Ziemer
Tranter
Fannin

SIGNALS & SYSTEMS *Continuou. and Discret*

PRENTICE HALL

Fourth Edition

SIGNALS & SYSTEMS

Continuous and Discrete

Rodger E. Ziemer
William H. Tranter
D. Ronald Fannin

Signals and Systems: Continuous and Discrete

FOURTH EDITION

Rodger E. Ziemer
University of Colorado—Colorado Springs

William H. Tranter
Virginia Polytechnic Institute and State University

D. Ronald Fannin
University of Missouri—Rolla

Prentice Hall
PRENTICE HALL
Upper Saddle River, NJ 07458

CHAPTER 1

Signal and System Modeling Concepts

1-1 Introduction

This book deals with *systems* and the interaction of *signals* in systems. A system, in its most general form, is defined as a combination and interconnection of several components to perform a desired task.† Such a task might be the measurement of the acceleration of a rocket or the transmission of a message from New York to Los Angeles. The measurement of the acceleration might make use of visual observation of its position versus time. An equally unsophisticated solution to the message delivery problem might use a horse and rider. Obviously, more complex solutions are possible (and probably better). Note, however, that our definition is sufficiently general to include them all.

We will be concerned primarily with *linear* systems. Such a restriction is reasonable because many systems of engineering interest are closely approximated by linear systems and very powerful techniques exist for analyzing them. We consider several methods for analyzing linear systems in this book. Although each of the methods to be considered is general, not all of them are equally convenient for any particular case. Therefore, we will attempt to point out the usefulness of each.

A *signal* may be considered to be a function of time that represents a physical variable of interest associated with a system. In electrical systems, signals usually represent currents and voltages, whereas in mechanical systems, they might represent forces and velocities (or positions).‡ In the example mentioned above, one of the signals of interest represents the acceleration, but this could be integrated to yield a signal proportional to velocity. Since electrical voltages and currents are relatively easy to process, the original signal representing acceleration, which is a mechanical signal, would probably be converted to an electrical one before further *signal processing* takes place. Examples illustrating these remarks will be given in the next section.

Just as there are several methods of systems analysis, there are several different ways of representing and analyzing signals. They are not all equally convenient in any particular situation. As we study methods of signal representation and system analysis we will attempt to point out useful applications of the techniques.

So far, the discussion has been rather general. To be more specific and to fix more clearly the ideas we have introduced, we will expand on the acceleration measurement and the message delivery problems already mentioned.

†The Institute of Electrical and Electronics Engineers dictionary defines a system as "an integrated whole even though composed of diverse, interacting structures or subjunctions."

‡More generally, a signal can be a function of more than one independent variable, such as the pressure on the surface of an airfoil, which is a function of three spatial variables *and* time.

1-2 Examples of Systems

EXAMPLE 1-1

An accelerometer, consisting of a spring-balanced weight on a frictionless slide, is shown schematically in Figure 1-1. It is to be used to measure the longitudinal acceleration of a rocket whose acceleration profile is shown in Figure 1-2a. (a) If the weight used to indicate acceleration is 2 g, determine the spring constant K such that the departure of the weight from its equilibrium position at the maximum acceleration of 40 m/s², achieved at $t = 72$ s from launch, is 1 cm. (b) If a minimum increment of 0.5 mm of movement by the weight can be detected, to what minimum increment of acceleration does this correspond? (c) Derive the profile for the longitudinal velocity of the rocket and sketch it.

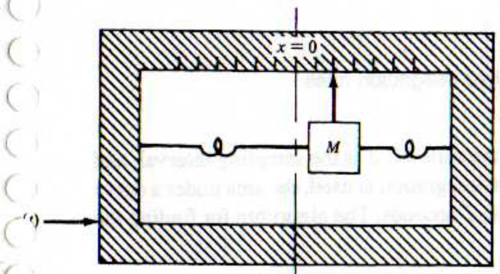


FIGURE 1-1. System for measuring the acceleration of a rocket.

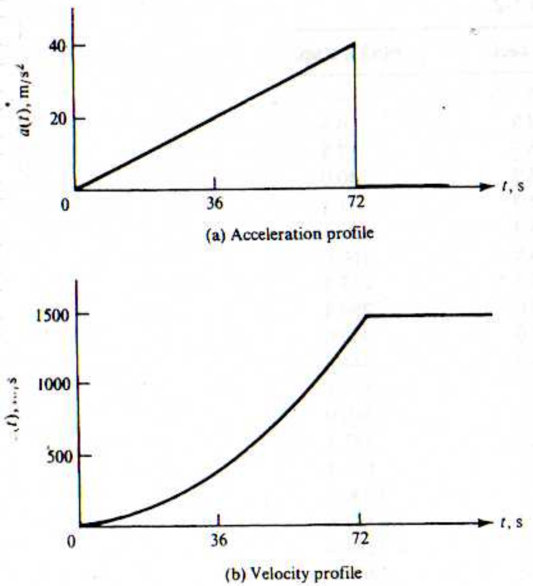


FIGURE 1-2. Acceleration and velocity profiles for rocket.

Solution:

(a) The force, Ma, on the weight due to acceleration is balanced by the force, Kx, due to the spring tension. Thus

$$Ma = Kx \tag{1-1}$$

where x is the departure from the equilibrium position, taken to be $x = 0$, M is the mass of the weight, K is the spring constant, and a represents acceleration of the rocket. Solving for K, we obtain

$$K = \frac{Ma_{max}}{x_{max}} = \frac{(0.002 \text{ kg})(40 \text{ m/s}^2)}{0.01 \text{ m}} = 8 \text{ kg/s}^2 \tag{1-2}$$

(b) With $\Delta x_{min} = 0.5$ mm $= 0.0005$ m, we have

$$\Delta a_{min} = \frac{K\,\Delta x_{min}}{M} = \frac{(8)(0.0005)}{0.002} = 2 \text{ m/s}^2 \tag{1-3}$$

(c) For $t > 0$, let the acceleration as a function of time be represented as†

$$a(t) = \begin{cases} \alpha t, & t \le t_0 \\ 0, & t > t_0 \end{cases} \tag{1-4}$$

where $t_0 = 72$ s and $\alpha = 5/9$ m/s³ for this example. The velocity is obtained by integration of $a(t)$, which yields

$$v_r(t) = \begin{cases} \alpha t^2/2, & t \le t_0 \\ \alpha t_o^2/2, & t > t_0 \end{cases} \tag{1-5}$$

Using the given value for α, the velocity at $t = t_0$ (burnout) is

$$v_r(72) = \frac{5}{9}\frac{(72)^2}{2} = 1{,}440 \text{ m/s} = 4{,}724 \text{ ft/s} = 3{,}221 \text{ mi/hr} \tag{1-6}$$

The velocity as a function of time is shown in Figure 1-2b.

EXAMPLE 1-2

To determine velocity by means of the accelerometer of the previous example, an operational amplifier integrating circuit is used as shown in Figure 1-3. The wiper of the potentiometer at the input is tied to the accelerometer weight and can provide a maximum input voltage of 0.1 V depending on whether the acceleration is positive or negative. The operational amplifier is ideal (infinite input resistance and infinite gain). However, its maximum output is limited to 10 V. Choose RC such that the operational amplifier output will not be overdriven for the maximum velocity expected (assume $R_p << R$).

Solution: Kirchhoff's current law at the node joining R, C, and the amplifier input can be expressed as

$$\frac{v_1 - v_s}{R} + C\frac{d}{dt}(v_1 - v_0) = 0 \tag{1-7}$$

†But for a few exceptions, the signals in this chapter are assumed to be zero for $t < 0$.

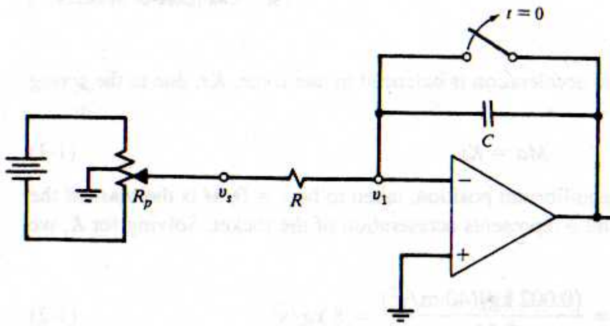


FIGURE 1-3. Integrator circuit used to determine velocity from the accelerometer analyzed in Example 1-1.

However, $v_1 \approx 0$ at the negative input of the operational amplifier since its infinite gain constrains voltages at its inverting and noninverting inputs to be approximately equal, and the positive input is grounded. Thus Kirchhoff's current law equation, when rearranged, simplifies to

$$v_0(t) = -\frac{1}{RC}\int_0^t v_s(\lambda)\, d\lambda \tag{1-8}$$

where the capacitor is assumed initially uncharged. From the previous example, we take the voltage at the potentiometer wiper to be

$$v_s(t) = \beta t, \qquad 0 \le t \le t_0 \tag{1-9}$$

where $v_s(t_0) = 0.1$ V, giving $\beta = 0.1/72 \approx 1.4 \times 10^{-3}$ V/s. Thus, at t_0, the operational amplifier output voltage is

$$v_0(t_0) = -\frac{1}{RC}\frac{\beta t_0^2}{2} \tag{1-10}$$

Setting this equal to -10 V, which is the constraint imposed by the operational amplifier at its output, and solving for RC, we obtain

$$RC = \frac{\beta t_0^2}{20} = \frac{(0.1/72)(72)^2}{20} = 0.36 \text{ s} \tag{1-11}$$

For $R = 50$ kΩ, the required value of C is 7.2 μF.

EXAMPLE 1-3

Another way to carry out the integration operation of the previous example (or any signal processing function, for that matter) is by sampling the voltage analog to the acceleration signal each T seconds, quantizing the samples so that they can be represented numerically, and performing the integration numerically. It will be shown in Chapter 8 that all the information present in a signal can be represented by sample values taken sufficiently often if the signal is *bandlimited* (a term to be precisely defined later). Two simple integration methods that might be used are the rectangular and trapezoidal rules. These are illustrated in Figure 1-4. The former approximates the area under a curve as the sum of a series of "boxcars" or rectangular areas, and can be carried out as the algorithm

$$\hat{v}[(n+1)T] = \hat{v}(nT) + Ta[(n+1)T] \tag{1-12}$$

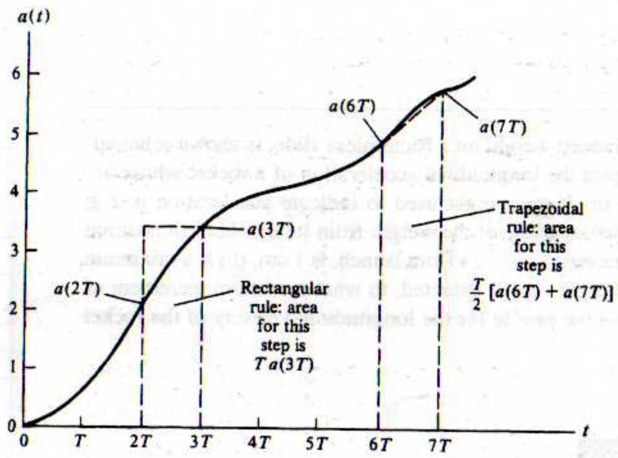


FIGURE 1-4. Illustration of discrete-time integration rules.

where $\hat{v}(nT)$ is the approximation to the velocity at sampling time nT, T is the sampling interval, and $a(nT)$ is the acceleration at sampling time nT. If trapezoidal integration is used, the area under a curve is approximated as a sum of contiguous trapezoids of width T seconds. The algorithm for finding the velocity from the acceleration in this case is given by

$$\hat{v}[(n+1)T] = \hat{v}(nT) + \{a(nT) + a[(n+1)T]\}(T/2) \tag{1-13}$$

TABLE 1-1
Numerical Results for Example 1-2

n	nT, s	$v(nT)$, rect.	$v(nT)$, trap.
0	0	0	0
1	4	8.9	4.4
2	8	26.7	17.8
3	12	53.3	40.0
4	16	88.9	71.1
5	20	133.3	111.1
6	24	186.7	160.0
7	28	248.9	217.8
8	32	320.0	284.4
9	36	400.0	360.0
10	40	488.9	444.4
11	44	586.7	537.8
12	48	693.3	640.0
13	52	808.9	751.1
14	56	933.3	871.1
15	60	1066.7	1000.0
16	64	1208.9	1137.8
17	68	1360.0	1284.4
18	72	1520.0	1440.0

Note that these algorithms are *recursive* in that the next approximate value for the velocity is computed from the old approximate value plus the sample value for the acceleration. The recursive structure can be removed by writing the equations for $n = 0, 1, 2, \ldots$ and doing a substitution of the one for $n = 0$ into the one for $n = 1$, then this result into the one for $n = 2$, and so on. In the case of the trapezoidal rule, two sample values for the acceleration are used—the present one and the immediate past one. Note, also, that the T in the argument of the various sampled signals is unnecessary—the index n is the independent variable. More will be said about integration algorithms in Chapter 9. To finish this example, we give a table of values for both algorithms. Note that for the acceleration profile shown in Figure 1-2, the trapezoidal rule gives exact results. Had the acceleration profile not been linear, this would not have been the case.

The starting point of any systems analysis or design problem is a *model* which, no matter how refined, is *always an idealization of a real-world (physical) system.* Hence the result of any systems analysis is an idealization of the true state of affairs. Nevertheless, if the model is sufficiently accurate, the results obtained will portray the operation of the actual system sufficiently accurately to be of use.

