_0 = 2.61313$$

$$a_2 = \frac{\cos(\pi/8)}{\sin(\pi/4)}\, a_1 = 3.41421$$

$$a_3 = \frac{\cos(\pi/4)}{\sin(3\pi/8)}\, a_2 = 2.61313$$

$$a_4 = \frac{\cos(3\pi/8)}{\sin(\pi/2)}\, a_3 = 1$$

Problem E-2

Note that complex conjugate pole pairs at $e^{\pm j\theta}$ yield polynomials of the form

$$\left(s-e^{j\theta}\right)\left(s-e^{-j\theta}\right)=s^2-2\cos\theta\, s+1$$

in the denominator of $H(s)$.

For $n = 1$ we have a single pole at $s = -1$. Thus

$$B_1(s) = s + 1$$

For $n = 2$ we have a pole pair at $s = e^{\pm j\theta}$ where $\theta = 90° + 45° = 135°$. Thus

$$\begin{aligned} B_2(s) &= s^2 - 2\cos(135°)s + 1 \\ &= s^2 + 1.41421s + 1 \end{aligned}$$

For $n = 3$ we have a pole pair at $s = e^{\pm j\theta}$ where $\theta = 90° + 30° = 120°$ and a pole at $s = -1$. Thus

$$\begin{aligned} B_3(s) &= (s+1)\left(s^2 - 2\cos(120°)s + 1\right) \\ &= (s+1)\left(s^2 + s + 1\right) \\ &= s^3 + 2s^2 + 2s + 1 \end{aligned}$$

The fourth-order Butterworth filter has four poles arranged as two complex-conjugate pole pairs. Since the angular separation between the poles is $180°/4 = 45°$ we have one pole-pair at $e^{\pm j\theta_1}$ where $\theta_1 = 90° + 22.5° = 112.5°$ and one pole-pair at $e^{\pm j\theta_2}$ where $\theta_2 = 90° + 22.5° + 60°$ or $\theta_2 = 157.5°$. Thus

$$\begin{aligned} B_4(s) &= \left(s^2 - 2\cos(112.5°)s + 1\right)\left(s^2 - 2\cos(157.5°)s + 1\right) \\ &= \left(s^2 + 0.76537s + 1\right)\left(s^2 + 1.84776s + 1\right) \\ &= s^4 + 2.61313s^3 + 3.41422s^2 + 2.61313s + 1 \end{aligned}$$

The fifth-order Butterworth filter has one real pole at $s = -1$ and two complex-conjugate pole pairs at $e^{\pm j\theta_1}$ and $e^{\pm j\theta_2}$. The angles are

$$\begin{aligned} \theta_1 &= 90° + 18° = 108° \\ \theta_2 &= 90° + 18° + 36° = 144° \end{aligned}$$

Thus

$$
\begin{aligned}
B_5(s) &= (s+1)\left(s^2 - 2\cos(108^\circ)s + 1\right)\left(s^2 - 2\cos(144^\circ)s + 1\right) \\
&= (s+1)\left(s^2 + 0.61803s + 1\right)\left(s^2 + 1.61803s + 1\right) \\
&= s^5 + 3.23606s^4 + 5.23605s^3 + 5.23605s^2 + 3.23606s + 1
\end{aligned}
$$

The sixth-order Butterworth filter has three complex pole pairs, $e^{\pm j\theta_1}$, $e^{\pm j\theta_2}$ and $e^{\pm j\theta_3}$. The angles are

$$
\begin{aligned}
\theta_1 &= 90^\circ + 15^\circ = 105^\circ \\
\theta_2 &= 90^\circ + 15^\circ + 30^\circ = 135^\circ \\
\theta_3 &= 90^\circ + 15^\circ + 2(30^\circ) = 165^\circ
\end{aligned}
$$

Thus

$$
\begin{aligned}
B_6(s) &= \left(s^2 - 2\cos(105^\circ)s + 1\right)\cdot\left(s^2 - 2\cos(135^\circ)s + 1\right)\left(s^2 - 2\cos(165^\circ)s + 1\right) \\
&= \left(s^2 + 0.51764s + 1\right)\cdot\left(s^2 + 1.41421s + 1\right)\left(s^2 + 1.93185s + 1\right)
\end{aligned}
$$

Performing the indicated polynomial multiplication yields

$$
B_6(s) = s^6 + 3.86370s^5 + 7.46410s^4 + 9.14162s^3 + 7.46410s^2 + 3.86370s + 1
$$

These results agree with Table D-1.

Problem E-3

(a) For a second-order Bessel filter

$$
H_{Be}(s) = \frac{k}{s^2 + 3s + 3}
$$

so that the steady-state frequency response is given by

$$
H_{Be}(j\omega) = \frac{k}{\left(3 - \omega^2\right) + j3\omega}
$$

The dc gain is $H_{Be}(0)$. Thus

$$
H_{Be}(0) = \frac{k}{3} = 1
$$

so that $k = 3$.

(b) Since the frequency response is defined by

$$H_{Be}(j\omega) = \frac{3}{\left(3-\omega^2\right)+j3\omega} = A(\omega)\ e^{j\theta(\omega)}$$

the amplitude response is defined by

$$A(\omega) = \frac{3}{\sqrt{\left(3-\omega^2\right)+(3\omega)^2}}$$

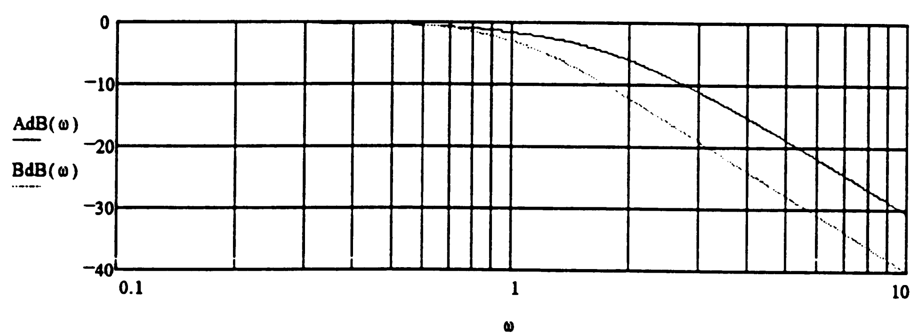
For the second-order Butterworth filter with a dc gain of 1 we have

$$H_{Bu}(j\omega) = \frac{1}{\left(1-\omega^2\right)+j\sqrt{2}\,\omega} = B(\omega)\ e^{j\phi(\omega)}$$

which yields the amplitude response

$$B(\omega) = \frac{1}{\sqrt{(1-\omega)^2+\left(\sqrt{2}\,\omega\right)^2}}$$

These are shown below.



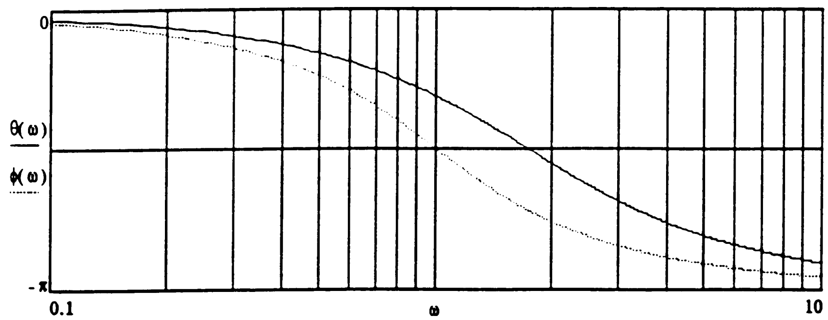
(c) The phase response for the Bessel filter is

$$\theta(\omega) = -\tan^{-1}\frac{3\omega}{3-\omega^2}$$

and the phase response for the Butterworth filter is

$$\phi(\omega) = -\tan^{-1}\frac{\sqrt{2}\,\omega}{1-\omega^2}$$

These are shown below



(d) The group delay is estimated using the initial slope of the phase characteristic. For $\omega << 1$ we have, for the Bessel filter

$$\theta(\omega) \approx -\tan^{-1}\omega \approx -\omega, \qquad \omega << 1$$

Thus, the nominal passband group delay is

$$T_g(\omega) \approx -\frac{d\theta(\omega)}{\omega} = 1 = t_o$$

The cutoff frequency of the Bessel filter is

$$f_c = \frac{1}{2\pi t_o} = \frac{1}{2\pi}$$

or

$$\omega_c = 1$$

This is the 3-dB cutoff frequency of the Butterworth filter and therefore the bandwidths are equal.

Problem E-4

(a) For the third-order Bessel filter

$$H_{Be}(s) = \frac{k}{s^3 + 6s^2 + 15s + 15}$$

so that the steady-state frequency response is given by

$$H_{Be}(j\omega) = \frac{k}{\left(15 - 6\omega^2\right) + j\left(15\omega - \omega^3\right)}$$

The dc gain is $H_{Be}(0)$. Thus

$$H_{Be}(0) = \frac{k}{15} = 1$$

so that $k = 15$.

(b) Since the frequency response is defined by

$$H_{Be}(j\omega) = \frac{15}{\left(15 - 6\omega^2\right) + j\left(15\omega - \omega^3\right)} = A(\omega)e^{j\theta(\omega)}$$

the amplitude-response is defined by

$$A(\omega) = \frac{15}{\sqrt{\left(15 - 6\omega^2\right)^2 + \left(15\omega - \omega^3\right)^2}}$$

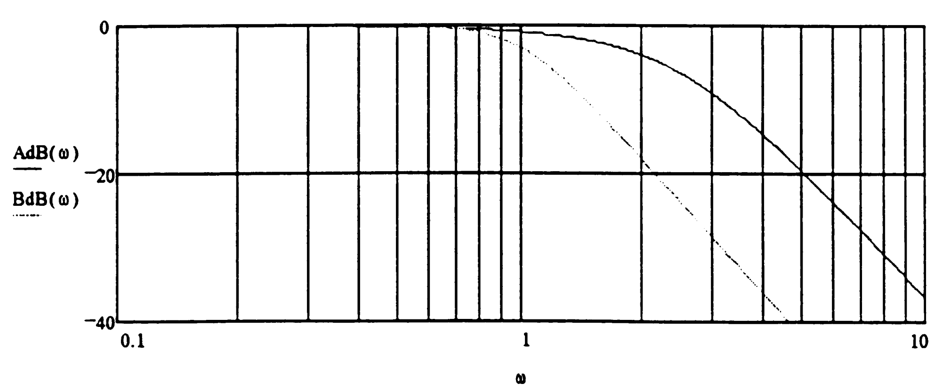
For the third-order Butterworth filter with a dc gain of 1 we have

$$H_{Bu}(j\omega) = \frac{1}{\left(1 - 2\omega^2\right) + j\left(2\omega - \omega^3\right)} = B(\omega)\, e^{j\phi(\omega)}$$

which yields the amplitude response

$$B(\omega) = \frac{1}{\sqrt{\left(1 - 2\omega^2\right)^2 + \left(2\omega - \omega^3\right)^2}}$$

These are shown below.



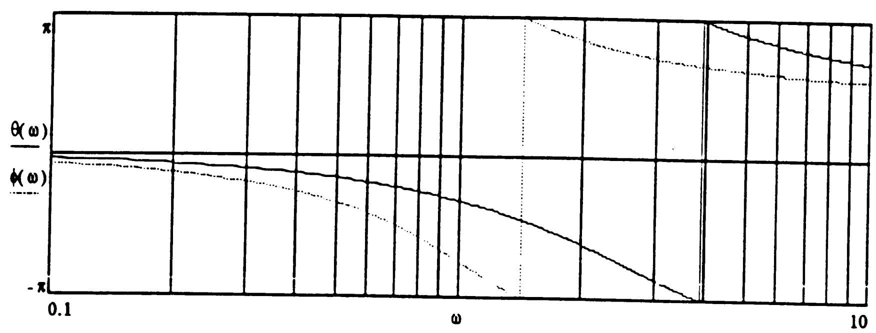
(c) The phase-response for the Bessel filter is

$$\theta(\omega) = -\tan^{-1} \frac{15\omega - \omega^3}{15 - 6\omega^2}$$

and the phase response for the Butterworth filter is

$$\phi(\omega) = -\tan^{-1} \frac{2\omega^2 - \omega^3}{1 - 2\omega^2}$$

These are shown below.



(d) The group delay is estimated using the initial slope of the phase characteristic. For $\omega << 1$ we have, for the Bessel filter,

$$\theta(\omega) = -\tan^{-1} \omega \approx -\omega, \qquad \omega << 1$$

Thus the nominal passband group delay is

$$T_g(\omega) = -\frac{d\theta(\omega)}{d\omega} = 1 = t_o$$

The cutoff frequency of the Bessel filter is

$$f_c = \frac{1}{2\pi t_o} = \frac{1}{2\pi}$$

or

$$\omega_c = 1$$

This is the 3-dB cutoff frequency of the Butterworth filter and therefore the bandwidths are equal.

Problem E-5

Letting *RdB* represent the passband ripple in dB gives

$$20\log_{10}\left[\sqrt{1+\varepsilon^2}\right] = RdB$$

From the above expression

$$\varepsilon = \sqrt{10^{RdB/10} - 1}$$

The MATLAB program to solve the problem is shown below

```
rdB = [0.2 0.5 2 2.5 3 5];          % Vector of input values
e = sqrt(10.^(rdB/10)-1);           % Compute epsilon
e                                   % Display episilon
```

Executing the program gives the six required results as a vector

```
 e =

    0.2171    0.3493    0.7648    0.8822    0.9976    1.4705
```

Problem E-6

This problem is solved using Equation (E-45). The MATLAB program is

```
n = 4;                                              % Filter order
rdB = [0.05 0.2 1 2 3 5];                           % Passband ripple
e = sqrt(10.^(rdB/10)-1);                           % Determine epsilon
w3 = zeros(1,length(n));                            % Initialize vector
for j = 1:length(e)
e1 = e(1,j);                                        % Select a value of e
if e < 1
        w3(j) = cosh((1/n)*acosh(1/e1));            % Equation (E-45)
else
        w3(j) = cos((1/n)*acos(1/e1));              % Eq. (E-45) for e > 1
end
end
w3                                                  % Display results
```

Executing the program yields the six required results as a vector.

```
w3 =

    1.2784    1.1563    1.0530    1.0184    1.0001    0.9789
```

Problem E-7

This problem is identical to the previous one except that the filter order is 5.

```
n = 5;                                          % Filter order
rdB = [0.05 0.2 1 2 3 5];                       % Passband ripple
e = sqrt(10.^(rdB/10)-1);                       % Determine epsilon
w3 = zeros(1,length(n));                        % Initialize vector
for j = 1:length(e)
e1 = e(1,j);                                    % Select a value of e
if e < 1
        w3(j) = cosh((1/n)*acosh(1/e1));        % Equation (E-45)
else
        w3(j) = cos((1/n)*acos(1/e1));          % Eq. (E-45) for e > 1
end
end
w3                                              % Display results
```

Executing the MATLAB program yields the six required results in vector form as follows

```
w3 =

    1.1754    1.0992    1.0338    1.0117    1.0001    0.9865
```

Problem E-8

This problem is easily solved using the following MATLAB script:

```
[b,a] = cheby1(3,0.75,1,'s');    % Numerator and denominator of H(s)
[r,p,k] = residue(b,a);          % Find roots residues
p                                % Display poles
b                                % Display numerator polynomial
a                                % Display denominator polynomial
```

Executing the program yields the pole locations, and the numerator, b, and the denominator, a, of H(s). These are

```
p =

  -0.2741 + 0.9876i
  -0.2741 - 0.9876i
  -0.5481


b =

         0         0         0    0.5758
```

```
a =

    1.0000    1.0963    1.3509    0.5758
```

Problem E-9

This problem is identical to the previous problem except that the filter is changed from 3 to 4 and the passband ripple is changed from 0.75 dB to 2 dB. The MATLAB script for solving this problem is

```
[b,a] = cheby1(4,2.0,1,'s');      % Numerator and denominator of H(s)
[r,p,k] = residue(b,a);           % Find roots residues
p                                 % Display poles
b                                 % Display numerator polynomial
a                                 % Display denominator polynomial
```

Executing the program yields

```
p =

 -0.1049 + 0.9580i
 -0.1049 - 0.9580i
 -0.2532 + 0.3968i
 -0.2532 - 0.3968i


b =

     0     0     0     0  0.1634


a =

  1.0000  0.7162  1.2565  0.5168  0.2058
```

Problem E-10

We now increase the filter order to 5. This gives the following MATLAB script:

```
[b,a] = cheby1(5,2.0,1,'s');      % Numerator and denominator of H(s)
[r,p,k] = residue(b,a);           % Find roots residues
p                                 % Display poles
b                                 % Display numerator polynomial
a                                 % Display denominator polynomial
```

Executing the program gives

```
p =

  -0.0675 + 0.9735i
  -0.0675 - 0.9735i
  -0.1766 + 0.6016i
  -0.1766 - 0.6016i
  -0.2183


b =

         0         0         0         0         0    0.0817


a =

    1.0000    0.7065    1.4995    0.6935    0.4593    0.0817
```

Problem E-11

This problem is solved using MathCAD as shown below.

First we define C(n,ω) for ω<1:

$$C1(n,\omega) := \cos(n \cdot \operatorname{acos}(\omega))$$

Next we define C(n,ω) for ω>1:

$$C2(n,\omega) := \cosh(n \cdot \operatorname{acosh}(\omega))$$

Now the Chebyshev polynomials can be defined:

$$C(n,\omega) := \text{if}(\omega \le 1, C1(n,\omega), C2(n,\omega))$$

For 3-dB passband ripple the value of ε is

$$\varepsilon := \sqrt{10^{0.3} - 1} \qquad \text{or} \qquad \varepsilon = 0.998$$

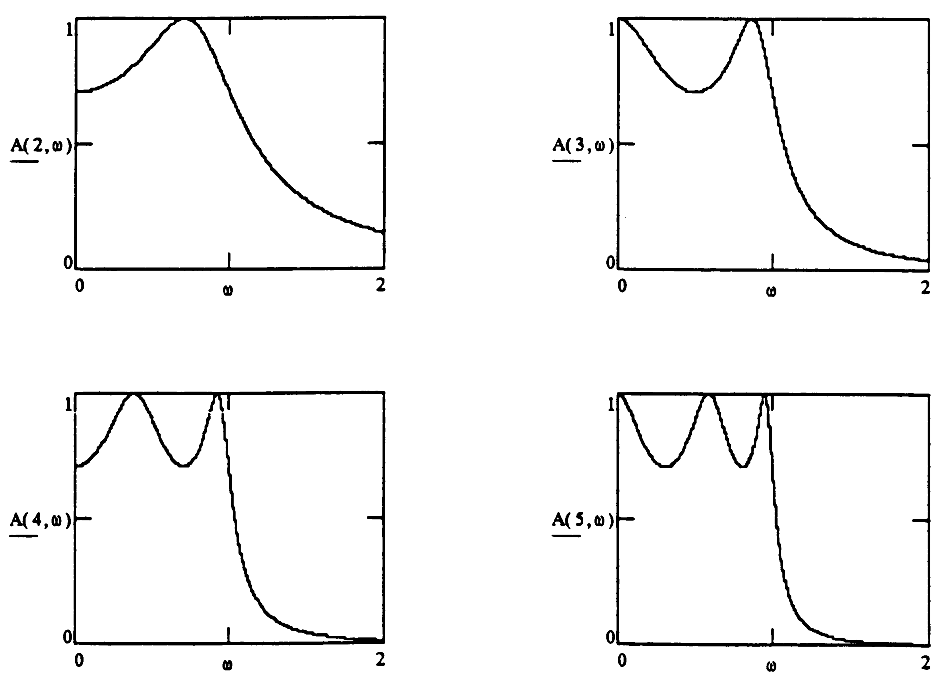
The amplitude responses of the Chebyshev filters are:

$$A(n,\omega) := \frac{1}{\sqrt{1 + \varepsilon^2 \cdot C(n,\omega)^2}}$$

The range is set as follows:

$$\omega := 0, 0.001 .. 2$$

The resulting plots are:



Problem E-12

This problem is solved using MathCAD as shown below.

First we define C(n,ω) for ω<1:

$$C1(n,\omega) := \cos(n \cdot \mathrm{acos}(\omega))$$

Next we define C(n,ω) for ω>1:

$$C2(n,\omega) := \cosh(n \cdot \mathrm{acosh}(\omega))$$

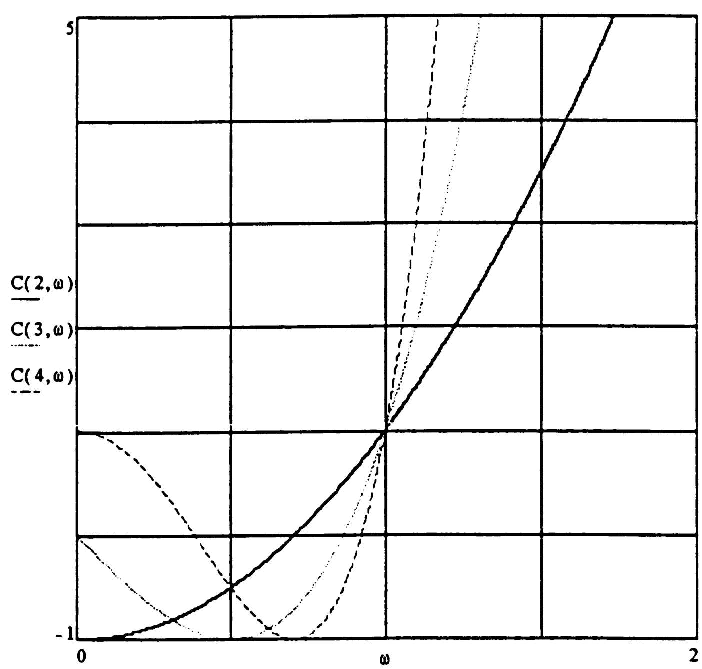
Now the Chebyshev polynomials can be defined:

$$C(n,\omega) := \mathrm{if}(\omega \le 1, C1(n,\omega), C2(n,\omega))$$

The range is set as follows:

$$\omega := 0, 0.001 .. 2$$

The resulting figure is:



Problem E-13

Assuming that the Butterworth filter has a 3 dB frequency of 100 Hertz, the attenuation at 400 Hertz is

$$A(400) = \frac{1}{\sqrt{1+\left(\frac{400}{100}\right)^{2n}}}$$

Thus we desire

$$20\log_{10}\left(\frac{1}{\sqrt{1+(4)^{2n}}}\right) = -45$$

Thus

$$10\log_{10}\left(1+(4)^{2n}\right) = 45$$

Solving for n we get

$$2n\ln(4) = \ln\left(10^{45/10}-1\right)$$

which yields

$$n = \frac{1}{2}\frac{\ln\left(10^{45/10}-1\right)}{\ln(4)} = 3.7$$

Thus, the necessary filter order is four.

Problem E-14

This problem differs significantly from the preceding problem in that the Chebyshev filter does not have an attenuation of 3 dB at the critical frequency. Rather, the attenuation at the critical frequency is equal to the passband ripple. Recognizing this, an iterative technique is used in which the 3 dB frequency and the critical frequency are assumed the same and the necessary filter order to give an attenuation of 45 dB at 400 Hertz is computed. The critical frequency is then adjusted so that the filter has the required attenuation of 3 dB (actually 3.01 dB) at 100 Hertz. The attenuation at 400 Hertz is then computed. The filter order is then reduced by one and the process is repeated. The reason for this step is that the assumption that the critical and 3 dB frequencies are the same may result in a conservative design in which the filter order is greater than necessary. Reducing the filter is continued until the attenuation at 400 Hertz is less than 45 dB so that the design specification is not satisfied.

The MATLAB program to perform the above process is shown below.

```
epidB = input('Enter epi in dB > ');      % Input passband ripple in dB
epi = sqrt((10^(epidB/10))-1);            % Convert from dB units

y = (1/epi)*sqrt((10^(45/10))-1);         % Interim value
n = acosh(y)/acosh(400/100);              % Initial estimate of n
n = ceil(n);                              % Round to next integer

HdB = 50;                                 % Initialize loop

while HdB>44.9                            % Test
w3 = cosh((1/n)*acosh(1/epi));            % Normalized 3 dB freq. (E-45)
wc = 2*pi*100/w3;                         % Adjust 3dB frequency

w = 2*pi*100/wc;                          % Set value of w for f=100
aep14a                                    % Go to subroutine
x1 = HdB;                                 % Attenuation in dB at f=100

w=(2*pi*400)/wc;                          % Set value of w for f=400
aep14a                                    % Go to subroutine
x2 = HdB;                                 % Attenuation in dB at f=400

sprintf('For n = %2i, Attn (f=100) %.2f dB,  Attn (f=400) %.2f
dB',n,x1,x2)

n = n-1;                                  % Reduce order by one
end                                       % End of loop
```

The command **aep14a** is a subroutine that computes the Chebyshev polynomial and the filter attenuation at a given normalized frequency, passband ripple and filter order. It is shown below.

```
% This script calculates the Chebyshev polynomial Cn(w)
% We also calculate the attenuation in dB

if w<1
        Cnw = cos(n*acos(w));
else
        Cnw = cosh(n*acosh(w));
end

HdB = 10*log10(1+epi*epi*Cnw*Cnw);
```

We now work the problem.

(a) The MATLAB session for a passband ripple of 0.5 dB follows.

```
Enter epi in dB > 0.5

ans =
```

```
For n =  4, Attn (f=100) 3.01 dB,  Attn (f=400) 59.72 dB

ans =

For n =  3, Attn (f=100) 3.01 dB,  Attn (f=400) 42.76 dB
```

From the above we see that the necessary filter order is four.

(b) For a passband ripple of 2 dB the MATLAB session follows.

```
Enter epi in dB > 2

ans =

For n =  3, Attn (f=100) 3.01 dB,  Attn (f=400) 46.28 dB

ans =

For n =  2, Attn (f=100) 3.01 dB,  Attn (f=400) 28.78 dB
```

We can see that the necessary filter order is three.

(c) For a passband ripple of 3 dB the MATLAB session follows.

```
Enter epi in dB > 3

ans =

For n =  3, Attn (f=100) 3.01 dB,  Attn (f=400) 47.73 dB

ans =

For n =  2, Attn (f=100) 3.01 dB,  Attn (f=400) 29.82 dB
```

For three dB passband ripple we see that again the necessary filter order is three.

Problem E-15

We'll solve this problem using MathCAD for a change.

The steady-state frequency resopnse for a second-order Butterworth filter is determined by writing the frequency scaled transfer function (as a function of s) and letting s equal j*2*pi*f. We then define j:

$$j := \sqrt{-1}$$

The sinusoidal steady state frequency response then becomes (note that there are two unknown parameters):

$$H1(f, a0, b1) := \frac{a0}{(j \cdot 2 \cdot \pi \cdot f)^2 + \sqrt{2} \cdot (j \cdot 2 \cdot \pi \cdot f) \cdot b1 + a0}$$

Define the constants:

$$a0 := (2 \cdot \pi \cdot 300)^2 \qquad b1 := 2 \cdot \pi \cdot 300$$

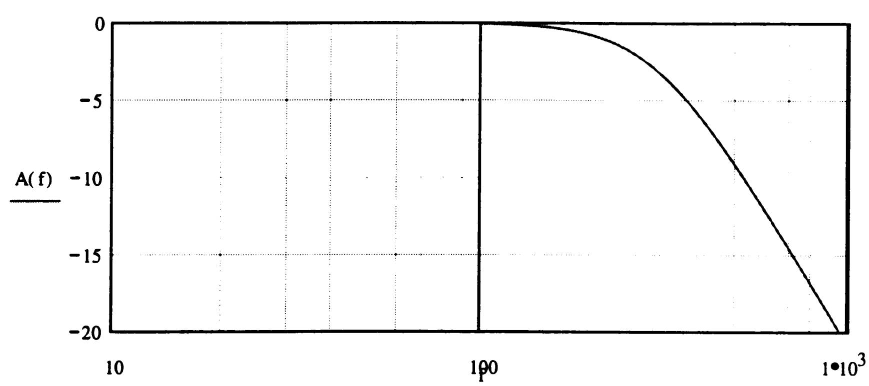
Define the amplitude and phase responses:

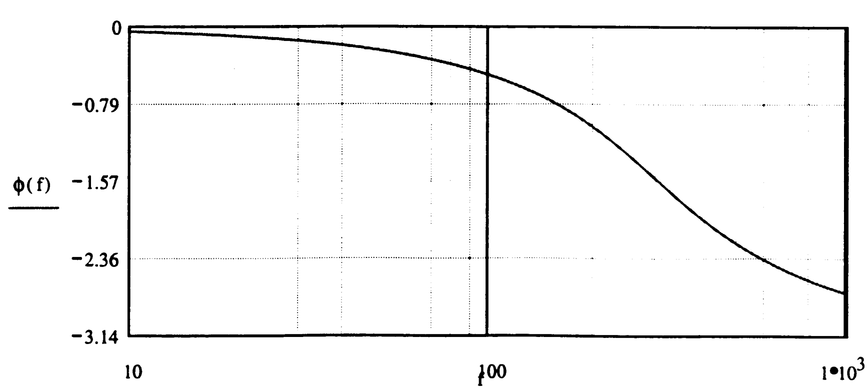
$$A(f) := 20 \cdot \log(|H1(f, a0, b1)|) \qquad \phi(f) := \arg(H1(f, a0, b1))$$

Define the range of f:

$$f := 10, 11 .. 1000$$

The plots of the amplitude and the phase responses are shown below.





Problem E-16

We'll also solve this problem using MathCAD.

We solve this problem by developing a general template that can be used for a variety of problems by entering only the appropriate transfer function. The first step is to define j. Thus

$$j := \sqrt{-1}$$

Next we define the center frequency, fc, and the bandwidth, fb. These are

$$fc := 3000$$

and

$$fb := 600$$

Since a bandpass filter is to be developed the lowpass to bandpass conversion characteristic is next defined. This gives

$$a(f) := \frac{(2\cdot\pi\cdot fc)^2 + (j\cdot 2\cdot\pi\cdot f)^2}{(j\cdot 2\cdot\pi\cdot f)\cdot(2\cdot\pi\cdot fb)}$$

Note that the (2*pi) squared term can be cancelled from the above expression. We shall, however, leave a(f) in the form defined above so that the expression is clearer. All that now need be done is to enter the transfer function of the prototype filter. For a second-order bandpass filter we use a first-order prototype filter. Thus

$$H(f) := \frac{1}{a(f) + 1}$$

The amplitude and phase response are next defined. This gives

$$A(f) := 20\cdot\log(|H(f)|)$$

and

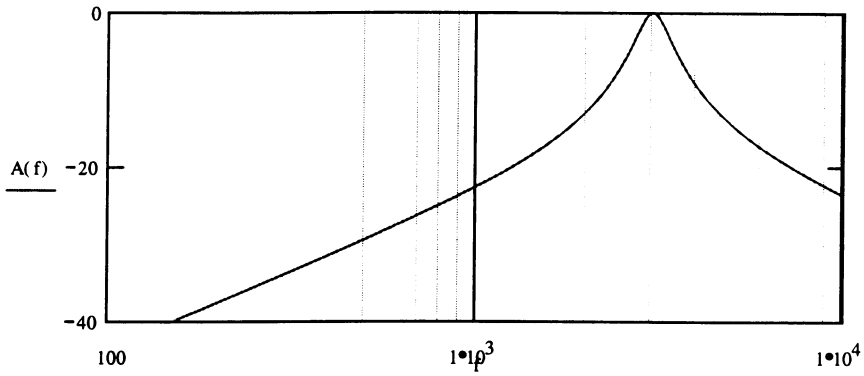
$$\phi(f) := \arg(H(f))$$

The iteration range for the frequency is specified as

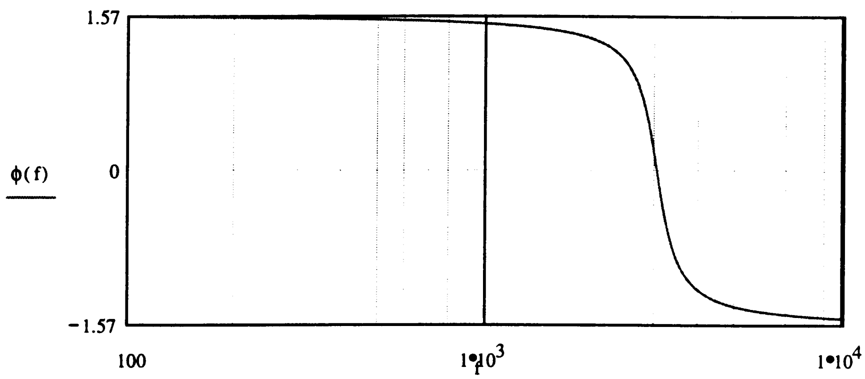
$$f := 100, 110 .. 10000$$

The amplitude and the phase responses follow and are shown at the top of the following page.

The amplitude response is



and the phase response is



Problem E-17

We'll also solve this problem using MathCAD.

We solve this problem by developing a general template that can be used for a variety of problems by entering only the appropriate transfer function. The first step is to define j. Thus

$$j := \sqrt{-1}$$

Next we define the center frequency, fc, and the bandwidth, fb. These are

$$fc := 100$$

and

$$fb := 50$$

Since a bandpass filter is to be developed the lowpass to bandpass conversion characteristic is next defined. This gives

$$a(f) := \frac{(2\cdot\pi\cdot fc)^2 + (j\cdot 2\cdot\pi\cdot f)^2}{(j\cdot 2\cdot\pi\cdot f)\cdot(2\cdot\pi\cdot fb)}$$

Note that the (2*pi) squared term can be cancelled from the above expression. We shall, however, leave a(f) in the form defined above so that the expression is clearer. All that now need be done is to enter the transfer function of the prototype filter. For a fourth-order bandpass filter we use a second-order prototype. Thus

$$H(f) := \frac{1}{a(f)^2 + \sqrt{2}\cdot a(f) + 1}$$

The amplitude and phase response are next defined. This gives

$$A(f) := 20\cdot\log(|H(f)|)$$

and

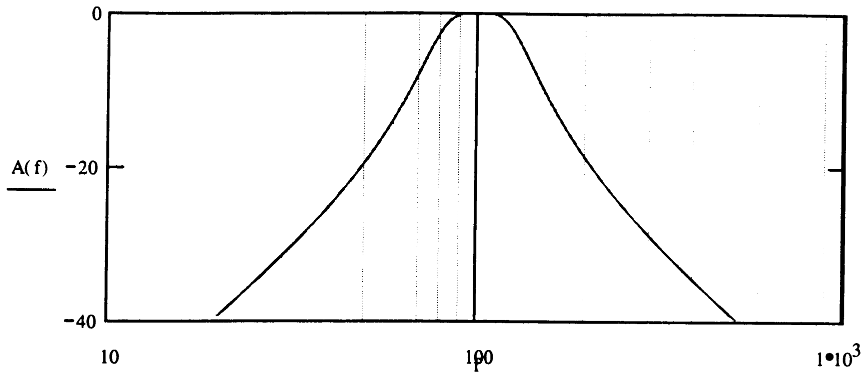
$$\phi(f) := \arg(H(f))$$

The iteration range for the frequency is specified as

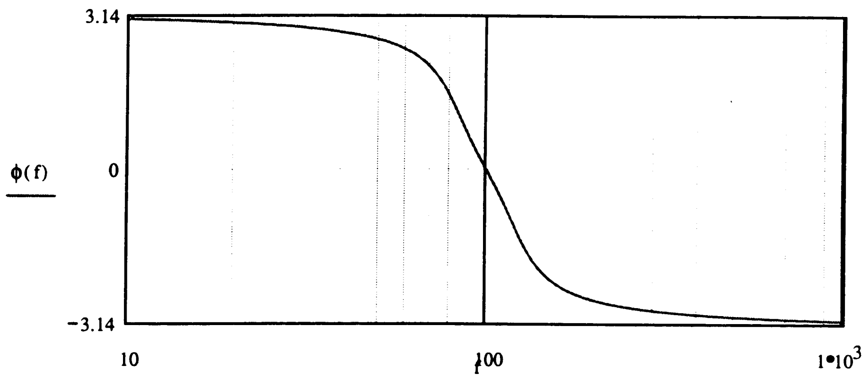
$$f := 10, 11 .. 1000$$

The amplitude and the phase responses are shown at the top of the following page.

The amplitude response (in dB) is



and the phase response is



We now determine the upper and lower half-power frequencies. From (E-71) we can write

$$fl := -0.5 \cdot fb + 0.5 \cdot \sqrt{fb^2 + 4 \cdot fc^2}$$

and

$$fu := fl + fb$$

This gives

$$fl = 78.078$$

and

$$fu = 128.078$$

We can check the results as shown below:

$$A(78.078) = -3.01 \qquad A(128.078) = -3.01$$

Problem E-18

We'll also solve this problem using MathCAD.

We solve this problem by developing a general template that can be used for a variety of problems by entering only the appropriate transfer function. The first step is to define j. Thus

$$j := \sqrt{-1}$$

Next we define the center frequency, fc, and the bandwidth, fb. These are

$$fc := 100$$

and

$$fb := 50$$

Since a bandpass filter is to be developed the lowpass to bandpass conversion characteristic is next defined. This gives

$$a(f) := \frac{(2\cdot\pi\cdot fc)^2 + (j\cdot 2\cdot\pi\cdot f)^2}{(j\cdot 2\cdot\pi\cdot f)\cdot(2\cdot\pi\cdot fb)}$$

Note that the (2*pi) squared term can be cancelled from the above expression. We shall, however, leave a(f) in the form defined above so that the expression is clearer. All that now need be done is to enter the transfer function of the prototype filter. For a sixth-order bandpass filtrer we use a third-order prototype. Thus

$$H(f) := \frac{1}{a(f)^3 + 2\cdot a(f)^2 + 2\cdot a(f) + 1}$$

The amplitude and phase response are next defined. This gives

$$A(f) := 20\cdot\log(|H(f)|)$$

and

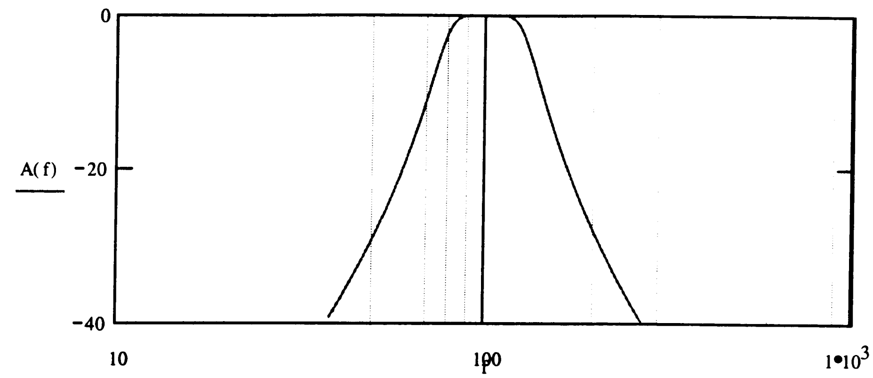
$$\phi(f) := \arg(H(f))$$

The iteration range for the frequency is specified as

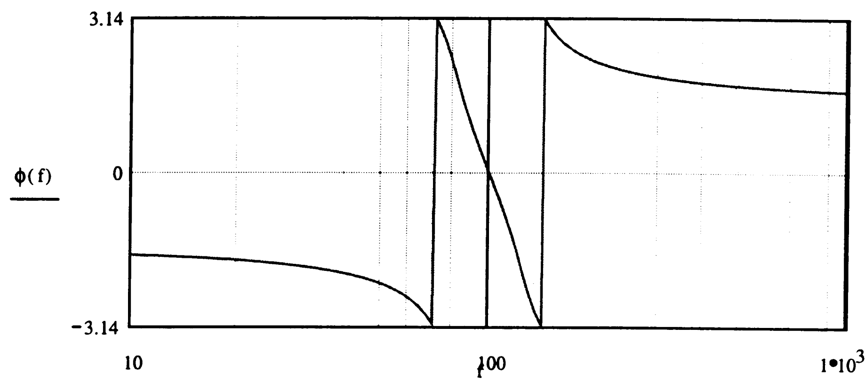
$$f := 10, 11 .. 1000$$

The amplitude and the phase responses are shown at the top of the following page.

The amplitude response is



and the phase response is



We now determine the upper and lower half-power frequencies. From (E-71) we can write

$$fl := -0.5 \cdot fb + 0.5 \cdot \sqrt{fb^2 + 4 \cdot fc^2}$$

and

$$fu := fl + fb$$

This gives

$$fl = 78.078$$

and

$$fu = 128.078$$

We can check the results as shown below:

$A(81.98) = -1.011$ $\qquad A(121.98) = -1.011$

Problem E-19

We'll also solve this problem using MathCAD.

The first step is to define j. Thus

$$j := \sqrt{-1}$$

Next we define the center frequency, fc, and the bandwidth, fb. These are

$$fc := 1000$$

and

$$fb := 300$$

Since a notch filter is to be developed the lowpass to notch filter conversion characteristic is next defined. This gives

$$a(f) := \frac{(j \cdot 2 \cdot \pi \cdot f) \cdot (2 \cdot \pi \cdot fb)}{(j \cdot 2 \cdot \pi \cdot f)^2 + (2 \cdot \pi \cdot fc)^2}$$

Note that the (2*pi) squared term can be cancelled from the above expression. We shall, however, leave a(f) in the form defined above so that the expression is clearer. All that now need be done is to enter the transfer function of the prototype filter. For a fourth-order notch filter we use a second-order prototype. Thus

$$H(f) := \frac{1}{a(f)^2 + \sqrt{2} \cdot a(f) + 1}$$

The amplitude and phase response are next defined. This gives

$$A(f) := 20 \cdot \log(|H(f)|)$$

and

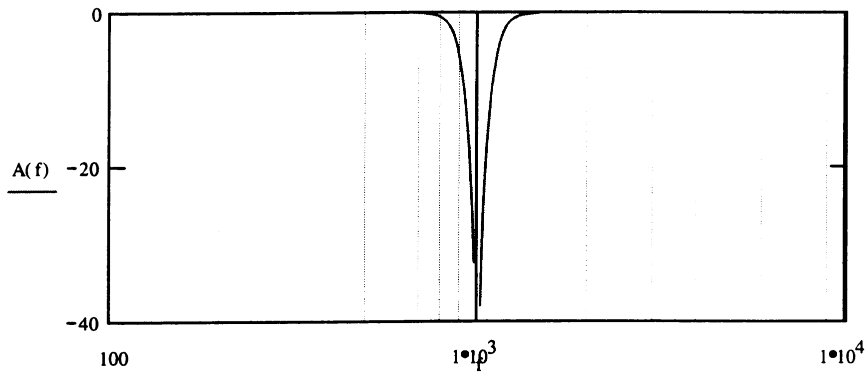
$$\phi(f) := \arg(H(f))$$

The iteration range for the frequency is specified as (we must choose values of f that avoid the point f=1000). Thus

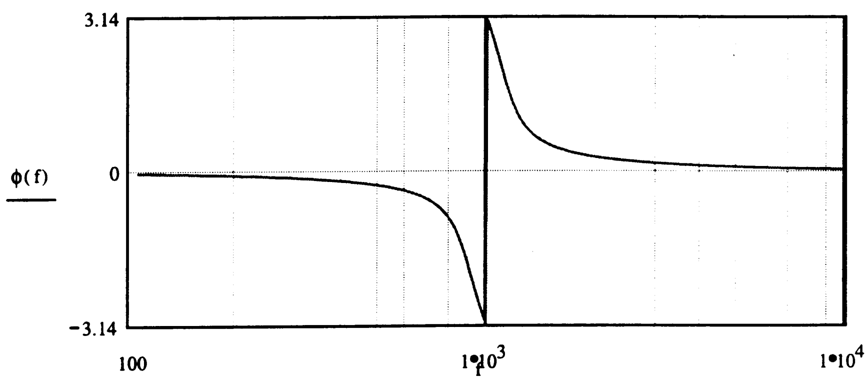
$$f := 107, 117 .. 10000$$

The amplitude and the phase responses follow and are shown at the top of the following page.

The amplitude response is



and the phase response is



As a check we now determine the upper and lower half-power frequencies. From (E-71) we can write

$$fl := -0.5 \cdot fb + 0.5 \cdot \sqrt{fb^2 + 4 \cdot fc^2}$$

and

$$fu := fl + fb$$

This gives

$$fl = 861.187421$$

and

$$fu = 1161.187421$$

We can check the results as shown below:

$A(861.187) = -3.010271$ $\quad$ $A(1161.187) = -3.010321$

Problem E-20

We'll also solve this problem using MathCAD.

First we define the center frequency and the bandwidth. This gives

$$fc := 900 \qquad\qquad fb := 600$$

From (D-54) the upper critical frequency is

$$fu := 0.5 \cdot fb + 0.5 \cdot \sqrt{fb^2 + 4 \cdot fc^2}$$

and the lower critical frequency is

$$fl := fu - fb$$

With the given values these yield

$fu = 1248.683$ Hertz

and

$fl = 648.683$ Hertz

As a check we compute

$$\sqrt{fl \cdot fu} = 900$$

Problem E-21

We'll now switch back to MATLAB. The script file for this problem is shown below.

```
fc = 1000;
fb = 100;
fu = 0.5*fb+0.5*sqrt(fb*fb+4*fc*fc);
fl = fu-fb;
w1 = 2*pi*fl;
w2 = 2*pi*fu;

[b,a] = butter(2,[w1 w2],'s');
b
a
```

Execution of the program yields the following:

```
b =

                0                 0   3.9478e+005                 0                 0


a =

  1.0000e+000   8.8858e+002   7.9352e+007   3.5080e+010   1.5585e+015
```

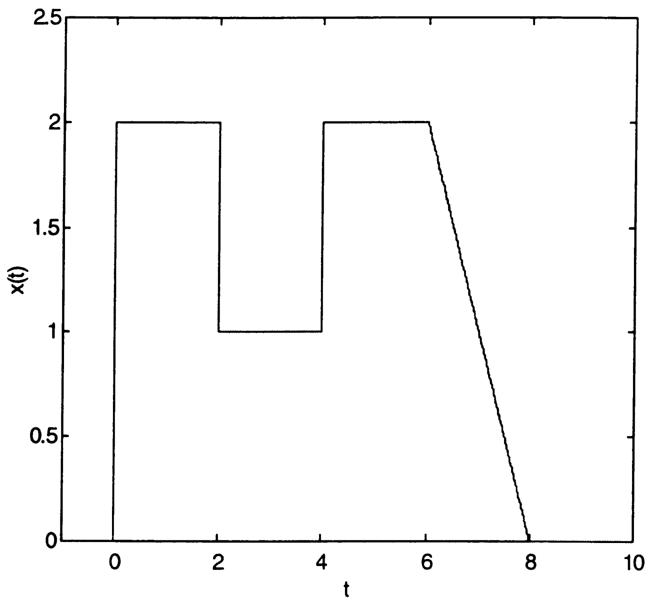
Comparison with the results of Example E-5 shows that the denominators of the bandpass filter and the notch filters to be the same.

PART II

SOLUTIONS TO COMPUTER EXERCISES

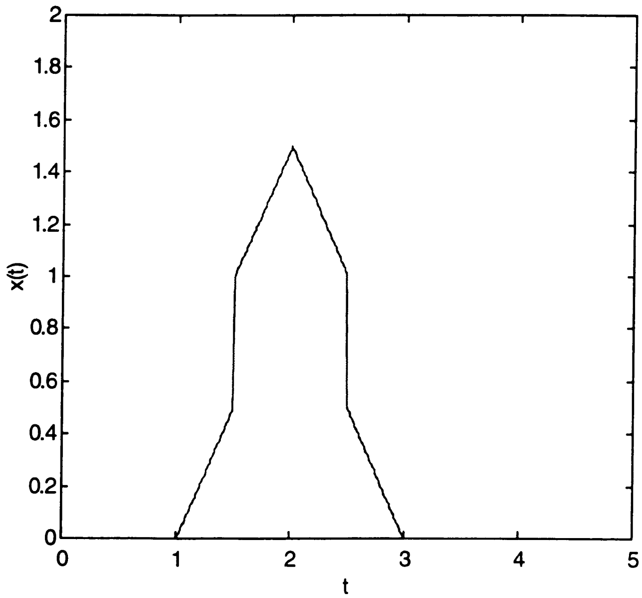
Computer Exercise 1-1

```
t=  -1:.01:10;
x = 2*stp_fn(t) - stp_fn(t-2) + stp_fn(t-4) - rmp_fn(t-6) + rmp_fn(t-8);
 plot(t, x), xlabel('t'), ylabel('x(t)'), axis([-1 10 0 2.5])
```



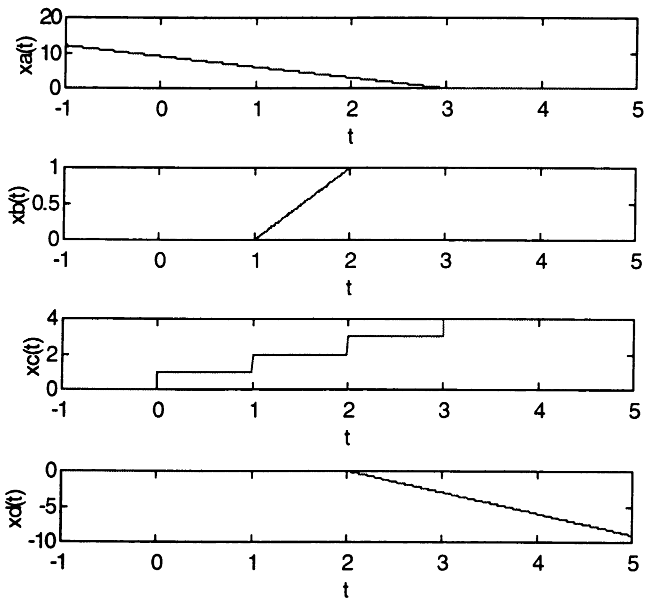
Computer Exercise 1-2

```
t = 0:.01:5;
x = rmp_fn(t-1)+0.5*stp_fn(t-1.5)-2*rmp_fn(t-2)-0.5*stp_fn(t-
2.5)+rmp_fn(t-3);
plot(t, x), xlabel('t'), ylabel('x(t)'), axis([0 5 0 2])
```



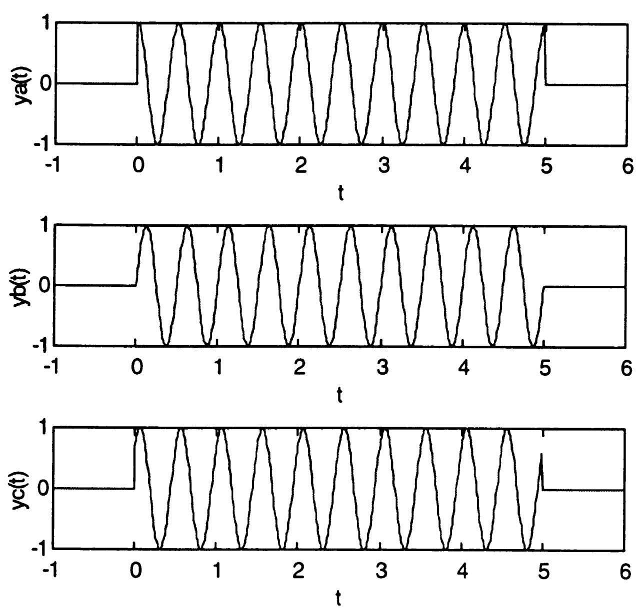
Computer Exercise 1-3

```
t = -1:.01:5;
xa = 3*rmp_fn(3-t);
xb = rmp_fn(t-1) - rmp_fn(t-2);
xc = stp_fn(t) + stp_fn(t-1) + stp_fn(t-2) + stp_fn(t-3);
xd = -3*rmp_fn(t-2);
subplot(4,1,1), plot(t, xa), xlabel('t'), ylabel('xa(t)')
subplot(4,1,2), plot(t, xb), xlabel('t'), ylabel('xb(t)')
subplot(4,1,3), plot(t, xc), xlabel('t'), ylabel('xc(t)')
subplot(4,1,4), plot(t, xd), xlabel('t'), ylabel('xd(t)')
```



<u>**Computer Exercise 1-4**</u>

```
t = -1:.01:6;
ya = cos(4*pi*t).*(stp_fn(t) - stp_fn(t-5));
yb = sin(4*pi*t).*(stp_fn(t) - stp_fn(t-5));
yc = 1/sqrt(2)*(ya + yb);
clg
subplot(3,1,1),plot(t, ya),xlabel('t'),ylabel('ya(t)')
subplot(3,1,2),plot(t, yb),xlabel('t'),ylabel('yb(t)')
subplot(3,1,3),plot(t, yc),xlabel('t'),ylabel('yc(t)')
```

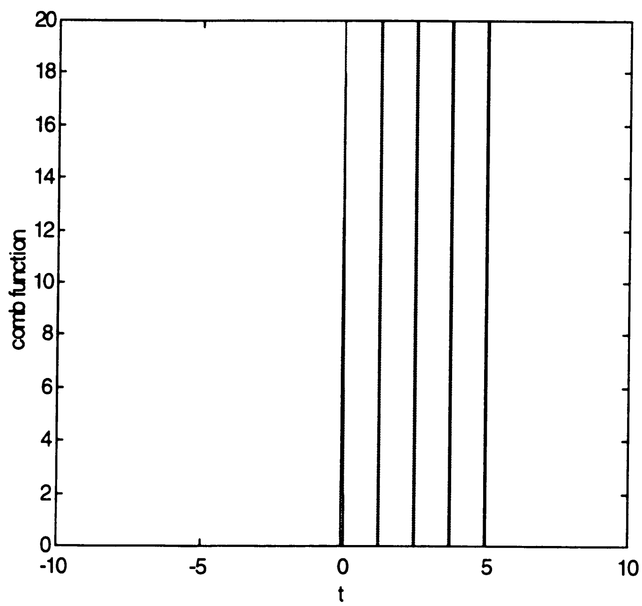


Computer Exercise 1-5

```
% cmb_fn(t,t_rep,t_width,delta). This function generates a comb of unit impulses centered
% at t = 0 spaced by t_rep and of total duration t_width.
%
function y = cmb_fn(t,t_rep,t_width,delta)
L = length(t);
N = round(t_width/t_rep+1);
y = zeros(size(t));
for n = 1:N
    y=y+impls_fn(t-(n-1)*t_rep+(N-1)*t_rep/2,delta);
end
```

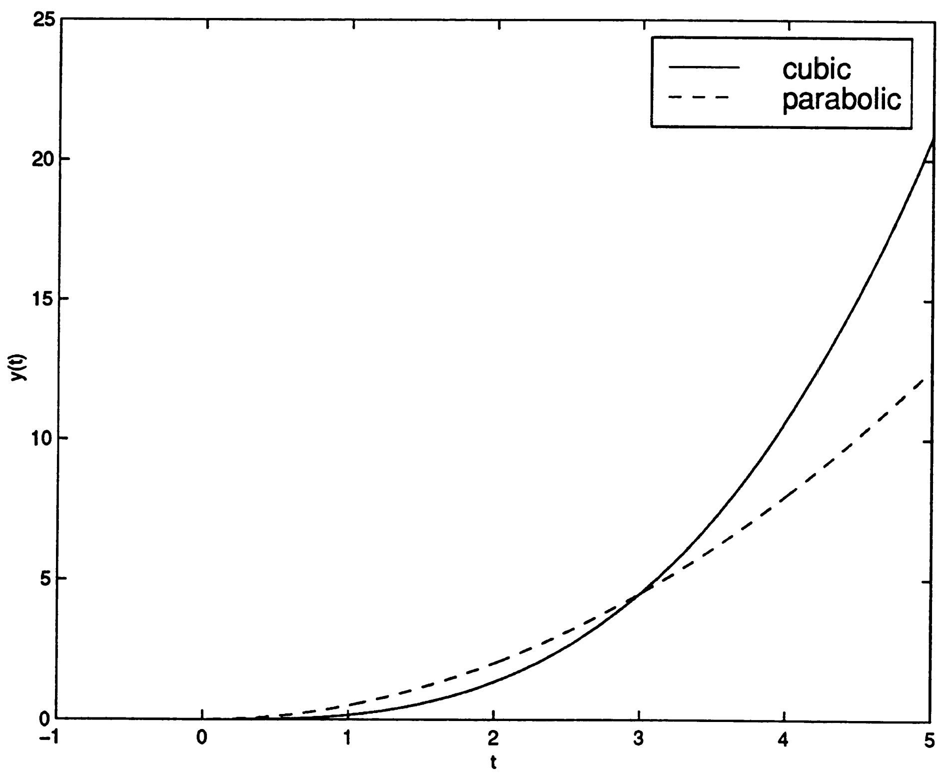
Sample run:

```
t = -10:.01:10;
y = cmb_fn(t - 2.5, 1.25, 5.1, .05);
plot(t, y), xlabel('t'), ylabel('comb function')
```



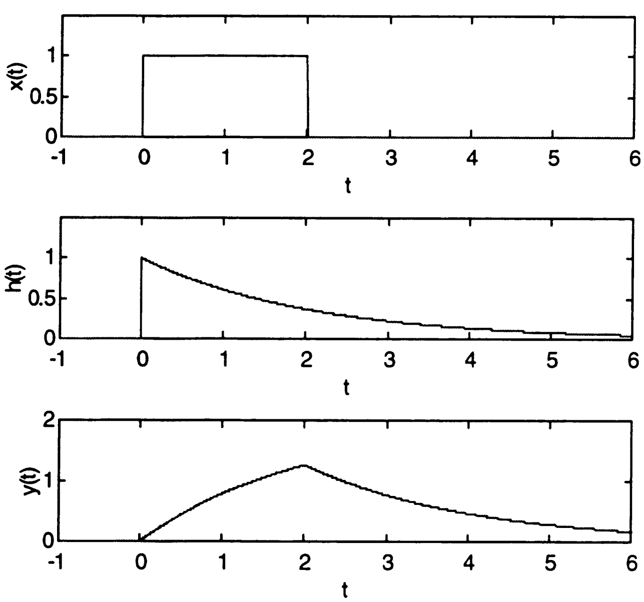
Computer Exercise 1-6

```
%   par_fn(t):  This function generates a unit parabola function
%
function y = par_fn(t)
y = 0.5*t.^2.*stp_fn(t);

%   cub_fn(t):    This functions generates a unit cubic function
%
function y = cub_fn(t)
y = (1/6)*t.^3.*stp_fn(t);

%   Sample use of the unit-parabola and unit-cubic functions
%
t = -1:.01:5;
y_par = par_fn(t);
y_cub = cub_fn(t);
plot(t, y_cub,'-'),xlabel('t'),ylabel('y(t)')
hold on
plot(t, y_par,'--')
legend('cubic', 'parabolic')
```



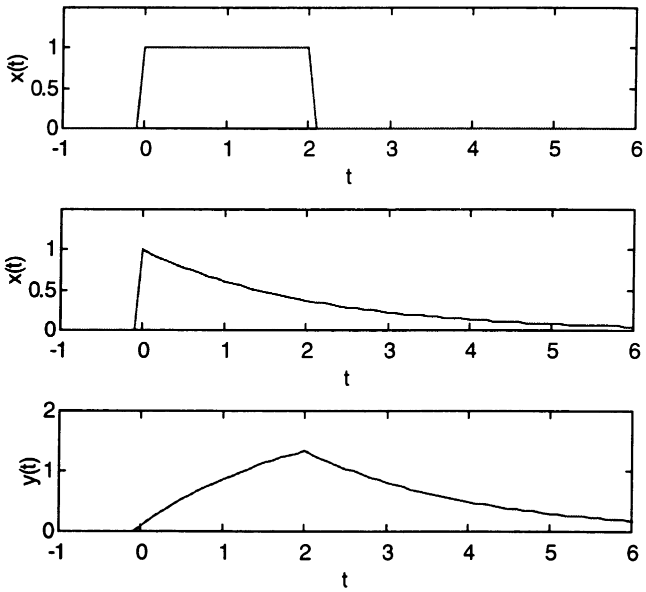
Computer Exercise 2-1

```
del_t = .005;
t = -1:del_t:6;
L = length(t);
tp = [2*t(1):del_t:2*t(L)];
x_2_1 = zeros(size(t));
h_2_1 = zeros(size(t));
y_2_1 = zeros(size(tp));
x_2_1 = pls_fn((t-1)/2);
h_2_1 = exp(-0.5*t).*stp_fn(t);
y_2_1 = del_t*conv(x_2_1, h_2_1);
subplot(3,1,1), plot(t, x_2_1), xlabel('t'), ylabel('x(t)'), axis([t(1)
t(L) 0 1.5])
subplot(3,1,2),
plot(t, h_2_1), xlabel('t'), ylabel('h(t)'), axis([t(1) t(L) 0 1.5])
subplot(3,1,3), plot(tp, y_2_1),xlabel('t'), ylabel('y(t)'), axis([t(1)
t(L) 0 2])
```



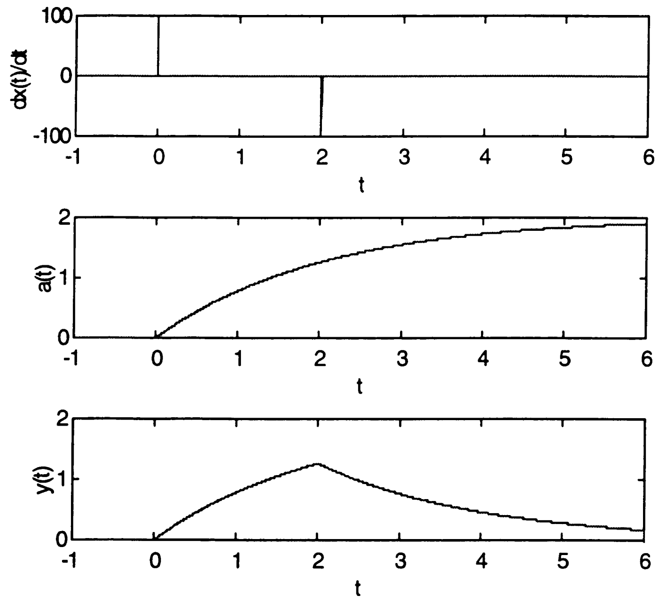
(b) Step size = 0.1; result for convolution is still not bad in this case.

```
del_t = .1;
t = -1:del_t:6;
L = length(t);
tp = [2*t(1):del_t:2*t(L)];
x_2_1 = zeros(size(t));
h_2_1 = zeros(size(t));
y_2_1 = zeros(size(tp));
x_2_1 = pls_fn((t-1)/2);
h_2_1 = exp(-0.5*t).*stp_fn(t);
y_2_1 = del_t*conv(x_2_1, h_2_1);
subplot(3,1,1), plot(t, x_2_1), xlabel('t'), ylabel('x(t)'), axis([t(1)
t(L) 0 1.5])
```



Computer Exercise 2-2

```
del_t = .005;
t = -1:del_t:6;
L = length(t);
tp = [2*t(1):del_t:2*t(L)];
dx_2_2 = zeros(size(t));
a_2_2 = zeros(size(t));
y_2_2 = zeros(size(tp));
dx_2_2 = impls_fn(t,.01) - impls_fn(t-2,.01);
a_2_2 = 2*(1-exp(-0.5*t)).*stp_fn(t);
y_2_2 = del_t*conv(dx_2_2, a_2_2);
subplot(3,1,1), plot(t, dx_2_2), xlabel('t'), ylabel('dx(t)/dt'),
axis([t(1) t(L) -100 100])
subplot(3,1,2), plot(t, a_2_2), xlabel('t'), ylabel('a(t)'), axis([t(1)
t(L) 0 2])
subplot(3,1,3), plot(tp, y_2_2),xlabel('t'), ylabel('y(t)'), axis([t(1)
t(L) 0 2])
```

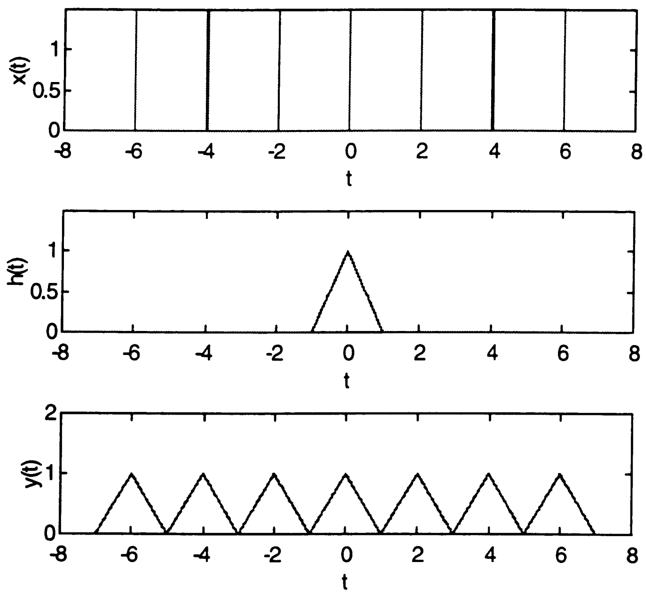


Computer Exercise 2-3

The following triangle function is defined for this exercise:

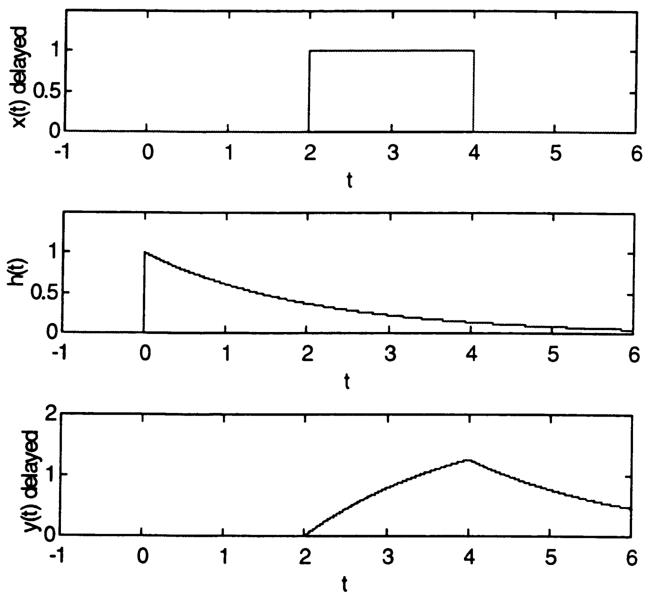
```
% This function generates a unit-high triangle centered
% at zero and extending from -1 to 1
%
function y = trgl_fn(t)
y = (1 - abs(t)).*pls_fn(t/2);
```

```
clg
del_t = .005;
t = -8:del_t:8;
L = length(t);
tp = [2*t(1):del_t:2*t(L)];
x_2_3 = zeros(size(t));
h_2_3 = zeros(size(t));
y_2_3 = zeros(size(tp));
x_2_3 = cmb_fn(t,2,12,.01);
h_2_3 = trgl_fn(t);
y_2_3 = del_t*conv(x_2_3, h_2_3);
subplot(3,1,1), plot(t, x_2_3), xlabel('t'), ylabel('x(t)'), axis([t(1)
t(L) 0 1.5])
subplot(3,1,2), plot(t, h_2_3), xlabel('t'), ylabel('h(t)'), axis([t(1)
t(L) 0 1.5])
subplot(3,1,3), plot(tp, y_2_3),xlabel('t'), ylabel('y(t)'), axis([t(1)
t(L) 0 2])
```



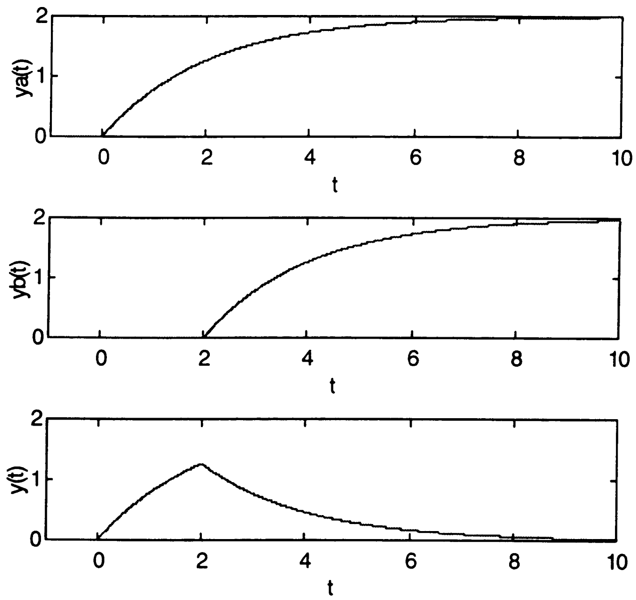
Computer Exercise 2-4

```
del_t = .005;
t = -1:del_t:6;
L = length(t);
tp = [2*t(1):del_t:2*t(L)];
x_2_4 = zeros(size(t));
h_2_4 = zeros(size(t));
y_2_4 = zeros(size(tp));
x_2_4_d = pls_fn((t-1-2)/2);        %An additional delay of 2
h_2_4 = exp(-0.5*t).*stp_fn(t);
y_2_4_d = del_t*conv(x_2_4_d, h_2_4);
subplot(3,1,1), plot(t, x_2_4_d), xlabel('t'), ylabel('x(t)'),
axis([t(1) t(L) 0 1.5])
subplot(3,1,2), plot(t, h_2_4), xlabel('t'), ylabel('h(t)'), axis([t(1)
t(L) 0 1.5])
subplot(3,1,3), plot(tp, y_2_4_d),xlabel('t'), ylabel('y(t)'),
axis([t(1) t(L) 0 2])
```



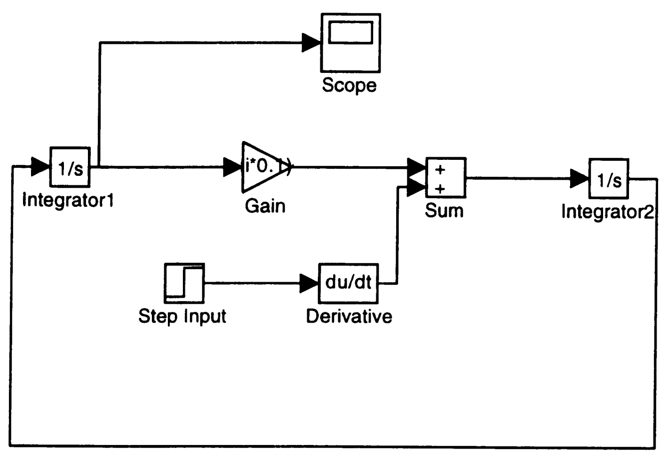
<u>**Computer Exercise 2-5**</u>

```
clg
del_t = .005;
t = -1:del_t:10;
L = length(t);
tp = [2*t(1):del_t:2*t(L)];
x_2_5a = zeros(size(t));
x_2_5b = zeros(size(t));
h_2_5 = zeros(size(t));
y_2_5 = zeros(size(tp));
y_2_5a = zeros(size(tp));
y_2_5b = zeros(size(tp));
x_2_5a = stp_fn(t);
x_2_5b = stp_fn(t-2);
h_2_5 = exp(-0.5*t).*stp_fn(t);
y_2_5a = del_t*conv(x_2_5a, h_2_5);
y_2_5b = del_t*conv(x_2_5b, h_2_5);
y_2_5 = y_2_5a - y_2_5b;
subplot(3,1,1), plot(tp, y_2_5a), xlabel('t'), ylabel('ya(t)'),
axis([t(1) t(L) 0 2])
subplot(3,1,2), plot(tp, y_2_5b), xlabel('t'), ylabel('yb(t)'),
axis([t(1) t(L) 0 2])
subplot(3,1,3), plot(tp, y_2_5), xlabel('t'), ylabel('y(t)'),
 axis([t(1) t(L) 0 2])
```



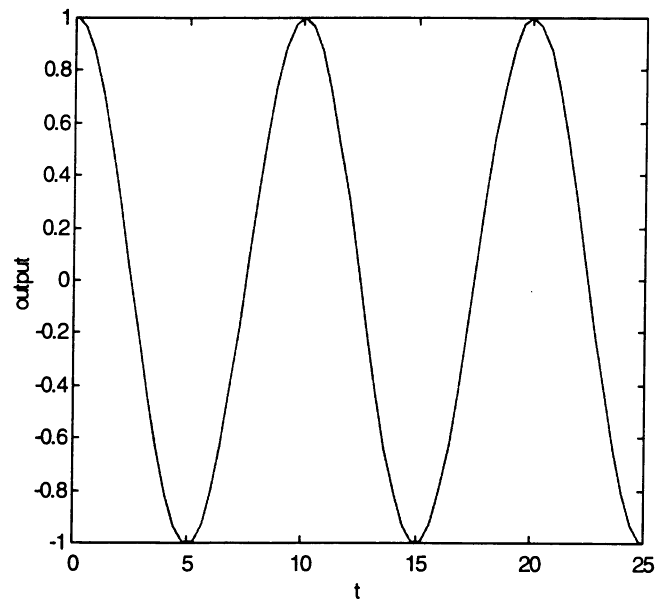
Computer Exercise 2-6

```
simulink
c_ex2_6
```



Simulation for 0.1 hertz frequency:

```
[t,q] =    lsim('c_ex2_6',25);
plot(t,q(:, 1)), xlabel('t'), ylabel('output')
```

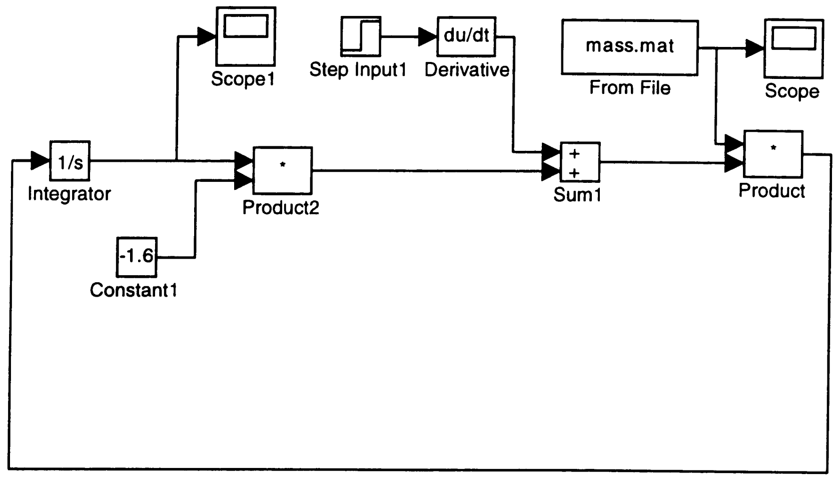


Computer Exercise 2-7

Rewrite the governing differential equation as

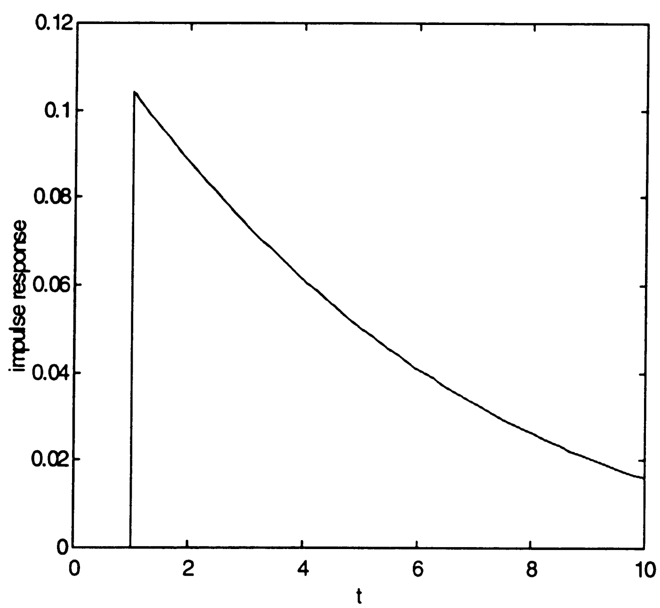
$$\frac{dh(t,\tau)}{dt} = \frac{1}{M_0 - kt}[\delta(t-\tau) - (\alpha - k)h(t,\tau)]$$

where the parameters are defined in the problem statement. The simulink diagram below implements this equation. The time varying mass term, $1/(M_0 - kt)$ is implemented as the lookup file mass.mat. The impulsive input is obtained by differentiating a step and applied at time $t = 1$. Other parameter values are $a = 2$, $k = 0.4$, and $M_0 = 10$.