The previous examples illustrate the concept of a system and the *design* of systems to accomplish desired tasks. Each example involved the concept of a *signal.* In Example 1-1, the signal was the displacement of the acceleration measuring weight. This was coupled to the operational amplifier system of Example 1-2 to produce *voltage signals*, one proportional to acceleration and the other proportional to velocity.

Another concept demonstrated by the first two examples is that of a subsystem. When taken separately, we can speak of each as a *system*, whereas both taken together also compose a system. To avoid confusion in such cases, we refer to the separate parts (accelerometer and signal-processing integrator in this case) as *subsystems.* The third example illustrated the idea of *digital signal processing,* a concept explored later in the book.

After considering one more example, which is somewhat different from the first three, the remainder of this chapter will be concerned with an introduction to *useful signal models.* In Chapter 2, methods for describing and analyzing systems in the *time domain* will be examined. Throughout the rest of the book we consider other methods of signal and system analysis.

EXAMPLE 1-4 Communications Link

Figure 1-5 is a pictorial representation of a two-way communications link as might exist, for example, between New York and Los Angeles. It might consist entirely of earth-based links such as wire lines and microwave links. Alternatively, a relay satellite could be employed. Regardless of the particular mechanization used, the systems analyst must decide on the most important aspects of the physical system for the application and attempt to represent them by a model.

Let us suppose that the link employs a synchronous-orbit satellite repeater, stationed 35,784 km above the earth's equator. At this altitude the satellite is stationary with respect to the earth's surface since the period of the satellite's orbit is equal to one day. Let us also suppose that the analyst is concerned primarily about echoes in the system due to reflections of the electromagnetic-wave carriers at the satellite and at the receiving ground station. Assume that the transmitter and receiver are equidistant at a slant distance $d = 40{,}000$ km from the satellite, and that the delay (up or down) is

$$\tau = \frac{d}{c} = \frac{(40{,}000\ \text{km})(1000\ \text{m/km})}{3 \times 10^8\ \text{m/s}} = 0.13\ \text{s}$$

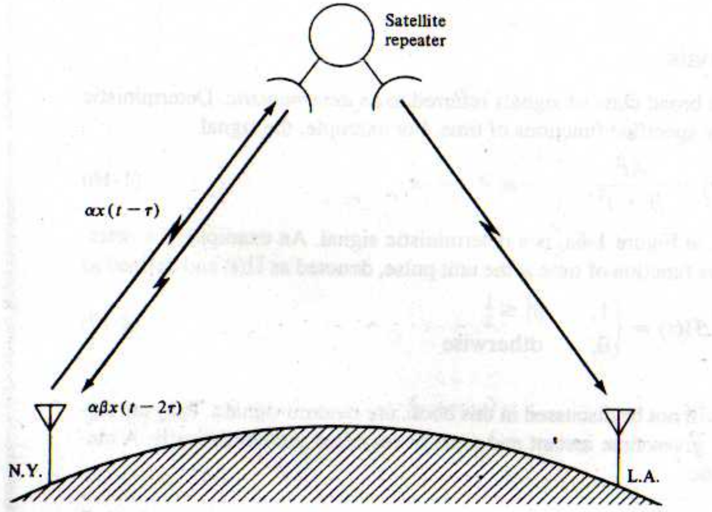


FIGURE 1-5. Satellite communications link.

where $c = 3 \times 10^8$ m/s is the velocity of electromagnetic propagation in free space. The slant distance d accounts for the satellite being located above the earth's equator midway between the ground stations, each of which is assumed to be at a latitude of 40° north. The signal received at the satellite is

$$s_{\text{sat}}(t) = \alpha x(t - \tau), \qquad \tau \le t \le T + \tau \tag{1-14a}$$

where $x(t)$ is the transmitted signal and $[0, T]$ is the interval of time over which the transmission takes place. The parameter α is the attenuation, which accounts for various system characteristics as well as the spreading of the electromagnetic wave as it propagates from the transmitter antenna. A portion of $s_{\text{sat}}(t)$ is reflected and returned to the New York ground station terminal to be received by the receiver at that end; we represent it as

$$s_{\text{refl}}(t) = \alpha\beta x(t - 2\tau), \qquad 2\tau \le t \le T + 2\tau \tag{1-14b}$$

where β is another attenuation factor. The remainder of $s_{\text{sat}}(t)$ is relayed to the Los Angeles ground station after processing by the satellite repeater. A portion of this signal may be reflected, but we will assume its effect to be negligible compared to that of $\alpha\beta x(t - 2\tau)$ when received at the New York station. Thus a speaker at New York will hear

$$s(t) \simeq x(t) + \alpha\beta x(t - 2\tau), \qquad 0 \le t \le T + 2\tau \tag{1-15}$$

That is, the speaker will hear his undelayed speech as well as an attenuated version of his speech delayed by approximately

$$\Delta T = 2\tau = 0.26\ \text{s}$$

Psychologically, the effect of this can be very disconcerting to a speaker if the delayed signal is sufficiently strong; it is virtually impossible for the person to avoid stuttering. Thus a systems analyst would proceed by relating α and β to more fundamental systems parameters and designing for an acceptably small value of $\alpha\beta$.

1-3 Signal Models

Examples of Deterministic Signals

In this book we are concerned with a broad class of signals referred to as *deterministic*. Deterministic signals can be modeled as completely specified functions of time. For example, the signal

$$x(t) = \frac{At^2}{B + t^2}, \qquad -\infty < t < \infty \tag{1-16}$$

where A and B are constants, shown in Figure 1-6a, is a deterministic signal. An example of a deterministic signal that is not a continuous function of time is the unit pulse, denoted as $\Pi(t)$ and defined as

$$\Pi(t) = \begin{cases} 1, & |t| \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{1-17}$$

It is shown in Figure 1-6b.

A second class of signals, which will not be discussed in this book, are random signals. They are signals taking on random values at any given time instant and must be modeled probabilistically. A random signal is illustrated in Figure 1-6c.

Continuous-Time Versus Discrete-Time Signals

The signals illustrated in Figure 1-6 are examples of *continuous-time signals.* It is important to note that "continuous time" does not imply that a signal is a mathematically continuous function, but rather that it is a function of a *continuous-time variable.*

In some systems the signals are represented only at discrete values of the independent variable (i.e., time). Between these discrete-time instants the value of the signal may be zero, undefined, or of no interest. An example of such a *discrete-time* or *sample-data signal* is shown in Figure 1-7a. Often, the intervals between signal values are the same, but they need not be.

A distinction between discrete-time and quantized signals is necessary. A *quantized signal* is one whose values may assume only a countable† number of values, or levels, but the changes from level to level may occur at any time. Figure 1-7b shows an example of a quantized signal. A real-world situation that can be modeled as a quantized signal is the opening and closing of a switch.

Discrete-time signals will be considered in Chapter 8. The following example provides an illustration of two such signals.

EXAMPLE 1-5

Two common discrete-time signals are the unit pulse and unit step signals. The first is defined by the equation*

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{1-18}$$

and the second by the equation

$$u[n] = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases} \tag{1-19}$$

where n takes on only integer values.

†A *countable* set is a set of objects whose members can be put into one-to-one correspondence with the positive integers. For example, the sets of all integers and of all rational numbers are countable, but the set of all real numbers is not.

*The brackets are used to indicate a discrete-time signal.

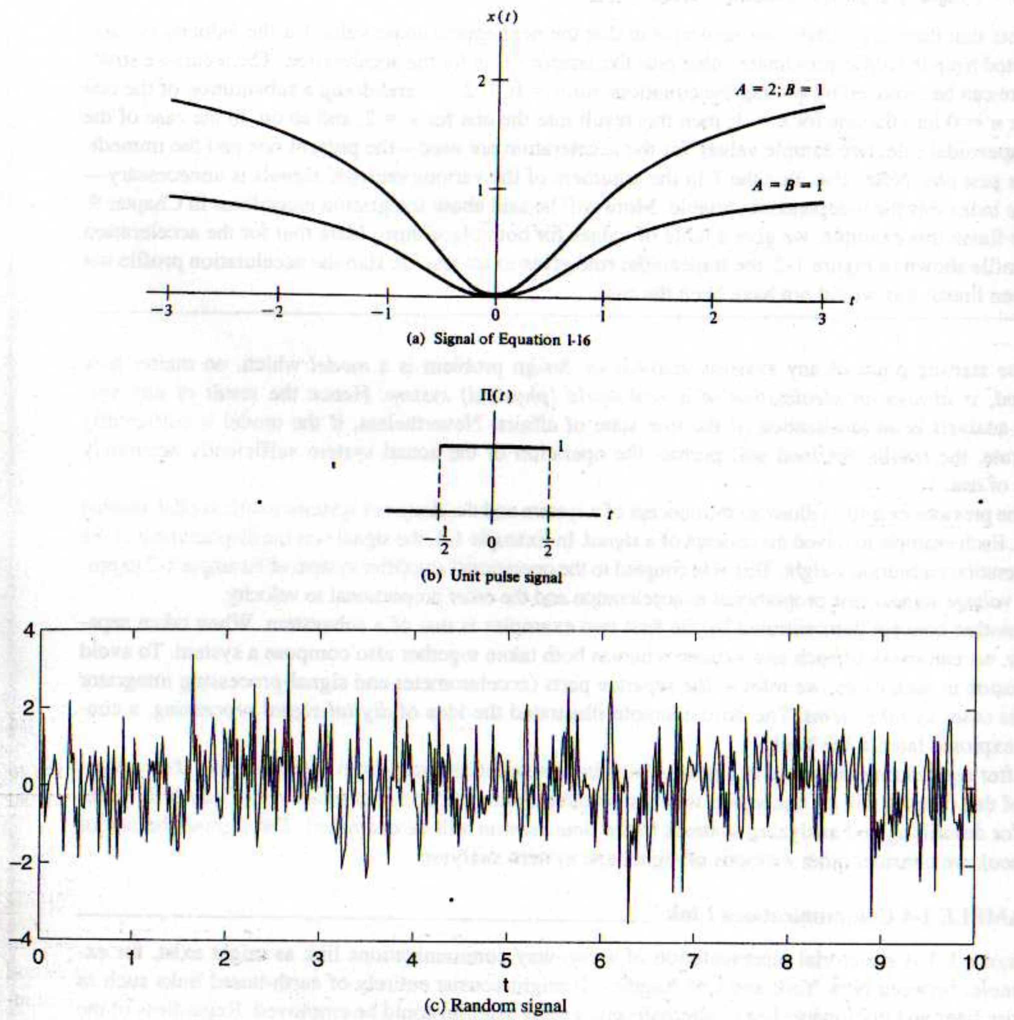
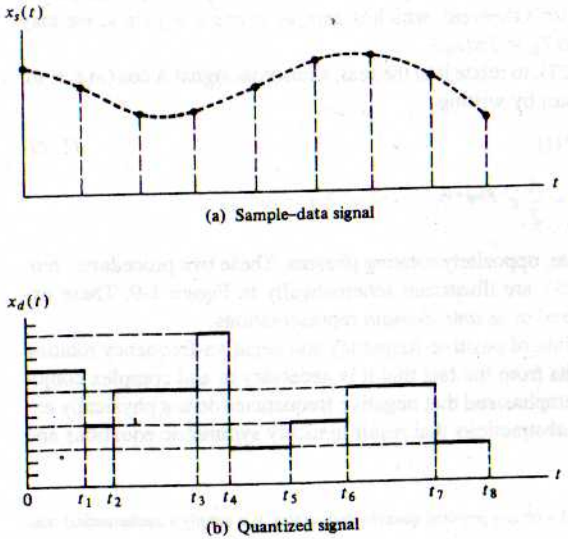


FIGURE 1-6. Graphical representation of two deterministic continuous-time signals and a random signal.

Other discrete-time signals may be built from these elementary signals. For example, a pulse consisting of five 1's in a row starting at $n = 0$ and ending at $n = 4$ and 0's elsewhere may be represented in terms of the discrete-time unit step as

$$p[n] = u[n] - u[n-5] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

The student should verify this by sketching $u[n]$, $u[n-5]$, and their difference.



(a) Sample-data signal

(b) Quantized signal

FIGURE 1-7. Sample-data, or discrete-time, and quantized signals.

Periodic and Aperiodic Signals

A signal $x(t)$ is *periodic* if and only if

$$x(t + T_0) = x(t), \qquad -\infty < t < \infty \tag{1-20}$$

where the constant T_0 is the period. The smallest value of T_0 such that (1-20) is satisfied is referred to as the *fundamental period*, and is hereafter simply referred to as the *period*. Any deterministic signal not satisfying (1-20) is called *aperiodic*.