The simulink implementation can be run directly or run from the command window as follows:

```
[t,v] =    lsim('c_ex2_7',10);
plot(t, v), xlabel('t'), ylabel('impulse response')
```

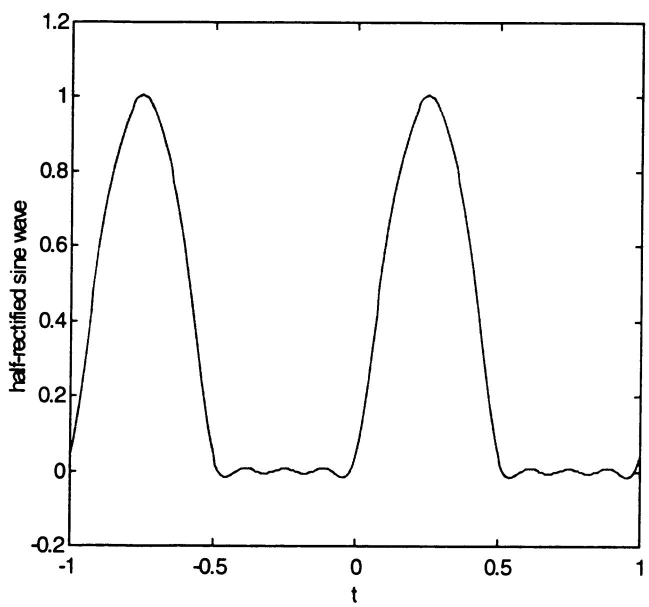


This is the same result as obtained by plotting the equation given in the problem statement for the impulse response.

Computer Exercise 3-1

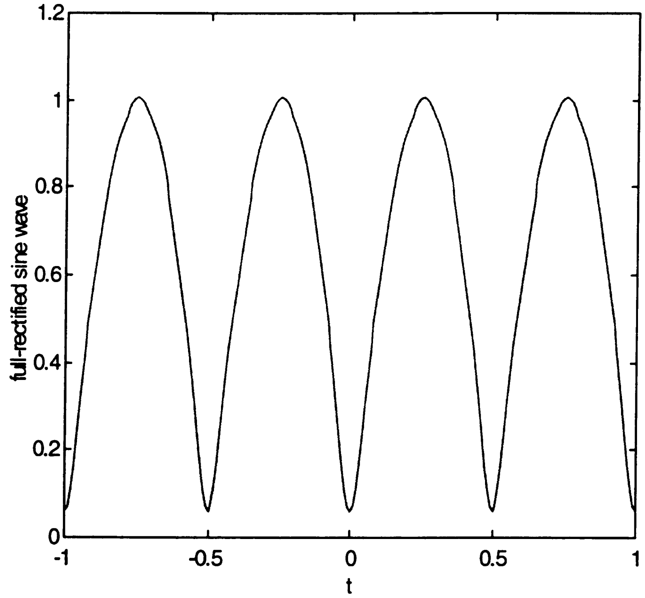
(a) A program for plotting the partial sum Fourier series for the half-rectified sine wave through the sixth harmonic is given below:

```
t = -1:.01:1;
n = [-6:1:6];
omega_0 = 2*pi;
X_0 = 1/pi;
X_1 = -0.25*j;
X_2 = -1/(3*pi);
X_4 = -1/(15*pi);
X_6 = -1/(35*pi);
X = [X_6 0 X_4 0 X_2 -X_1 X_0 X_1 X_2 0 X_4 0 X_6];
x_har_6 = X*exp(j*omega_0*n'*t);  % Matrix multiply of the vector X
                                    % times the matrix exp( ) efficiently
                                    % implements the Fourier sum.
plot(t, real(x_har_6)), xlabel('t'), ylabel('half-rectified sine wave')
```



(b) A program for the full-wave rectified sine wave is similar. Calculation through the tenth harmonic is implemented. Note that many harmonics are required to reproduce the sharp nulls in the waveform.

```
t = -1:.01:1;
n = [-10:1:10];
omega_0 = 2*pi;
X_0 = 2/pi;
X_2 = -2/(3*pi);
X_4 = -2/(15*pi);
X_6 = -2/(35*pi);
X_8 = -2/(63*pi);
X_10 = -2/(99*pi);
X = [X_10 0 X_8 0 X_6 0 X_4 0 X_2 0 X_0 0 X_2 0 X_4 0 X_6 0 X_8 0 X_10];
x_har_10 = X*exp(j*omega_0*n'*t);
plot(t, real(x_har_10)), xlabel('t'), ylabel('full-rectified sine wave')
```



The student should be able to implement the other Fourier series calculations using the above as a guide.

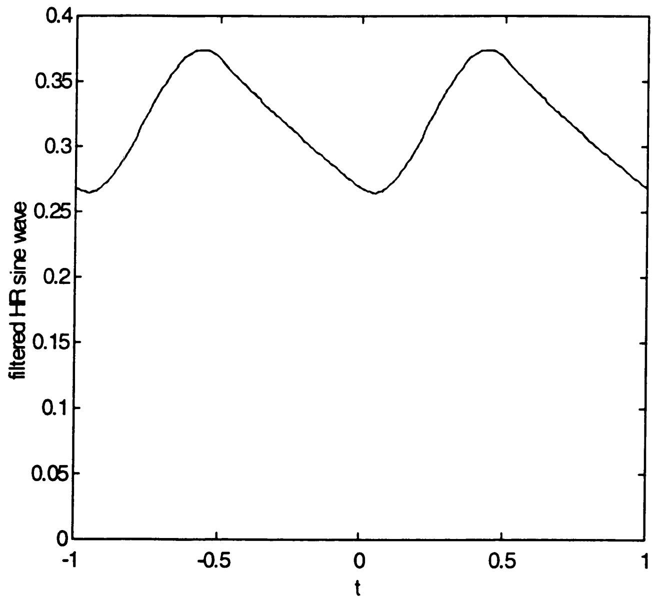
Computer Exercise 3-2

The lowpass filter has frequency response function

$$H(j\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega / \omega_3},\ \ \omega_3 = 1/RC$$

The Fourier coeficients of the output of the filter with a half-rectified sine wave as input are equal to the Fourier coefficients of the input times the value of the frequency response function at the frequency of the respective harmonic. The MATLAB program is an extension of the one given in the solution to Computer Exercise 2-1, and is given below:

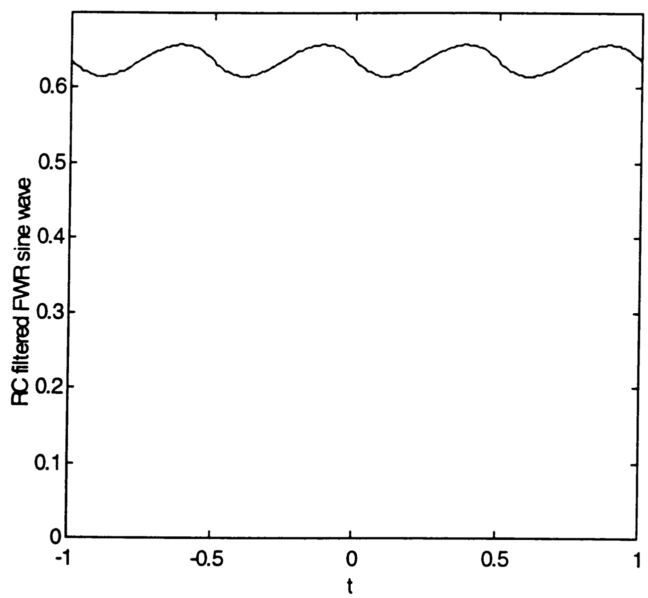
```
t = -1:.01:1;
n = [-6:1:6];
omega_0 = 2*pi;
omega_3 = 0.1*omega_0;  % Set 3-dB frequency of filter = 0.1*omega_0
X_0 = 1/pi;
X_1 = -0.25*j;
X_2 = -1/(3*pi);
X_4 = -1/(15*pi);
X_6 = -1/(35*pi);
X = [X_6 0 X_4 0 X_2 -X_1 X_0 X_1 X_2 0 X_4 0 X_6];
H = zeros(1, 6);
for k = 1:1:6
  H(k) = 1/(1+j*k*omega_0/omega_3);
end
H = [fliplr(conj(H)) 1 H];
Y = X.*H;
y_har_6 = Y*exp(j*omega_0*n'*t);
plot(t, real(y_har_6)), xlabel('t'), ylabel('filtered HR sine wave'),...
axis([-1 1 0 .4])
```



Computer Exercise 3-3

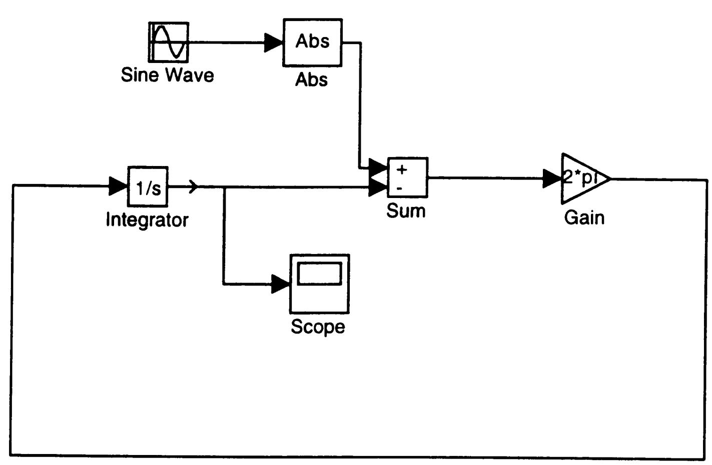
The program is modeled after the one in the previous computer exercise. Note that the ripple frequency is twice that of the filtered half-wave rectifier.

```
t = -1:.01:1;
n = [-10:1:10];
omega_0 = 2*pi;
X_0 = 2/pi;
X_2 = -2/(3*pi);
X_4 = -2/(15*pi);
X_6 = -2/(35*pi);
X_8 = -2/(63*pi);
X_10 = -2/(99*pi);
X = [X_10 0 X_8 0 X_6 0 X_4 0 X_2 0 X_0 0 X_2 0 X_4 0 X_6 0 X_8 0 X_10];
H = zeros(1, 10);
for k = 1:1:10
   H(k) = 1/(1+j*k*omega_0/omega_3);
end
H = [fliplr(conj(H)) 1 H];
Y = X.*H;
y_har_10 = Y*exp(j*omega_0*n'*t);
plot(t, real(y_har_10)), xlabel('t'), ylabel('RC filtered FWR sine
wave'), axis([-1 1 0 0.7])
```



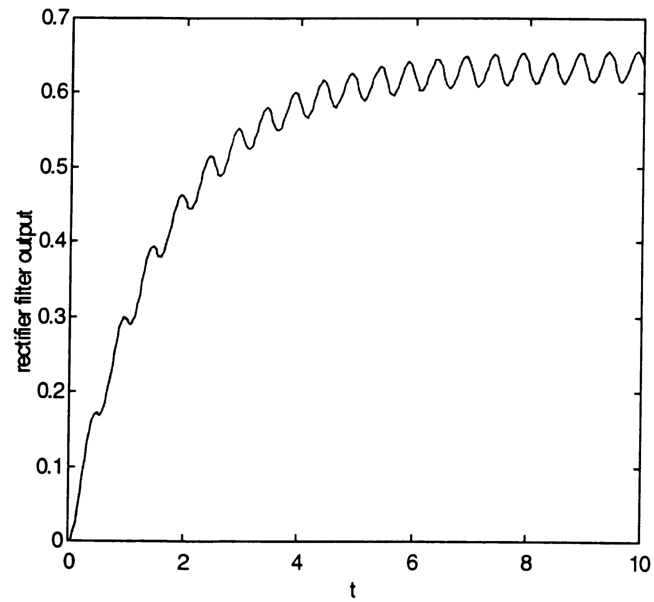
<u>**Comptuter Exercise 3-4**</u>

A SIMULINK block diagram is shown below:



It can be run directly in SIMULINK or with the command window statements below:

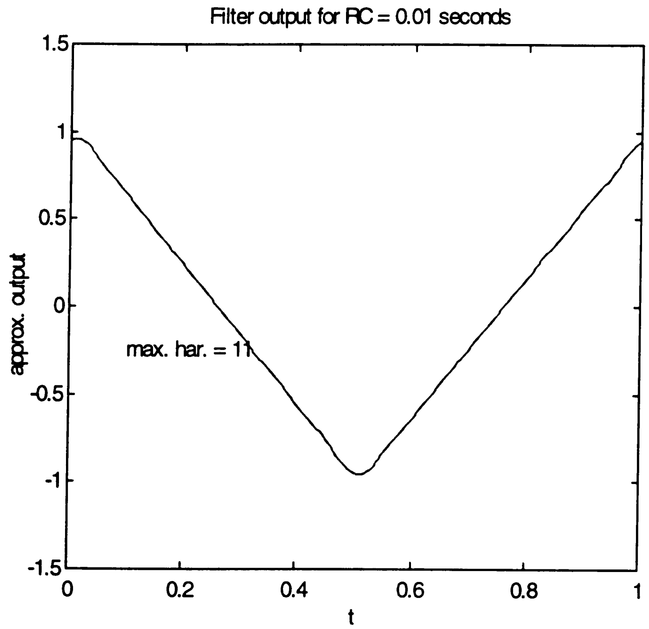
```
[t,y] =    lsim('c_ex3_4', 10);
plot(t, y), xlabel('t'), ylabel('rectifier filter output')
```



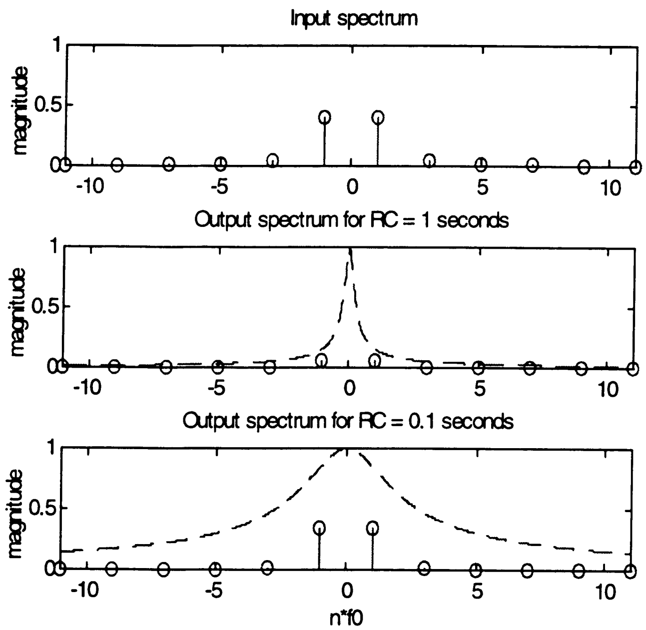
Computer Exercise 3-5

Rework of Example 3-10 for a triangular waverform:

```
% Plots for Example 3-10 modified for a triangular waveform
%
RC = .01;
n_max = 11;           % This can be a vector
N = length(n_max);
t = 0:.005:1;
omega_0 = 2*pi;
for k = 1:N
  n = [-n_max(k):2:n_max(k)];
  L_n = length(n);
  X_n = (4./(pi^2*n.^2));
  x = X_n*exp(j*omega_0*n'*t);
  Y_n = X_n.*(1./(1+j*n*omega_0*RC));
  y = Y_n*exp(j*omega_0*n'*t); % The output
  subplot(N,1,k),plot(t, real(y)), xlabel('t'), ylabel('approx.
output'),...
  axis([0 1 -1.5 1.5]), text(.1,-.25, ['max. har. = ',
num2str(n_max(k))])
  if k == 1
    title(['Filter output for RC = ', num2str(RC), ' seconds'])
  end
end
```




```
% Plots for Example 3-11 modified for a triangular wave form
%
clg
RC = [1 0.1];
n_max = 11;
N_RC = length(RC);
omega_0 = 2*pi;
f0 = 1;
for k = 1:N_RC
  n = [-n_max:2:n_max];
  f = [-n_max*f0:.02:n_max*f0];
  H = 1./(1+j*2*pi*f*RC(k));
  L_n = length(n);
  nn = 2:L_n/2+1;
  sgn = (-1).^nn;
  X_n = (4./(pi^2*n.^2));
  Y_n = X_n.*(1./(1+j*n*omega_0*RC(k)));
  if k == 1
  subplot(N_RC+1,1,1),stem(n*f0, abs(X_n)), ylabel('magnitude'),...
  title('Input spectrum'),axis([-n_max*f0 n_max*f0 0 1])
  end
  subplot(N_RC+1,1,k+1),stem(n*f0, abs(Y_n)), ylabel('magnitude'),...
  title(['Output spectrum for RC = ', num2str(RC(k)), ' seconds']),
  axis([-n_max*f0 n_max*f0 0 1]),hold on
  subplot(N_RC+1,1,k+1), plot(f,abs(H),'--')
  if k == N_RC
    xlabel('n*f0')
  end
end
```



The Fourier coefficients for a triangular waveform approach 0 faster with n increasing than those for a square wave and therefore the filter bandwidth can be less for a given fidelity of the output waveform for a triangular waveform than for a square wave.

Computer Exercise 4-1

(a) **x1 = sym('exp(-t^2)')**

```
x1 =
exp(-t^2)
```

X1 = fourier(x1)

```
X1 =
pi^(1/2)*exp(-1/4*w^2)
```

(b) **x2 = sym('t*exp(-t^2)')**

```
x2 =
t*exp(-t^2)
```

X2 = fourier(x2)

```
X2 =
-1/2*i*pi^(1/2)*w*exp(-1/4*w^2)
```

(c) **x3 = sym('(exp(-3*t)+exp(-2*t))*Heaviside(t)')**

```
x3 =
(exp(-3*t)+exp(-2*t))*Heaviside(t)
```

X3 = fourier(x3)

```
X3 =
1/(3+i*w)+1/(2+i*w)
```

Computer Exercise 4-2

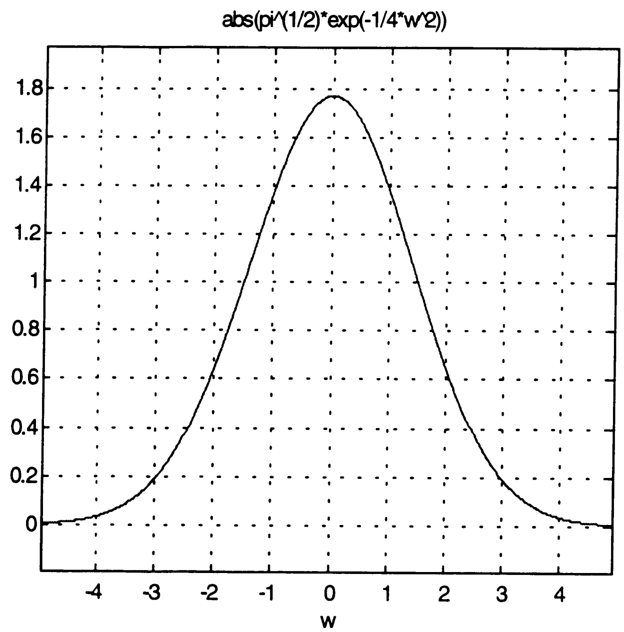
(a) **X1_mag ='abs(pi^(1/2)*exp(-1/4*w^2))'**

```
X1_mag =
abs(pi^(1/2)*exp(-1/4*w^2))
```

X1_arg = 'angle(pi^(1/2)*exp(-1/4*w^2))'

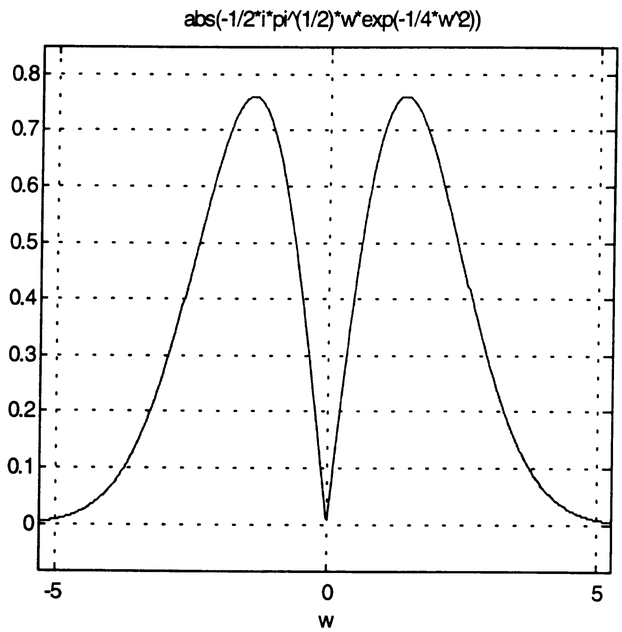
```
X1_arg =
angle(pi^(1/2)*exp(-1/4*w^2))
```

ezplot(X1_mag)



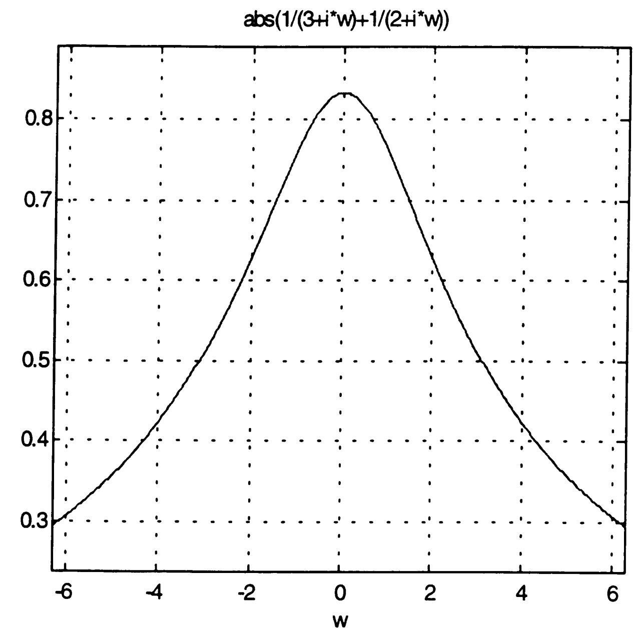
The phase is identically zero.

```
X2_mag =

X2_mag =
abs(-1/2*i*pi^(1/2)*w*exp(-1/4*w^2))

X2_arg = 'angle(-1/2*i*pi^(1/2)*w*exp(-1/4*w^2))'

X2_arg =
angle(-1/2*i*pi^(1/2)*w*exp(-1/4*w^2))

ezplot(X2_mag)
```

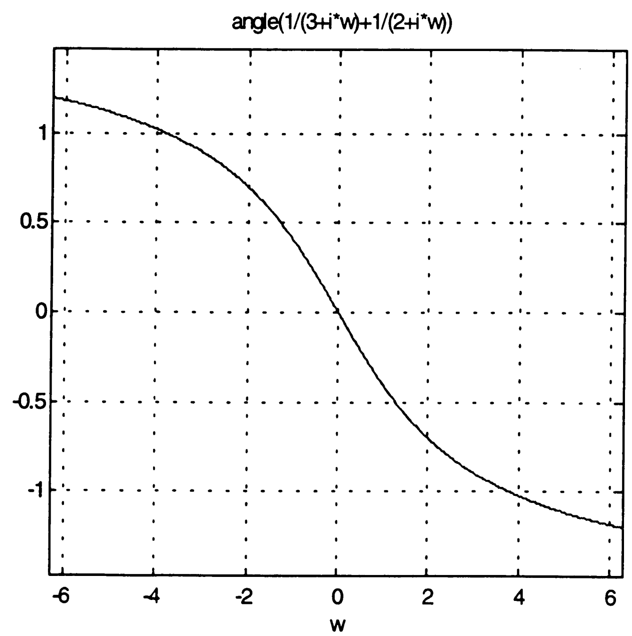


The phase of X2 is -90 degrees for negative frequencies and 90 degrees for positive frequencies. MATLAB will not plot this correctly.

```
X3_mag = 'abs(1/(3+i*w)+1/(2+i*w))'

X3_mag =
abs(1/(3+i*w)+1/(2+i*w))

X3_arg = 'angle(1/(3+i*w)+1/(2+i*w))'

X3_arg =
angle(1/(3+i*w)+1/(2+i*w))

ezplot(X3_mag)
```



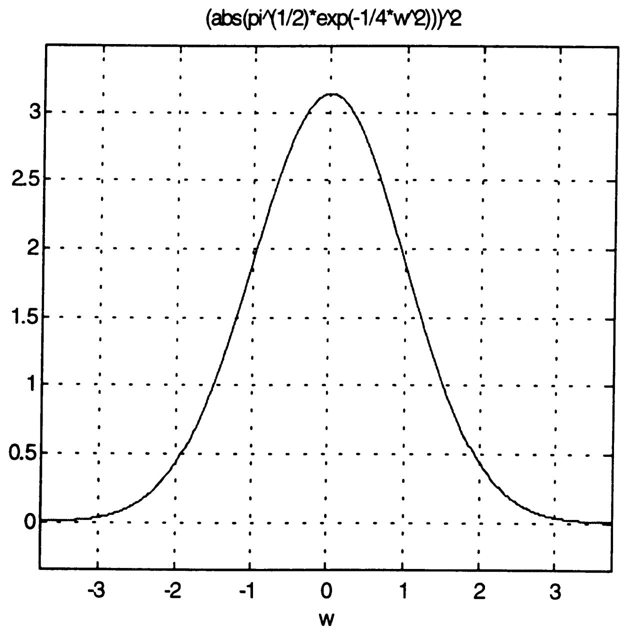

```
ezplot(X3_arg)
```



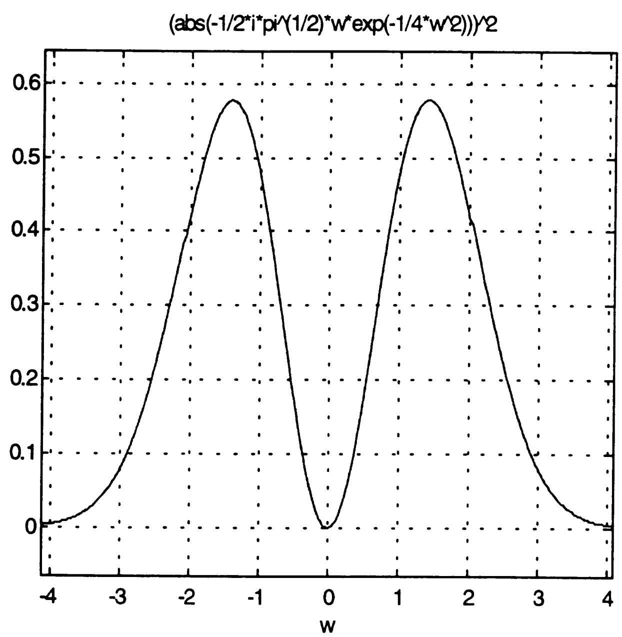
(b) **G1 = '(abs(pi^(1/2)*exp(-1/4*w^2)))^2'**

```
G1 =
(abs(pi^(1/2)*exp(-1/4*w^2)))^2
```

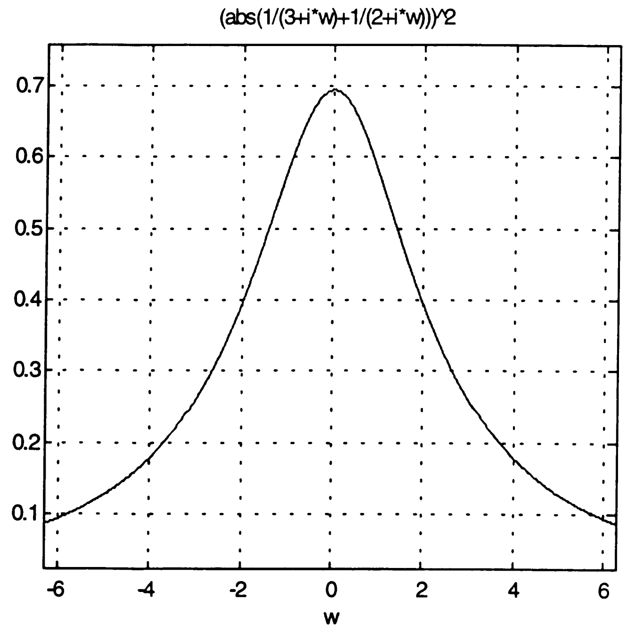
```
ezplot(G1)
```




```
G2 = '(abs(-1/2*i*pi^(1/2)*w*exp(-1/4*w^2)))^2'

G2 =
(abs(-1/2*i*pi^(1/2)*w*exp(-1/4*w^2)))^2

ezplot(G2)
```



```
G3 = '(abs(1/(3+i*w)+1/(2+i*w)))^2'

G3 =
(abs(1/(3+i*w)+1/(2+i*w)))^2

ezplot(G3)
```

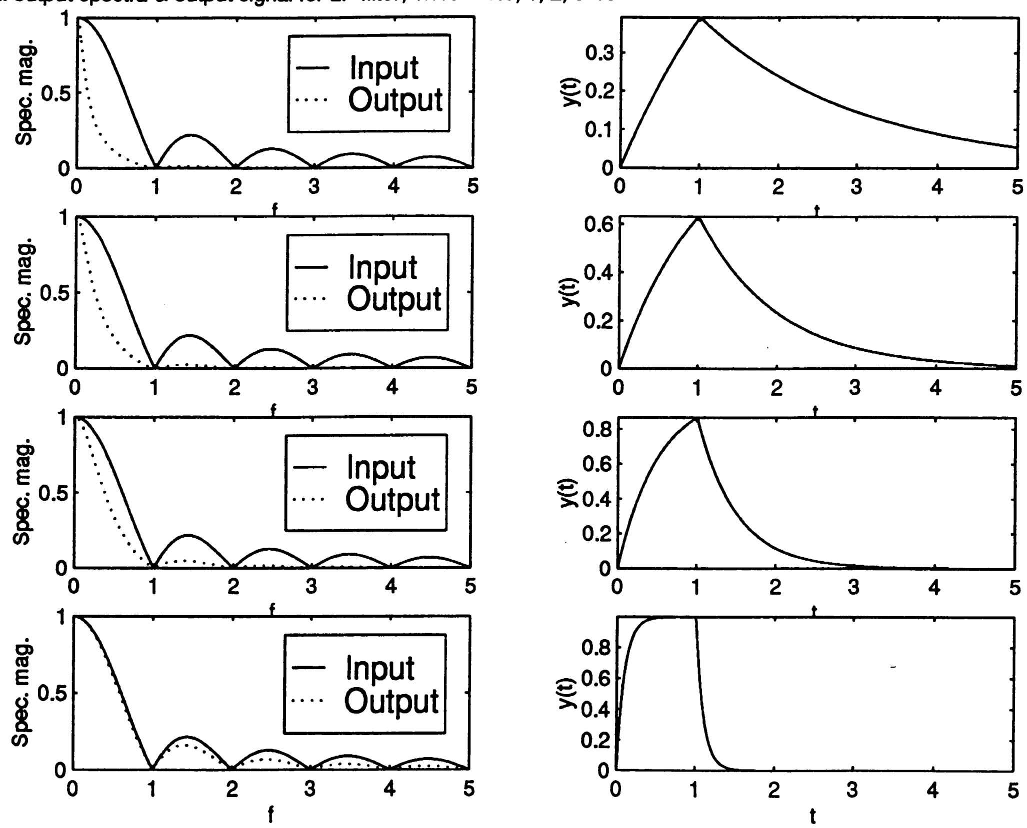


Computer Exercise 4-3

```
%       Plots of Figure 4-12
%
clf
RC = 4;
T = 1;
A = 1;
f = 0:.01:5;
t = 0:.01:5;
X = A*T*sinc(T*f).*exp(-j*pi*f*T);
for k = 1:4
  RC = RC/2;
  if k == 4
    RC = .1;
  end
  f3 = 1/(2*pi*RC);
        H = 1./(1+j*f/f3);
  k_even = 2*k;
  k_odd = 2*k-1;
        Y = H.*X;
        x = A*(1-exp(-t/RC)).*stp_fn(t)-A*(1-exp(-(t-T)/RC)).*stp_fn(t-T);
  subplot(4,2,k_odd),plot(f,abs(X)),xlabel('f'),ylabel('Spec. mag.'),...
  if k == 1
  axis([0 5 0 inf]),...
  title('Input & output spectra & output signal for LP filter; T/RC = 0.5, 1, 2, & 10')
```

```
    end
    hold on
    subplot(4,2,k_odd),plot(f,abs(Y),':'),legend('Input','Output'),...
    axis([0 5 0 inf])
    hold off
    subplot(4,2,k_even),plot(t,x),xlabel('t'),ylabel('y(t)'),axis([0 5 0 inf])
end
```