A familiar example of a periodic signal is a sinusoidal signal that may be expressed as

$$x(t) = A\sin(2\pi f_0 t + \theta), \qquad -\infty < t < \infty \tag{1-21}$$

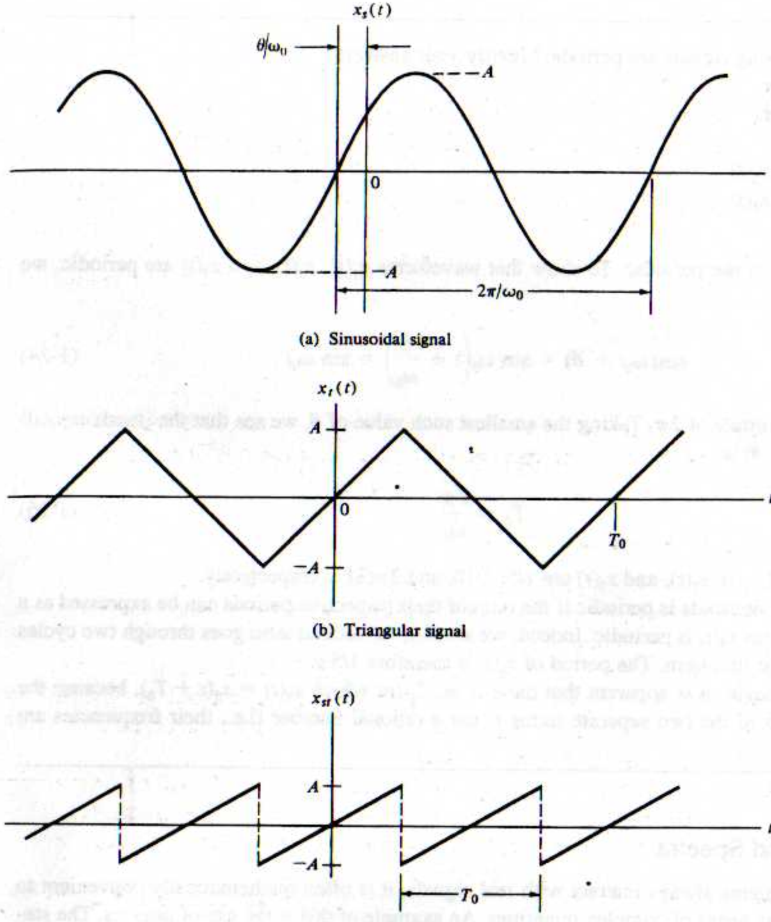
where A, f_0, and θ are constants referred to as the amplitude, frequency in hertz,† and relative phase, respectively. For simplicity, $\omega_0 = 2\pi f_0$ is sometimes used, where ω_0 is the frequency in rad/s. The period of this signal is $T_0 = 2\pi/\omega_0 = 1/f_0$, which can be verified by direct substitution and use of trigonometric identities as follows. We wish to verify that

$$x\left(t + \frac{2\pi}{\omega_0}\right) = x(t), \qquad \text{all } t \tag{1-22}$$

But

$$\begin{aligned} x\left(t + \frac{2\pi}{\omega_0}\right) &= A\sin\left[\omega_0\left(t + \frac{2\pi}{\omega_0}\right) + \theta\right] \\ &= A\sin(\omega_0 t + 2\pi + \theta) \\ &= A[\sin(\omega_0 t + \theta)\cos 2\pi + \cos(\omega_0 t + \theta)\sin 2\pi] \end{aligned} \tag{1-23}$$

†One hertz is one cycle per second.



(a) Sinusoidal signal

(b) Triangular signal

(c) Sawtooth signal

FIGURE 1-8. Various types of periodic signals.

Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, the required relationship for periodicity is verified. Note that $T_0 = 2\pi/\omega_0$ is the smallest value of T_0 such that $x(t) = x(t + T_0)$. Several types of periodic signals are illustrated in Figure 1-8 together with the sinusoidal signal.

The sum of two or more sinusoids may or may not be periodic, depending on the relationships between their respective periods or frequencies. If the ratio of their periods can be expressed as a rational number, or their frequencies are commensurable,‡ their sum will be a periodic signal.

‡Two frequencies, f_1 and f_2, are commensurable if they have a common measure. That is, there is a number f_0 contained in each an integral number of times. Thus, if f_0 is the largest such number,

$$f_1 = n_1 f_0 \qquad \text{and} \qquad f_2 = n_2 f_0$$

where n_1 and n_2 are integers; f_0 is called the *fundamental frequency*. The periods, T_1 and T_2, corresponding to f_1 and f_2, are therefore related by $T_1/T_2 = n_2/n_1$.

EXAMPLE 1-6

Which of the following signals are periodic? Justify your answers.

(a) $x_1(t) = \sin 10\pi t$
(b) $x_2(t) = \sin 20\pi t$
(c) $x_3(t) = \sin 31t$
(d) $x_4(t) = x_1(t) + x_2(t)$
(e) $x_5(t) = x_1(t) + x_3(t)$

Solution: Only (e) is not periodic. To show that waveforms $x_1(t)$, $x_2(t)$, and $x_3(t)$ are periodic, we note that

$$\sin(\omega_0 t + \theta) = \sin \omega_0\left(t + \frac{\theta}{\omega_0}\right) = \sin \omega_0 t \tag{1-24}$$

if θ is an integer multiple of 2π. Taking the smallest such value of θ, we see that the (fundamental) period of $\sin(\omega_0 t + \theta)$ is

$$T_0 = \frac{2\pi}{\omega_0} \tag{1-25}$$

Thus the periods of $x_1(t)$, $x_2(t)$, and $x_3(t)$ are 1/5, 1/10, and $2\pi/31$ s, respectively.

The sum of two sinusoids is periodic if the ratio of their respective periods can be expressed as a rational number. Thus $x_4(t)$ is periodic. Indeed, we see that the second term goes through two cycles for each cycle of the first term. The period of $x_4(t)$ is therefore 1/5 s.

After a little thought, it is apparent that there is no T_0 for which $x_5(t) = x_5(t + T_0)$, because the ratio of the periods of the two separate terms is not a rational number (i.e., their frequencies are incommensurable).

Phasor Signals and Spectra

Although physical systems always interact with real signals, it is often mathematically convenient to represent real signals in terms of complex quantities. An example of this is the use of phasors. The student should recall from circuits courses that the phasor quantity

$$\tilde{X} = Ae^{j\theta} = A\underline{/\theta} \tag{1-26}$$

is a shorthand notation for the real, sinusoidal signal†

$$x(t) = \text{Re}(\tilde{X}e^{j\omega_0 t}) = A\cos(\omega_0 t + \theta), \qquad -\infty < t < \infty \tag{1-27}$$

We refer to the complex signal

$$\begin{aligned}\tilde{x}(t) &= Ae^{j(\omega_0 t + \theta)}, \qquad -\infty < t < \infty \\ &= \tilde{X}e^{j\omega_0 t}\end{aligned} \tag{1-28}$$

†We could also project the sinusoidal signal onto the imaginary axis. Projection onto the real axis is used throughout this book, however.

as the rotating phasor signal.† It is characterized by the three parameters A, the amplitude; θ, the phase; and $\omega_0 > 0$, the radian frequency. Using Euler's theorem, which is $\exp(ju) = \cos u + j \sin u$, we may readily show that $\tilde{x}(t)$ is periodic with period $T_0 = 2\pi/\omega_0$.‡

In addition to taking its real part, as in (1-27), to relate it to the real, sinusoidal signal $A\cos(\omega_0 t + \theta)$, we may relate $\tilde{x}(t)$ to its sinusoidal counterpart by writing

$$\begin{aligned}x(t) &= \tfrac{1}{2}\tilde{x}(t) + \tfrac{1}{2}\tilde{x}^*(t) \\ &= \frac{A}{2}e^{j(\omega_0 t + \theta)} + \frac{A}{2}e^{-j(\omega_0 t + \theta)}, \qquad -\infty < t < \infty\end{aligned} \tag{1-29}$$

which is a representation in terms of conjugate, oppositely rotating phasors. These two procedures, represented mathematically by (1-27) and (1-29), are illustrated schematically in Figure 1-9. These expressions for $x(t) = A\cos(\omega_0 t + \theta)$ are referred to as *time-domain* representations.

Note that (1-29) can be thought of as the sum of positive-frequency and negative-frequency rotating phasors. This mathematical abstraction results from the fact that it is necessary to add complex conjugate quantities to obtain a real quantity. It is emphasized that negative frequencies do not physically exist, but are merely convenient mathematical abstractions that result in nicely symmetric equations and figures such as (1-29) and Figure 1-9b.

†A rotating phasor signal is not necessarily associated with any physical quantity that rotates. It is simply a mathematical convenience to represent sinusoids by the Re(·) operation.

‡Complex algebra basics: Many times students will do more work than necessary in complex algebra manipulations. This footnote hopefully will give some pointers to minimize such manipulations. A complex number may be represented in the alternative forms of cartesian and polar:

$$X = a + jb = Re^{j\theta} = R\cos\theta + jR\sin\theta$$

where a is its real part, b its imaginary part, R its magnitude or modulus, and θ its argument or angle. The latter equation follows by virtue of Euler's relationship, or $\exp(j\theta) = \cos(\theta) + j\sin(\theta)$. Matching real and imaginary parts, we see that $\text{Re}(X) = a = R\cos(\theta)$ and $\text{Im}(X) = b = R\sin(\theta)$, where Re(·) and Im(·) denote the real and imaginary parts, respectively. The inverse relationships are $R = (a^2 + b^2)^{1/2}$ and $\theta = \tan^{-1}(b/a)$. Most programmable calculators have built-in routines for going back and forth between the cartesian and polar forms. Consider a second complex number, defined by

$$Y = c + jd = Se^{j\phi} = S\cos\phi + jS\sin\phi$$

Addition and subtraction are facilitated with cartesian representation; i.e.,

$$X \pm Y = (a + jb) \pm (c + jd) = (a \pm c) + j(b \pm d)$$

Multiplication and division may be carried in either cartesian or polar form, but usually the polar form is most convenient. For example, multiplication in cartesian form becomes

$$XY = (a + jb)(c + jd) = (ac + j^2bd) + j(bc + ad) = (ac - bd) + j(bc + ad)$$

where $j^2 = -1$ has been used. In polar form, multiplication is more simply performed as

$$XY = (Re^{j\theta})(Se^{j\phi}) = RSe^{j(\theta+\phi)} = RS\cos(\theta + \phi) + jRS\sin(\theta + \phi)$$

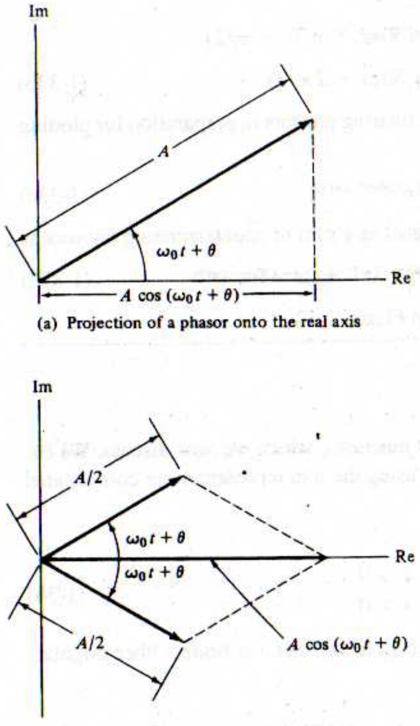
Division in cartesian form is facilitated by the process of rationalization as

$$\frac{X}{Y} = \frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}$$

This fairly lengthy process is shortened considerably by representation of the numbers in polar form:

$$\frac{X}{Y} = \frac{Re^{j\theta}}{Se^{j\phi}} = \frac{R}{S}e^{j(\theta-\phi)} = \frac{R}{S}\cos(\theta - \phi) + j\frac{R}{S}\sin(\theta - \phi)$$

As a numerical example, consider $X = 8 + j6$ and $Y = 3 + j4$. Their sum is $(8 + j6) + (3 + j4) = 11 + j10$ and their difference is $(8 + j6) - (3 + j4) = 5 + j2$. Their product computed in cartesian form is $(8 + j6)(3 + j4) = (24 - 24) + j(18 + 32) = j50$. In polar form it is $\{10\exp[\tan^{-1}(3/4)]\}\{5\exp[\tan^{-1}(4/3)]\} = 50\exp(j\pi/2) = j50$. Their quotient in polar form is $\{10\exp[\tan^{-1}(3/4)]\}/\{5\exp[\tan^{-1}(4/3)]\} = 2\exp\{j[\tan^{-1}(3/4) - \tan^{-1}(4/3)]\} = 2\exp[-j0.284] = 1.92 - j0.56$, which follows by Euler's relation. If the rationalization procedure is used, the same result in cartesian form is obtained.



(a) Projection of a phasor onto the real axis

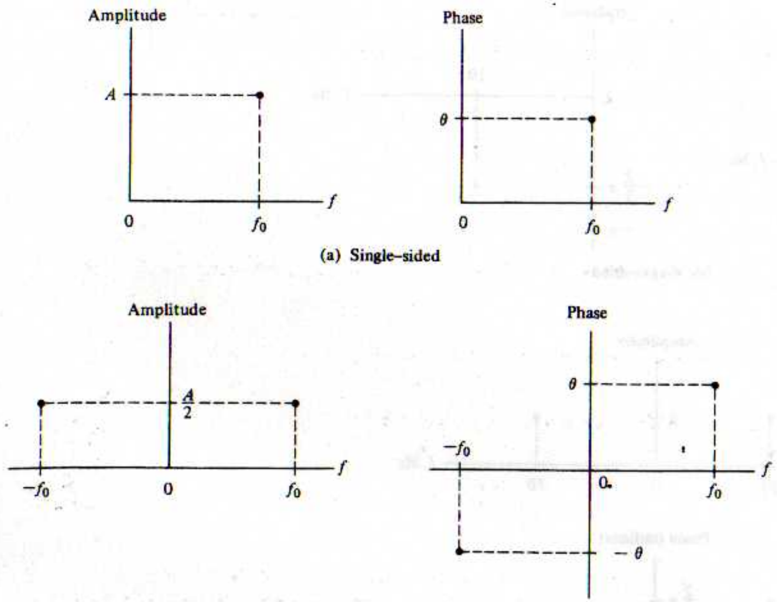
(b) Addition of complex conjugate phasors

FIGURE 1-9. Two ways of relating a phasor signal to a sinusoidal signal.