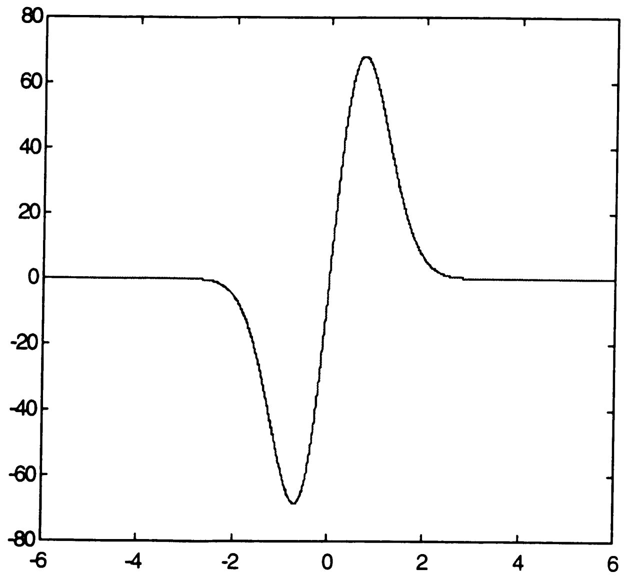
Input & output spectra & output signal for LP filter; T/RC = 0.5, 1, 2, & 10



Computer Exercise 4-4

The MATLAB program below will approximately compute the Hilbert transform of a signal numerically using convolution. The Hilbert transform of the signal of Computer Exercise 4-1(b) is computed and plotted. For the others, move the comment signs.

```
% Hilbert transform computation
%
del_t = .005;
t = -3:del_t:3;
L = length(t);
tp = [2*t(1):del_t:2*t(L)];
h = 1./(pi*(t + .00001));
%x = pls_fn(t);
%x = trgl_fn(t);
%x = exp(-t.^2);
x = t.* exp(-t.^2);
y = del_t*conv(x, h);
plot(tp, y)
```

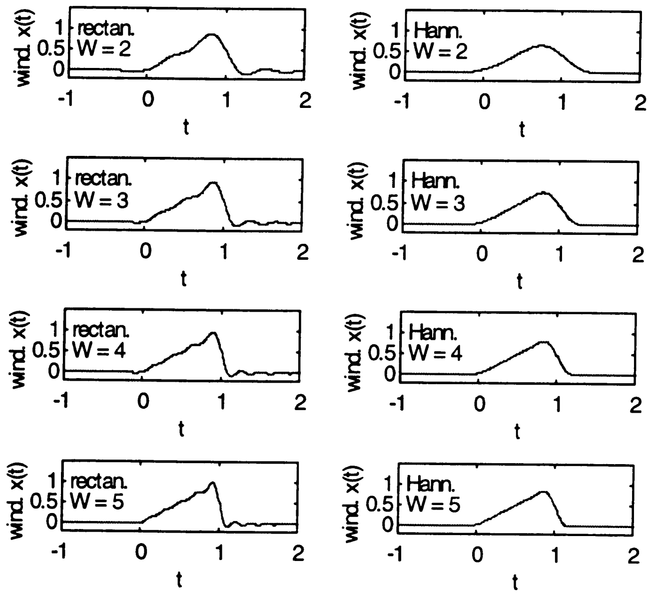


<u>**Computer Exercise 4-5**</u>

The sawtooth is shown as an example.

```
% Generalization of Example 4-19 to include, in addition to a square
pulse (1), a sawtooth (2),
% trapezoid (3), and raised cosine (4)
%
pulse_type = 2;
t_max = 2;
t = -t_max+1:.01:t_max;
L = length(t);
for k = 1:4          % Loop to do 4 values of window width
k_even = 2*k;
k_odd = 2*k - 1;
W = 2 + (k-1);
tp = [2*t(1):.01:2*t(L)];
w_r = 2*W*sinc(2*W*t);        % Inverse transform of rectangular window
w_h = W*(sinc(2*W*t)+.5*sinc(2*W*t-1)+.5*sinc(2*W*t+1)); % Hanning
window
if pulse_type == 1
  x = pls_fn(t-.5);
elseif pulse_type == 2
  x = rmp_fn(t).*stp_fn(1 - t);
elseif pulse_type == 3
  x = rmp_fn(t) - rmp_fn(t - 0.5) - rmp_fn(t - 1) + rmp_fn(t - 1.5);
elseif pulse_type == 4
  x = rsd_cos(t - 1);
end
x_tilde_r = .01*conv(w_r, x);     % Convolve rectangular window with
pulse
x_tilde_h = .01*conv(w_h, x);     % Convolve Hanning window with pulse
subplot(4,2,k_odd),plot(tp, x_tilde_r),xlabel('t'),ylabel('wind.
x(t)'),...
```

```
   text(-.9,1,'rectan.'),text(-.9,.5, ['W = ',num2str(W)]),axis([-
t_max+1 t_max -.2 1.5])
subplot(4,2,k_even),plot(tp, x_tilde_h),xlabel('t'),ylabel('wind.
x(t)'),...
   text(-.9,1,'Hann.'),text(-.9,.5,['W = ',num2str(W)]),axis([-t_max+1
t_max -.2 1.5])
end
```

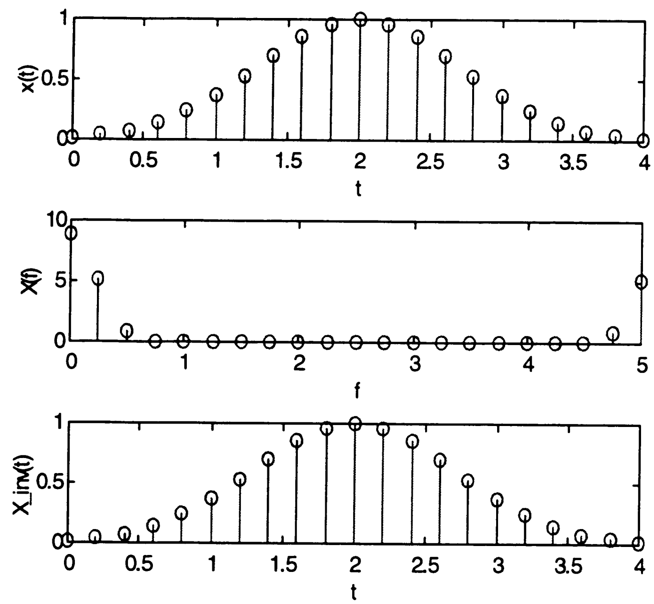


Computer Exercise 4-6

The Gaussian pulse is used as an example.

```
% FFT of the signals of Computer Exercise 4-1:  (1) Gaussian pulse; (2)
t*Gaussian pulse; (3) sum of
%  exponentials
%
clg
pulse_type = 1;
T = 4;
del_t = .2;
t = 0:del_t:T;
L_t = length(t);
del_f = 1/T;
f_max = (L_t-1)*del_f;
f = 0:del_f:f_max;
if pulse_type == 1
  x = exp(-(t-T/2).^2);
elseif pulse_type == 2
  x = t.*exp(-(t-T/2).^2);
elseif pulse_type == 3
  x = (exp(-3*t) + exp(-2*t)).*stp_fn(t);
end
X = fft(x);
```

```
X_inv = ifft(X);
subplot(3,1,1), stem(t,x), axis([0 T 0 1]),...
  xlabel('t'), ylabel('x(t)')
subplot(3,1,2), stem(f,abs(X)), axis([0 f_max 0 10]),...
  xlabel('f'), ylabel('X(f)')
subplot(3,1,3), stem(t,abs(X_inv)), axis([0 T 0 1]),...
  xlabel('t'), ylabel('X_inv(t)')
```



The following exercises were run from an M-file with echo on.

```
» c_ex5_1
%         Computer Exercise 5-1
%
sym  s;
X1 = sym('s^3+6*s^2+11*s+6');
X2 = sym('(s^2+4)^2');
X3 = sym('s^2+4*s+4');
%
%         (a)
%
Y1 = X1/X2

Y1 =

(s^3+6*s^2+11*s+6)/(s^2+4)^2

y1 = ilaplace(Y1)

y1 =

9/4*t*cos(2*t)+7/4*t*sin(2*t)+15/8*sin(2*t)+cos(2*t)

%
%         (b)
%
Y2 = X1/X3

Y2 =

(s^3+6*s^2+11*s+6)/(s^2+4*s+4)

y2 = ilaplace(Y2)

y2 =

Dirac(1,t)+2*Dirac(t)-exp(-2*t)

%
%         (c)
%
Y3 = 1/(X1*X2)

Y3 =

1/(s^3+6*s^2+11*s+6)/(s^2+4)^2

y3 = ilaplace(Y3)

y3 =

1/338*exp(-3*t)-1/64*exp(-2*t)+1/50*exp(-t)-1983/270400*cos(2*t)-453/135200*sin(2*t)+9/2080*t*cos(2*t)-7/
2080*t*sin(2*t)
```

```
%
%       (d)
%
Y4 = X1/(X2*X3)

Y4 =

(s^3+6*s^2+11*s+6)/(s^2+4)^2/(s^2+4*s+4)

y4 = ilaplace(Y4)

y4 =

-1/64*exp(-2*t)+1/64*cos(2*t)+3/32*sin(2*t)-7/32*t*cos(2*t)+9/32*t*sin(2*t)
```

```
%
%       Computer Exercise 5-2
%
%       (a)
%
syms t s;
ya = ilaplace(Y1*Y2)

ya =

Dirac(t)+63/8*t*cos(2*t)-2*t*sin(2*t)+49/16*sin(2*t)+8*cos(2*t)

%
%       (b)
%
yb = ilaplace(Y3*Y4)

yb =

1/4096*t*exp(-2*t)+1/2048*exp(-2*t)-1/2048*cos(2*t)+13/16384*sin(2*t)-7/8192*t*cos(2*t)-9/8192*t*sin(2*t)
+3/4096*t^2*cos(2*t)-1/4096*t^2*sin(2*t)+1/6144*t^3*sin(2*t)

%
%       (c)
%
yc = ilaplace(Y2*Y4)

yc =

1/64*t*exp(-2*t)+1/64*exp(-2*t)-1/64*cos(2*t)+49/128*sin(2*t)+1/4*t*cos(2*t)+63/64*t*sin(2*t)

%
%       (d)
%
yd = ilaplace(Y1*Y4)

yd =

1/48*t^3*sin(2*t)+7/256*t^2*sin(2*t)+7/64*t*sin(2*t)+149/2048*sin(2*t)-21/256*t^3*cos(2*t)-7/32*t^2*cos(2
*t)-149/1024*t*cos(2*t)
```

```
» c_ex5_3
%       Computer Exercise 5-3
%
%       (a)     The MATLAB 5 symbolic processor cannot handle the Laplace
%               transform of powers of sines or cosines, so this computer
%               exercise has been modified from that given in the book.
%
x1 = sym('cos(omega*t)*exp(-alpha*t)*Heaviside(t)')

x1 =

cos(omega*t)*exp(-alpha*t)*Heaviside(t)

X1 = laplace(x1)

X1 =

(s+alpha)/((s+alpha)^2+omega^2)

x1_chk = ilaplace(X1)

x1_chk =

exp(-alpha*t)*cos(omega*t)

%
%       (b)
%
x2 = sym('t^2*cos(omega*t)*exp(-alpha*t)*Heaviside(t)')

x2 =

t^2*cos(omega*t)*exp(-alpha*t)*Heaviside(t)

X2 = laplace(x2)

X2 =

-6/((s+alpha)^2+omega^2)^2*(s+alpha)+8*(s+alpha)^3/((s+alpha)^2+omega^2)^3

x2_chk = ilaplace(X2)

x2_chk =

exp(-alpha*t)*t^2*cos(omega*t)

%
%       (c)
%
x3 = x1 + x2

x3 =

cos(omega*t)*exp(-alpha*t)*Heaviside(t)+t^2*cos(omega*t)*exp(-alpha*t)*Heaviside(t)
```

```
X3 = laplace(x3)

X3 =

(s+alpha)/((s+alpha)^2+omega^2)-6/((s+alpha)^2+omega^2)^2*(s+alpha)+8*(s+alpha)^3/((s+alpha)^2+omega^2)
^3

x3_chk = ilaplace(X3)

x3_chk =

exp(-alpha*t)*cos(omega*t)+exp(-alpha*t)*t^2*cos(omega*t)

» c_ex5_4
%       Computer Exercise 5-4
%
%       First, get the numerator and denominator polynomials; use the expand function to
%       expand powers of polynomials.
%
syms s t;
X1 = sym('s^3+6*s^2+11*s+6');
X2 = expand(sym('(s^2+4)^2'));
X3 = sym('s^2+4*s+4');
%
%       (a)
%
Y1 = X1/X2

Y1 =

(s^3+6*s^2+11*s+6)/(s^4+8*s^2+16)

pretty(Y1)

                     3     2
                    s  + 6 s  + 11 s + 6
                    --------------------
                       4     2
                      s  + 8 s  + 16
%
%       (b)
%
Y2 = X1/X3

Y2 =

(s^3+6*s^2+11*s+6)/(s^2+4*s+4)

pretty(Y2)

                     3     2
                    s  + 6 s  + 11 s + 6
                    --------------------
                       2
                      s  + 4 s + 4
```

```
%       (c)
%
Y3 = 1/(expand(X1*X2))

Y3 =

1/(s^7+19*s^5+104*s^3+6*s^6+54*s^4+144*s^2+176*s+96)

pretty(Y3)

                         1
      ---------------------------------------------------------
       7        5         3     6       4         2
      s  + 19 s  + 104 s  + 6 s  + 54 s  + 144 s  + 176 s + 96
%
%       (d)
%
Y4 = X1/(expand(X2*X3))

Y4 =

(s^3+6*s^2+11*s+6)/(s^6+4*s^5+12*s^4+32*s^3+48*s^2+64*s+64)

pretty(Y4)

                 3      2
                s  + 6 s  + 11 s + 6
         ---------------------------------------------
          6      5       4       3       2
         s  + 4 s  + 12 s  + 32 s  + 48 s  + 64 s + 64

» num_a = [0 1 6 11 6];
» num_b = [1 0 8 0 16];
» den_a = [1 0 8 0 16];
» [R_a P_a K_a] = residue(num_a, den_a)
R_a =
  1.0e+007 *
  -1.8008 - 2.3168i
  -1.8008 + 2.3168i
   1.8008 + 2.3168i
   1.8008 - 2.3168i
P_a =
   0.0000 + 2.0000i
   0.0000 - 2.0000i
   0.0000 + 2.0000i
   0.0000 - 2.0000i
K_a =
    []
```

The R_a's are the residues (expansion coefficients), the P_a's are the poles, and K_a is the remainder, which is empty.

For Y2, we have an improper rational function. Long division gives

$$Y2 = s + 2 - \frac{s+2}{s^2 + 4s + 4}$$

We do a partial fraction expansion of the last term as follows:

```
» num_b = [ 0 1 2];
» den_b = [1 4 4];
» [R_b P_b K_b] = residue(num_b, den_b)
R_b =
    1
    0
P_b =
   -2
   -2
K_b =
    []
```

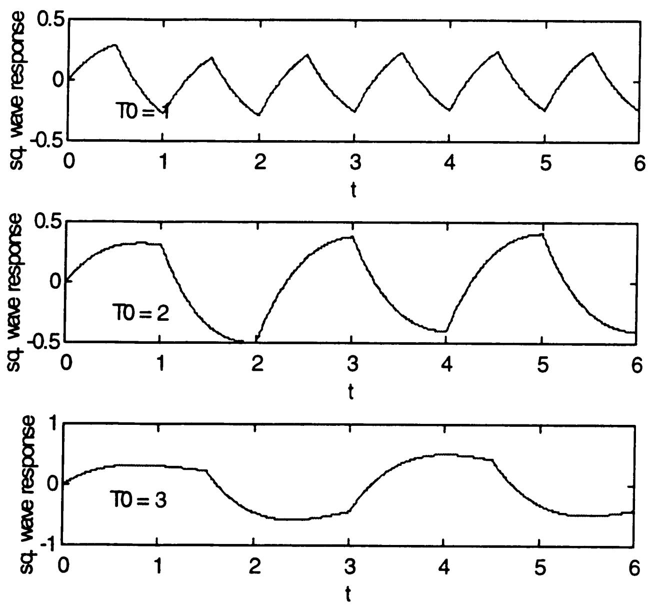
Note that the original expression had $s + 2$ in the numerator, which is a factor of the denominator. The output of the residue function tells us that the last term is really $1/(s + 2)$.

```
» num_c = [0 0 0 0 0 0 0 1];
» den_c = [1 6 19 54 104 144 176 96];
» [R_c P_c K_c] = residue(num_c, den_c)
R_c =
  1.0e+004 *
   0.0000
  -1.0931 - 3.5619i
  -1.0931 + 3.5619i
   0.0000
   1.0931 + 3.5619i
   1.0931 - 3.5619i
   0.0000
P_c =
  -3.0000
   0.0000 + 2.0000i
   0.0000 - 2.0000i
  -2.0000
   0.0000 + 2.0000i
   0.0000 - 2.0000i
  -1.0000
K_c =
    []

» num_d = [0 0 0 1 6 11 6];
» den_d = [1 4 12 32 48 64 64];
» [R_d P_d K_d] = residue(num_d, den_d)
R_d =
  1.0e+006 *
  -2.5849 + 0.2552i
  -2.5849 - 0.2552i
   0.0000
        0
   2.5849 - 0.2552i
```

2.5849 + 0.2552i
P_d =
0.0000 + 2.0000i
0.0000 - 2.0000i
-2.0000
-2.0000
0.0000 + 2.0000i
0.0000 - 2.0000i
K_d =
[]

Computer Exercise 5-5

```
% Plots for Problem 5-35
%
t = 0:.01:6;
L = length(t);
n_max = 12;
T = zeros(n_max,L);
for m = 1:3
T0 = m;
for n = 1:n_max
  T(n,:) = (-1)^n*exp(-(t-n*T0/2)).*sin(t-n*T0/2).*stp_fn(t-n*T0/2);
end
i = exp(-t).*sin(t).*stp_fn(t)+2*sum(T);
subplot(3,1,m), plot(t,i), xlabel('t'),...
  ylabel('sq. wave response'),...
  text(.5, -.25, ['T0 = ', num2str(T0)])
end
```



Computer Exercise 5-6

The purpose of the following discussion is to set up the equations for this computer exercise. The given system has transfer function

$$\frac{Y(s)}{X(s)} = \frac{1}{s + 1/\tau_0}$$

To convert from the continuous-time domain to discrete time, we substitute

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

This gives

$$\frac{Y(z)}{X(z)} = A\frac{1+z^{-1}}{1+Bz^{-1}} \text{ where } A = \frac{T}{2}\frac{1}{1+T/(2\tau_0)} = \frac{\tau_0}{1+2\tau_0/T} \text{ and } B = \frac{1-2\tau_0/T}{1+2\tau_0/T}$$

In the z-domain, this is equivalent to

$$Y(z) = -Bz^{-1}Y(z) + AX(z) + Az^{-1}X(z)$$

and in the time domain, it is equivalent to

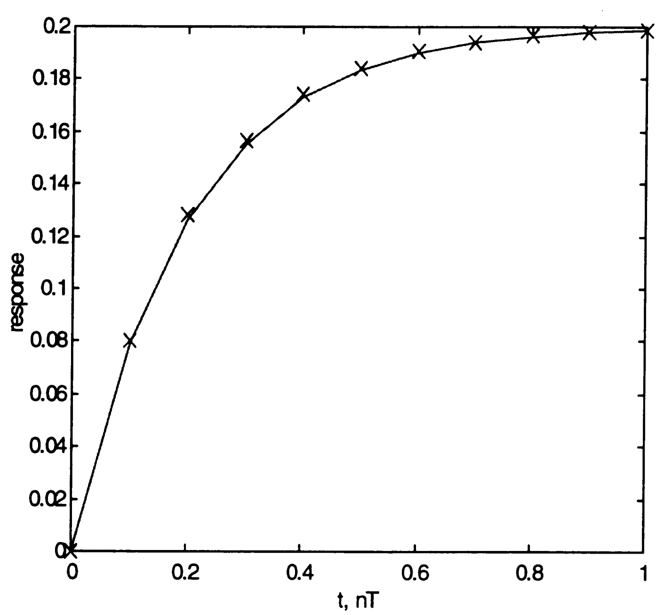
$$y(n) = -By(n-1) + Ax(n) + Ax(n-1)$$

as the equation to be used in implementing the discrete-time approximation for the response of the continuous-time system. We will find the response to a step input. For comparison, the continuous-time system has step response

$$y_a(t) = \tau_0[1 - \exp(-t/\tau_0)]u(t)$$

where $u(t)$ is the unit step. We sample this at the time instants $\{nT\}$ to compare with the discrete-time system response.

```
% Program for computing the discrete-time and continuous-time
% system response of Computer Exercise 5-6
%
clg
n_max = 10;
n = 0:1:n_max;
tau0 = 0.2;
T = 0.1;
t = T*n;
A = tau0/(1+2*tau0/T);
B = (1-2*tau0/T)/(1+2*tau0/T);
x = stp_fn(n);
yn = zeros(size(n));
yo = 0;
for nn = 1:n_max
   yn(nn+1) = -B*yn(nn) + A*x(nn+1) + A*x(nn);
   yo = yn(nn+1);
end
ya = tau0*(1 - exp(-t/tau0)).*stp_fn(t);
plot(n*T,yn,'x'),xlabel('t, nT'), ylabel('response')
hold on
plot(t,ya)
```



The following exercises were run with echo on.

```
» c_ex6_1
%         Computer Exercise 6-1
%
sym s;

Z = sym('[1+6/s, -6/s;-6/s, 6/s+3/(5*s)+3*s/5]')

Z =

[          1+6/s,          -6/s]
[           -6/s, 6/s+3/(5*s)+3*s/5]

E = sym('[1/s; 0]');
Vi = sym('[0; -1/(2*s)+6/5]');
E_plus_Vi = E+Vi

E_plus_Vi =

[      1/s]
[ -1/2/s+6/5]

Z_inv = inv(Z);
I = Z_inv*E_plus_Vi

I =

[ (11+s^2)/(11*s+s^3+6+6*s^2)+10*s/(11*s+s^3+6+6*s^2)*(-1/2/s+6/5)]
[ 10/(11*s+s^3+6+6*s^2)+5/3*(s+6)*s/(11*s+s^3+6+6*s^2)*(-1/2/s+6/5)]
```

Since MATLAB can't inverse Laplace transform a matrix, copy I1 onto the command window and define it as a symbolic expression:

```
» I1 = sym('(11+s^2)/(11*s+s^3+6+6*s^2)+10*s/(11*s+s^3+6+6*s^2)*(-1/2/s+6/5)')

I1 =

(11+s^2)/(11*s+s^3+6+6*s^2)+10*s/(11*s+s^3+6+6*s^2)*(-1/2/s+6/5)

» i1 = ilaplace(I1)

i1 =

-21/2*exp(-3*t)+14*exp(-2*t)-5/2*exp(-t)

» c_ex6_2
%         Computer Exercise 6-2
%
syms t s;
y = sym('((t^2/13+11*t/169-217/4394)*exp(-t)+(865*sin(5*t)/4394+217*cos(5*t)/4394)*exp(-2*t))')

y =

((t^2/13+11*t/169-217/4394)*exp(-t)+(865*sin(5*t)/4394+217*cos(5*t)/4394)*exp(-2*t))
```

```
Y = laplace(y)

Y =

2/13/(s+1)^3+11/169/(s+1)^2-217/4394/(s+1)+4325/4394/((s+2)^2+25)+217/4394*(s+2)/((s+2)^2+25)

H = sym('(s^2+3)/(s^3+3*s^2+3*s+1)')

H =

(s^2+3)/(s^3+3*s^2+3*s+1)

X = Y/H

X =

(2/13/(s+1)^3+11/169/(s+1)^2-217/4394/(s+1)+4325/4394/((s+2)^2+25)+217/4394*(s+2)/((s+2)^2+25))/(s^2+3)
*(s^3+3*s^2+3*s+1)

x = ilaplace(X)

x =

exp(-2*t)*cos(5*t)

» c_ex6_3
%       Computer Exercise 6-3
%
%       Solution of Example 6-5 for currents using MATLAB
%       symbolic toolbox
%
syms t s;
Z=sym('[2+s, -1, 0; -1, 1+2/s,-2/s; 0, -2/s, 2+2/s]');
Vs = laplace(sym('cos(2*t)'));
Vc = laplace(sym('2*cos(2*t)'));
E = sym('[s/(s^2+4); 0; 2*s/(s^2+4)]');
Vi = sym('[2; 0; 0]');
E_plus_Vi = E+Vi;
Z_inv = inv(Z);
I = Z_inv*E_plus_Vi

I =

[     (s+3)/(s^2+4*s+5)*(s/(s^2+4)+2)+2/(s^2+4*s+5)*s/(s^2+4)]
[ (s+1)/(s^2+4*s+5)*(s/(s^2+4)+2)+2*(s+2)/(s^2+4*s+5)*s/(s^2+4)]
[ 1/(s^2+4*s+5)*(s/(s^2+4)+2)+(s^2+3*s+4)/(s^2+4*s+5)*s/(s^2+4)]
```

The Laplace transforms in the second two rows were copied into the command window and inverse Laplace transformed. See Computer Exercise 6-2 for an explanation.

```
» I2 = ilaplace(sym('(5*s^2+2*s^3+13*s+8)/(s^2+4*s+5)/(s^2+4)'))

I2 =
```

77/65*exp(-2*t)*cos(t)-109/65*exp(-2*t)*sin(t)+53/65*cos(2*t)+34/65*sin(2*t)

```
» I3 = ilaplace(sym('(5*s+5*s^2+8+s^3)/(s^2+4*s+5)/(s^2+4)'))
```

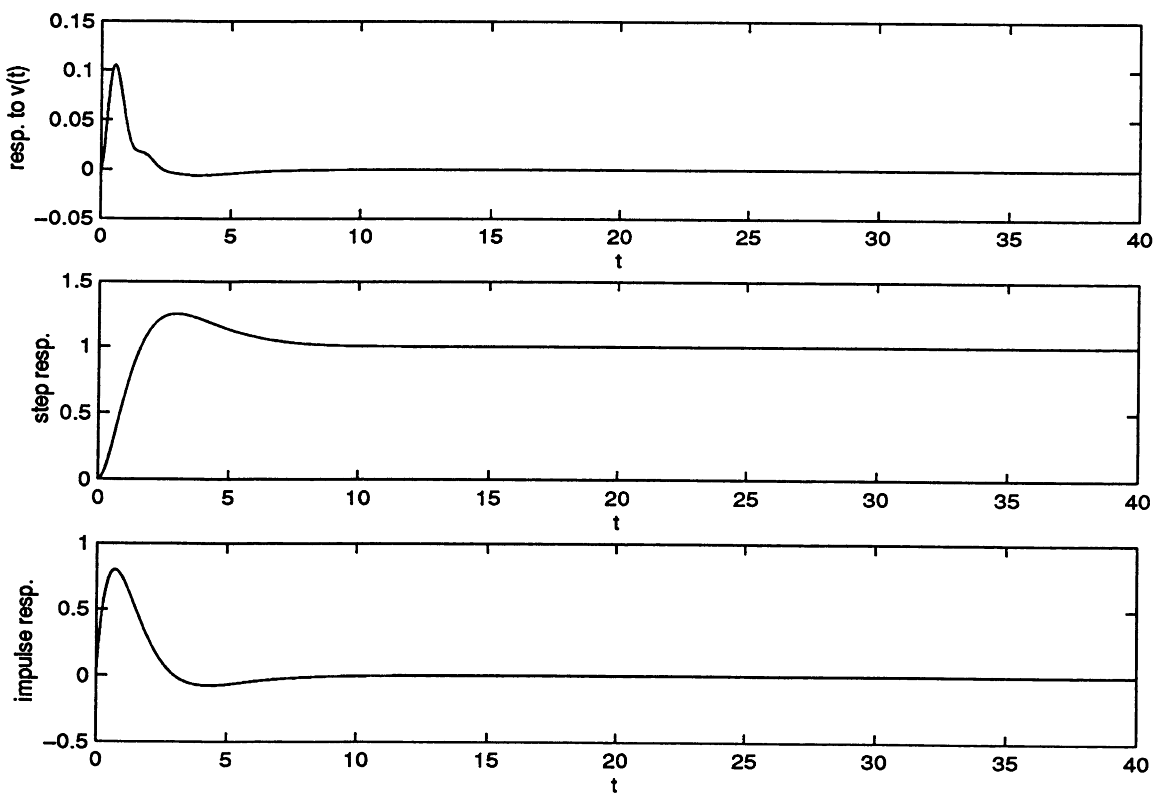
I3 =

16/65*exp(-2*t)*cos(t)+93/65*exp(-2*t)*sin(t)+49/65*cos(2*t)+2/65*sin(2*t)

Computer Exercise 6-4

The following program does parts (a) - (c):

```
t = 0:.01:40;              % Define time variable
num = [0 0 3 1];           % Define numerator and
den = [1 3 3 1];           % denominator polynomials
v = exp(-2*t).*cos(5*t);   % Define input in time  domain
y_v = lsim(num, den, v, t); % Use lsim( ) to get output.  Without y_v = lsim( ), plot is done.
y_step = step(num,den,t);  % Step response
y_imp = impulse(num,den,t);          % Impulse response
% Plot response to given input, step & impulse responses.
subplot(3,1,1),plot(t, y_v),xlabel('t'),ylabel('resp. to v(t)')
subplot(3,1,2),plot(t, y_step),xlabel('t'),ylabel('step resp.')
subplot(3,1,3),plot(t, y_imp),xlabel('t'),ylabel('impulse resp.')
```



(d) The impulse response is the inverse Laplace transform of $H(s)$. Use the residue function of MATLAB to find the partial fraction expansion:

```
» num = [0 0 3 1];
» den = [1 3 3 1];
» [R,P,k] = residue(num,den)
```

```
R =
     0
   3.0000
  -2.0000

P =
  -1.0000
  -1.0000
  -1.0000

k =
   []
```

This corresponds to the partial fraction expansion

$$H(s) = \frac{3}{(s+1)^2} - \frac{2}{(s+1)^3}$$

Using the MATLAB symbolic processor, this can be inverse Laplace transformed to give the impulse response:

```
» h = ilaplace(sym('3/(s+1)^2 - 2/(s+1)^3'))

h =

3*t*exp(-t)-t^2*exp(-t)
```

For the step response, we inverse Laplace transform $H(s)/s$. This corresponds to the same numerator polynomial as before but with a multiplicative s in the denominator polynomial of the $H(s)$. Using the residue function we obtain

```
» num = [0 0 3 1];
» den = [1 3 3 1 0];
» [R,P,k] = residue(num,den)

R =
  -1.0000
  -1.0000
   2.0000
   1.0000

P =
  -1.0000
  -1.0000
  -1.0000
     0

k =
   []
```

This corresponds to the partial fraction expansion

$$Y_{step}(s) = -\frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} + \frac{1}{s}$$

Again using the symbolic inverse Laplace transform capability of MATLAB, we obtain

» y_step = ilaplace(sym('-1/(s+1)-1/(s+1)^2+2/(s+1)^3+1/s'))

y_step =

-exp(-t)-t*exp(-t)+t^2*exp(-t)+1

Now do the damped cosine. Using the symbolic capability of MATLAB, we have

» V = laplace(sym('exp(-2*t)*cos(5*t)'))

V =

(s+2)/((s+2)^2+25)

» H = sym('3/(s+1)^2-2/(s+1)^3')

H =

3/(s+1)^2-2/(s+1)^3

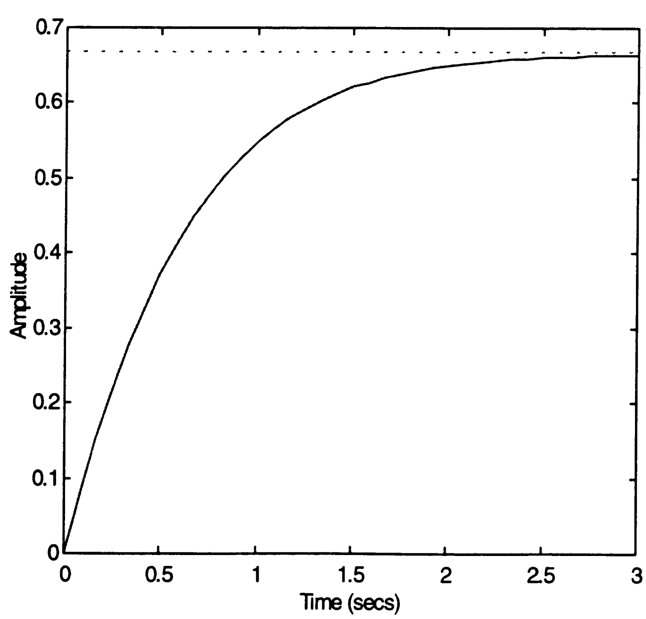
» y_v = ilaplace(V*H)

y_v =

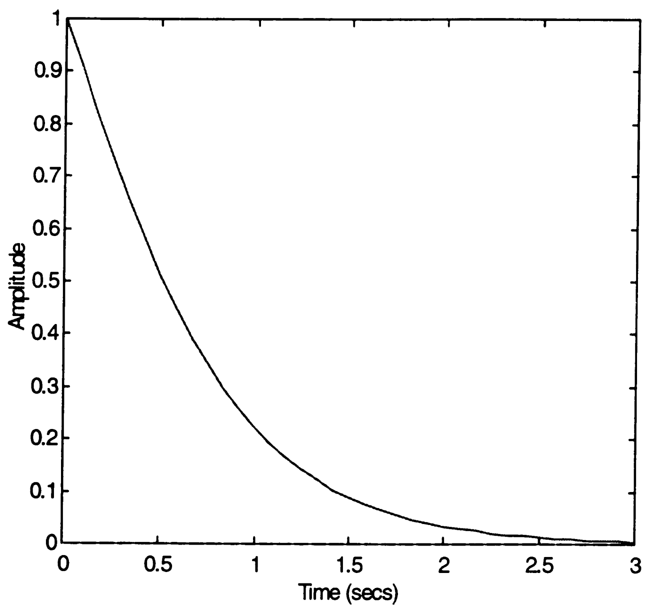
-1/26*t^2*exp(-t)+15/338*t*exp(-t)+505/4394*exp(-t)-505/4394*exp(-2*t)*cos(5*t)-70/2197*exp(-2*t)*sin(5*t)