An alternative representation for $x(t)$ is provided in the *frequency domain.* Since $\tilde{x}(t) = A \exp[j(\omega_0 t + \theta)]$ is completely specified by A and θ for a given value of f_0, this alternative frequency-domain representation can take the form of two plots, one showing the amplitude A as a function of frequency f, and the other showing θ as a function of f. Because $\tilde{x}(t)$ depends only on the single frequency f_0, the resulting plots each consist of a single point or "line" at $f = f_0$, as illustrated in Figure 1-10a. Had we considered a signal that is the sum of two phasors, each plot would have had two points or lines present. The plot of amplitude versus frequency is referred to as the *single-sided amplitude spectrum.* The modifier "single-sided" is used because these spectral plots have points or lines only for positive frequencies.

If a spectral plot corresponding to (1-29) is made, spectral lines will be present at $f = f_0$ *and* at $f = -f_0$, since $x(t)$ is obtained as the sum of oppositely rotating phasors. Spectral plots for the single sinusoidal signal under consideration here are shown in Figure 1-10b. Such plots are referred to as *double-sided* amplitude and phase spectra. Had a sum of sinusoidal components been present, the spectral plots would have consisted of multiple lines. Note that because of the convention of taking the real part [see (1-27)], any signal expressed as a sine function must first be converted to a cosine function before obtaining the spectrum. For this purpose, the identity $\sin(\omega_0 t + \theta) = \cos(\omega_0 t + \theta - \pi/2)$ is useful.

It is important to emphasize two points about double-sided spectra. First, the lines at negative frequencies are present precisely because it is necessary to add complex conjugate phasors to obtain the real sinusoidal signal $A \cos(\omega_0 t + \theta)$. Second, we note that the amplitude spectrum has *even* symmetry about



(a) Single-sided

(b) Double-sided

FIGURE 1-10. Amplitude and phase spectra for the signal $A \cos(\omega_0 t + \theta)$.

the origin, and the phase spectrum has *odd* symmetry. This symmetry is again a consequence of $x(t)$ being a real signal, which implies conjugate rotating phasors must be added to obtain a real quantity.

Figures 1-10a and b serve as equivalent spectral representations for the signal $A \cos(\omega_0 t + \theta)$ if we are careful to identify the spectra plotted as single-sided or double-sided, as the case may be. We will find later that the Fourier series and Fourier transform lead to spectral representations for more complex signals.

EXAMPLE 1-7

We wish to sketch the single-sided and double-sided amplitude and phase spectra of the signal

$$x(t) = 4 \sin\left(20\pi t - \frac{\pi}{6}\right), \qquad -\infty < t < \infty \tag{1-30}$$

To sketch the single-sided spectra, we write $x(t)$ as the real part of a rotating phasor and plot the amplitude and phase of this phasor as a function of frequency for $t = 0$. Noting that $\cos(u - \pi/2) = \sin u$, we find that

$$\begin{aligned} x(t) &= 4 \cos\left(20\pi t - \frac{\pi}{6} - \frac{\pi}{2}\right) \\ &= 4 \cos\left(20\pi t - \frac{2\pi}{3}\right) \\ &= \operatorname{Re}\left\{4 \exp\left[j\left(20\pi t - \frac{2\pi}{3}\right)\right]\right\} \end{aligned} \tag{1-31}$$

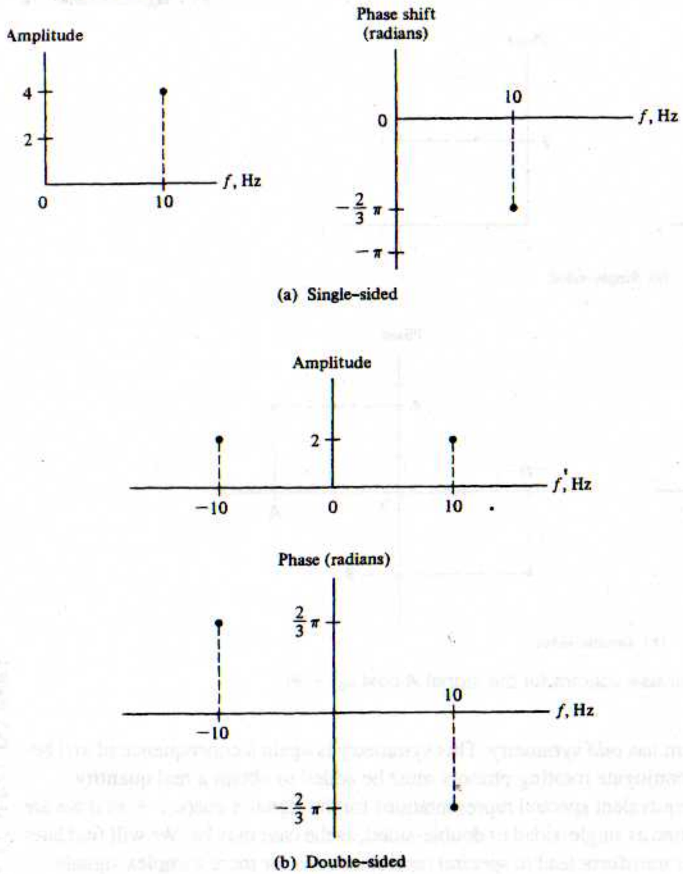


FIGURE 1-11. Amplitude and phase spectra for Example 1-6.

which results in the amplitude and phase spectral plots shown in Figure 1-11a. To plot the double-sided amplitude and phase spectra, we write $x(t)$ as the sum of complex conjugate rotating phasors. Recalling that $2 \cos u = \exp(ju) + \exp(-ju)$, we obtain

$$x(t) = 2 \exp\left[j\left(20\pi t - \frac{2\pi}{3}\right)\right] + 2 \exp\left[-j\left(20\pi t - \frac{2\pi}{3}\right)\right] \tag{1-32}$$

from which the double-sided amplitude and phase spectral plots of Figure 1-11b result.

EXAMPLE 1-8

As an example involving a sum of two sinusoids, consider the spectrum of the signal

$$x(t) = 2 \cos(10\pi t + \pi/4) + 4 \sin(30\pi t - \pi/6) \tag{1-33a}$$

Changing the second term to a cosine function gives

$$\begin{aligned} x(t) &= 2 \cos(10\pi t + \pi/4) + 4 \cos(30\pi t - \pi/6 - \pi/2) \\ &= 2 \cos(10\pi t + \pi/4) + 4 \cos(30\pi t - 2\pi/3) \end{aligned} \tag{1-33b}$$

This can be written in terms of the real part of the sum of rotating phasors in preparation for plotting the single-sided spectra as

$$x(t) = \mathrm{Re}[2e^{j(10\pi t + \pi/4)} + 4e^{j(30\pi t - 2\pi/3)}] \tag{1-33c}$$

In order to plot the double-sided spectra, we write the signal as a sum of counterrotating phasors as

$$x(t) = e^{j(10\pi t + \pi/4)} + e^{-j(10\pi t + \pi/4)} + 2e^{j(30\pi t - 2\pi/3)} + 2e^{-j(30\pi t - 2\pi/3)} \tag{1-33d}$$

Single-sided and double-sided spectral plots are shown in Figure 1-12.

Singularity Functions

An important subclass of aperiodic signals is the *singularity functions,* which we now discuss. We begin by introducing the unit step and unit ramp functions and using them to represent more complicated signals, such as in the acceleration in Example 1-1.

Consider the *unit step function,* $u_{-1}(t)$, defined as

$$u(t) = u_{-1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{1-34}$$

The value of $u(t)$ at $t = 0$ will not be specified at this time except to say that it is finite. Other singularity functions are defined in terms of $u_{-1}(t)$ by the relations

$$u_{i-1}(t) = \int_{-\infty}^{t} u_i(\lambda)\, d\lambda, \qquad i = \ldots, -2, -1, 0, 1, 2, \ldots \tag{1-35a}$$

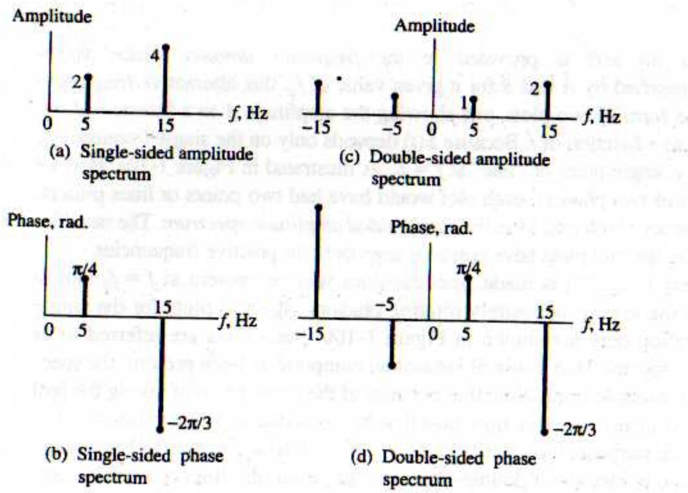


FIGURE 1-12. Spectra for Example 1-7.

or

$$u_{i+1}(t) = \frac{du_i(t)}{dt} \tag{1-35b}$$

Figure 1-13a shows the unit step. We may graphically integrate it to obtain $u_{-2}(t)$, shown in Figure 1-13b. This singularity function is referred to as the *unit ramp function,* $r(t)$, and can be expressed algebraically as[†]

$$r(t) = u_{-2}(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1-36}$$

Carrying this discussion one step further, we see that a *unit parabolic function* is given by

$$u_{-3}(t) = \begin{cases} t^2/2, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1-37}$$

This singularity function is sketched in Figure 1-13c. It is clear that the unit ramp and unit parabolic functions would be convenient for representing the acceleration and velocity of Example 1-1.

We may shift any signal on the time axis simply by replacing t by $t - t_0$. If $t_0 > 0$, the signal is shifted to the right; if $t_0 < 0$, it is shifted to the left. For example, replacing t by $t - \frac{1}{2}$ in the definition of the unit step, (1-34), we obtain

$$u(t - \tfrac{1}{2}) = \begin{cases} 0, & t - \frac{1}{2} < 0 \\ 1, & t - \frac{1}{2} > 0 \end{cases} \tag{1-38a}$$

or

$$u(t - \tfrac{1}{2}) = \begin{cases} 0, & t < \frac{1}{2} \\ 1, & t > \frac{1}{2} \end{cases} \tag{1-38b}$$

Similarly, replacing t by $t + \frac{1}{2}$ results in a unit step that "turns on" at $t = -\frac{1}{2}$. Using this technique, we can represent other signals in terms of singularity functions. For example, the *unit pulse function,* defined by (1-17), can be represented as

$$\Pi(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2})$$

For the definition of $\Pi(t)$ given by (1-17), it would be consistent to define $u(0) = 1$.

We note, also, that replacing t by $-t$ turns a function around.[‡] Doing this in (1-36), we obtain

$$r(-t) = \begin{cases} -t, & t \leq 0 \\ 0, & t > 0 \end{cases} \tag{1-40}$$

which is a ramp that starts at $t = 0$ and *increases* linearly as *t decreases.*

Also, we can obtain a change of scale on the abscissa simply by multiplying t by a constant, say, β. If $\beta > 1$, the signal is compressed. If $\beta < 1$, the signal is expanded.

To summarize, consider an arbitrary signal represented as $x(\beta t + \alpha)$. We rewrite it as

$$x(\beta t + \alpha) = x[\beta(t + \alpha/\beta)] \tag{1-41}$$

[†]The subscripts are needed on (1-35a) and (1-35b) to identify different members of the class of singularity functions. Since the step and ramp are very commonly used members, they are often denoted by the special symbols $u(t)$ and $r(t)$, respectively.

[‡]The negation of the independent variable is often referred to as *folding,* since the resulting reversal of the signal in time can be viewed as folding a paper on which the signal has been plotted along its ordinate and viewing it from behind the paper.

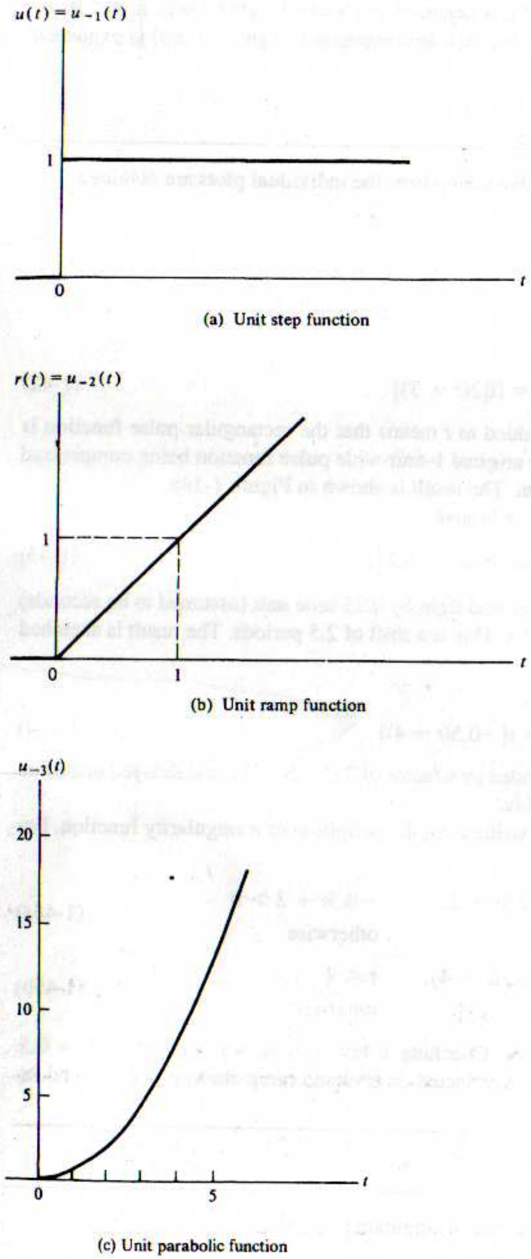


FIGURE 1-13. Examples of three singularity functions.

If $t_0 \triangleq \alpha/\beta$ is positive, $x(t)$ is shifted *left* by α/β; if t_0 is negative, it is shifted *right* by α/β. If $\beta < 0$, $x(t)$ is *reversed* or *reflected* through the origin. If $|\beta| > 1$, $x(t)$ is compressed; if $|\beta| < 1$, $x(t)$ is expanded. An example will illustrate these remarks.

EXAMPLE 1-9

Consider the following signals. Sketch each, discussing how the individual plots are obtained.

(a) $x_1(t) = \Pi(2t + 6)$
(b) $x_2(t) = \cos(20\pi t - 5\pi)$
(c) $x_3(t) = r(-0.5t + 2)$

Solution: For (a), we write

$$x_1(t) = \Pi[2(t + 3)] \tag{1-42}$$

by factoring out the 2 multiplying t. The 3 added to t means that the rectangular pulse function is shifted left by 3 units, and the 2 results in the original 1-unit-wide pulse function being compressed by a factor of 2 to a $\frac{1}{2}$-unit-wide pulse function. The result is shown in Figure 1-14a.

For (b), the 20π multiplying t is factored out to give

$$x_2(t) = \cos[20\pi(t - 0.25)] \tag{1-43}$$

which shows that $x_2(t)$ is a cosine waveform shifted right by 0.25 time unit (assumed to be seconds) with a period of $T_0 = 2\pi/\omega_0 = 2\pi/20\pi = 0.1$ s. This is a shift of 2.5 periods. The result is sketched in Figure 1-14b.

For $x_3(t)$, we write

$$x_3(t) = r[-0.5(t - 4)] \tag{1-44}$$

which says $r(t)$ is reflected about $t = 0$, expanded by a factor of 2 ($1/0.5 = 2$), and delayed or shifted right 4 units. A sketch is shown in Figure 1-14c.

It is sometimes useful to check results by writing out the definition of a singularity function. For part (c), this takes the form

$$r(-0.5t + 2) = \begin{cases} -0.5t + 2, & -0.5t + 2 > 0 \\ 0, & \text{otherwise} \end{cases} \tag{1-45a}$$

$$= \begin{cases} -0.5(t - 4), & t < 4 \\ 0, & \text{otherwise} \end{cases} \tag{1-45b}$$

From this, it is clear that $x_3(t) = 0$ for $t \geq 4$. Checking a few values, we see that $x_3(3) = 0.5$, $x_3(2) = 1$, and $x_3(0) = 2$ so that the result is a reflected or reversed ramp starting at $t = 4$ and increasing 1 unit for each 2-unit decrease in t.

EXAMPLE 1-10

Express the signals shown in Figure 1-15 in terms of singularity functions.

Solution: One possible representation for $x_a(t)$ is

$$x_a(t) = u(t) - r(t - 1) + 2r(t - 2) - r(t - 3) + u(t - 4) - 2u(t - 5) \tag{1-46}$$

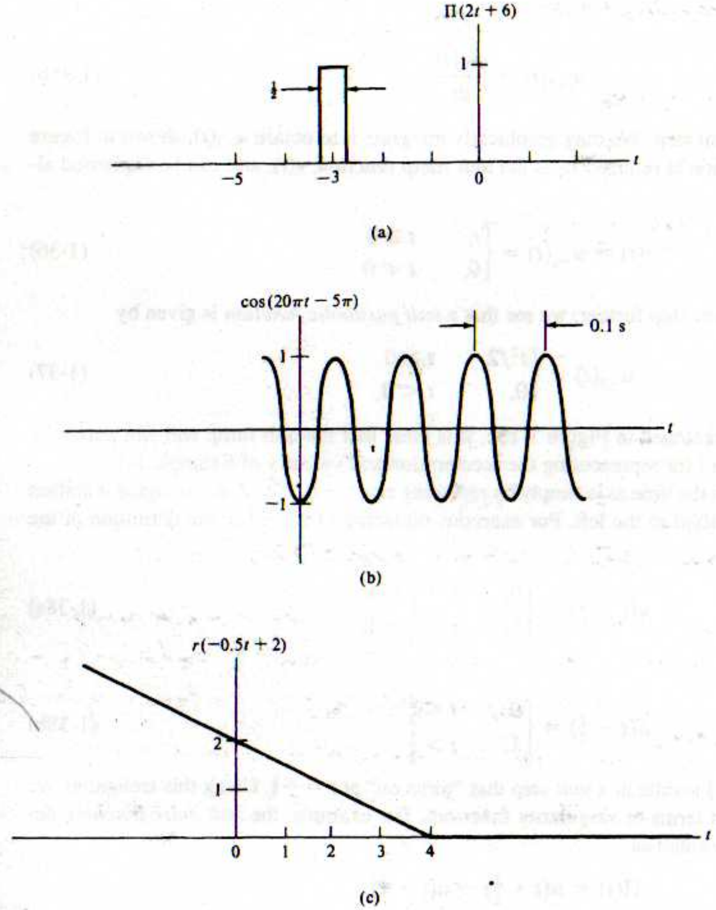


FIGURE 1-14. Signals relating to Example 1-9.

The step function $u(t)$ is unity for $t > 0$. At $t = 1$, the ramp function $-r(t - 1)$ goes downward with slope -1. The sum of ramps $-r(t - 1) + 2r(t - 2)$ goes upward with slope $+1$ starting at $t = 2$. The ramp $-r(t - 3)$ then flattens out this upward ramp starting at $t = 3$, and the step $u(t - 4)$ increases the plateau of unity height to a value of 2. The final step, $2u(t - 5)$, "shuts off" the preceding sum of singularity functions so that $x(t) = 0$ for $t > 5$.

For $x_b(t)$ we use a product representation. A possible representation is

$$x_b(t) = 2u(t)u(2 - t) + u(t - 3)u(5 - t) \tag{1-47}$$

The product of unit step functions $2u(t)u(2 - t)$ forms the first pulse, and the product $u(t - 3)u(5 - t)$ forms the second pulse. Note that $u(2 - t)$ and $u(5 - t)$ are step functions that are unity from $t = -\infty$ to $t = 2$ and $t = 5$, respectively, and zero thereafter.

The student should attempt to find other possible representations for $x_a(t)$ and $x_b(t)$.

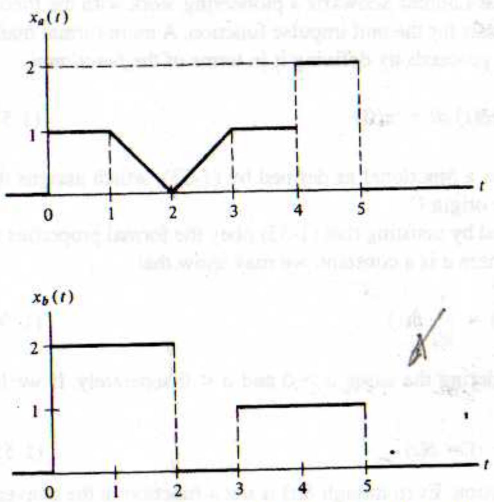


FIGURE 1-15. Signal to be expressed in terms of singularity functions.

EXERCISE

Represent the acceleration and velocity of the rocket of Example 1-1 in terms of singularity functions.

Answers: $a(t) = 40u_{-2}(t/72)u_{-1}(72 - t)$ m/s². $v(t) = \frac{5}{9}u_{-3}(t)u_{-1}(72 - t) + 1440u_{-1}(t - 72)$ m/s. (Other answers are possible.)

We turn next to the *unit impulse function* or *delta function,*† $\delta(t)$, which has the properties

$$\delta(t) = 0, \qquad t \neq 0 \tag{1-48a}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{1-48b}$$

Equation (1-48b) states that the area of a unit impulse function is unity, and (1-46a) indicates that this unity area is obtained in an infinitesimal interval on the t-axis.

The motivation for defining a function with these properties stems from the need to represent phenomena that happen in time intervals short compared with the resolution capability of any measuring apparatus used, but which produce an almost instantaneous change in a measured quantity. Examples are the nearly instantaneous increase in voltage across a capacitor placed across the terminals of a battery, or the nearly instantaneous change in velocity of a billiard ball struck by a rapidly moving cue ball. In the first example, the voltage across the capacitor is the result of an extremely large current flowing for a very short time interval, whereas the billiard ball's velocity changes almost instantaneously due to the transfer of momentum from the cue ball at the moment of impact. The current flowing into the capacitor and the force transmitted by the cue ball are examples of quantities that are usefully modeled

†The two names stem from the use of this function in engineering and physics; the term "delta function" is associated with Dirac, a physicist, who introduced the notation $\delta(t)$ into quantum mechanics. We will refer to $\delta(t)$ simply as a unit impulse function.

by impulse functions. In neither case are we able to measure, nor are we particularly interested in, what happens at exactly the moment the action takes place. Rather we are able to observe only the conditions beforehand and afterward.

Other examples of physical quantities that are modeled by mathematical entities having infinitely small dimensions and infinite "weight" are point masses and point charges.

Since it is impossible for any conventional function to have the properties (1-48), we attempt to picture the unit impulse function by considering the limit of a conventional function as some parameter approaches zero.† For example, consider the signal

$$\delta_\epsilon(t) = \frac{1}{2\epsilon}\Pi\left(\frac{t}{2\epsilon}\right) = \begin{cases} \dfrac{1}{2\epsilon}, & |t| \leq \epsilon \\ 0, & |t| > \epsilon \end{cases} \tag{1-49}$$

which is shown in Figure 1-16. We see that, no matter how small ϵ is, $\delta_\epsilon(t)$ always has unit area. Furthermore, this area is obtained as the integration is carried out from $t = -\epsilon$ to $t = \epsilon$. As $\epsilon \to 0$ this area is obtained in an infinitesimally small width, thus implying that the height of $\delta_\epsilon(t) \to \infty$ such that *unity area lies between the function and the t-axis.* This limit is shown as a heavy arrow in the figure.

Many other signals exist that provide heuristic visualizations for $\delta(t)$ in the limit as some parameter approaches zero. Examples are

$$\delta_\epsilon(t) = \frac{1}{\epsilon}\left[\frac{\sin(\pi t/\epsilon)}{\pi t/\epsilon}\right]^2 \tag{1-50}$$

which is sketched in Figure 1-17a, and

$$\delta_\epsilon(t) = \begin{cases} (1 - |t|/\epsilon)/\epsilon, & |t| < \epsilon \\ 0, & \text{otherwise} \end{cases} \tag{1-51}$$

†The present discussion is aimed at providing an intuitive understanding of the unit impulse function. We will give a more formal mathematical development shortly.

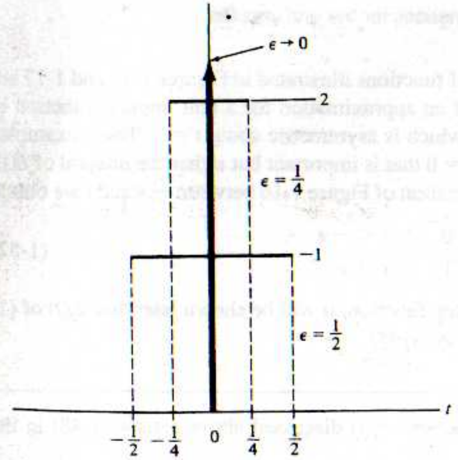


FIGURE 1-16. Square-pulse approximations for the unit impulse function.

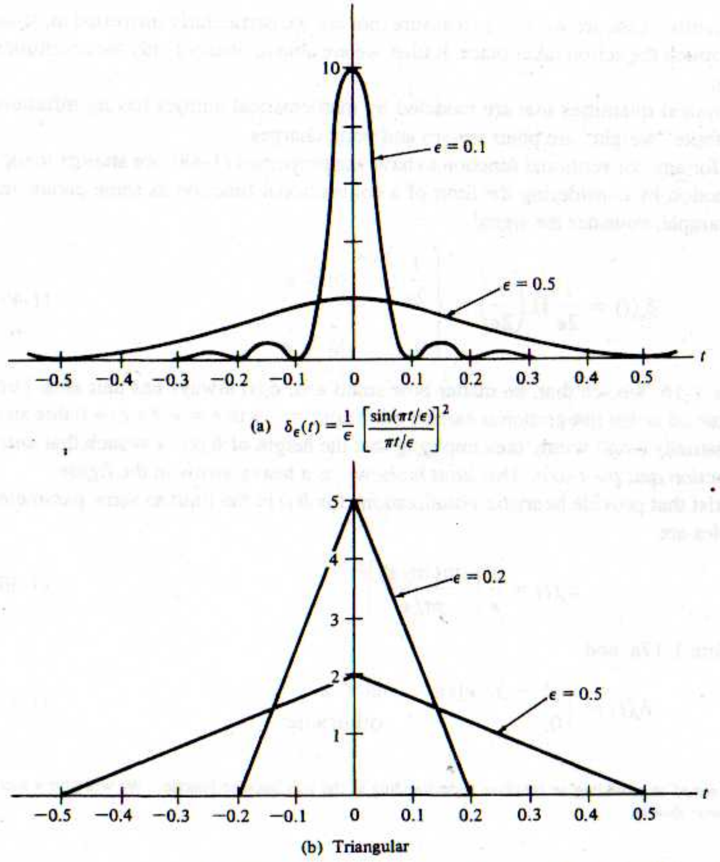


FIGURE 1-17. Two approximations for the unit impulse.

shown in Figure 1-17b. Although the three families of functions illustrated in Figures 1-16 and 1-17 are symmetric about $t = 0$, an equally good example of an approximation for a unit impulse function in terms of the properties (1-48) is $\epsilon^{-1} \exp(-t/\epsilon)u(t)$, which is asymmetric about $t = 0$. These examples illustrate that it is not the actual behavior of $\delta(t)$ at $t = 0$ that is important but rather the integral of $\delta(t)$.