Computer Exercise 6-5

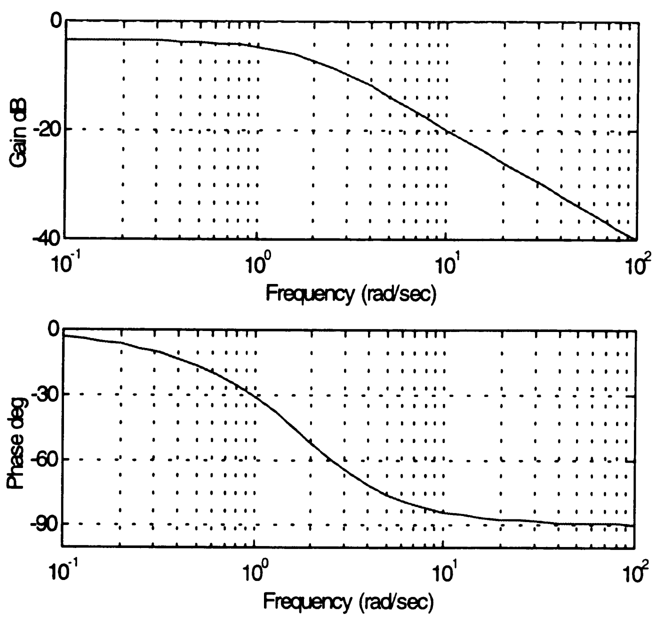
(a)
```
n1 = [0 1 4];
d1 = [1 5 6];
step(n1, d1)
```



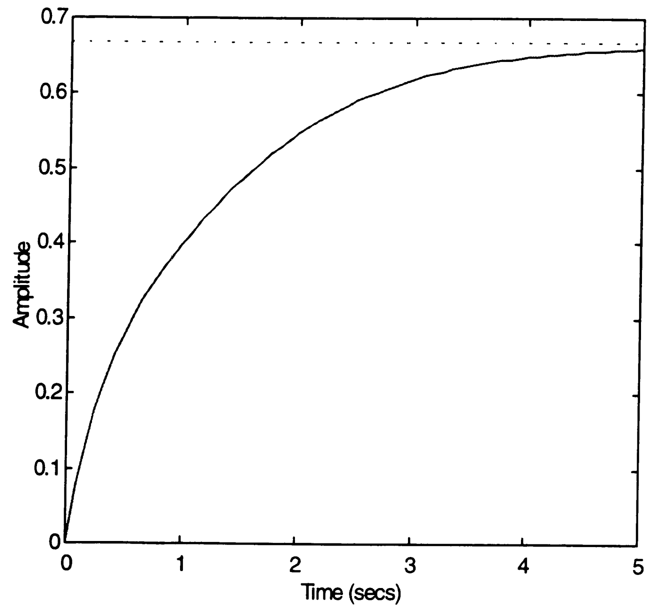

```
impulse(n1, d1)
```



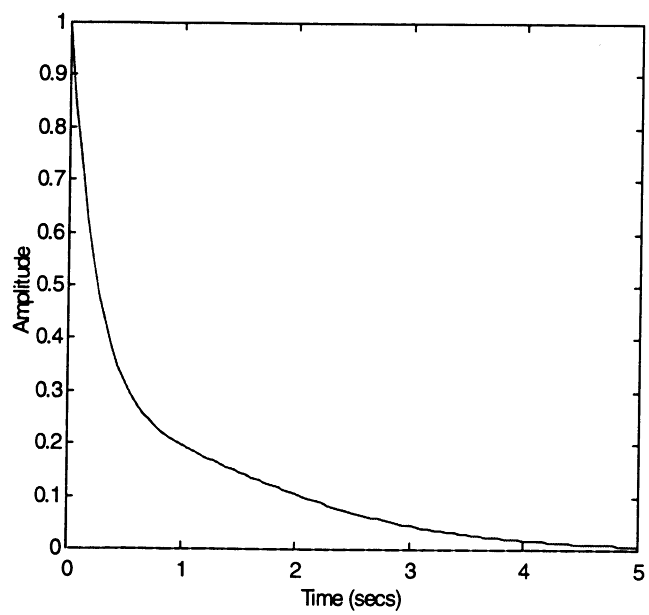

```
bode(n1, d1)
```



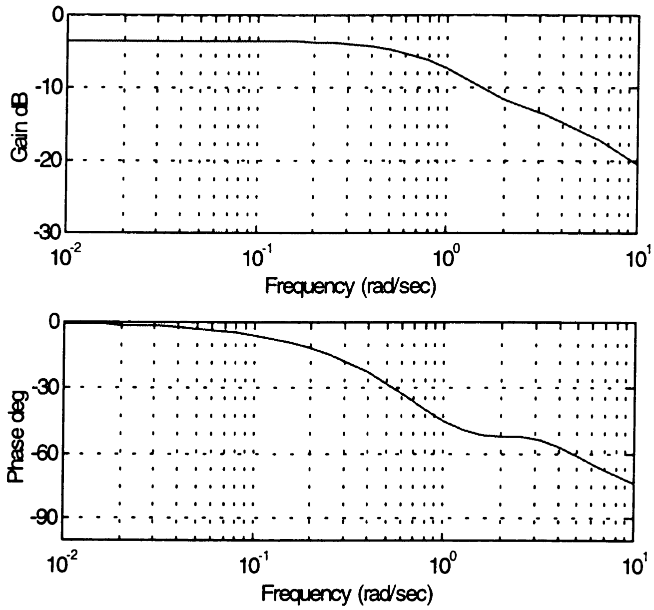
(b)
```
n2 = [0 1 3 4];
d2 = [1 6 11 6];
step(n2, d2)
```



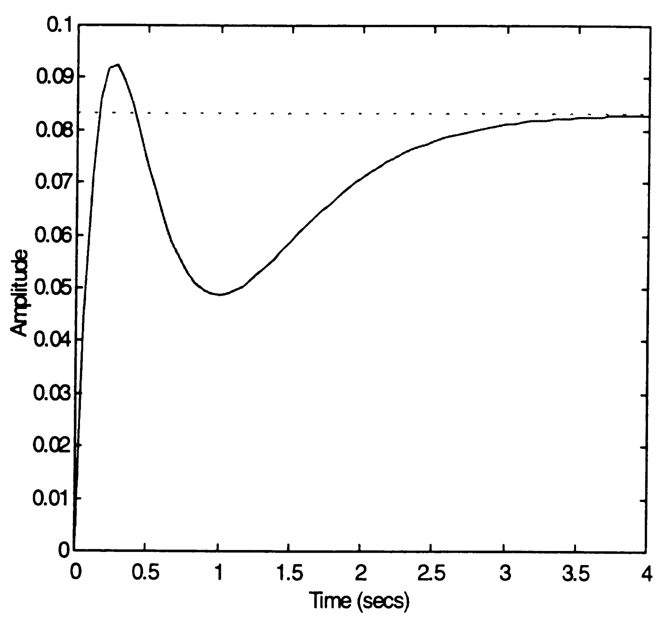
impulse(n2, d2)



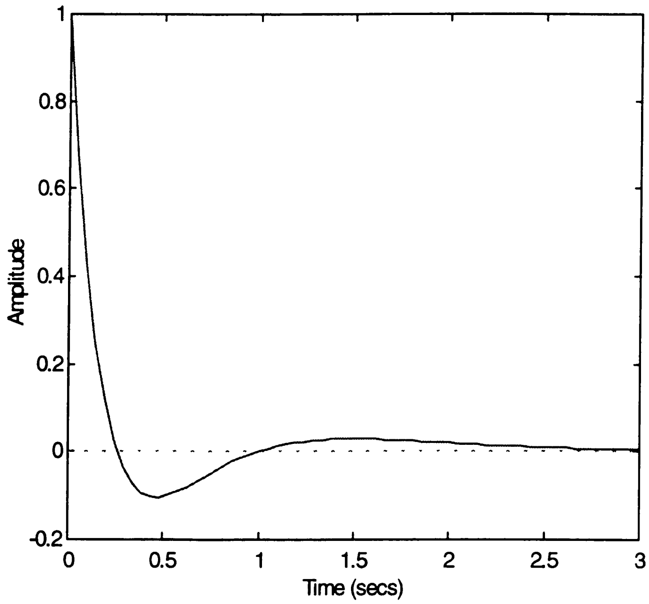
bode(n2, d2)



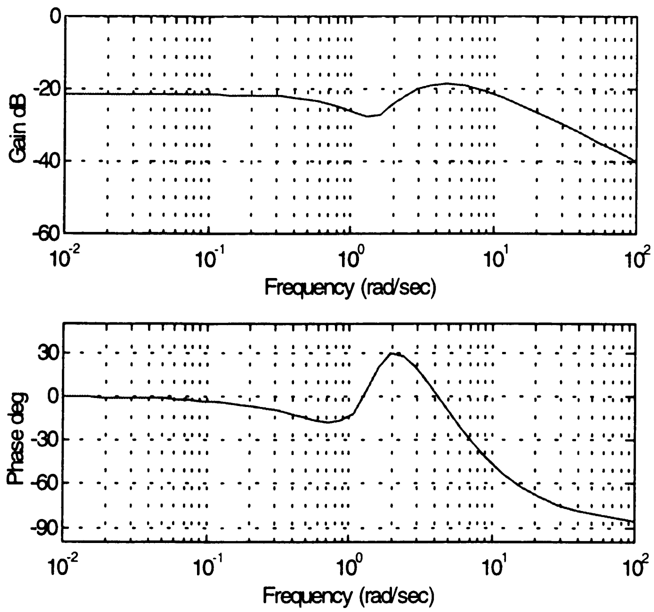
(c)
```
n3 = [0 1 2 3 2];
d3 = [1 10 35 50 24];
step(n3, d3)
```




```
impulse(n3, d3)
```




```
bode(n3, d3)
```



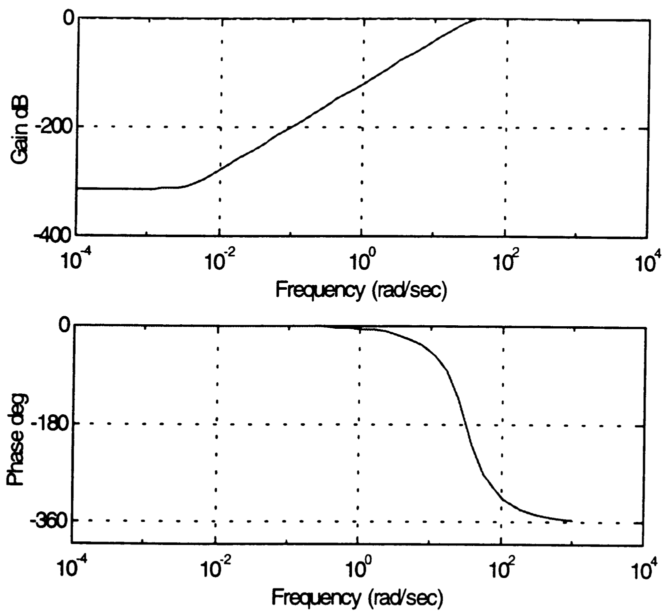
Problem 6-6

(a) [na, da] = **butter(4,10*pi,'high','s')**

```
na =
   1.0000     0.0000     0.0000     0.0000     0.0000
da =
  1.0e+005 *
```

```
     0.0000     0.0008     0.0337     0.8102     9.7409
```

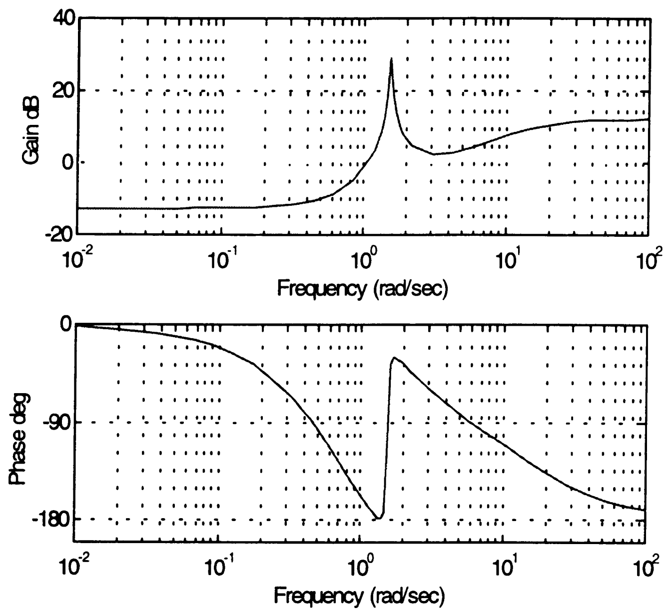
bode(na, da)



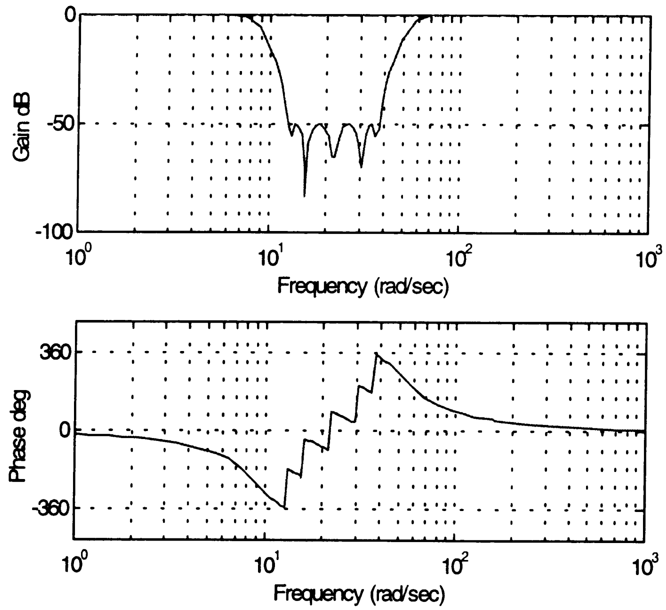
(b) **[nb, db] = cheby1(3, 0.5, [8*pi 18*pi])**

```
nb =
  Columns 1 through 4
  -4.1221              0.0000 - 0.0000i  12.3663 - 0.0000i   0.0000 -
0.0000i
  Columns 5 through 7
 -12.3663 - 0.0000i   0.0000 + 0.0000i   4.1221 + 0.0000i
db =
  Columns 1 through 4
   1.0000            17.4957 + 0.0000i  38.4714 + 0.0000i  67.8303 +
0.0000i
  Columns 5 through 7
  92.6748 - 0.0000i  66.3233 - 0.0000i  17.7011 - 0.0000i
```

bode(nb, db)




```
[nc dc] = cheby2(5, 50, [4*pi 12*pi], 'stop', 's')

nc =
  1.0e+013 *
  Columns 1 through 7
    0.0000    0.0000    0.0000    0.0000    0.0000    0.0000    0.0002
  Columns 8 through 11
    0.0000    0.0336    0.0000    2.3862
dc =
  1.0e+013 *
  Columns 1 through 7
    0.0000    0.0000    0.0000    0.0000    0.0000    0.0001    0.0014
  Columns 8 through 11
    0.0164    0.1327    0.6877    2.3862

bode(nc, dc)
```


0
-50
-100
Gain dB
10^0
10^1
10^2
10^3
Frequency (rad/sec)
360
0
-360
Phase deg
10^0
10^1
10^2
10^3
Frequency (rad/sec)

Computer Exercise 7-1

First we consider Example 7-5. The first step is to modify the script file for Exercise 7-5 so that we can enter the script file with a user-designated sampling interval, Ts, and also suppress plotting. The modified script file is

```
% This script file is the modified version of c7ex5.m and is
% used for Computer Exercise 7-1.

a = [0 1; -2 -3];                % Define a matrix
b = [0; 1];                      % Define b matrix
c = [1 0; 1 1];                  % Define c matrix
nf = (5/Ts)+1;                   % Define number of samples
x = zeros(2,nf);                 % Initialize x matrix
x(:,1) = [1; 1];                 % Establish initial condition
u = ones(1,nf);                  % Define input signal
[ad,bd] = c2d(a,b,Ts);           % Compute discrete system
% The next three lines perform the simulation
for n=1:nf-1
        x(:,n+1) = ad*x(:,n)+bd*u(n);
end
y = c*x;                         % Compute system output
y1 = y(1,:);                     % Determine y1(t)
y2 = y(2,:);                     % Determine y2(t)
t = (0:nf-1)*Ts;                 % Establish time vector
tf = (nf-1)*Ts;                  % Establish 'finish' time
```

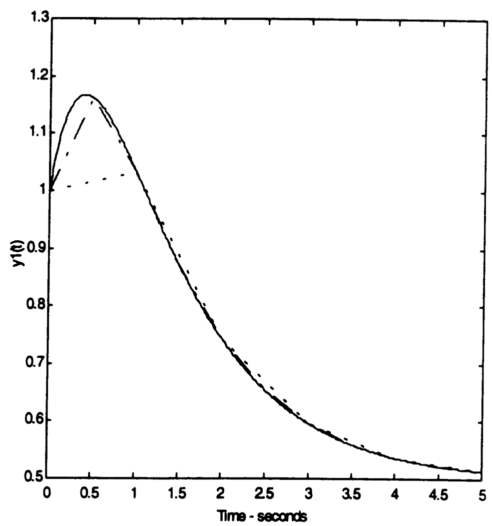
A script is then developed that will execute the above code for three different sampling frequencies. The sampling frequencies chosen are Ts = 1.0, Ts = 0.5 and Ts = 0.01. The script file is shown below for the first system output, y1(t).

```
Ts = 1.0;
c7ex5a
plot(t,y1,':')
hold on
Ts = 0.5;
c7ex5a
plot(t,y1,'-.')
Ts = 0.01;
c7ex5a
plot(t,y1)
y1 = 0.5+2*exp(-t)-(3/2)*exp(-2*t);
plot(t,y1)
hold off
xlabel('Time - seconds')
ylabel('y1(t)')
```

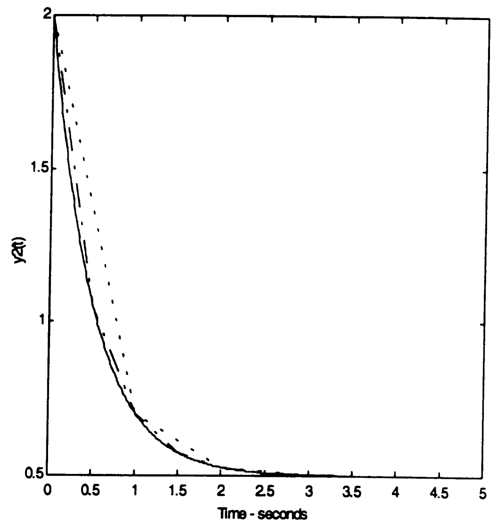
A second script file is developed for the second output. This file follows.

```
Ts = 1.0;
c7ex5a
plot(t,y2,':')
hold on
Ts = 0.5;
c7ex5a
plot(t,y2,'-.')
Ts = 0.01;
c7ex5a
plot(t,y2)
y2 = 0.5+(3/2)*exp(-2*t);
plot(t,y2)
hold off
xlabel('Time - seconds')
ylabel('y2(t)')
```

Executing the first file generates the plot for y1(t), which is shown below.



The sampling at Ts = 1, Ts = 0.5 and Ts = 0.01 can clearly be seen. Next we run the second file in order to generate the second output. The result is shown at the top of the following page.



The sampling can clearly be seen. Examination of the script files and the resulting plots for both y1(t) and y2(t) show that four curves are plotted but only three curves can be seen in the plots. The reason for this is that the theoretical results, (7-85), and the numerical results for Ts = 0.01 fall on top of each other. This indicates close agreement.

First we consider Example 7-6. The first step, as before, is to modify the script file for Exercise 7-6 so that we can enter the script file with a user-designated sampling interval, Ts, and also suppress plotting. The modified script file is as shown below:

```
a = [0 1; -2 -3];               % Define a matrix
b = [2 1; 0 1];                 % Define b matrix
c = [1 0; 1 1];                 % Define c matrix
nf = (5/Ts)+1;                  % Define number of samples
x = zeros(2,nf);                % Initialize x matrix
x(:,1) = [0; 0];                % Establish initial conditions
t = (0:nf-1)*Ts;                % Define sampling point vector
u = ones(2,nf);                 % Initialize u and define first signal
u(2,:) = exp(-3*t);             % Define second input signal
[ad,bd] = c2d(a,b,Ts);          % Compute discrete equivalent
% The next three lines perform the simulation
for n=1:nf-1
        x(:,n+1) = ad*x(:,n)+bd*u(:,n);
end
y = c*x;                        % Compute system output
y1 = y(1,:);                    % Determine y1(t)
y2 = y(2,:);                    % Determine y2(t)
t = (0:nf-1)*Ts;                % Establish time vector
tf = (nf-1)*Ts;                 % Establish 'finish' time
```

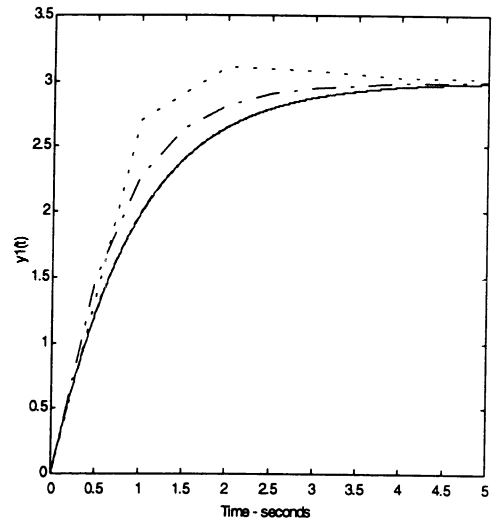
As before, a script is developed that will execute the above code for three different sampling frequencies. The sampling frequencies chosen are Ts = 1.0, Ts = 0.5 and Ts = 0.01. The script file for the first system output, y1(t), follows.

```
Ts = 1.0;
c7ex6a
plot(t,y1,':')
hold on
clear all
Ts = 0.5;
c7ex6a
plot(t,y1,'-.')
clear all
Ts = 0.01;
c7ex6a
plot(t,y1)
y1 = 3-(5/2)*exp(-t)-exp(-2*t)+0.5*exp(-3*t);
plot(t,y1)
hold off
xlabel('Time - seconds')
ylabel('y1(t)')
```

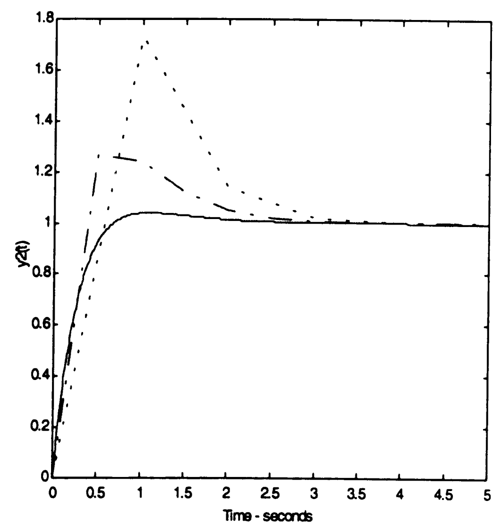
The corresponding script file for y2(t) is given below.

```
Ts = 1.0;
c7ex6a
plot(t,y2,':')
hold on
clear all
Ts = 0.5;
c7ex6a
plot(t,y2,'-.')
clear all
Ts = 0.01;
c7ex6a
plot(t,y2)
y1 = 1+exp(-2*t)-2*exp(-3*t);
plot(t,y2)
hold off
xlabel('Time - seconds')
ylabel('y2(t)')
```

Executing the first script file to generate y1(t) gives the plot shown at the top of the following page.



Executing the second script file yields the plot below for y2(t).



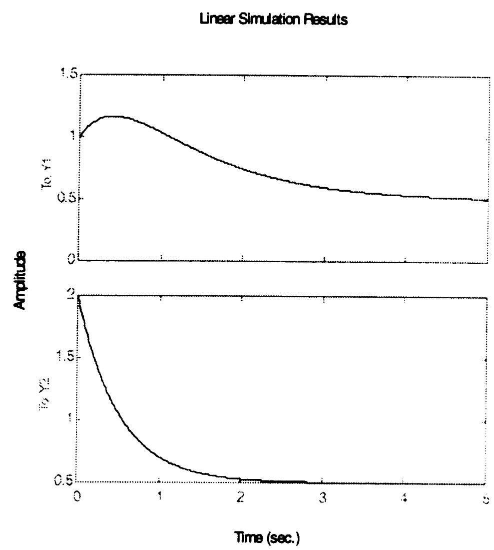
Once again, the samples taken at Ts = 1.0, Ts = 0.5 and Ts = 0.01 can be clearly seen. The fact that the theoretical results fall on top of the numerical results for Ts = 0.01 show close agreement between the theoretical and the numerical results.

Computer Exercise 7-2

For Example 7-5 we have

```
» a = [0 1; -2 -3];
» b = [0; 1];
» c = [1 0; 1 1];
» d = [0; 0];
» t = (0:500)*0.01;
» x0 = [1, 1];
» u = ones(501,1);
» lsim(a,b,c,d,u,t,x0)
```

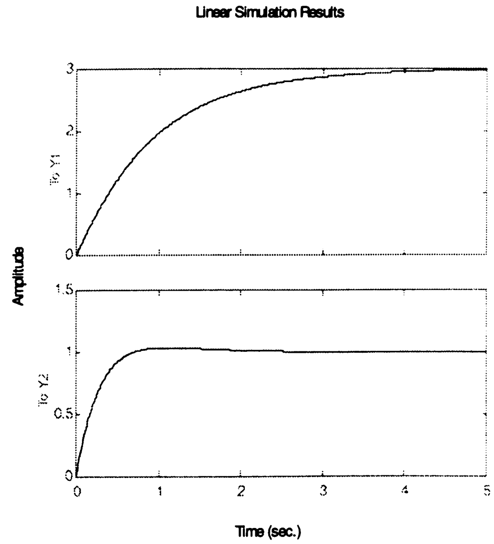
which gives the plot



For Exercise 7-6 we have

```
» a = [0 1; -2 -3];
» b = [2 1; 0 1];
» c = [1 0; 1 1];
» d = [0 0; 0 0];
» t = (0:500)*0.01;
» x0 = [0; 0];
» u1 = ones(501,1);
» u2 = exp(-3*t);
» u = [u1,u2'];
» lsim(a,b,c,d,u,t,x0)
```

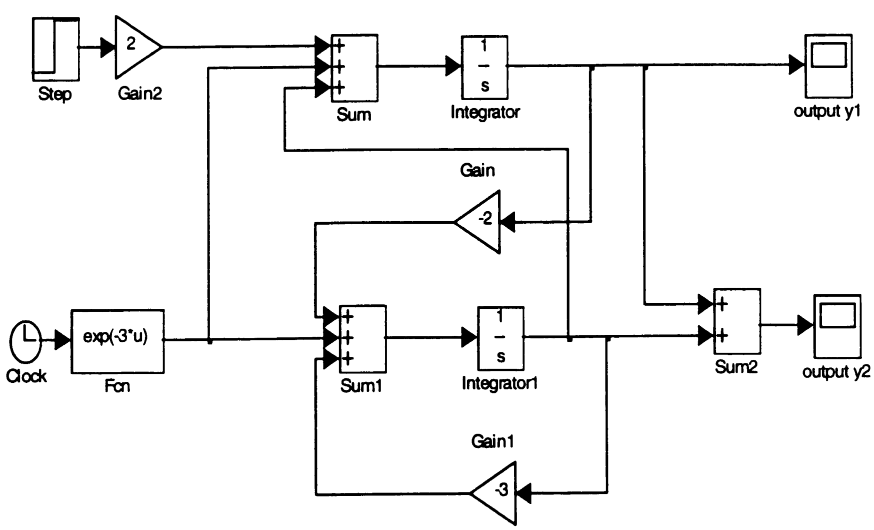
which yields the result



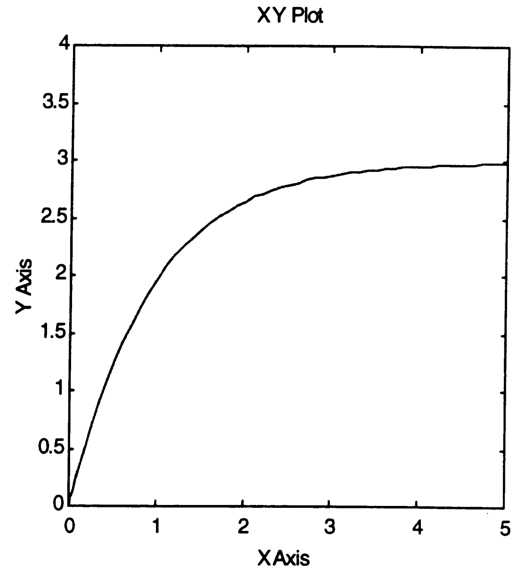
It can be seen that the results are in good agreement with the results of Examples 7-5 and 7-6, respectively.

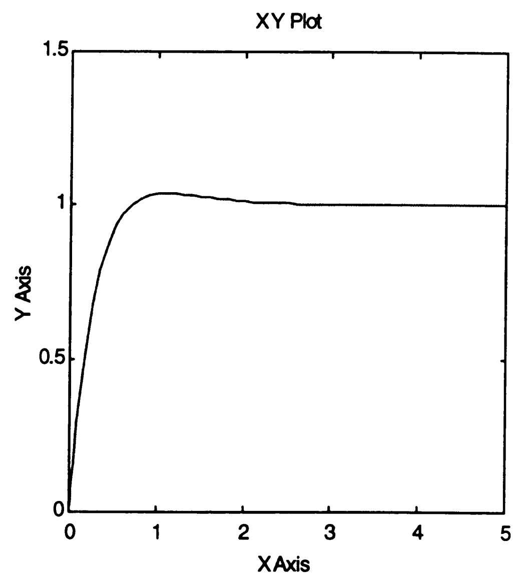
Computer Exercise 7-3

The SIMULINK block diagram is as follows



The two outputs are





The results are consistent with those given for Example 7-6 in the textbook. SIMULINK is clearly a convenient computational environment for working problems of this type.

Computer Exercise 7-4

The script file for this problem is shown below. The technique is to convert the transfer function to state-space form since a state-space formulation is asked for in the problem statement. A discrete version is then formed using the MATLAB command **c2dm**. The step response is then generated using the MATLAB command **dstep**.

The result is verified by first forming (using the **conv** command) the step response

$$Y(s) = \frac{1}{s}\frac{1}{s^5 + 14s^4 + 56s^3 + 85s^2 + 46s + 8}$$

Note that the denominator is 6th order. Since the roots of the denominator are distinct, it follows that

$$Y(s) = \sum_{n=1}^{6} \frac{R_n}{s - p_n}$$

where R_n is the residue of *Y(s)* at the pole at $s = p_n$. Thus, *y(t),* the step response, is given by

$$y(t) = \sum_{n=1}^{6} R_n e^{p_n t}$$

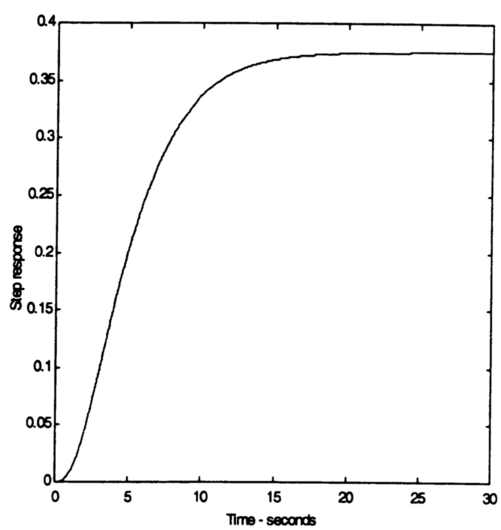
The MATLAB command **residue** is used to determine the residues and the pole locations.

```
b = [1 3];                                  % Numerator
a = [1 14 56 85 46 8];                      % Denominator
Ts = 0.01;                                  % Sampling period
Tf = 30;                                    % Final Time
Nsamp = Tf/Ts+1;                            % Number of samples
t = 0:Ts:Tf;                                % Time vector for plotting

[A,B,C,D] = tf2ss(b,a);                     % Convert to state-space
[Ad,Bd,Cd,Dd] = c2dm(A,B,C,D,Ts,'zoh');     % Convert to discrete time
[y,x] = dstep(Ad,Bd,Cd,Dd,1,Nsamp);         % Compute step response
plot(t,y')                                  % Plot step response
xlabel('Time - seconds')                    % Label x-axis
ylabel('Step response')                     % Label y-axis
pause                                       % Pause to look

% The remaining code generates the step response be calculating the
% inverse Laplace transform of H(s)/s by generating the residues and
% the poles of H(s)/s

a1 = conv([1 0],a);                         % Generate denominator of H(s)/s
```

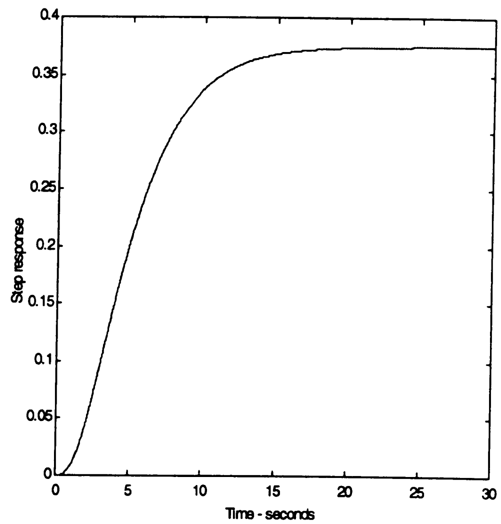
```
[r,p,k] = residue(b,a1);                % Generate poles and residues

h1 = r(1)*exp(p(1)*t);                  % First term
h2 = r(2)*exp(p(2)*t);                  % Second term
h3 = r(3)*exp(p(3)*t);                  % Third term
h4 = r(4)*exp(p(4)*t);                  % Fourth term
h5 = r(5)*exp(p(5)*t);                  % Fifth term
h6 = r(6)*exp(p(6)*t);                  % Sixth term

h = h1+h2+h3+h4+h5+h6;                  % step response

%plot(t,h1,t,h2,t,h3,t,h4,t,h)          % Look at all terms if desired

plot(t,h)                               % Plot result
xlabel('Time - seconds')                % Label x-axis
ylabel('Step response')                 % Label y-axis
```

Note that we plot the numerical result and then pause. After the pause the theoretical result is plotted.

Executing the program first generates the numerical result shown below.



We then plot the theoretical result.



Note that they match, as was expected.