Note that by integrating the square-pulse approximation of Figure 1-16 between $-\infty$ and t we obtain

$$\int_{-\infty}^{t} \delta_\epsilon(\lambda)\, d\lambda = \begin{cases} 0, & t < -\epsilon \\ 1, & t > \epsilon \end{cases} \tag{1-52}$$

In the limit as $\epsilon \to 0$, the right side becomes a unit step function. It will be shown later that $u_0(t)$ of (1-35b) and $\delta(t)$ have the same properties as expressed by (1-48).

EXERCISE

Verify that all the families of approximating functions $\delta_\epsilon(t)$ discussed above satisfy (1-48) in the limit as $\epsilon \to 0$.

The considerations above illustrate that the unit impulse function is not really a function in the normal sense. Indeed, it was not until the 1950s that Laurent Schwartz's pioneering work with the theory of distributions provided a firm mathematical basis for the unit impulse function. A more formal mathematical definition of the unit impulse function proceeds by defining it in terms of the *functional*

$$\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = x(0) \tag{1-53}$$

where $x(t)$ is continuous at $t = 0$. That is, $\delta(t)$ is a functional as defined by (1-53), which assigns the value $x(0)$ to any function $x(t)$ continuous at the origin.‡

Several useful properties of $\delta(t)$ can be proved by insisting that (1-53) obey the formal properties of integrals. For example, by considering $\delta(at)$, where a is a constant, we may show that

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{1-54}$$

by changing variables of integration and considering the cases $a > 0$ and $a < 0$ separately. If we let $a = -1$ in (1-54), we have

$$\delta(-t) = \delta(t) \tag{1-55}$$

which agrees with the definition of an even function. Even though $\delta(t)$ is not a function in the conventional sense, we may make use of (1-55) in manipulations involving unit impulse functions as long as we remember that it was a consequence of the defining relation (1-53) and the formal properties of integrals.

A second useful property of the delta function is the *sifting property*, which is

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0) \tag{1-56a}$$

where $x(t)$ is assumed to be continuous at $t = t_0$. This property can be proved from (1-53) by letting $\lambda = t - t_0$ in the integral (1-56a) to obtain

$$\int_{-\infty}^{\infty} x(\lambda + t_0)\delta(\lambda)\, d\lambda = x(t_0) \tag{1-56b}$$

If we define $y(\lambda) = x(\lambda + t_0)$, this integral is the same as (1-53); $y(\lambda)$ is continuous at $\lambda = 0$ as required because $x(t)$ is continuous at $t = t_0$.

An alternative form of (1-53) which will prove useful in Chapters 2 and 3 is

$$\int_{-\infty}^{\infty} x(\lambda)\delta(t - \lambda)\, d\lambda = x(t) \tag{1-57}$$

which is obtained by simple renaming of variables in (1-56a) and by using the "even" property of the delta function expressed by (1-55). The form of the integral in (1-57), known as a *convolution*, is dealt with in more detail in Chapter 2.

Integrals with finite limits can be considered as special cases of (1-56a) by defining $x(t)$ to be zero outside a certain interval, say $t_1 < t < t_2$. Thus (1-56a) becomes

$$\int_{t_1}^{t_2} x(t)\delta(t - t_0)\, dt = \begin{cases} x(t_0), & t_1 < t_0 < t_2 \\ 0, & \text{otherwise} \end{cases} \tag{1-58}$$

‡For a further discussion, see A. Papoulis, *Signal Analysis* (New York: McGraw-Hill, 1977), or R. N. Bracewell, *The Fourier Transform and Its Applications*, 2nd ed. (New York: McGraw-Hill, 1978).

A fourth property of $\delta(t)$ is that

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0) \tag{1-59}$$

if $x(t)$ is continuous at $t = t_0$, which follows intuitively from $\delta(t - t_0) = 0$ everywhere except at $t = t_0$.

We may find integrals similar to (1-56a) involving derivatives of $x(t)$. For example, assuming $x(t)$ to have a first derivative that is continuous at $t = t_0$ and using the chain rule for differentiation, we obtain

$$\frac{d}{dt}[x(t)\delta(t - t_0)] = \dot{x}(t)\delta(t - t_0) + x(t)\dot{\delta}(t - t_0) \tag{1-60}$$

where the overdot denotes differentiation with respect to time.† Assuming $\dot{x}(t)$ to be continuous at $t = t_0$ and using (1-59), we have

$$\frac{d}{dt}[x(t)\delta(t - t_0)] = \dot{x}(t_0)\delta(t - t_0) + x(t)\dot{\delta}(t - t_0) \tag{1-61}$$

Integrating from $t = t_1$ to $t = t_2$, where $t_1 < t_0 < t_2$, we obtain

$$\int_{t_1}^{t_2} \frac{d}{dt}[x(t)\delta(t - t_0)]\,dt = \int_{t_1}^{t_2} \dot{x}(t_0)\delta(t - t_0)\,dt + \int_{t_1}^{t_2} x(t)\dot{\delta}(t - t_0)\,dt, \qquad t_1 < t_0 < t_2$$

or

$$0 = \dot{x}(t_0) + \int_{t_1}^{t_2} x(t)\dot{\delta}(t - t_0)\,dt, \qquad t_1 < t_0 < t_2 \tag{1-62}$$

where we have used

$$\int_{t_1}^{t_2} \frac{d}{dt}[x(t)\delta(t - t_0)]\,dt = x(t)\delta(t - t_0)\Big|_{t_1}^{t_2} = 0, \qquad t_1 < t_0 < t_2 \tag{1-63}$$

and

$$\int_{t_1}^{t_2} \dot{x}(t)\delta(t - t_0)\,dt = \dot{x}(t_0)\int_{t_1}^{t_2} \delta(t - t_0)\,dt = \dot{x}(t_0), \qquad t_1 < t_0 < t_2 \tag{1-64}$$

which follow because $\delta(t - t_0) = 0$ everywhere except at $t = t_0$ and $\dot{x}(t)$ is continuous at $t = t_0$, so that $\dot{x}(t_0)$ is defined. Rearranging (1-62), we obtain

$$\int_{t_1}^{t_2} x(t)\dot{\delta}(t - t_0)\,dt = -\dot{x}(t_0), \qquad t_1 < t_0 < t_2 \tag{1-65}$$

In general, if the nth derivative of $x(t)$ exists and is continuous at $t = t_0$, it can be shown that

$$\int_{t_1}^{t_2} x(t)\delta^{(n)}(t - t_0)\,dt = (-1)^n x^{(n)}(t_0), \qquad t_1 < t_0 < t_2 \tag{1-66}$$

†The derivative of the unit impulse function is sometimes referred to as the *unit doublet*, or simply doublet. It can be visualized by graphically differentiating the triangular approximation for the unit impulse shown in Figure 1-17b. Again, we emphasize that $\dot{\delta}(t)$ is to be interpreted in the context of an integral similar to (1-53). It is not simply two opposite-sign impulses located at the origin.

where

$$x^{(n)}(t_0) \triangleq \frac{d^n x(t)}{dt^n}\Big|_{t=t_0} \quad \text{and} \quad \delta^{(n)}(t - t_0) \triangleq \frac{d^n}{dt^n}[\delta(t - t_0)]$$

A unit impulse function and its derivatives may be treated as generalized functions in the sense that if

$$f(t) = a_0\delta(t) + a_1\dot{\delta}(t) + \cdots + a_n\delta^{(n)}(t) \tag{1-67a}$$

and

$$g(t) = b_0\delta(t) + b_1\dot{\delta}(t) + \cdots + b_n\delta^{(n)}(t) \tag{1-67b}$$

then

$$f(t) + g(t) = (a_0 + b_0)\delta(t) + (a_1 + b_1)\dot{\delta}(t) + \cdots \tag{1-68}$$

Equation (1-68) applies also if $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ are functions that are continuous at $t = 0$.†

EXERCISE

Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} e^{-\alpha t^2}\delta(t - 10)\,dt$

(b) $\int_{0}^{\infty} e^{-\alpha t^2}\delta(t + 10)\,dt$

(c) $\int_{-\infty}^{\infty} e^{-\alpha t^2}\dot{\delta}(t - 10)\,dt$

(d) $\int_{-\infty}^{\infty} [5\delta(t) + e^{-(t-1)}\delta(t) + \cos 5\pi t\,\delta(t) + e^{-t^2}\dot{\delta}(t)]\,dt$

Answers: (a) $e^{-100\alpha}$; (b) 0; (c) $20\alpha e^{-100\alpha}$; (d) $6 + e$.

We now return to the assertion made earlier that

$$u(t) = u_{-1}(t) = \int_{-\infty}^{t} u_0(\lambda)\,d\lambda = \int_{-\infty}^{t} \delta(\lambda)\,d\lambda \tag{1-69}$$

or, equivalently, that

$$\delta(t) = \frac{du(t)}{dt} \tag{1-70}$$

†This, again, is formally proved by using (1-53). Another question that may vex the student is, "What about $0 \cdot \delta(t)$?" To show that this is zero, use (1-53) to write

$$\int_{-\infty}^{\infty} 0 \cdot \delta(t)x(t)\,dt = 0 \cdot \int_{-\infty}^{\infty} \delta(t)x(t) = 0 \cdot x(0) = 0$$

where $x(t)$ is continuous at $t = 0$.

which is (1-35b) with $i = -1$. Letting $x(t)$ be continuous at $t = 0$, we consider

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)\frac{du(t)}{dt}\,dt &= x(t)u(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\frac{dx(t)}{dt}\,dt \\ &= x(\infty) - \int_0^{\infty}\frac{dx(t)}{dt}\,dt \\ &= x(\infty) - x(t)\big|_0^{\infty} \\ &= x(\infty) - x(\infty) + x(0) \\ &= x(0)\end{aligned} \tag{1-71}$$

which was obtained through integration by parts. That is, $du(t)/dt$ has the same property as $\delta(t)$ as expressed by (1-53).

1-4 Energy and Power Signals

Quite often the particular representation used for a signal depends on the type of signal involved. It is therefore convenient to introduce a method for signal classification at this point. It is useful to classify signals as those having finite energy and those having finite average power. Some signals have neither finite average power nor finite energy.

To introduce these signal classes, suppose that $e(t)$ is the voltage across a resistance R producing a current $i(t)$. The instantaneous power per ohm is

$$p(t) = \frac{e(t)i(t)}{R} = i^2(t) \tag{1-72}$$

Integrating over the interval $|t| \le T$, we define the total energy and the average power on a per ohm basis as the limits

$$E = \lim_{T\to\infty}\int_{-T}^{T} i^2(t)\,dt \quad \text{joules} \tag{1-73}$$

and

$$P = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} i^2(t)\,dt \quad \text{watts} \tag{1-74}$$

respectively.

For an arbitrary signal $x(t)$, which may, in general, be complex, the total energy normalized to unit resistance is defined as

$$E \triangleq \lim_{T\to\infty}\int_{-T}^{T} |x(t)|^2\,dt \quad \text{joules} \tag{1-75}$$

and the average power normalized to unit resistance is defined as

$$P \triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt \quad \text{watts} \tag{1-76}$$

Based on the definitions (1-75) and (1-76), the following classes of signals are defined:

1. $x(t)$ is an energy signal if and only if $0 < E < \infty$, so that $P = 0$.
2. $x(t)$ is a power signal if and only if $0 < P < \infty$, thus implying that $E = \infty$.
3. Signals that satisfy neither property are therefore neither energy nor power signals.†

EXAMPLE 1-11

Consider the signal

$$x_1(t) = Ae^{-\alpha t}u(t), \qquad \alpha > 0, \tag{1-77}$$

where A and α are constants. Using (1-75), we can verify that $x_1(t)$ has energy

$$E = \frac{A^2}{2\alpha} \tag{1-78}$$

Letting $\alpha \to 0$, we obtain the signal

$$x_2(t) = Au(t). \tag{1-79}$$

which has infinite energy but finite power, as can be verified by applying (1-76):

$$P = \frac{A^2}{2} \tag{1-80}$$

Thus $x_2(t)$ is a power signal.

EXAMPLE 1-12

Consider the periodic, sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \theta) \tag{1-81}$$

The normalized average power of this signal is

$$P = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A^2\cos^2(\omega_0 t + \theta)\,dt \tag{1-82a}$$

Using appropriate trigonometric identities, we may rewrite this integral as

$$P = \lim_{T\to\infty}\frac{A^2}{2T}\int_{-T}^{T}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2(\omega_0 t + \theta)\right]dt = \frac{A^2}{2} \tag{1-82b}$$

which follows because

$$\lim_{T\to\infty}\frac{A^2}{4T}\int_{-T}^{T}\cos 2(\omega_0 t + \theta)\,dt = 0 \tag{1-83}$$

†It is easy to contrive examples of signals with infinite energy and zero average power that are nonzero over a finite range of t, but we classify such signals as neither energy nor power. An example is $x(t) = t^{-1/4}$, $t \ge 1$, and zero otherwise.