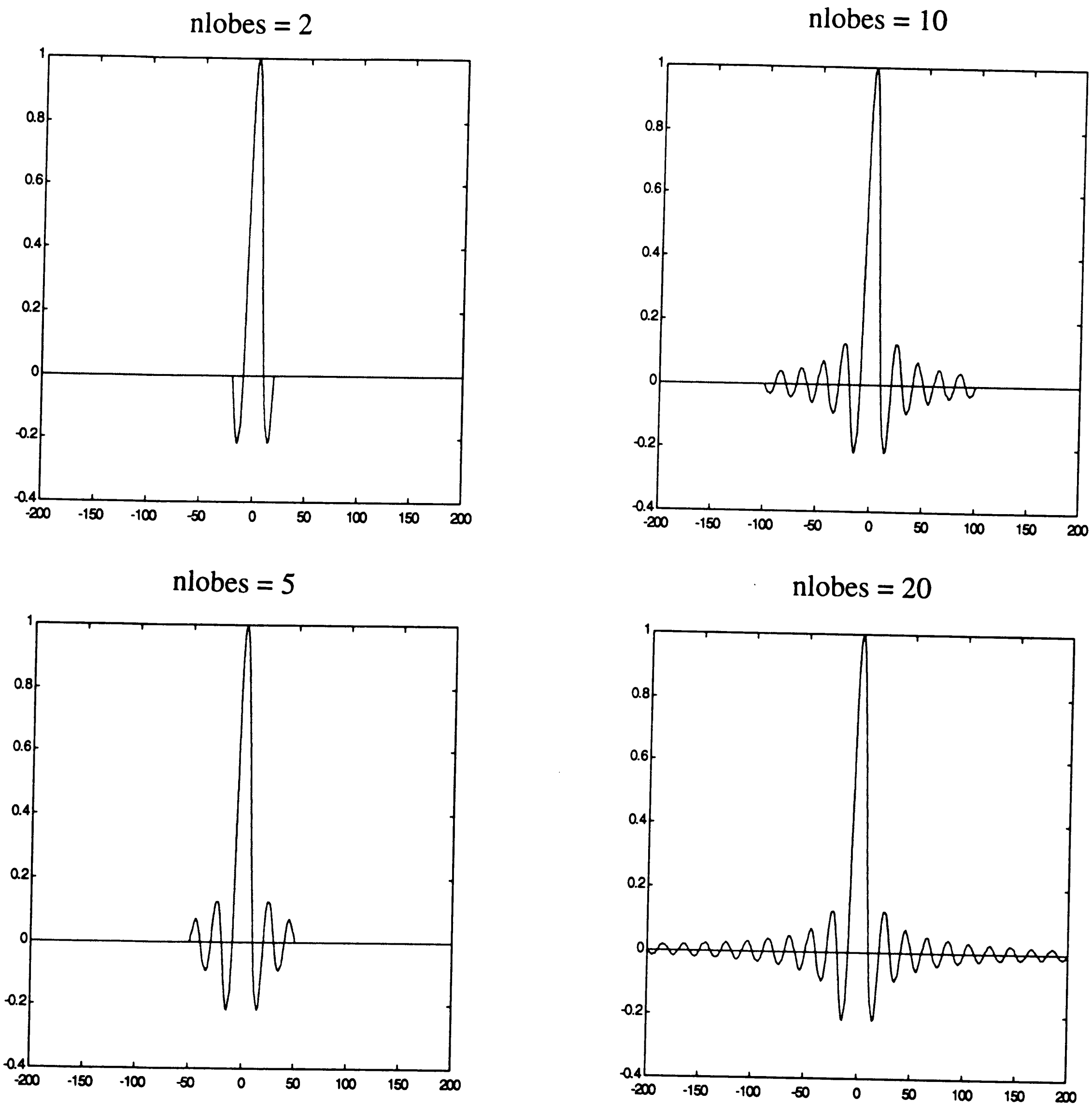
Computer Exercise 8-1

The MATLAB code used to solve this computer exercise is shown below.

```
nlobes = 20;                              % Number of lobes of the sinc fcn.

fs = 1000;                                % Sampling frequency
nf = 2501;                                % Total number of samples
f1 = 0.005*fs;                            % Define f1 as required
delt = 1/fs;                              % Define sampling period
t = delt*(0:nf-1);                        % Define time vector
nsinc = 10*nlobes;                        % Define number of sinc samples

phi = [0.0484 2.4091 0.4200];             % Define phases

x1 = 1.2*sin(2*pi*f1*t+phi(1,1));         % Define first term of test signal
x2 = 0.7*sin(2*pi*2*f1*t+phi(1,2));       % Second term of test signal
x3 = 0.9*sin(2*pi*3*f1*t+phi(1,3));       % Third term of test signal
x = x1+x2+x3;                             % Test signal

xd1 = zeros(1,nf);                        % Initialize vector
xd1(1,1:10:nf) = x(1,1:10:nf);            % Decimated version of x

j = -nsinc:nsinc;                  % Sample locations for sinc function
xj = sinc(j/10);                   % Compute sinc function values

y = conv(xj,xd1);                         % Interpolate xd1

nstart = 1400;                            % Start time for plotting
nstop = 1600;                              % Stop time for plotting
npts = nstop-nstart+1;                    % Number of points to plot
x1 = x(1,nstart-nsinc:nstop-nsinc);       % x array for plotting
y1 = y(1,nstart:nstop);                   % y array for plotting
n1 = 1:npts;                              % Independent variable

plot(n1,x1,n1,y1)                         % plot results
```

There are several items in the above code worthy of discussion. First, the parameter **nlobes** establishes the extent of the **sinc** function used for interpolation. The value of **nlobes** is actually the number of zero crossings of *sinc(x)* for $x > 0$. Since we interpolate by taking every 10^{th} sample (note the MATLAB statement **xj = sinc(j/10)**), for **nlobes** equal to 20 we interpolate using 41 samples of the sinc function. This corresponds to 20 samples of *sinc(x)* for $x > 0$, 20 samples for $x < 0$, and one sample for $x = 0$. The following page shows several plots of *sinc(x)* for differing values of **nlobes**.

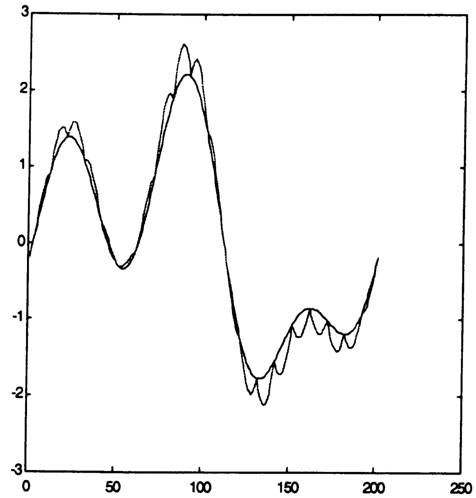
The phases were chosen by generating waveforms with random phases using the MATLAB statement **phi = 2*pi*rand(1,3)** and observing the resulting waveform. When an "interesting" waveform appeared, the phases were set equal to the values yielding that waveform.



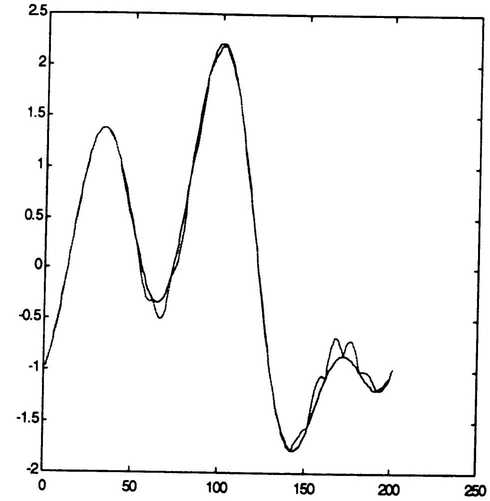
Note that since f1 is 0.005*fs, where fs is the sampling frequency, the highest frequency present in the waveform is 3*f1 or 0.015*fs. Since this waveform is decimated by a factor of 10, the decimated waveform os sampled at 0.15*fs. Since this is less than one-half of the sampling frequency, "perfect" reconstruction can still take place.

The program is executed using four different values of **nlobes**; 1, 2, 5 and 20. The results are shown on the following page. Both the actual and the interpolated waveforms are shown as solid lines so that the plots are readable. It should be easy, however, to tell which curve is which since the waveform being interpolated (the actual *x(t)*) is the "smooth" curve and is the waveform to which the interpolated waveform converges for large values of **nlobes**.

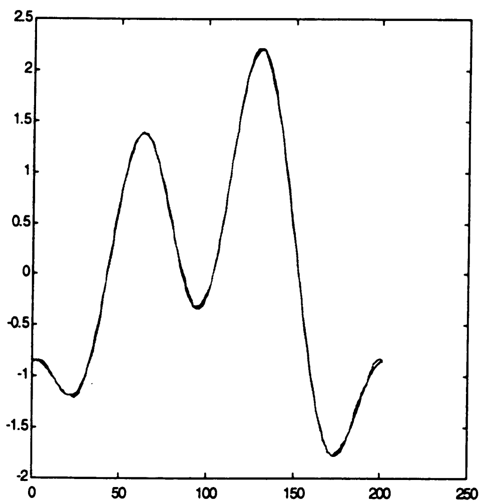
nlobes = 1



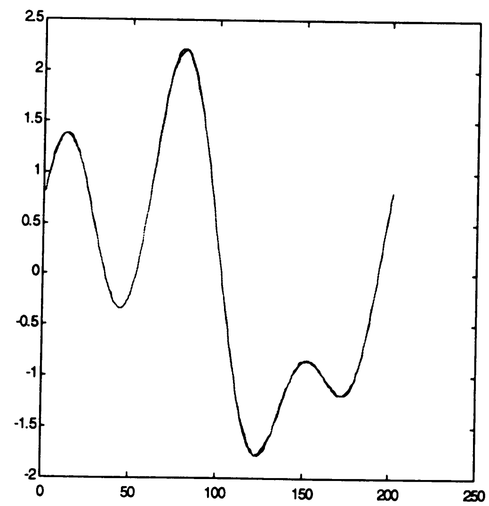
nlobes = 2



nlobes = 5



nlobes = 20



Computer Exercise 8-2

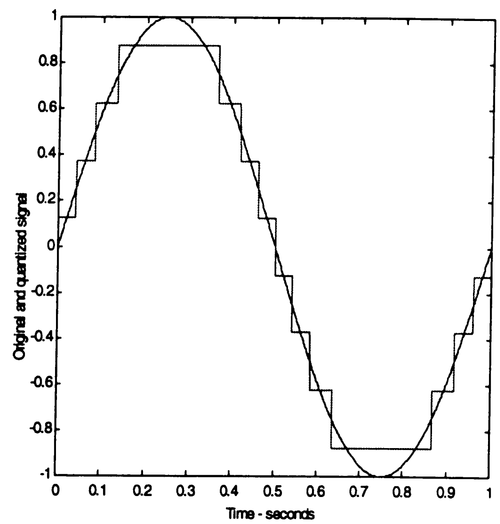
The script for solving this computer example is shown below. Note that the quantizer is realized with one line of computer code. It is not necessary to write different quantizer code for positive input and for negative input since the MATLAB command **floor** rounds towards minus infinity.

```
n = 3;                                    % Set wordlength

f = 1;                                    % Frequency of sinusoid
fs = 1000;                                % Sampling frequency
Ts = 1/f;                                 % Time duration of signal
delt = 1/fs;                              % Sampling time
nf = Ts/delt;                             % Total number of samples

t = delt*(0:nf);                          % Sampling time vector
sig = sin(2*pi*f*t);                      % Generate sinusoidal signal
D = max(sig)-min(sig);                    % Compute dynamic range
delq = D/2^n;                             % Quantizing step size

sq = zeros(1,nf+1);                       % Initialize sq vector

for nn=1:nf+1
sq(nn) = delq*floor((sig(nn))/(delq+eps))+delq/2;  % The quantizer
end

plot(t,sig,t,sq)                          % Plot sig and sq
xlabel('Time - seconds')                  % Label x axis
ylabel('Original and quantized signal')   % Label yaxis

qe = sig-sq;                              % Compute quantizing error
qnp = (qe*qe')/nf;                        % Compute quantizing noise power
sp = (sig*sig')/nf;                       % Compute signal power
snrdbe = 10*log10(sp/qnp)                 % Compute SNR in dB
snrdbt = 1.76+6.02*n                      % Theoretical value
```

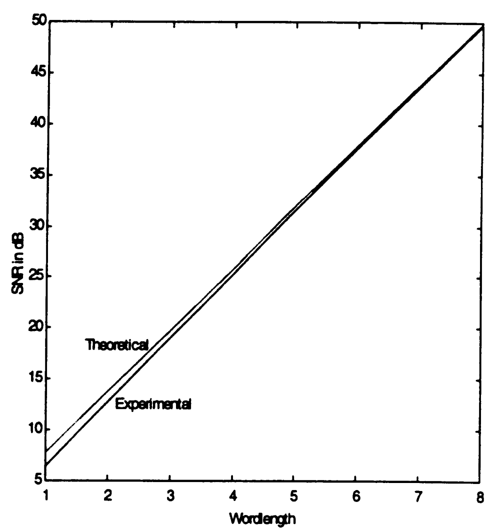
Following is the plot of the original signal and the quantized signal for a wordlength of 3 bits. Note that there are eight quantizing levels. As can be seen from the last two lines of the MATLAB code, two results are computed. The "experimental" result is based on the actual quantized version of the sinusoidal test signal. The "theoretical" result is the result derived in the textbook (Equation (8-58)). The theoretical result is

$$(SNR)_{dB} = 1.76 + 6.02n$$

where n is the wordlength. As pointed out in the text, the above result is valid for large n.



If the MATLAB code shown on the previous page is placed inside a loop to iterate on the wordlength, *n*, for *n* = 1, 2, 3, 4, 5, 6, 7 and 8, the plot shown below results. Note that the "theoretical" SNR exceeds the "experimental" SNR. Comparison of the signal and the quantized signal shows why the power in the sinusoidal signal exceeds the power in the quantized version of the signal. Note that the two results converge as the wordlength increases as is expected.



Computer Exercise 8-3

The MATLAB script for solving this Computer Exercise is shown below.

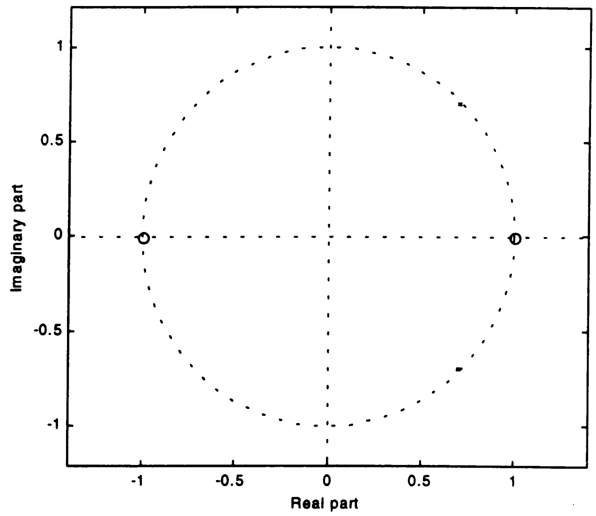
```
clear all                                   % Start with a clean slate

np = input('Enter number of poles > ');

jp = 1;
while jp<=np
        disp('Data for pole'), jp
        realp = input('Enter real part of pole location  > ');
        imagp = input('Enter imaginary part of pole location > ');
        pole(jp) = realp+i*imagp;
        if imagp == 0
                jp = jp+1;
        else
                pole(jp+1) = conj(pole(jp));
                jp = jp+2;
        end
end

disp(' ')                                   % Insert blank line
disp('The pole locations are'), pole        % List pole locations

nz = input('Enter number of zeros > ');

jz = 1;
while jz<=nz
        disp('Data for zero'); jz
        realz = input('Enter real part of zero location  > ');
        imagz = input('Enter imaginary part of zero location > ');
        zero(jz) = realz+i*imagz;
        if imagz == 0
                jz = jz+1;
        else
                zero(jz+1) = conj(zero(jz));
                jz = jz+2;
        end
end

disp(' ')                                   % Insert blank line
disp('The zero locations are'), zero        % List zero locations

zplane(zero',pole')                         % Display pole-zero plot
pause                                       % Pause to look

[num,den] = zp2tf(zero',pole',1);           % Convert to transfer function
[h,w] = freqz(num,den,512);                 % Compute frequency response
r = w/(2*pi);                               % Normalized frequency for plots
amp = 20*log10(abs(h));                     % Amplitude response in dB
phase = (180/pi)*angle(h);                  % Phase response

semilogy(r,amp)                             % Plot amplitude response
xlabel('Normalized frequency, r = f/fs')% Label x axis
ylabel('Amplitude response in dB')          % Label y axis
pause                                       % Pause to look

plot(r,phase)                                % Plot phase response
xlabel('Normalized frequency, r = f/fs') % Label x axis
```

```
ylabel('Phase response - degrees')      % Label y axis
pause                                   % Pause to look

Gd = grpdelay(num,den,w);               % Compute group delay
plot(r,Gd)                              % Plot group delay
xlabel('Normalized frequency, r = f/fs')% Label x axis
ylabel('Group delay - seconds')         % Label y axis
```

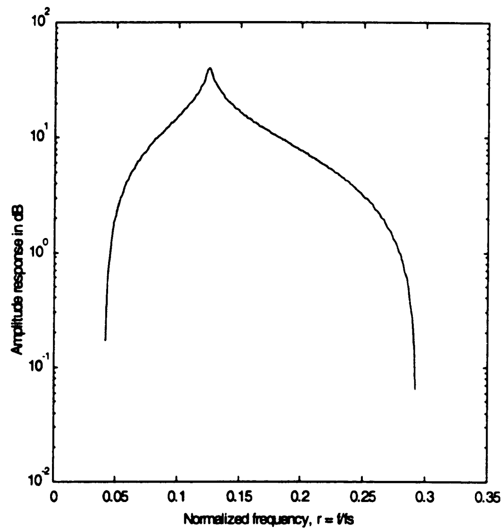
We will test the program using a bandpass filter with a maximum gain at 1/8 the sampling frequency. This will require poles close to the unit circle in the region of 45 degrees. The MATLAB dialog is as follows:

```
Enter number of poles > 2
Data for pole

jp =

     1

Enter real part of pole location  > 0.7
Enter imaginary part of pole location > 0.7

The pole locations are

pole =

 7.0000e-001 +7.0000e-001i 7.0000e-001 -7.0000e-001i

Enter number of zeros > 2
Data for zero

jz =

     1

Enter real part of zero location  > 1
Enter imaginary part of zero location > 0
Data for zero

jz =

     2

Enter real part of zero location  > -1
Enter imaginary part of zero location > 0

The zero locations are

zero =

     1    -1
```

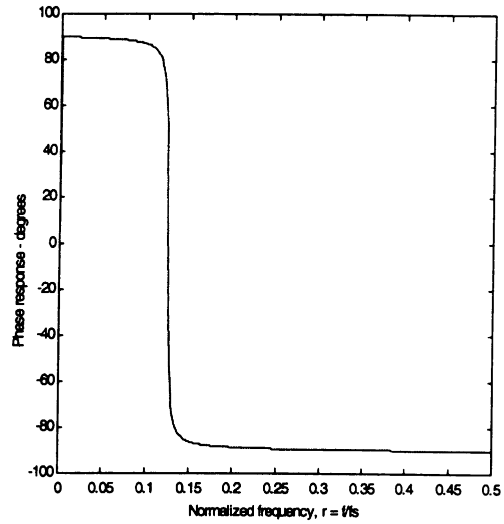
The pole-zero plot is shown at the top of the following page.



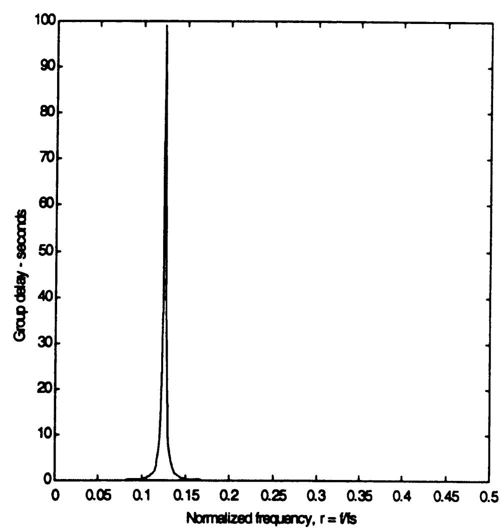
Next we have the amplitude response.



At the top of the following page the phase response is illustrated. Note that the phase jumps to +90 degrees for small values of frequency because of the pole at +1.



Finally we have the group delay. The group delay is long for frequency groups in the region of the bandpass because of the steep phase response in the region of r = 1/8.



The student should experiment with placing poles and zeros using the mouse. This is accomplished with the **ginput** command.

Computer Exercise 8-4

This computer exercise can be solved by using the MATLAB Application in Example 8-13 as a guide and using the **input** statement to provide a user interface with the program. The script file for convolving two strings, x1(n) and x2(n), appears below.

```
syms n z                                    % Declare symbolic
x1n = input('Enter x1(n) > ','s');          % Input x1(n)
x2n = input('Enter x2(n) > ','s');          % Input x2(n)
x1z = ztrans(x1n,n,z);                      % z-transform x1(n)
x2z = ztrans(x2n,n,z);                      % z-transform x2(n)
yz = x1z*x2z;                               % Multiply X1(z) and X2(z)
yn = iztrans(yz,z,n);                       % Inverse z-transform product
yn                                          % Display result
```

The above code is tested using the two sequences in Exercise 8-13. The resulting dialog with MATLAB appears below.

```
» c8ce4
Enter x1(n) > (1/2)^n
Enter x2(n) > (1/3)^n

yn =

3*(1/2)^n-2*(1/3)^n
```

We see that the results are in agreement with Exercise 8-13.

Computer Exercise 8-5

The MATLAB code is shown below.

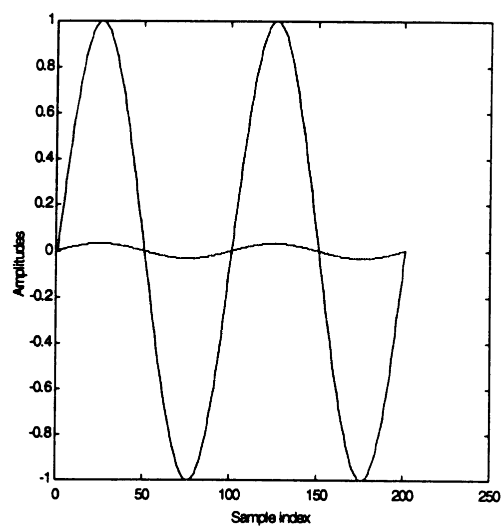
```
f = 1;                              % Reference (sampling) frequency
r = 0.01;                           % Value of r for signal
A = 1;                              % Amplitude
a = 15/16;                          % Filter constant
oma = 1-a;                          % Filter constant
nf = 1000;                          % Number of points to generate
n = 1:nf;

x = A*sin(2*pi*n*r);                % Signal vector

y = zeros(1,nf);                    % Initialize output vector

for k=1:nf                          % For loop to implement filter
        if k == 1                   % No initial condition for k = 1
        y(k) = oma*x(k);
        else
        y(k) = oma*x(k)-a*y(k-1);
```

```
        end
end                                     % Done filtering

x1 = x(1,nf-200:nf);                    % Display last 200 input samples
y1 = y(1,nf-200:nf);                    % Display last 200 output samples

lnx1 = length(x1);                      % Determine length of vector
nn = 1:lnx1;                            % Independent variable for plotting
plot(nn,x1,nn,y1)                       % Plot results
xlabel('Sample index')                  % Label x axis
ylabel('Amplitudes')

maxx = max(x1);                         % Maximum input
minx = min(x1);                         % Minimum input
xrange = maxx - minx;                   % Dynamic range of input
maxy = max(y1);                         % Maximum output
miny = min(y1);                         % Minimum output
yrange = maxy - miny;                   % Dynamic range of output
GdB = 20*log10(yrange/xrange);          % Gain in dB
GdB                                     % Display gain
```

Note that a routine is included to determine the gain in dB at the frequency of the test signal. The gain for this example is

```
GdB =

 -2.9827e+001
```

which is reasonable (See Figure 8-25). The phase response may be determined by looking at zero crossings in the plot of the input and out signals which are shown below.



Computer Exercise 8-6

The development of the MATLAB script for this problem is straightforward. The contour of radius r_c is sampled at 1000 points. This defines the increments in θ. The integral is realized using the trapezoidal approximation. The trapezoidal approximation hopefully yields small errors for 1000 samples. This can (and will) be checked as the routine is verified.

The form of the integral indicates that

$$X(z) = \frac{1}{1 - Kz^{-1}}$$

so that

$$x(nT) = (K)^n$$

for $n \le 0$.

The MATLAB script for the computer exercise is

```
theta = linspace(0,(2*pi),1000);                   % Sample phase angle
rc = input('Enter radius of contour (rc) > ');     % Recommend rc=1
n = input('Enter value of n > ' );                 % Index in time domain
k = input('Enter value of k > ');                  % Pole location (z=k)
num = (rc^(n+1))*exp(i*(n+1)*theta);               % Calculate numerator
den = (rc*exp(i*theta))-k;                         % Calculate denominator
f = (num ./ den)/(2*pi);                           % Calculate integrand
fr = real(f);                                      % Real part of integrand
fi = imag(f);                                      % Im. part of integrand
frout = trapz(theta,fr);                           % Real part of integral
frout                                              % Output real result
fiout = trapz(theta,fi);                           % Im. part of integral
fiout                                              % Output im. result
```

Note that we output both the real and the imaginary parts of the result. Since the result, *x(nT)*, is real, the imaginary part should be zero. This is part of the verification process.

In order to verify the result we let *K* = ½. For this case $x(nT) = (1/2)^n$ for $n \le 0$. First we execute the program for *n* = 0 and let the contour be the unit circle. The MATLAB dialog is as follows.

```
» c8ce6
Enter radius of contour (rc) > 1
Enter value of n > 0
Enter value of k > 1/2

frout =
```

```
    1.0000


fiout =

 -8.4914e-017
```

We see that the result is correct. The imaginary part is zero to within quantization errors. We now let $n = 1$. This gives

```
» c8ce6
Enter radius of contour (rc) > 1
Enter value of n > 1
Enter value of k > 1/2

frout =

    0.5000


fiout =

  3.4414e-017
```

Once again, we have a correct result. We now let $n = 2$. This gives

```
» c8ce6
Enter radius of contour (rc) > 1
Enter value of n > 2
Enter value of k > 1/2

frout =

    0.2500


fiout =

 -1.2128e-016
```

We should be able to make the contour smaller and still get the correct result as long as we enclose the pole. To test this we let $r_c = 0.6$. This gives

```
» c8ce6
Enter radius of contour (rc) > 0.6
Enter value of n > 2
Enter value of k > 1/2

frout =

    0.2500
```

```
fiout =

  2.7281e-017
```

If we let $r_c = 0.4$, the pole is not enclosed in the contour and we should obtain zero. The corresponding MATLAB dialog is

```
» c8ce6
Enter radius of contour (rc) > 0.4
Enter value of n > 2
Enter value of k > 1/2

frout =

  2.8840e-017


fiout =

  1.3583e-017
```

As one final check, let's go back to $r_c = 1$, and use the long format in order to gain a feel for the accuracy of the result. The corresponding MATLAB dialog is

```
» format long
» c8ce6
Enter radius of contour (rc) > 1
Enter value of n > 2
Enter value of k > 1/2

frout =

   0.25000000000000


fiout =

   -1.212849536514488e-016
```

Observation of the real part should give us considerable confidence in the result.

Computer Exercise 9-1

This example makes use of the symbolic capabilities of MATLAB. The program should be clear by examining the comments. Note the use of convolution to perform polynomial multiplication and to form Ha(s)/s from Ha(s). Perhaps the most confusing part of the program is the "if block" near the end of the program. This is made necessary because of the fact that MATLAB expresses polynomials in terms of z rather than in terms of z^{-1}.

```
num1 = 0.5*[1 4];                  % Define numerator of Ha(s)
den1 = conv([1 1],[1 2]);          % Define denominator of Ha(s)
den2 = conv(den1,[1 0]);           % Place pole at origin to form Ha(s)/s

Fs = 20/(2*(22/7));        % Define samp. freq. in Hz
T = 1/Fs;                  % Define samp. period

num = num1;                % Numerator of Ha(s)/s
den = den2;                % Denominator of Ha(s)/s

syms z s n t hos hoz hot honT hoz1 hoz    % Define symbolic variables

yy = n*T;                  % Define nT as a variable

nums = poly2sym(num,s); % Define symbolic numerator
dens = poly2sym(den,s); % Define symbolic denominator

hos = nums/dens;           % Define Ha(s)/s as a symbolic function
hot = ilaplace(hos);       % Define h(t) for Ha(s)/s

honT = subs(hot,t,yy);     % Define symbolic function h(nT)

hoz1 = ztrans(honT);       % Define H1(z)

hoz = ((z-1)/z)*hoz1;      % Desired H(z) is H1(z)*((z-1)/z)

[nn,dd] = numden(hoz);     % Symbolic numerator and denominator of H(z)

bn = sym2poly(nn);         % Numerator coefficients of H(z)
ad = sym2poly(dd);         % Denominator coefficients of H(z)

nout = bn/ad(1);           % Normalize numerator
dout = ad/ad(1);           % Normalize denominator

ln = length(nout);         % Length of numerator vector
ld = length(dout);         % Length of denominator vector

% The following "if" block removes the leading zeros of the
% numerator of H(z). The is important so that the resulting
% numerator polynomial has the correct order.

if ld > ln
      nout = [zeros(1,ld-ln),nout];
```

```
end

% End of "if" block.

nout                          % Output numerator of H(z)
dout                          % Output denominator of H(z)
```

Execution of the program yields the following output.

```
nout =

         0    0.1712   -0.0454


dout =

    1.0000   -1.2637    0.3895
```

Comparing this with Equation (9-87) shows that we have agreement.

Computer Exercise 9-2

This example makes use of the symbolic capabilities of MATLAB and especially the capability for making symbolic substitutions. The symbolic substitution capability is especially useful in making the lowpass to bandpass transformation.

```
% For the example filter we have:

        fs = 5000;                % Sampling frequency
        ru = 1000/fs;             % Normalized upper critical frequency
        rl = 500/fs;              % Normalized lower critical frequency

        a = [1 1];                % Denominator coefficients of prototype
        b = [0 1];                % Numerator of prototype

        A = cot(pi*(ru-rl));                          % Equation (9-137)
        B = 2*(cos(pi*(ru+rl)))/cos(pi*(ru-rl));      % Equation (9-144)


syms z s hos hoz           % Decalre symbolic

% The input to this portion of the script file are the
% numerator and denominator coefficient vectors.

nums = poly2sym(b,s);      % Make b vector symbolic
```

```
dens = poly2sym(a,s);        % Make a vector symbolic

hos = nums/dens;             %Symbolic H(s)

yy = 'A*(1-B*z^(-1)+z^(-2))/(1-z^(-2))';    % lp2bp transformation

hoz = subs(hos,s,yy);        % Symbolic H(z)

[nn,dd] = numden(hoz);       % Extract num and den of H(z)

nne = eval(nn);              % Evaluate numerator
dde = eval(dd);              % Evaluate denominator

bn = sym2poly(nne);          % Make numerator a vector
ad = sym2poly(dde);          % Make denominator a vector

numout = bn/ad(1);           % Normalize numerator coefficients of H(z)
denout = ad/ad(1);           % Normalize denominator coefficients of H(z)

numout                       % Output numerator
denout                       % Output denominator
```

Execution of the program yields the following output.

```
numout =

    0.2452         0   -0.2452         0         0


denout =

    1.0000   -0.9329    0.5095         0         0
```

Comparing this with the results of Example 9-9 show that we have agreement.

Computer Exercise 9-3

The MATLAB program for this Computer Exercise is rather lengthly. It is, however, divided into several sections, with each section corresponding to the problem statement. The program follows. Even though the problem is lengthly, the individual parts are straightforward.

```
clear all                              % Start with clean slate

n = 4;                                 % Filter order
r = 1;                                 % Passband ripple in dB
```

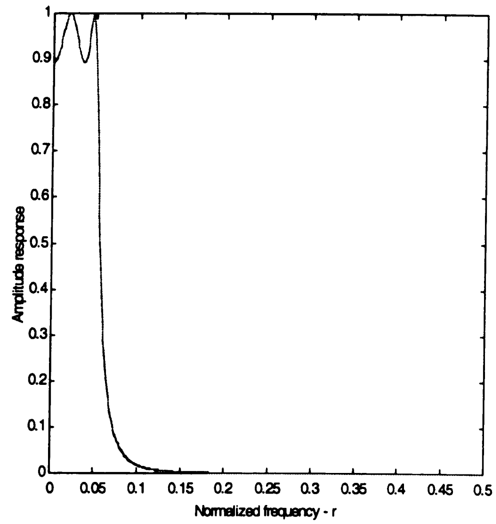
```
fs = 2000;                                  % Sampling frequency
fn = 100;                                   % Cutoff frequency
Wn = 2*pi*fn;                               % Compute break freq in rad/sec

% Part (a) - digital filter transfer function

[b,a] = cheby1(n,r,Wn,'s');                 % Compute
[numd,dend] = bilinear(b,a,fs,fn);          % Compute bilinear z-transform
numd                                        % Display numerator of H(z)
dend                                        % Display denominator of H(z)

% Part (b) - amplitude responses

[hd,wd] = freqz(numd,dend);                 % Response of digital filter
wa = wd*fs;                                 % Compute analog frequencies
ha = freqs(b,a,wa);                         % Response of analog filter
ampd = abs(hd);                             % Digital filter amp. response
ampa = abs(ha);                             % Analog filter amp. response
r = wd/(2*pi);                              % Normalized frequency vector
plot(r,ampd,r,ampa)                         % Plot amplitude response
xlabel('Normalized frequency - r')          % Label x axis
ylabel('Amplitude response')                % Label y axis
pause                                       % Pause to look

% Part (c) - phase responses

phid = angle(hd);                           % Phase resp. of digital filter
phia = angle(ha);                           % Phase resp. of analog filter
plot(r,(180/pi)*phid,r,(180/pi)*phia)       % Plot phase responses
xlabel('Normalized frequency - r')          % Label x axis
ylabel('Phase response - degrees')          % Label y axis
pause                                       % Pause to look

% Part (d1) - group delays

[Gdd,F] = grpdelay(numd,dend,512,fs);       % Compute digitalgroup delay
Gda = -diff(unwrap(angle(hd)))/(pi/512);% Compute analog group delay
Gda(512) = Gda(511);                        % Extrapolate last point
plot(r,Gdd,r,Gda)                           % Plot group delays
xlabel('Normalized frequency - r')          % Label x axis
ylabel('Group delay - seconds')             % Label y axis
pause                                       % Pause to look

% Part (d2) -  phase delays

Pdd = -(1/(2*pi))*unwrap(phid)./wd;         % Digital filter phase delay
Pda = -(1/(2*pi))*unwrap(phia)./wd;         % Analog filter phase delay
plot(r,Pdd,r,Pda)                           % Plot phase delays
xlabel('Normalized frequency - r')          % Label x axis
ylabel('Phase delay - seconds')             % Label y axis
pause                                       % Pause to look

% Part (e1) - pole-zero plot for digital filter
```

```
zplane(numd,dend)                         % Plot pole-zero plt
pause

% Part (e2) - pole-zero plot for analog filter

aa = roots(a);                            % Find poles
ba = roots(b);                            % Find zeros
reaa = real(aa);                          % Real parts of poles
imaa = imag(aa);                          % Imaginary parts of poles
reba = real(ba);                          % Real parts of zeros
imba = imag(ba);                          % Imaginary parts of zeros
plot(reaa,imaa,'x')                       % Plot poles
max1 = max(axis);                         % Maximum axis value
axis([-max1 max1 -max1 max1]);            % Scale
axis('square')                            % Make square
hold on                                   % Hold to draw axes
plot([-max1 max1],[0 0])                  % Draw x axis
plot([0 0],[-max1 max1])                  % Draw y axis
hold off                                  % Done plotting
xlabel('Real')                            % Label x axis
ylabel('Imaginary')                       % Label y axis

% Note: For the Chebyshev I filter all zeros (at least theoretically)
% are at infinity.  Therefore zeros are not plotted.
```

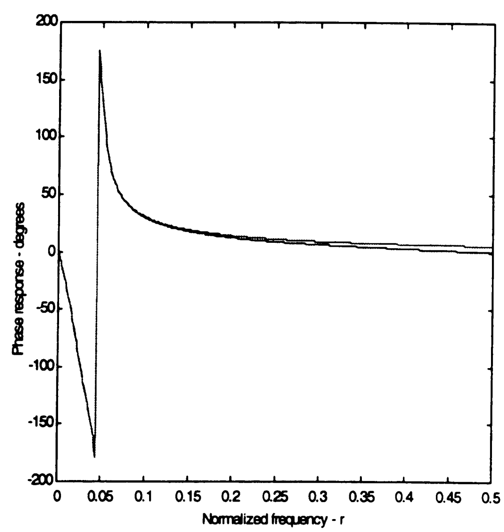
Execution of the program first yields the numerator and denominator polynomials of the bilinear z-transform digital filter.

```
numd =

  1.0e-003 *

    0.1298    0.5194    0.7791    0.5194    0.1298


dend =

    1.0000   -3.6079    4.9795   -3.1108    0.7415
```

The amplitude responses of the digital filter and the corresponding filter are next displayed. Note that even though two curves are plotted, only one is visible. This results because the sampling frequency is 20 times the critical frequency of the analog filter. As a result a very good approximation results from the synthesis process. The amplitude responses are shown at the top of the following page.