Average Power of a Periodic Signal

We note that there is no need to carry out the limiting operation to find P for a periodic signal, since an average carried out over a single period gives the same result as (1-76); that is, for a periodic signal, $x_p(t)$,

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x_p(t)|^2\, dt \tag{1-84}$$

where T_0 is the period. The proof of (1-84) is left to the problems.

EXAMPLE 1-13

From (1-84), the power of a rotating phasor signal of the form (1-28) is

$$P = \frac{\omega_0}{2\pi}\int_{t_0}^{t_0+2\pi/\omega_0} |Ae^{j(\omega_0 t+\theta)}|^2\, dt = A^2 \tag{1-85}$$

Using Euler's theorem, we may write

$$Ae^{j(\omega_0 t+\theta)} = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta) \tag{1-86}$$

The power of the real and imaginary components is $A^2/2$. Thus we may associate half the power in a rotating phasor with the real component and half with the imaginary component.

1-5 Energy and Power Spectral Densities

It is useful for some applications to define functions of frequency that when integrated over all frequencies give total energy or total power, depending on whether the signal under consideration is, respectively, an energy signal or a power signal. For an energy signal, a function of frequency when integrated that gives total energy is referred to as an *energy spectral density*. Denoting the energy spectral density of a signal $x(t)$ by $G(f)$, we have, by definition, that

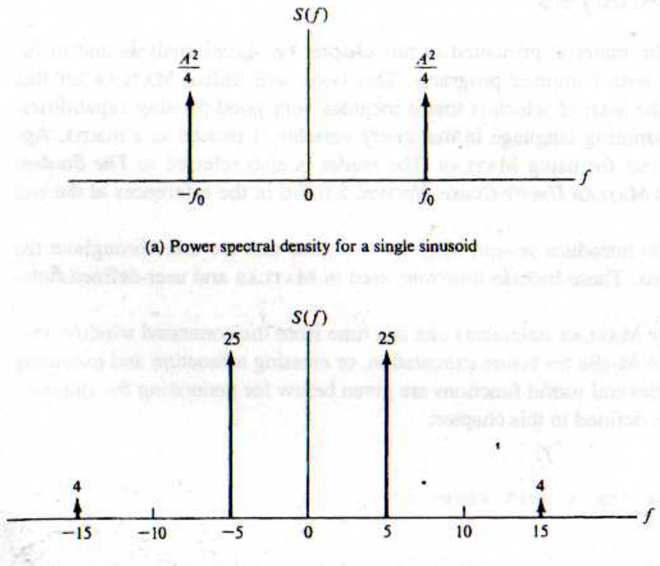
$$E = \int_{-\infty}^{\infty} G(f)\, df \tag{1-87}$$

where E is the signal's total energy. We will give a means for obtaining $G(f)$ for an arbitrary energy signal in Chapter 4.

Denoting the *power spectral density* of a power signal $x(t)$ by $S(f)$, we have, by definition, that

$$P = \int_{-\infty}^{\infty} S(f)\, df \tag{1-88}$$

where P is the average power of the signal. From our consideration of amplitude spectra for periodic signals, we can deduce what their power spectral densities are like. Since, for a single sinusoid of amplitude A and frequency f_0, the two-sided amplitude spectrum has a line at $-f_0$ of amplitude $A/2$ and a line at f_0 of amplitude $A/2$, we associate half of the power of the sinusoid, or $A^2/4$, with the frequency $-f_0$ and the other half, or $A^2/4$, with the frequency f_0. Since the power spectral density is integrated to give total average power, its representation at $f = -f_0$ must be $(A^2/4)\ \delta(f + f_0)$ and that at $f = f_0$ must be $(A^2/4)\ \delta(f - f_0)$. A plot is shown in Figure 1-18a. Generalizing, *for any signal possessing a two-sided line (amplitude) spectrum, we obtain the corresponding power spectral density by taking each line of the amplitude spectrum, squaring its value, and multiplying it by a unit impulse function located at that particular frequency* to get its power spectral density at that frequency. Note that the power spectral density of any signal is an even function of frequency, and it possesses no phase information about the signal.



(a) Power spectral density for a single sinusoid

(b) Power spectral density for the sinusoidal sum of Example 1-14

FIGURE 1-18. Power spectral densities for sinusoidal signals (numbers by impulses show weights).

EXAMPLE 1-14

Consider the signal

$$x(t) = 10\cos(10\pi t + \pi/7) + 4\sin(30\pi t + \pi/8) \tag{1-89}$$

(a) Plot its power spectral density.
(b) Compute the power lying within a frequency band from 10 Hz to 20 Hz.

Solution:

(a) The power spectral density is shown in Figure 1-18b. Analytically,

$$S(f) = 25\delta(f+5) + 25\delta(f-5) + 4\delta(f-15) + 4\delta(f+15) \tag{1-90}$$

(b) The power in the frequency band from 10 to 20 Hz is obtained by integrating $S(f)$ from $f = -20$ Hz to -10 Hz and from $f = 10$ Hz to 20 Hz since two-sided spectra show equal portions of the signal's spectrum on either side of $f = 0$. The result is 8 W. The total power of the signal is

$$P = \int_{-\infty}^{\infty} S(f)\, df = 50 + 8 = 58\ \text{W} \tag{1-91}$$

1-6 MATLAB in Signal Analysis

It is handy to be able to verify the material presented in this chapter on signal analysis and in future chapters on system analysis with computer programs. This book will utilize MATLAB for this purpose for several reasons, not the least of which is that it includes very good plotting capabilities. MATLAB is an array-based programming language in that every variable is treated as a matrix. Appendix A gives several helpful hints for using MATLAB. The reader is also referred to *The Student Edition of MATLAB*, Version 5 and *MATLAB User's Guide*, Version 5 listed in the references at the end of this chapter.

The purpose of this section is to introduce several MATLAB functions that are used throughout the book for signal and system analysis. These include functions used in MATLAB and user-defined functions to be given here.

One has the option of executing MATLAB statements one at a time from the command window, creating a program and saving it in an M-file for future executation, or creating a function and executing that from the command window. Several useful functions are given below for generating the singularity and other elementary functions defined in this chapter:

```
%       Function for generating a unit step
%
function u = stp_fn(t)
u = 0.5*(sign(t+eps) + 1);              % The constant eps is used to ensure
                                        % against the sign argument being 0,
                                        % where it's value is 0 giving u = 0.5
```

Note that a sign, or signum, function is used to generate this step because it can handle vectors. The same result could have been achieved by using a for loop, but since MATLAB is an interpretive language, this is considerably less efficient than using vector operations. The function stp_fn may be used for generating several other signals defined in this chapter. For example, function programs for generating a unit impulse approximation, a ramp function, and a square pulse signal are given below:

```
%       A function generating a rectangular approximation for
%       the unit impulse function; width is delta
%
function imp = impls_fn(t, delta)
imp = (stp_fn(t+delta/2)-stp_fn(t-delta/2))/delta;

%       Function for generating a unit ramp
%
function r = rmp_fn(t)
r=0.5*t.*(sign(t)+1);

%       This function generates a unit-high pulse centered at zero
%       and extending from -1/2 to 1/2
%
function y = pls_fn(t)
y = stp_fn(t+0.5) - stp_fn(t-0.5-eps);
```


EXAMPLE 1-15

To demonstrate the use of these functions we generate and plot the following:

1. A step starting at $t = 2$ and going to the right;
2. A unit impulse occuring at $t = -3.5$;
3. A ramp of slope 2 going backward from $t = 4$.

Note that to give the appearance of continuous-time signals in many cases we must choose the independent variable values sufficiently close together so that the plots appear to be smooth curves. The normal plot routine for MATLAB connects adjacent points with straight lines. In the cases considered here, the signals chosen will appear as they should when plotted because they consist of straight line segments. The MATLAB program listing below is in the form of an M-file. What is given below is exactly what one sees in the command window of *The Student Version of MATLAB* if the "echo on" switch command is executed first (a switch command stays in effect until the off switch command is executed, in this case "echo off" where the quotation marks are not typed but are used here to set the command off from the rest of the text). The results of the program are given in the plots shown in Figure 1-19.


```
EDU»clex15
%    M-file for Example 1-15
%
t=-10:.005:10;
x=stp_fn(t - 2);           % Generate step starting at t=2
y=impls_fn(t+3.5, 0.05); % Generate unit impulse at t=-3.5
z=2*rmp_fn(4 - t); % Generate backwards ramp starting at t = 4
subplot(3,1,1),plot(t,x),axis([-10 10 0 1.5]), xlabel('t'),
ylabel('u(t - 2)')
```

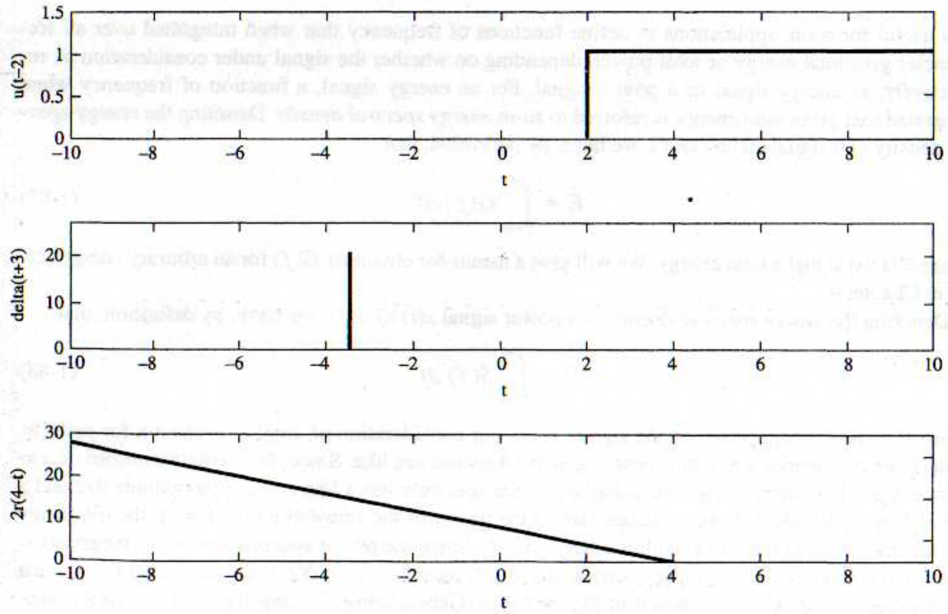


FIGURE 1-19. Various elementary signals computed and plotted with the aid of MATLAB.

```
subplot(3,1,2),plot(t,y),axis([-10 10 0 25]),xlabel('t'),
ylabel('delta(t + 3)')
subplot(3,1,3),plot(t,z),xlabel('t'),ylabel('2r(4 - t)')
```

Further examples of elementary signal generation and the building up of more complex signals from these are suggested in the computer exercises at the end of the chapter.

EXAMPLE 1-16

As a final example, we plot a sample-data signal using a special plotting feature of MATLAB. The chosen signal is a sinewave of frequency 0.5 hertz sampled at 0.2 second intervals. What appears in the MATLAB command window when the program is executed is given below (assuming "echo on" has been executed first). The output plot, which uses a stem plot function, is shown in Figure 1-20.

```
EDU», clex16
%   M-file for Example 1-16; plots a sample-data sinewave
%
del_t = 0.2;
T0 = 2;
n = 0:10;
x = sin(2*pi*n*del_t/T0);
stem(n*del_t,x),xlabel('n*del_t'), ylabel('x(n*del_t)')
```

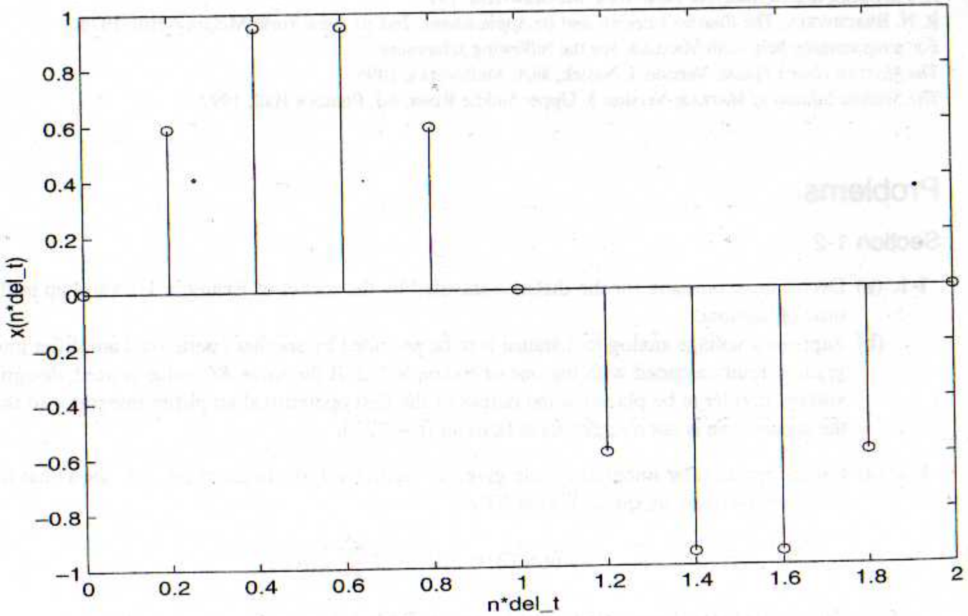


FIGURE 1-20. A sample-data sinewave plotted with the aid of MATLAB.

Summary

In this chapter, the concepts of a *signal* and a *system* were introduced. Two simple examples of systems were given. The remainder of the chapter then concentrated on models for signals. The following are the main points made in this chapter.