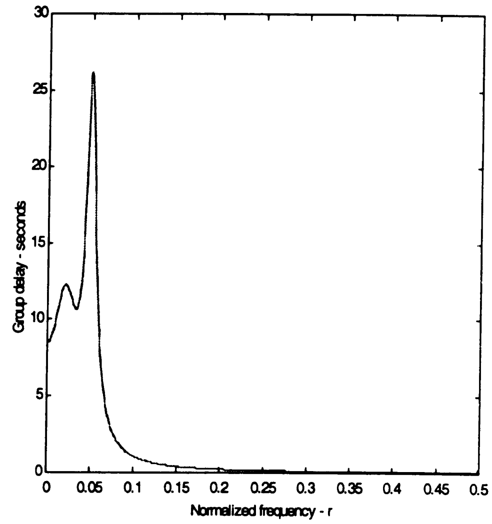


Next we plot the phase responses which follow.

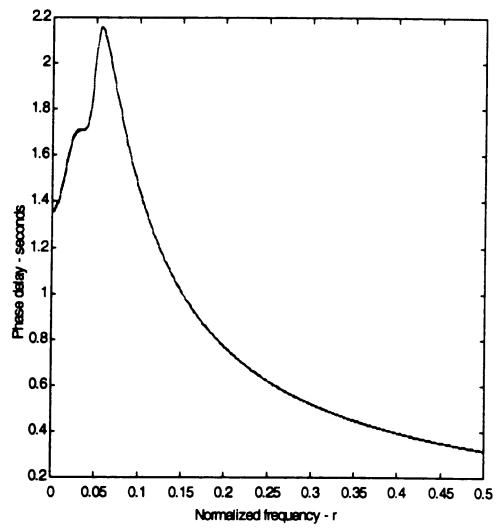


Note that we can see a little divergence of the two curves for very high frequences.

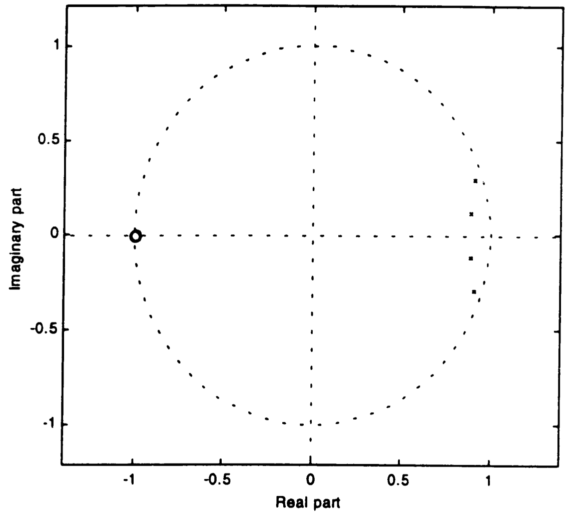
The group delays are shown at the top of the following page.



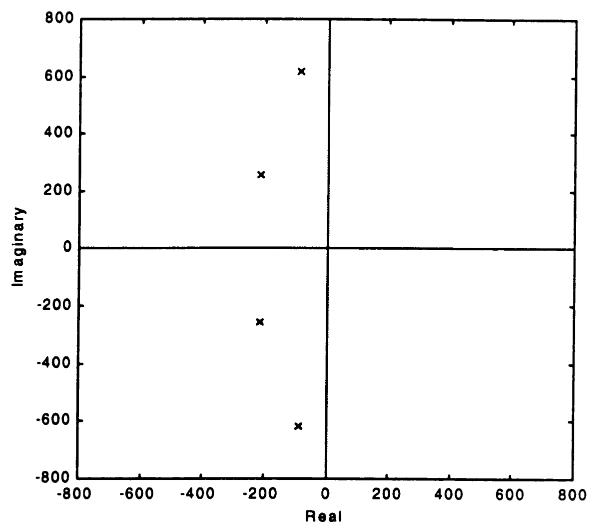
Next we have the phase delays. Since the phase responses match over a wide range of frequencies, we would expect to see excellent agreement in the group and phase delays also.



The pole-zero plot of the digital filters appear at the top of the following page.



Finally, we have the pole-zero plot of the analog filters.



<u>Computer Exercise 9-4</u>

Basically the solution of this Computer Exercise follows by simply substituting the specifications for the elliptic filter for the specifications of the Chebyshev I filter in the previous Computer Exercise. The MATLAB program starts on the following page. Note that we go to great pains to "unwrap" the phase.

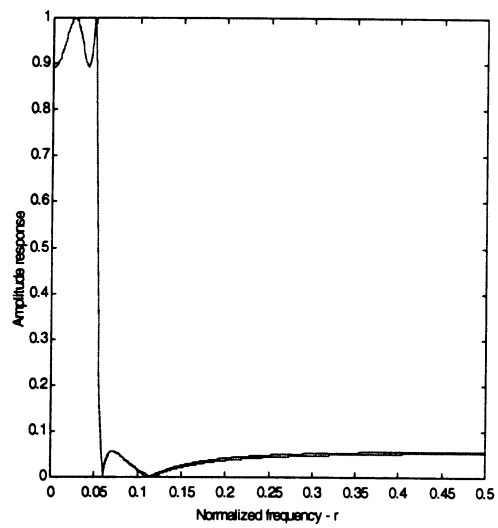
```
clear all                                    % Start with clean slate

n = 4;                                       % Filter order
rp = 1;                                      % Passband ripple in dB
rs = 25;
fs = 2000;                                   % Sampling frequency
fn = 100;                                    % Cutoff frequency
Wn = 2*pi*fn;                                % Compute break freq in rad/sec

% Part (a) - digital filter transfer function

[b,a] = ellip(n,rp,rs,Wn,'s');                 % Compute
[numd,dend] = bilinear(b,a,fs,fn);           % Compute bilinear z-transform
numd                                         % Display numerator of H(z)
dend                                         % Display denominator of H(z)

% Part (b) - amplitude responses

[hd,wd] = freqz(numd,dend);                  % Response of digital filter
wa = wd*fs;                                  % Compute analog frequencies
ha = freqs(b,a,wa);                          % Response of analog filter
ampd = abs(hd);                              % Digital filter amp. response
ampa = abs(ha);                              % Analog filter amp. response
r = wd/(2*pi);                               % Normalized frequency vector
plot(r,ampd,r,ampa)                          % Plot amplitude response
xlabel('Normalized frequency - r')           % Label x axis
ylabel('Amplitude response')                 % Label y axis
pause                                        % Pause to look

% Part (c) - phase responses

phid = unwrap(angle(hd));                    % Phase response of digital
filter
phia = unwrap(angle(ha));                    % Phase response of analog
filter

% The unwrap function removes 2*pi discontinuities from the phase
% response.  Because of the finite zeros in the elliptic filter
response,
% there are pi radian discontinuities in the phase response.  The
% following 10 lines of code removes these.  We do it twice since there
% are two zeros.

for jj=1:2
for j=1:511
        if phid(j+1) - phid(j) > 3
        phid(j+1) = phid(j+1) - pi;
        end
        if phia(j+1) - phia(j) > 3
        phia(j+1) = phia(j+1) - pi;
        end
end
end
```

```
plot(r,(180/pi)*phid,r,(180/pi)*phia)   % Plot phase responses
xlabel('Normalized frequency - r')      % Label x axis
ylabel('Phase response - degrees')      % Label y axis
pause                                   % Pause to look

% Part (d1) - group delays

[Gdd,F] = grpdelay(numd,dend,512,fs);   % Compute digitalgroup delay
Gda = -diff(phid)/(pi/512);             % Compute analog group delay
Gda(512) = Gda(511);                    % Extrapolate last point
plot(r,Gdd,r,Gda)                       % Plot group delays
xlabel('Normalized frequency - r')      % Label x axis
ylabel('Group delay - seconds')         % Label y axis
pause                                   % Pause to look

% Part (d2) -  phase delays

Pdd = -(1/(2*pi))*unwrap(phid)./wd;     % Digital filter phase delay
Pda = -(1/(2*pi))*unwrap(phia)./wd;     % Analog filter phase delay
plot(r,Pdd,r,Pda)                       % Plot phase delays
xlabel('Normalized frequency - r')      % Label x axis
ylabel('Phase delay - seconds')         % Label y axis
pause                                   % Pause to look

% Part (e1) - pole-zero plot for digital filter

zplane(numd,dend)                       % Plot poles and zeros
pause                                   % Pause to look

% Part (e2) - pole-zero plot for analog filter

aa = roots(a);                          % Find poles
ba = roots(b);                          % Find zeros
reaa = real(aa);                        % Real parts of poles
imaa = imag(aa);                        % Imaginary parts of poles
reba = real(ba);                        % Real parts of zeros
imba = imag(ba);                        % Imaginary parts of zeros
plot(reba,imba,'o')                     % Plot zeros
hold on                                 % Hold to add poles and axes
plot(reaa,imaa,'x')                     % Add poles to plot
max1 = max(axis);                       % Maximum axis value
plot([-max1 max1],[0 0])                % Draw x axis
plot([0 0],[-max1 max1])                % Draw y axis
hold off                                % Done plotting
axis([-max1 max1 -max1 max1]);          % Scale
axis('square')                          % Make square
title('Analog Filter Pole-Zero Plot')   % Add title
xlabel('Real')                          % Label x axis
ylabel('Imaginary')                     % Label y axis
```

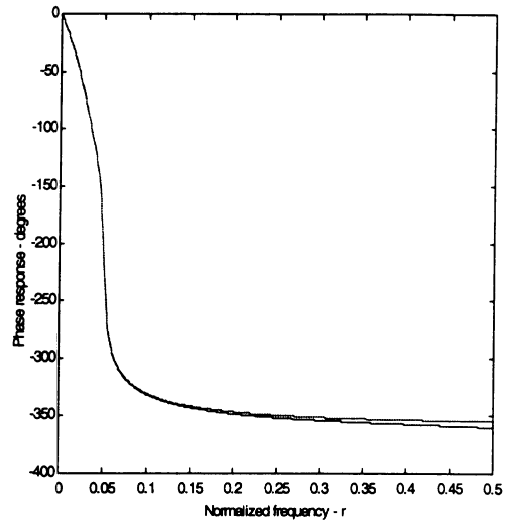
Execution of the program initially yields the numerator and denominator polynomials of the transfer function of the digital filter, These follow.

```
numd =

    0.0557   -0.1880    0.2685   -0.1880    0.0557

dend =

    1.0000   -3.5998    4.9761   -3.1213    0.7493
```

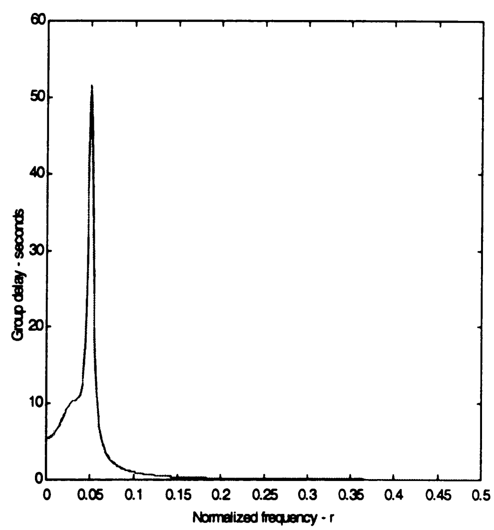
Next we have the amplitude responses of the digital and analog filters. Once again, the match is so good that we see only one curve.



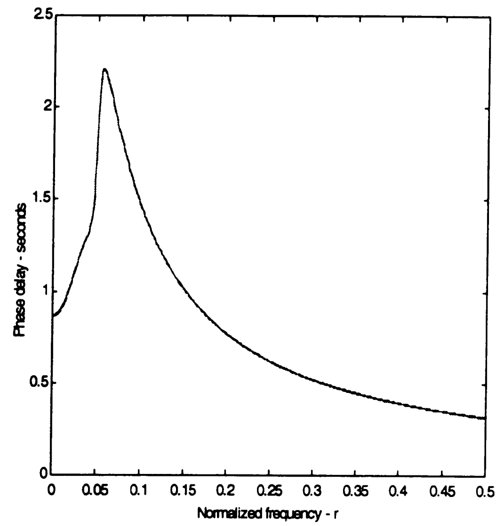
The phase responses appear at the top of the following page. Comparison with the phase responses of the previous Computer Exercise we see an absence of the 2*pi jumps. This occurs because we "unwrap" the phase as can be seen in the program. Note that, once again, we have good agreement.



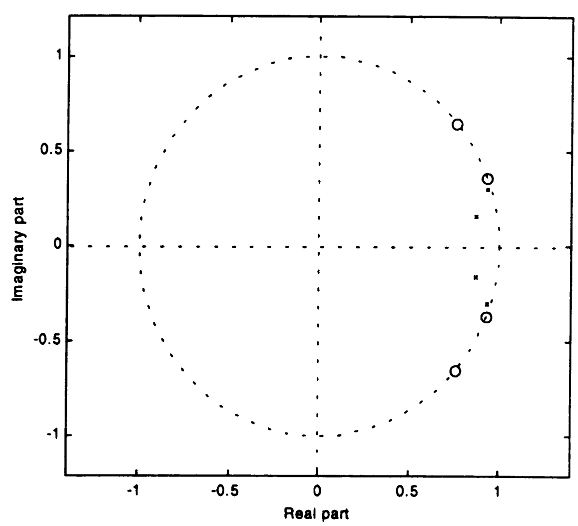
The group delays follow.



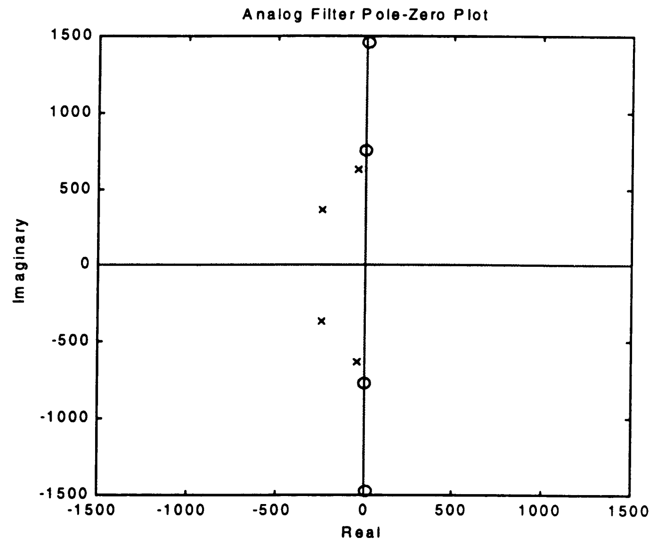
The phase delays are illustrated at the top of the following page.



The pole-zero plot for the digital filter is shown below.



The pole-zero plot for the analog filter is shown at the top of the following page. Note that we have show the finite zeros. The origin of these zeros can be seen by observing the amplitude responses.



Computer Exercise 9-5

The program for solving Computer Exercise 9-5 follows. Note that the filter order varies from 3 to 30 in steps of 3. In order to calculate the mean square error, a function is used to calculate the desired amplitude response at all kk = 512 points used to determine the frequency responses of the FIR and IIR digital filters. This function is designated **c9ce5a**.

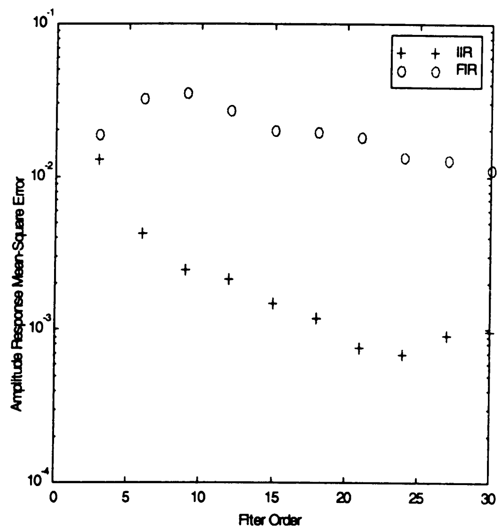
```
kk = 512;                                   % Determine resp. at kk points
f = [0 0.2 0.22 0.5 0.52 1];                % Vector of critical frequencies
m = [1 1 0.4 0.4 0 0];                      % Vector of amplitudes

w = (0:kk)*pi/kk;                           % Frequency set for error calc.
r = w/(2*pi);                               % Normalize frequency set
amp = zeros(1,(kk+1));                      % Initialize amplitude vector
for n=1:(kk+1)
        amp(1,n) = c9ce5a(r(n));            % Compute desired response
end

nn = 3*(1:10);                              % Vector of filter orders

for j=1:length(nn)                          % Iterate filter order
n = nn(j);                                  % Filter order for jth iteration
[b1,a1] = yulewalk(n,f,m);                  % IIR filter coefficients
b2 = remez(n,f,m);                          % FIR filter coefficients
h1 = freqz(b1,a1,w);                        % Complex resp. for IIR filter
h2 = freqz(b2,1,w);                         % Complex resp. for FIR filter
amp1= abs(h1);                              % Amplitude resp. for IIR filter
amp2 = abs(h2);                             % Amplitude resp. for FIR filter

error1 = amp1-amp;                          % Error for IIR filter
error2 = amp2-amp;                          % Error for FIR filter
```

```
mse1(1,j) = (error1*error1')/kk;          % MS error for IIR filter
mse2(1,j) = (error2*error2')/kk;          % MS error for FIR filter
clear a1 b1 b2                            % Be safe
end                                       % End iteration on filter order

semilogy(nn,rmse1,'+',nn,rmse2,'o')       % Plot errors
xlabel('Flter Order')                     % Label abscissa
ylabel('Amplitude Response Mean-Square Error')        % Label ordinate
legend('IIR','FIR')                       % Place legend on graph
```

The function for calculating the desired frequency response at all frequency points used to calculate the frequencey responses of the FIR and IIR filters follos.

```
function y=c9ce5a(r)

% The following five lines define the desired filter

B = 0.4;
r1 = 0.1;
r2 = 0.11;
r3 = 0.25;
r4 = 0.26;

y = 0;                          %  Default response

% The following if blocks establish the amplitude response
% at frequency r

if r<=r1
        y = 1;
end
if (r>r1)&(r<=r2)
        y = 1+(B-1)*((r1-r)/(r1-r2));
end
if (r>r2)&(r<=r3)
        y = B;
end
if (r>r3)&(r<r4)
        y = B-B*((r3-r)/(r3-r4));
end
```

Executing the program yields the following plot. It can clearly be seen that the mean-square error, for a given filter order, is larger for the FIR filter than for the IIR filter.

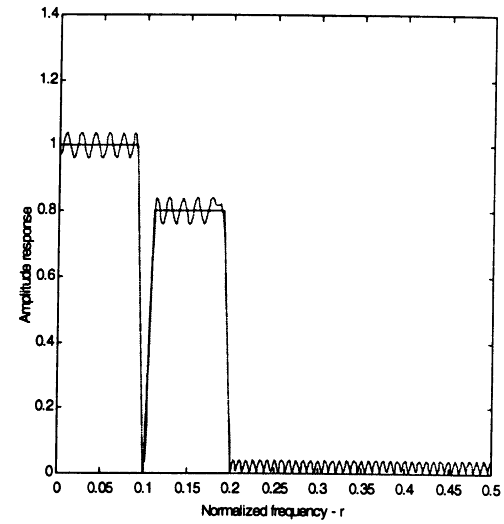


Computer Exercise 9-6

The amplitude response characteristic specified in Computer Exercise 9-6 is more difficult to satisfy than the amplitude response given in Examples 9-14 and 9-15 because of the notch at r = 0.1. As a matter of fact, the FIR realization requires that the notch be assigned a finite width since the FIR filter design program requires that the desired filter characteristic be given in bands of finite width. The following program, which determines the FIR filter, specifies the width of the notch to be 2e. The interested student should experiment with the notch width.

```
e = 0.001;                                        % Set width of notch
n = 120;                                          % Filter order
f = [0 0.18 0.2-e 0.2+e 0.22 0.38  0.4 1.0];      % Frequency vector
m = [1 1 0 0 0.8 0.8 0 0];                        % Magnitude vector
b = remez(n,f,m);                                 % Determine filter
[h,w] = freqz(b,1,512);                           % Compute freq. response
amp = abs(h);                                     % Compute amp. response
plot(f/2,m,w/(2*pi),amp)                          % Plot results
xlabel('Normalized frequency - r')                % Label x axis
ylabel('Amplitude response')                      % Label y axis
```

Execution of the program yields the result shown at the top of the following page.

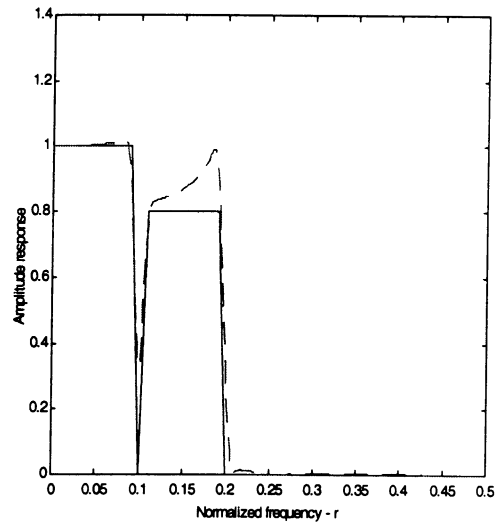


The magnitude of the ripple can be reduced by increasing the order of the filter.

The IIR filter is developed using the following program. Note that the IIR filter design algorithm does not require that the notch be assigned a finite width. The FIR algorithm requires frequency <u>bands</u> while the IIR algorithm requires only <u>breakpoints</u>.

```
e = 0.01;                                   % Set width of notch
n = 25;                                     % Filter order
f = [0 0.18 0.2 0.22 0.38  0.4 1.0];        % Frequency vector
m = [1 1 0 0.8 0.8 0 0];                    % Magnitude vector
[b,a] = yulewalk(n,f,m);                    % Compute H(z)
[h,w] = freqz(b,a,512);                     % Determine response
amp = abs(h);                               % Compute amp. response
plot(f/2,m,w/(2*pi),amp,'--')               % Plot result
xlabel('Normalized frequency - r')          % Label x axis
ylabel('Amplitude response')                % Label y axis
```

Also, for the IIR filter we have the following result.



Note that the IIR filter does a better job except for the relatively large error in the second passband. Also note that the IIR filter order is much lower than the FIR filter order.

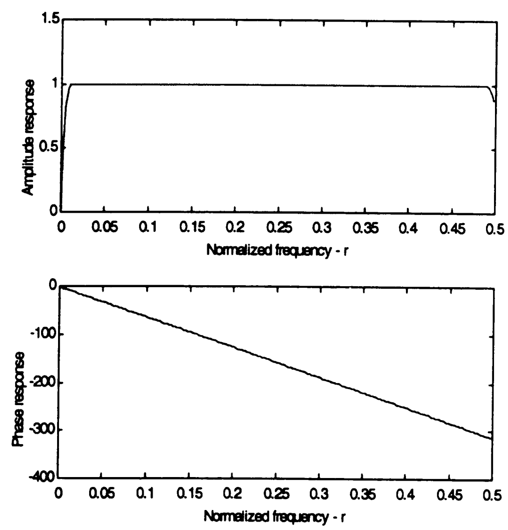
Computer Exercise 9-7

The MATLAB program for this Computer Exercise is shown below.

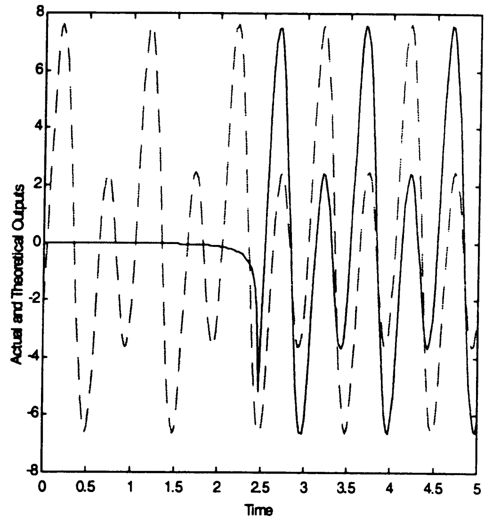
```
f = [0.02 0.98];                          % Frequency vector
a = [1 1];                                % Amplitude vector
h = remez(199,f,a,'Hilbert');             % Generate filter
[hf,w] = freqz(h,1,199);                  % Frequency response
amp = abs(hf);                            % Amplitude response
phase = unwrap(angle(hf));                % Phase response
subplot(2,1,1), plot(w/(2*pi),amp)        % Plot amplitude response
xlabel('Normalized frequency - r')        % Label x axis
ylabel('Amplitude response')              % Label y axis
subplot(2,1,2), plot(w/(2*pi),phase)      % Plot phase response
xlabel('Normalized frequency - r')        % Label x axis
ylabel('Phase response')                  % Label y axis
pause                                     % Pause to look

fs = 40;                                  % Sampling frequency
n = 0:200;                                % Vector of sample indices
T = 1/fs;                                 % Sampling period
nT = n*T;                                 % Vector of sampling times
x1 = 3*cos(2*pi*nT+pi/4);                 % First input term
x2 = 5*sin(4*pi*nT+pi/6);                 % Second input term
x = x1+x2;                                % Input signal
```

```
y = filter(h,1,x);                        % Computed output
yt1 = 3*cos(2*pi*nT+pi/4-pi/2);           % 1st term of theoretical output
yt2 = 5*sin(4*pi*nT+pi/6-pi/2);           % 2nd term of theoretical output
yt = yt1+yt2;                             % Theoretical output
subplot(1,1,1)                            % Restore plot window
plot(nT,y,nT,yt,'--')                     % Plot outputs
xlabel('Time')                            % Label x axis
ylabel('Actual and Theoretical Outputs')%Label y axis
```

Executing the program generates the amplitude and the phase responses. They follow.



We can see that the amplitude response and the phase response are consistent with those of a Hilbert transform network. After the pause, the computed output using the Hilbert transform network and the theoretical output are generated. Note from the MATLAB program that the theoretical output is generated by simply phase shifting input terms by an appropriate amount. The results follow. In the plot the theoretical output is shown by the dashed line and the output resulting from the Hilbert transform filter is shown by the solid line. Note the delay and the transient response of the Hilbert transform filter. Note also that the results agree except for the delay.



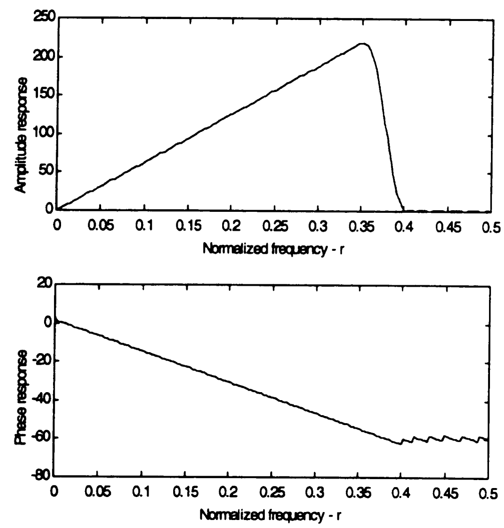
Computer Exercise 9-8

This Computer Exercise is identical to the preceding Computer Exercise except that the filter is a differentiator rather than an integrator. The MATLAB program is shown below.

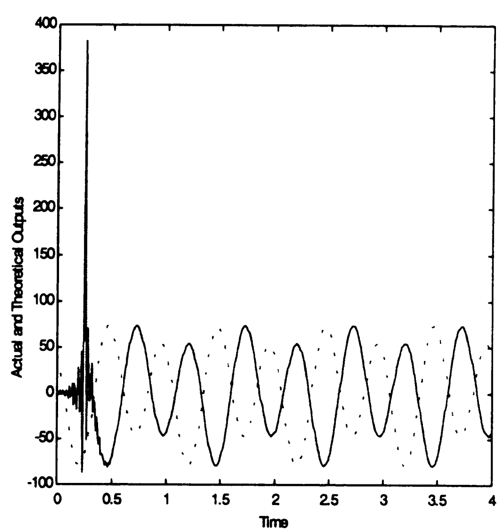
```
fs = 100;
f = [0 0.7 0.8 1];                                  % Frequency vector
a = [0 0.7*fs*pi 0 0];                              % Amplitude vector
h = remez(51,f,a,'differentiator');                 % Determine coefficients
[hf,w] = freqz(h,1,199);                            % Find freq. response
amp = abs(hf); phase = unwrap(angle(hf));           % Amp and phase response
subplot(2,1,1), plot(w/(2*pi),amp)                  % Plot amp. response
xlabel('Normalized frequency - r')                  % Label x axis
ylabel('Amplitude response')                        % Label y axis
subplot(2,1,2), plot(w/(2*pi),phase)                % Plot phase response
xlabel('Normalized frequency - r')                  % Label x axis
ylabel('Phase response')                            % Label y axis
pause                                               % Pause to look

n = 0:400;                                          % Vector of index values
T = 1/fs;                                           % Sampling period
nT = n*T;                                           % Form time vector
x1 = 3*cos(2*pi*nT+pi/4);                           % First term of input
x2 = 5*sin(4*pi*nT+pi/6);                           % Second term of input
x = x1+x2;                                          % Input

y = filter(h,1,x);                                  % Differentiation filter
yt1 = -3*(2*pi)*sin(2*pi*nT+pi/4);                  % First term
yt2 = 5*(4*pi)*cos(4*pi*nT+pi/6);                   % Second term
yt = yt1+yt2;                                       % Theoretical output
subplot(1,1,1)                                      % Reset plot window
```

```
plot(nT,y,'-',nT,yt,':')                    % plot outputs
xlabel('Time')                              % Label x axis
ylabel('Actual and Theoretical Outputs')    % Label y axis
```

Executing the program yields the amplitude and the phase responses shown.



We see that these correspond to a differentiator. After the pause the output using the differentiating filter and the theoretical output are generated as shown.



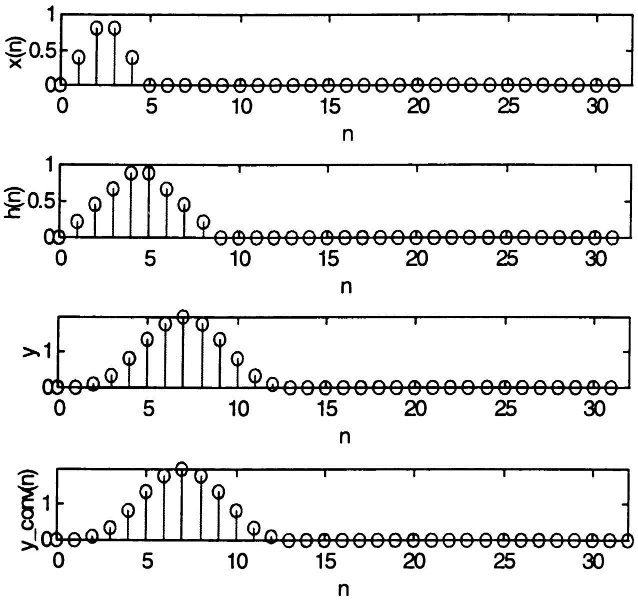
At first glance these may appear rather strange. Note the transient response of the differentiating filter and also note the delay. The reason for the delay should be clear from observation of the structure of the FIR filter or from the phase response. The "noise spike" on the first part of the filter output results because of the fact that the filter input is not continuous. Differentiating this discontinuity results in the "noise spike" and the "ringing". (Recall that the derivative of a discontinuity is an impulse function located at the point of the discontinuity and having a weight equal to the magnitude of the discontinuity.) This can, and should in most applications, be removed using a lowpass filter.

Computer Exercise 10-1

The following program will any combination of a square and a triangle. The circular convolution is compared with the linear convolution result.

```
% Computer Exercise 10-1; computes circular convolution
% between two short signals using the FFT
%
clg
I_x = 2;     % Input 1 for rectangle; 2 for triangle.
I_y = 2;     % Input 1 for rectangle; 2 for triangle.
NFFT = 32;   % Number of FFT points.
N1 = 5;      % Width in points of x(n).
N2 = 9;   % Width in points of y(n).
x = zeros(1,NFFT);
y = zeros(1,NFFT);
k =0:NFFT-1;
if I_x == 1
   x = pls_fn((k-(N1-1)/2)/(N1-1));
elseif I_x == 2
   x = trgl_fn((k-(N1/2))/(N1/2));
end
if I_y == 1
   h = pls_fn((k-(N2-1)/2)/(N2-1));
elseif I_y == 2
   h = trgl_fn((k-(N2/2))/(N2/2));
end
X = fft(x);
H = fft(h);
Y = X.*H;
y = ifft(Z);
y1 = conv(x,h);
kk = 0:2*(NFFT-1);
subplot(4,1,1), stem(k,x), axis([0 NFFT 0 1]),...
   xlabel('n'),ylabel('x(n)')
subplot(4,1,2), stem(k,h), axis([0 NFFT 0 1]),...
   xlabel('n'),ylabel('h(n)')
subplot(4,1,3), stem(k,real(y)), axis([0 NFFT 0 max(abs(y))]),...
   xlabel('n'),ylabel('y')
subplot(4,1,4), stem(kk,y1), axis([0 NFFT 0 max(abs(y1))]),...
   xlabel('n'),ylabel('y_conv(n)')
```

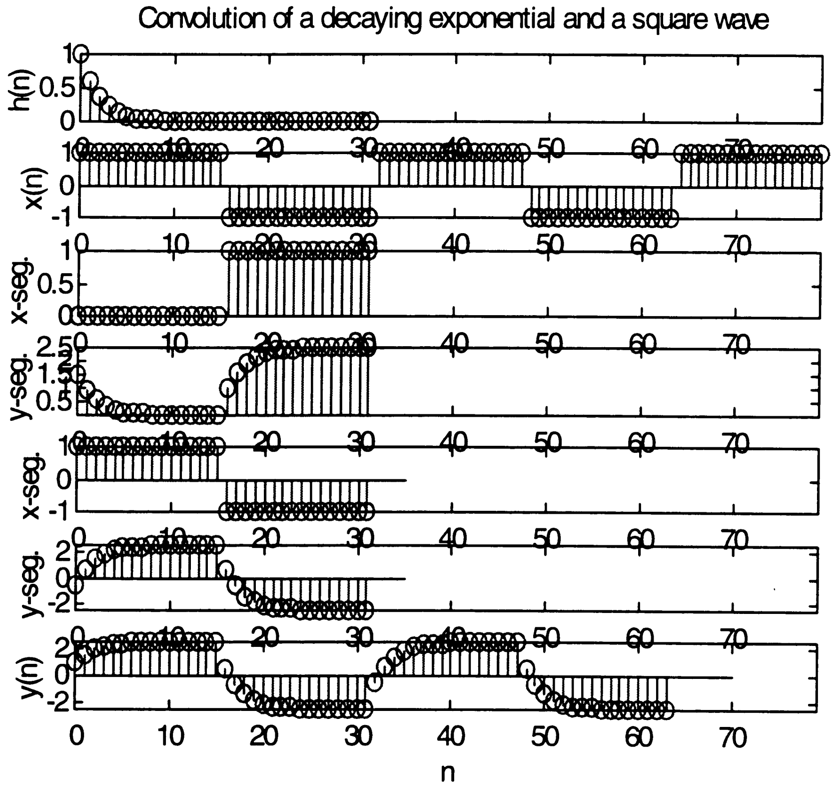
```
%       This function generates a triangle centered at zero
%       and extending from -1 to 1
%
function tri = trngl_fn(t)
tri = rmp_fn(t+1)-2*rmp_fn(t)+rmp_fn(t-1);
```



<u>**Computer Exercise 10-2**</u>

```
% Computer Exercise 10-2; computes convolution
% between long and short signal using the FFT
%
NFFT = 32;
N_seg = 5;
N_h =  2;       % Enter type of impulse response: 1 = square; 2 =
exponential
if N_h == 2
  alpha =  0.5;    %Enter alpha, the decay constant
end
N_x = 1;        % Enter type of signal: 1 = square; 2 = sinusoid
clg
y_t = [];
h = zeros(1,NFFT);
for n = 1:NFFT/2
  if N_h == 1
  h(n) = 1;
  elseif N_h == 2
  h(n) = exp(-alpha*(n-1));
  end
end
p = [0:NFFT-1];
l = [0:NFFT*N_seg/2-1];
subplot(7,1,1),stem(p,h),ylabel('h(n)'),axis([0 NFFT*N_seg/2-1 0 1]),...
if N_x == 1 & N_h == 1
  title('Convolution of a square pulse and a square wave')
  elseif N_x == 2 & N_h == 1
  title('Convolution of a square pulse and a sinewave')
  elseif N_x == 1 & N_h == 2
  title('Convolution of a decaying exponential and a square wave')
  elseif N_x == 2 & N_h == 2
  title('Convolution of a decaying exponential and a sinewave')
```

```
end
H = fft(h);
x = [];
if N_x == 1
   for i = 1:N_seg
   x_t = -(-ones(size(1:NFFT/2))).^i;
   x = [x x_t];
   end
elseif N_x == 2
   f_0 = 10;
   del_t = .01;
   for n = 1:NFFT*N_seg/2
   x(n) = sin(2*pi*f_0*(n-1)*del_t);
   end
end
subplot(7,1,2),stem(l,x),ylabel('x(n)'),axis([0 NFFT*N_seg/2-1 -1
1]),...
y = [];
for m = 1:N_seg-1
if m == 1
   x_1 = [zeros(size(1:NFFT/2)) x(1:NFFT/2)];
else
   x_1 = x((m-2)*NFFT/2+1:m*NFFT/2);
end
X_1 = fft(x_1);
Y_1 = H.*X_1;
y_1 = ifft(Y_1);
if m == 1
subplot(7,1,3),stem(p,x_1),ylabel('x-seg.'),axis([0 NFFT*N_seg/2-1
min(x_1) max(x_1) ]),...
subplot(7,1,4),stem(p,real(y_1)),ylabel('y-seg.'),...
   axis([0 NFFT*N_seg/2-1 min(real(y_1)) max(real(y_1)) ]),...
elseif m == 2
subplot(7,1,5),stem(p,x_1),ylabel('x-seg.'),...
   axis([0 NFFT*N_seg/2-1 min(x_1) max(x_1)]),...
subplot(7,1,6),stem(p,real(y_1)),ylabel('y-seg.'),...
   axis([0 NFFT*N_seg/2-1 min(real(y_1)) max(real(y_1))]),...
end
y = [y y_1(NFFT/2+1:NFFT)];
end
q = [0:(N_seg-1)*NFFT/2-1];
subplot(7,1,7),stem(q,real(y)),xlabel('n'),ylabel('y(n)'),...
   axis([0 N_seg*NFFT/2-1 min(real(y)) max(real(y))]),...
```

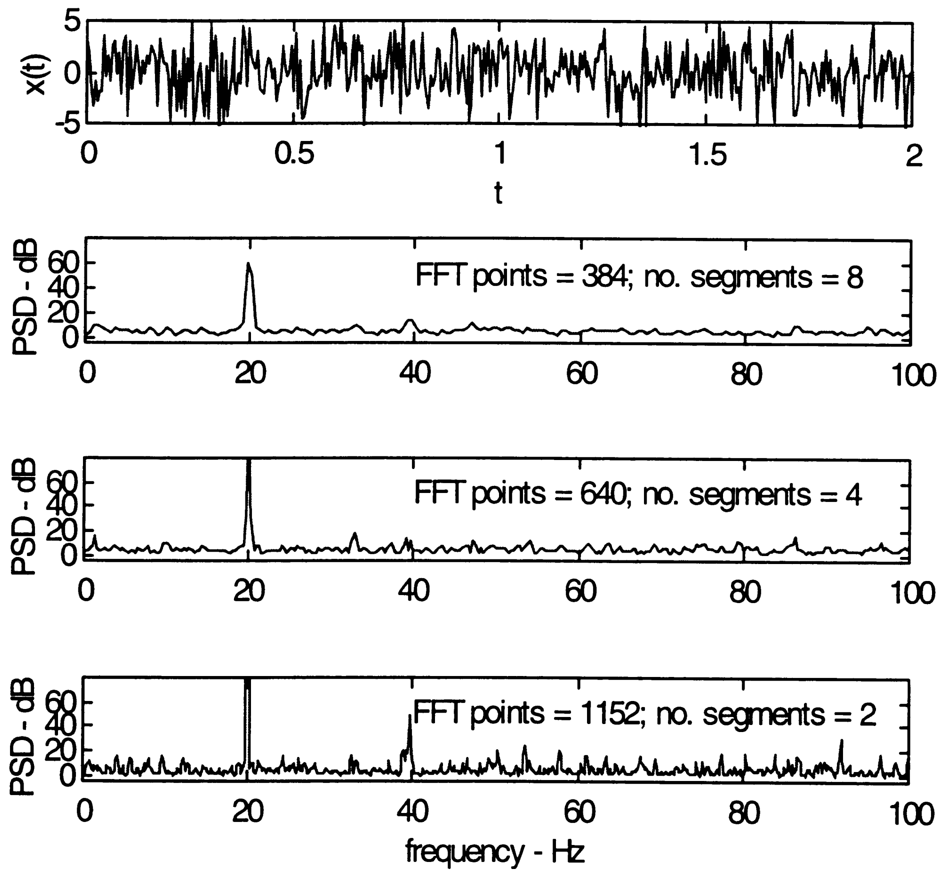


Computer Exercise 10-3

[P, F] = PSD(x, NFFT, f_samp,window,N_overlap) gives the psd of the vector x by using an FFT of NFFT points, with sampling frequency f_samp (determines scale on output), a Hanning window of length window, and each FFT overlapped by N_overlap.

```
% This program utilizes the psd function to plot the
% power spectral density of a cosine signal in Gaussian
% noise for various combinations of FFT size and number of
% segments averaged
%
clg
I_noise = 1;        % Enter 1 for Gaussian noise of variance 5 added to
signal
N_tot = 2048;
N_seg_0 = 8;
N_overlap = 128
k = 1:N_tot;
f_0 = 20;
del_t = 0.005;
f_samp = 1/del_t;
t = del_t*(k-1);
max_t = max(t);
x = cos(2*pi*f_0*(k-1)*del_t);
n = sqrt(5)*randn(size(x));
if I_noise == 1
```

```
   x = x+n;
else
   x = x;
end
subplot(4,1,1),plot(t,x),axis([0 2 -5 5]),xlabel('t'),ylabel('x(t)')
for l = 1:3
N_seg = N_seg_0/2^(l-1);
NFFT = N_tot/N_seg+N_overlap;
[P, F] = PSD(x,NFFT,f_samp,[],N_overlap);
subplot(4,1,l+1),plot(F,P),axis([0 100 -5 80]),ylabel('PSD - dB'),...
   text(40, 50, ['FFT points = ',num2str(NFFT),'; no. segments =
',num2str(N_seg)])
   if l == 3
      xlabel('frequency - Hz')
   end
end

N_overlap =
   128
```



Note that the overlap of 128 points has given increased resolution over the example in the text, but it also appears to have introduced some artifacts in the last spectral plot.

Computer Exercise 10-4

See Example 10-16 of the text for a MATLAB program in which various parameters can be adjusted.

Computer Exercise 10-5

See Example 10-13. The student can adapt this program for the additional window functions.

Computer Exercise E-1

The solution to this Computer Exercise makes use of the fact that the denominator of the normalized Butterworth filter transfer function is the Butterworth polynomial. We therefore have the MATLAB program shown below.

```
for n=1:6
        [b,a] = butter(n,1,'s');  % Generate den and num polynomials
        a                         % Display den (Butterworth polynomial)
end
```

Execution of the program yields the following output.

```
a =

     1     1


a =

    1.0000    1.4142    1.0000


a =

    1.0000    2.0000    2.0000    1.0000


a =

    1.0000    2.6131    3.4142    2.6131    1.0000


a =

    1.0000    3.2361    5.2361    5.2361    3.2361    1.0000
```

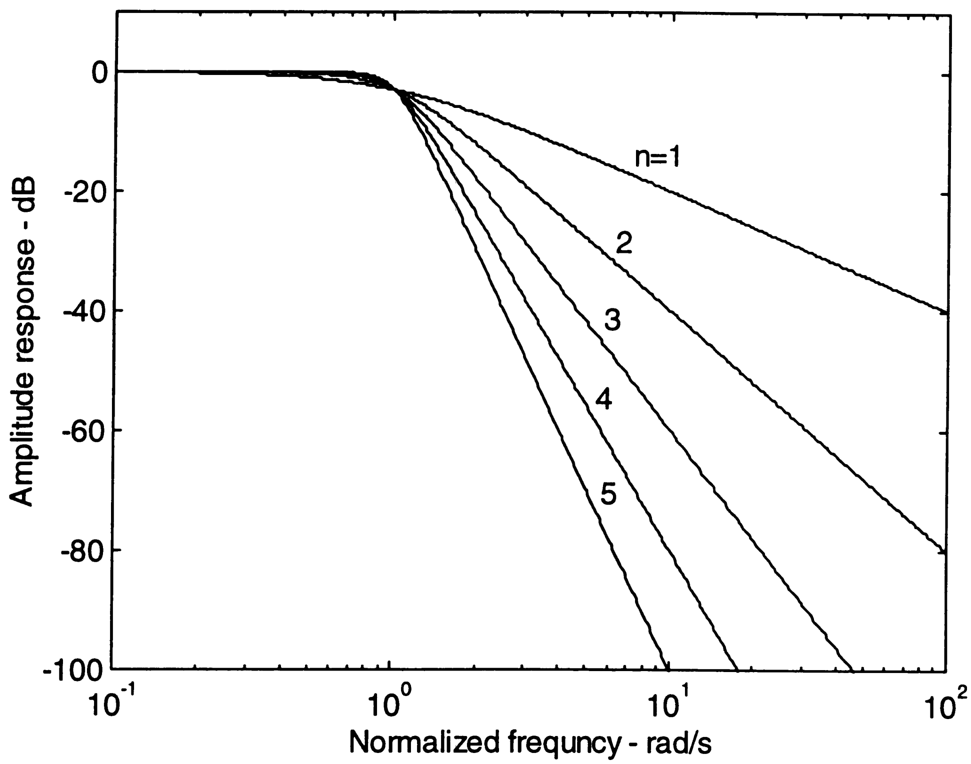
We see that we have agreement with Table E-1. By using the MATLAB **format** statement, the polynomial coefficients can be displayed to a greater precision if required.

Computer Exercise E-2

The solution of this Computer Exercise is accomplished using the MATLAB code illustrated at the top of the following page.

```
ord = [5 4 3 2 1];                          % Vector of desired orders
w = logspace(-1,2,500);                     % Frequency samples
for j=1:length(ord)                         % For loop beginning
        [b,a] = butter(ord(j),1,'s');       % Define filter
        if j==2, hold on, end               % Hold to overlay
        h = freqs(b,a,w);                   % Compute frequency response
        amp = abs(h);                       % Compute amplitude response
        ampdb = 20*log10(abs(h));           % Express in dB
        semilogx(w,ampdb);                  % Plot on proper scale
end                                         % End for loop
hold off                                    % No more overlays
axis([0.1 100 -100 10])                     % Scale to suit
xlabel('Normalized frequncy - rad/s')       % Label x axis
ylabel('Amplitude response - dB')           % Label y axis
```

Execution of the program yields the following output. The annotations on the plot were placed using the MATLAB command **gtext**.

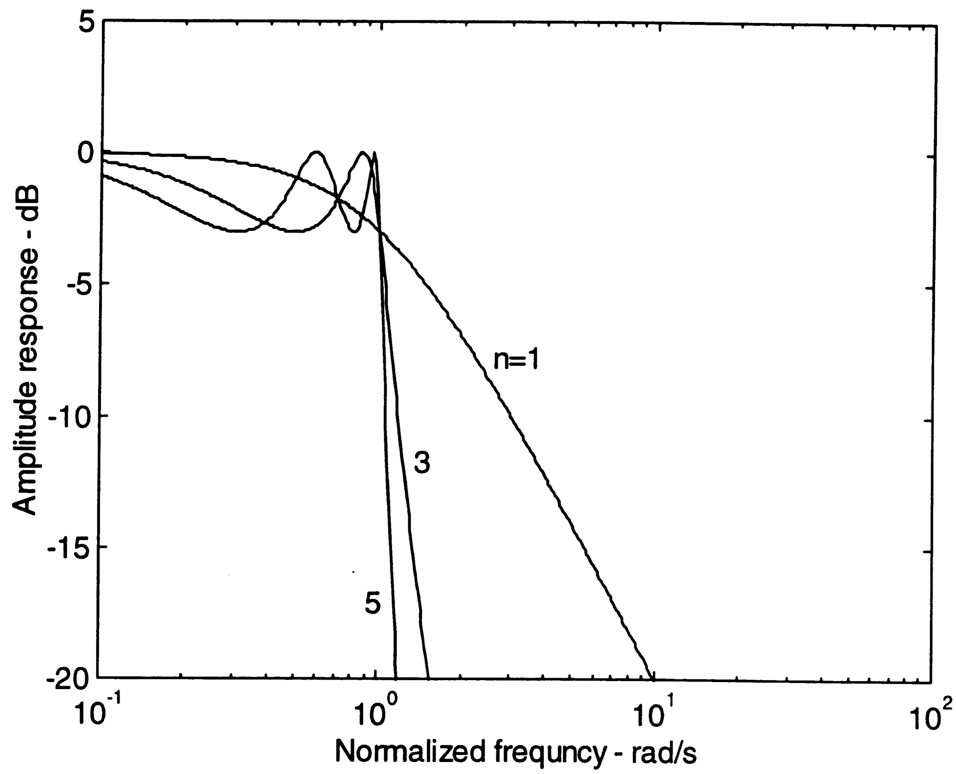


<u>Computer Exercise E-3</u>

This Computer Exercise is similar to the previous problem except that more flexibility is achieved by using a matrix for entering the data. This is especially useful for a Chebyshev filter since a Chebyshev filter involves more parameters than a Butterworth filter. Note that the vector shown plots amplitude responses for Chebyshev I filters having orders of 1, 3 and 5. All three filters have 3 dB passband ripple.

```
ord = [5 3 1; 3 3 3];                              % Input data

w = logspace(-1,2,500);                            % Set scale for plot
for j=1:length(ord)                                % Start of for loop
    [b,a] = cheby1(ord(1,j),ord(2,j),1,'s');       % Design filter
    if j==2, hold on, end                          % Hold for overlays
    h = freqs(b,a,w);                              % Frequency response
    amp = abs(h);                                  % Amplitude response
    ampdb = 20*log10(abs(h));                      % Amp. response in dB
    semilogx(w,ampdb);                             % Plot amp. responses
end                                                % End of for loop
hold off                                           % No more plots
axis([0.1 100 -20 5]);                             % Set axes
xlabel('Normalized frequncy - rad/s')              % Label x axis
ylabel('Amplitude response - dB')                  % Label y axis
```

Execution of the program yields the following plot. As was the case with the previous example, the annotations on the plot (filter orders) were placed on the plot using the MATLAB command **gtext**.



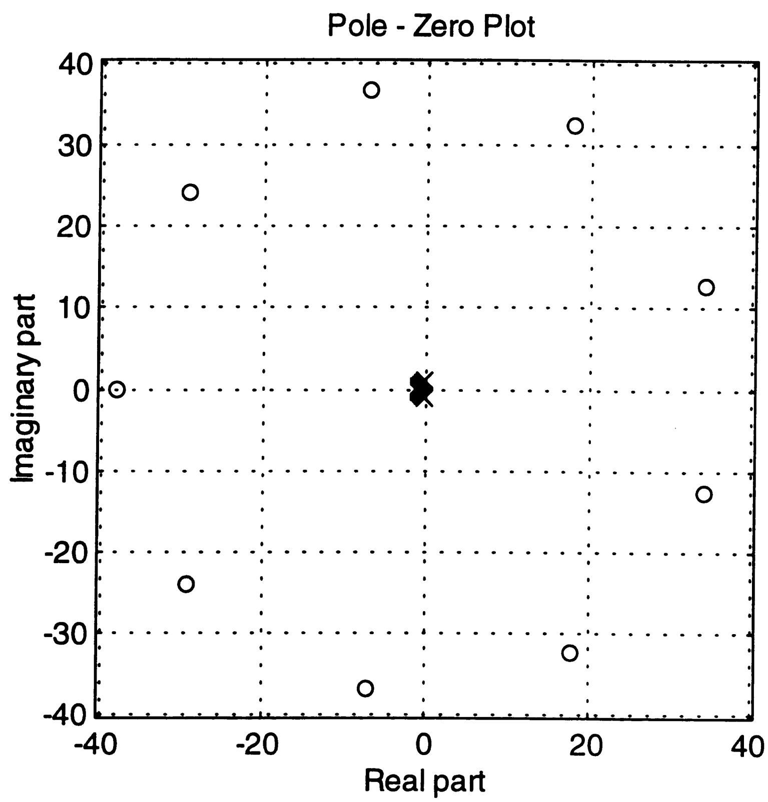
Computer Exercise E-4

All parts of this problem are solved using a single MATLAB program. It was decided for the solution shown here to have the maximum and minimum values of the real and imaginary parts of the poles and zeros determined automatically. Important parts of the display can then be selected for additional examination using the **zoom** command. The program is shown below.

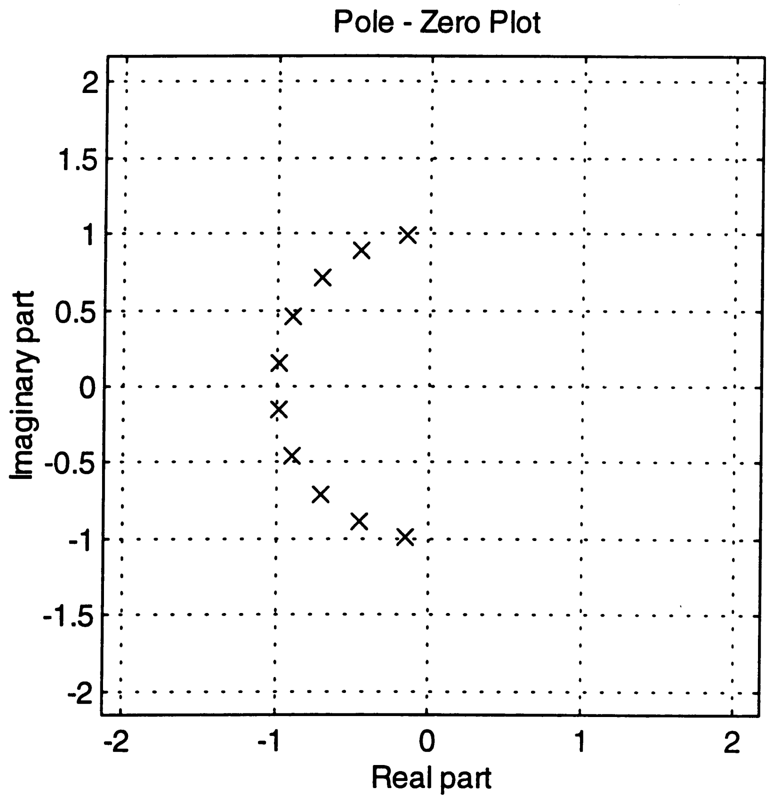
```
n = 10;                                                  % Ptototype filter order

stg1 = input('Enter lowpass or nonlowpass > ','s');  % Lp. or nonlp.?
if strcmp(stg1,'nonlowpass')
stg2 = input('Enter bandpass or notch > ','s');      % Bandpass or notch?
end

Wn = 1;                                                  % Bandwidth of prototype
Wo = 2*pi*10;                                            % Center frequency
Wb = 2*pi*2;                                             % Bandwidth
[b,a] = butter(n,Wn,'s');                                % Design prototype

if strcmp(stg1,'lowpass')
        bt = b;
        at = a;
else
if strcmp(stg2,'notch')
        [bt,at] = lp2bs(b,a,Wo,Wb);                      % Define notch filter
else
        [bt,at] = lp2bp(b,a,Wo,Wb);                      % Define bandpass filter
end
end

p = roots(at);                                           % Determine ploes
por = real(p);                                           % Real part of poles
poi = imag(p);                                           % Imag. part of poles
z = roots(bt);                                           % Determine zeros
zr = real(z);                                            % Real part of zeros
zi = imag(z);                                            % Imag. part of zeros

% The following lines of code provide axis scaling for plot

mxzr = max(zr);
mxzi = max(zi);
mxpr = max(por);
mxpi = max(poi);
mx = 1.1*max([mxzr mxzi mxpr mxpi]);

% Plot results

plot(por,poi,'X')                                        % Plot poles
axis([-mx mx -mx mx])                                    % Set axis
axis square                                              % Make plot square
hold on                                                  % Hold for more data
plot(zr,zi,'O')                                          % Plot zeros
hold off                                                 % Done plotting
grid                                                     % Plot grid
xlabel('Real part')                                      % Label x axis
ylabel('Imaginary part')                                 % Label y axis
title('Pole - Zero Plot')                                % Title plot
```

In responding to the prompts, care must be used to type correctly since no error traps are included.

If we execute the 'lowpass' option, the following plot results.



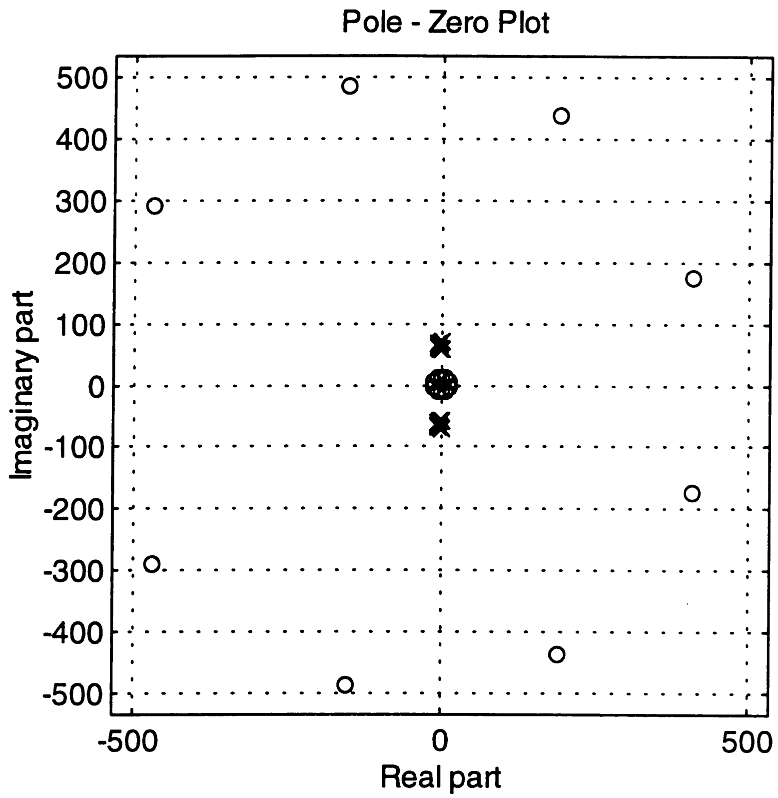
Note that a "classical' nth order Butterworth filter has an nth order zero at infinity and that MATLAB maps these zeros to a large non-infinite value. We know that the nth order Butterworth filter has n poles distributed on the left-hand part of the unit circle. Using the MATLAB **zoom** command, these poles can be clearly seen as illustrated below.



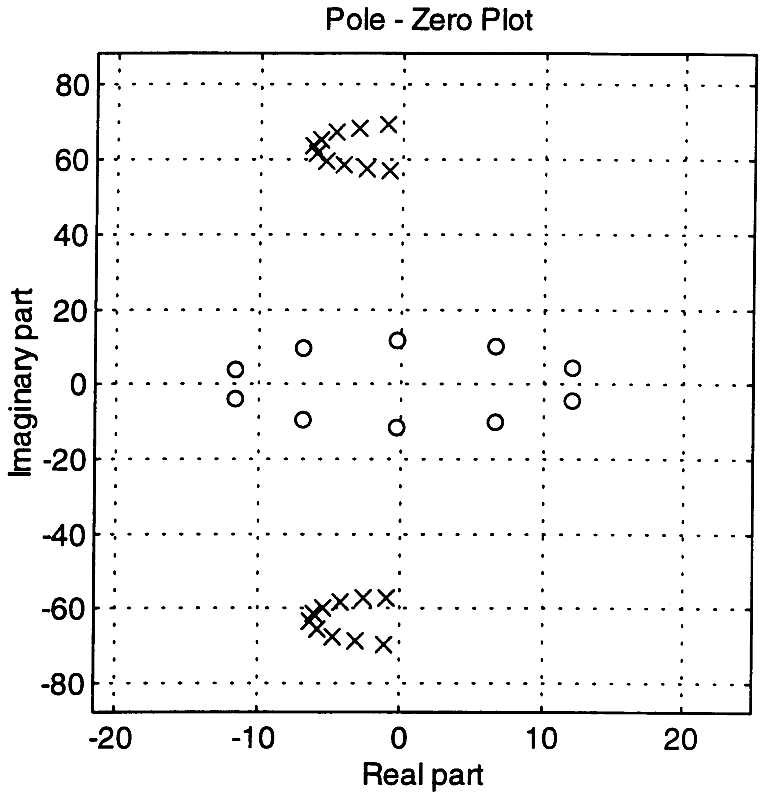
We now execute the program again to examine the bandpass filter. The response to the prompts are shown below.

```
Enter lowpass or nonlowpass > nonlowpass

Enter bandpass or notch > bandpass
```

This gives the plot shown at the top of the following page.



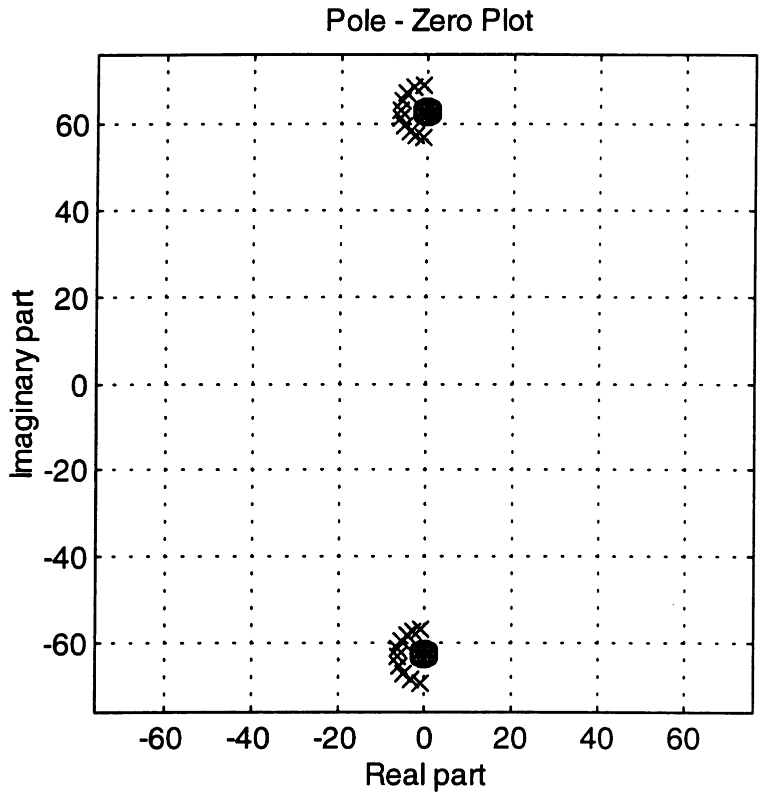
Using the **zoom** command results in the plot shown at the top of the following page.



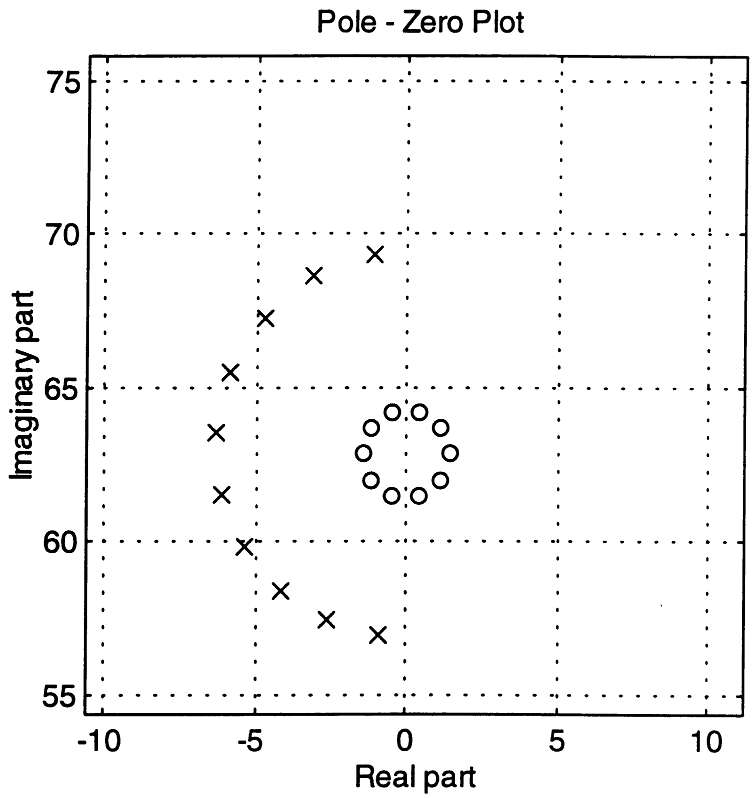
Both the poles and the zeros fall on circles having a radius determined by the bandwidth of the filter. These circles appear elliptical on the above plot because of the scaling.

We now consider the notch, or band reject, filter. Executing the program again with the following prompts gives the plot shown at the top of the following page.

```
Enter lowpass or nonlowpass > nonlowpass

Enter bandpass or notch > notch
```



It is interesting to examine one of the clusters of poles and zeros. Using the **zoom** command results in the following plot.



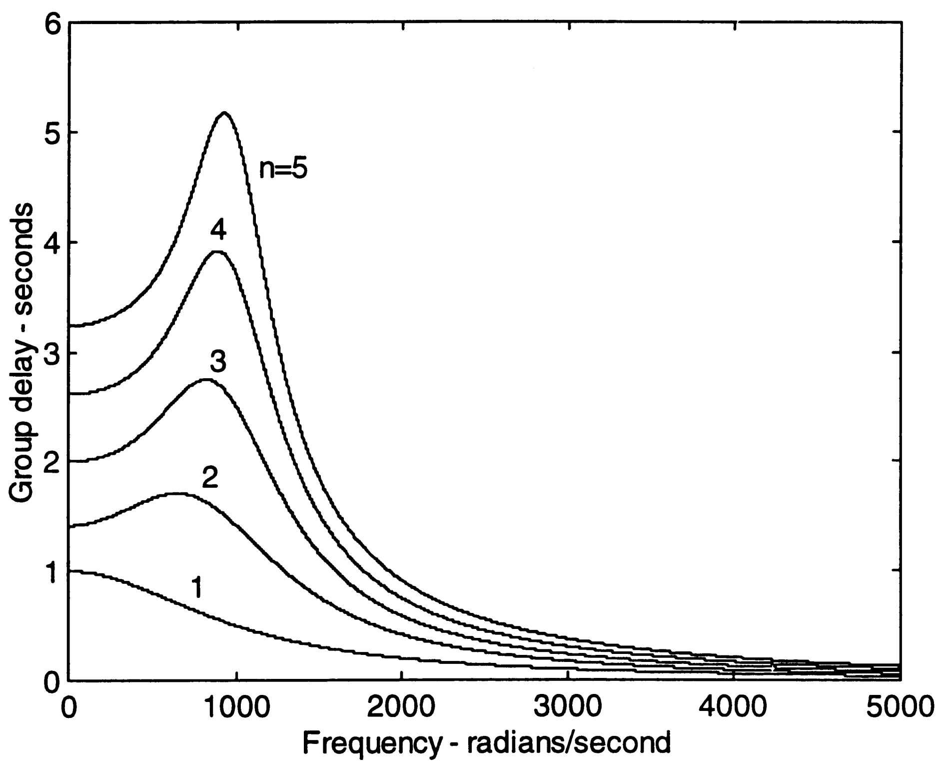
Computer Exercise E-5

This Computer Exercise is straightforward. The MATLAB code is shown below. There are basically three points of confusion that should be considered. First, in evaluating the equation for group delay, note that the first, m = 0, term in the sumand is zero for all values of frequency since each frequency value is raised to the zero value. While not strictly necessary, it helps to evaluate this term separately and, therefore, the sum runs from 1 to n-1 rather than from 0 to n-1. The other two points relate to the differentiation of the phase response. Note that the phase is unwrapped, using the **unwrap** command prior to differentiation. In addition, numerical differentiation reduces the number of points in the vector by one. Thus the "answer matrix" is resized.

```
n = [5 4 3 2 1];                       % Vector of filter orders
ans = zeros(5000,5);                   % Initialize ans matrix
wfinal = 5;                            % Final value of group delay
w = linspace(0,wfinal,5000);           % Select frequency increments
delw = wfinal/5000;                    % Increment for differentiation

for j=1:length(n)                      % Loop to iterate on order
nj = n(1,j);                           % Select filter order
term1 = 1 ./ (1 + w .^ (2*nj));        % Multiplier of sum
sum = 1/sin(pi/(2*nj));                % m=0 term in sum
   for m=1 :nj-1                       % Iterate sum
   term2 = sin((2*m+1)*(pi/(2*nj)));   % Term 2
   term3 = (w .^ (2*m));               % Term 3
   term4 = term3 ./ term2;             % Term 4
```

```
   sum = sum + term4;                       % Sum
end                                         % End of for loop
tgw = sum .* term1;                         % Group delay
ans(:,j) = tgw';                            % Make column vector
end                                         % End of outside loop
plot(ans,'w')                               % Plot results
xlabel('Frequency - radians/second')        % Label x axis
ylabel('Group delay - seconds')             % Label y axis

pause                                       % Pause to look

ans1 = zeros(4999,5);                       % Resize result matrix
for j=1:length(n)                           % Iterate on the filter order
        [b,a] = butter(n(j),1,'s');         % Generate Butterworth prototype
        h = freqs(b,a,w);                   % Frequency response
        phase = unwrap(angle(h));           % Generate and unwrap phase
        tgd1 = -(diff(phase)/delw);         % Differiate phase
        ans1(:,j) = tgd1';                  % Build matrix of results
end                                         % End of for loop
plot(ans1,'w')                              % Plot results
xlabel('Frequency - radians/second')        % Label x axis
ylabel('Group delay - seconds')             % Label y axis
```

Executing the program yields the two estimates of the group delay. The first result, found by evaluating equation equation (E-17) follows.



The annotations showing the filter order, were placed on the graph using the **gtext** command. The next result, numerical differentiation of the phase, which follows the pause is identical and is therefore not plotted.