1. A *system* is a combination and interconnection of several components to perform a desired task. Systems may be interconnected with each other to form other systems. In such cases the component systems are referred to as *subsystems*.
2. A *signal* is a function of time that represents a physical variable of interest. Signals may represent voltages, currents, forces, velocities, displacements, etc.
3. *Signal processing* is necessary to convert signals to more convenient forms and to produce desired quantities from measured quantities. An example was provided with the accelerometer, where the displacement of the mass, which was proportional to acceleration, was converted to an electrical voltage proportional to the acceleration, and this voltage was integrated to produce a signal proportional to velocity. A second integration would have produced a signal proportional to distance.
4. Signals can be *deterministic* or *random*. A deterministic signal is modeled as a completely specified function of time, whereas a random signal takes on a random value at any given time instant. This book is concerned only with deterministic signals.
5. Signals can be further categorized as *continuous time* or *discrete time*. The former type of signal is a function of a continuous-time variable, and the latter is a function of an independent variable assuming values from a discrete set. A *quantized* signal is one whose values are taken from a discrete set and should not be confused with a discrete-time signal. All signals discussed in this paragraph may be deterministic or random.
6. Signals may also be categorized as *periodic* or *aperiodic*. A periodic signal is one for which

   $$x(t + T_0) = x(t), \qquad -\infty < t < \infty$$

   where T_0 is termed the *period*. The smallest value of T_0 for which the above equation holds is called the *fundamental period*. (It should be clear that if T_0 is the fundamental period, then any integer multiple of T_0 is also a period.)
7. A useful periodic signal is the *rotating phasor signal*, defined as

   $$x(t) = Ae^{j(\omega_0 t + \theta)}, \qquad -\infty < t < \infty$$

   where A is the amplitude, ω_0 is the frequency in radians per second, and θ is the phase angle in radians. The quantity $A \exp(j\theta)$ is called a *phasor*. Rotating phasor signals are convenient in that a cosinusoidal signal can be represented either as the real part of a rotating phasor signal or as one-half the rotating phasor plus one-half its complex conjugate. A signal which is the sum of sinusoids can be represented as the sum of rotating phasor signals, with either the real part or half the rotating phasor sum plus half its complex conjugate taken.
8. Representation of sums of sinusoids in terms of rotating phasor sums allows one to plot signal spectra easily. *Single-sided spectra* are obtained by representing the sum of sinusoids as the real part of the corresponding rotating phasor sum; the phasor amplitudes are plotted versus frequency to obtain the *amplitude spectrum*, while the phasor phase angles are plotted versus frequency to obtain the *phase spectrum*. Since only positive-frequency components are present, the spectra exist only for frequencies greater than zero, which is the reason for the adjective "single-sided." *Double-sided spectra* are obtained by representing the sum of sinusoids as one-half the corresponding rotating phasor sum plus one-half its complex conjugate; the phasor

amplitudes are plotted versus frequency to obtain the *amplitude spectrum*, while the phasor phase angles are plotted versus frequency to obtain the *phase spectrum*. Since both positive- and negative- (due to the conjugation) frequency components are present, the spectra exist for both positive and negative frequencies, which is the reason for the adjective "double-sided."

9. Important examples of *singularity* functions are the *unit impulse*, the *unit step*, and the *unit ramp*. The unit impulse is defined by the property that

$$\int_{-\infty}^{\infty} \delta(t)x(t)\,dt = x(0)$$

where $x(t)$ is any signal continuous at $t = 0$. From this property, it can be deduced that a unit impulse as a "function" can be viewed as occurring at the origin and being infinitesimally narrow and infinitely high such that its area is unity. The unit step is defined to be unity for $t > 0$ and zero for $t < 0$, with the value at zero immaterial as long as it is finite. The unit ramp is a function that is zero for $t < 0$ and that increases linearly with unit slope for $t > 0$. The unit step is the derivative of the unit ramp, and the unit impulse is the derivative of the unit step. Likewise, a "unit doublet" could be defined as the derivative of the unit impulse, and a "unit parabola" could be defined as the integral of the unit ramp. The differentiation process or integration process can be carried out indefinitely to obtain a doubly infinite class of singularity functions. For purposes of the analyses carried out in this book, the unit impulse, step, and ramp functions will suffice. More complex functions can be built up of sums or products of singularity functions.

10. The sifting property of the unit impulse function mentioned above can be generalized to

$$\int_{t_1}^{t_2} x(t)\delta^{(n)}(t - t_0)\,dt = (-1)^n x^{(n)}(t_0), \qquad t_1 < t_0 < t_2$$

where the superscript (n) denotes the nth derivative, and the nth derivative of $x(t)$ is assumed to exist and be continuous at $t = t_0$.

11. Another classification of signals is that of energy, power, or neither. To define this classification, we first define the energy of a signal to be

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\,dt$$

and the power of a signal to be

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt$$

The following classes of signals can then be defined:

1. $x(t)$ is an *energy signal* if and only if $0 < E < \infty$, so that $P = 0$.
2. $x(t)$ is a *power signal* if and only if $0 < P < \infty$, thus implying that $E = \infty$.
3. Signals satisfying neither property are neither power nor energy signals.

12. The *energy spectral density* of a signal $G(f)$, is a function which, when integrated over frequency from $-\infty < f < \infty$ (f in hertz), gives the total energy in the signal.

13. The *power spectral density* of a signal, $S(f)$, is a function which, when integrated over frequency from $-\infty < f < \infty$, gives the total power in the signal. For any signal possessing a two-sided line (amplitude) spectrum, we obtain the corresponding power spectral density by taking each line of the amplitude spectrum, squaring its value, and multiplying it by a unit impulse function located at that particular frequency.

Further Reading

Many circuits books have a simplified treatment of some of the material in this chapter. Recent ones are:

L. S. Bobrow, *Elementary Linear Circuit Analysis*. New York: Holt, Rinehart, and Winston, 1987

W. H. Hayt, Jr. and J. E. Kemmerly, *Engineering Circuit Analysis*. New York: McGraw-Hill, 1993

D. R. Cunningham and J. A. Stuller, *Circuit Analysis*. 2nd ed. New York: John Wiley, 1995

L. P. Huelsman, *Basic Circuit Theory*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 1991

The spectrum analysis ideas introduced in this chapter are also treated in various communications texts. One example is:

R. E. Ziemer and W. H. Tranter, *Principles of Communications: Systems, Modulation, and Noise*, 4th ed. New York: Wiley, 1995

There are many other books treating continuous- and discrete-time signal and system theory with the order of topics not necessarily the same as used here. Recently published examples are:

A. V. Oppenheim and A.S. Willsky, *Signals and Systems*, 2nd ed. Upper Saddle River, NJ: Prentice Hall, 1997

E. W. Kamen and B. S. Heck, *Fundamentals of Signals and Systems Using Matlab*. Upper Saddle River, NJ: Prentice Hall, 1997

C. L. Phillips and J. M. Parr, *Signals, Systems, and Transforms*. Upper Saddle River, NJ: Prentice Hall, 1995

Several older books treating signal and system theory should be mentioned to recognize the pioneering efforts in this field. A by-no-means-complete list is:

R. A. Gabel and R. A. Roberts, *Signals and Linear Systems*, 3rd ed. New York: Wiley 1987

W. M. Siebert, *Circuits, Signals, and Systems*. New York: McGraw-Hill, 1986

C. D. McGillem and G. R. Cooper, *Continuous and Discrete Signal and System Analysis*. New York: Holt, Rinehart, and Winston, 1974

R. J. Schwarz and B. Friedland, *Linear Systems*. New York: McGraw-Hill, 1965

S. J. Mason and H. J. Zimmerman, *Electronic Circuits, Signals, and Systems*. New York: John Wiley, 1960

The following books provide further reading on the treatment of singularity functions from the standpoint of generalized function theory:

A. Papoulis, *Signal Analysis*. New York: McGraw-Hill, 1977

R. N. Bracewell, *The Fourier Integral and Its Applications*, 2nd ed. New York: McGraw-Hill, 1978.

For programming help with Matlab, see the following references:

The Matlab User's Guide, Version 5, Natick, MA: Mathworks, 1996

The Student Edition of Matlab, Version 5. Upper Saddle River, NJ: Prentice Hall, 1997

Problems

Section 1-2

1-1. **(a)** Derive an expression for the distance traveled by the rocket of Example 1-1 valid up to the time of burnout.

(b) Suppose a voltage analog to distance is to be provided by another operational amplifier integrator circuit cascaded with the one of Example 1-2. If the same *RC* value is used, design a voltage divider to be placed at the output of the first operational amplifier integrator so that the second one is not overdriven at burnout (t = 72 s).

1-2. **(a)** For the rectangular integration rule given in recursive form in Example 1-3, show that the nonrecursive form up through time *NT* is

$$\hat{v}(NT) = \hat{v}(0) + T\sum_{n=1}^{N} a(nT)$$

Check this using some of the values given in Table 1-1 (page 5).

(b) For the trapezoidal integration rule given in recursive form in Example 1-3, show that the nonrecursive form up through time NT is

$$\hat{v}(NT) = \hat{v}(0) + \left[a(0) + a(NT) + 2\sum_{n=1}^{N-1} a(nT)\right]\left(\frac{T}{2}\right)$$

Check this, using values from Example 1-3.

1-3. Rework Example 1-1 if the acceleration profile for the rocket is

$$a(t) = 20 \text{ m/s}^2, \quad 0 \le t \le 50 \text{ s}$$

and zero otherwise. Assume the same parameter values as given in Example 1-1. Find and plot the velocity profile.

1-4. Rework Example 1-1 with the acceleration profile shown. Sketch the resulting velocities.

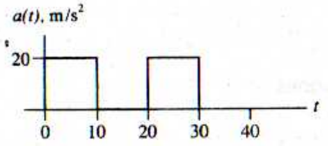


FIGURE P1-4

(a) Assume an initial velocity of zero.
(b) Assume an initial velocity of 1500 m/s.

1-5. In the satellite communications example, Example 1-4, let the transmitted signal be $x(t) = \cos\omega_0 t$.

(a) Show that the received signal can be put into the form

$$y(t) = \sqrt{1 + 2\alpha\beta\cos 2\omega_0\tau + (\alpha\beta)^2}\,\cos\left(\omega_0 t - \tan^{-1}\frac{\alpha\beta\sin 2\omega_0\tau}{1 + \alpha\beta\cos 2\omega_0\tau}\right)$$

(b) Plot the envelope of the cosine in the above equation versus $\omega_0\tau$ for $\alpha\beta = 0.1$. Note that the received signal will "fade" as τ varies due to satellite motion.

Section 1-3

1-6. **(a)** A sample-data signal derived by taking samples of a continuous-time signal, $m(t)$, is often represented in terms of an infinite sequence of rectangular pulse signals by multiplication. That is,

$$x_{sd}(t) = m(t)\sum_{n=-\infty}^{\infty}\Pi\left(\frac{t - nt_0}{\tau}\right)$$

If $m(t) = \exp(-t/10)u(t)$, $\tau = 0.5$ s, and $t_0 = 2$ s, sketch $x_{sd}(t)$ for $-1 \le t \le 10$ s.

(b) Flat-top sample representation of a signal is sometimes preferred over the above scheme. In this case, the sample-data representation of a continuous-time signal is given by

$$x_{ft}(t) = \sum_{n=-\infty}^{\infty} m(nt_0)\,\Pi\left(\frac{t - nt_0}{\tau}\right)$$

Sketch this sample-data representation for the same signal and parameters as given in part (a). Discuss the major difference between the two representations.

1-7. Ideal sampling is represented in two ways depending on whether the signal is considered to be continuous-time or discrete-time.

(a) The continuous-time signal $\cos(2\pi t)$ is to be represented in terms of samples by multiplying by a train of impulses spaced by 0.1 seconds. Let the impulse train be written as

$$\sum_{n=-\infty}^{\infty}\delta(t - 0.1n)$$

and show that the ideal impulse-sampled waveform is given by

$$\sum_{n=-\infty}^{\infty}\cos(0.2\pi n)\delta(t - 0.1n)$$

by using appropriate properties of the unit impulse.

(b) Show that the discrete-time representation is the same, except that the continuous-time unit impulse is replaced by a discrete-time unit pulse function, $\delta[n]$, appropriately shifted. Sketch for the signal $\cos(2\pi t)$ sampled each 0.1 seconds

1-8. Sketch the following signals:
(a) $\Pi(0.1t)$
(b) $\Pi(10t)$
(c) $\Pi(t - 1/2)$
(d) $\Pi[(t - 2)/5]$
(e) $\Pi[(t - 1)/2] + \Pi(t - 1)$

1-9. What are the fundamental periods of the signals given below? (Assume units of seconds for the t-variable.)
(a) $\sin 50\pi t$
(b) $\cos 60\pi t$
(c) $\cos 70\pi t$
(d) $\sin 50\pi t + \cos 60\pi t$
(e) $\sin 50\pi t + \cos 70\pi t$

1-10. Given the two complex numbers $A = 3 + j3$ and $B = 10\exp(j\pi/3)$.
(a) Put A into polar form and B into cartesian form. What is the magnitude of A? The argument of A? The real part of B? The imaginary part of B?
(b) Compute their sum. Show as a vector in the complex plane along with A and B.
(c) Compute their difference. Show as a vector in the complex plane.
(d) Compute their product in two ways: by multiplying both numbers in cartesian form and by multiplying in polar form. Show that both answers are equivalent.
(e) Compute the quotient A/B in two ways: by dividing with both numbers expressed in cartesian form and by dividing with both numbers expressed in polar form. Show that both answers are equivalent.

1-11. Find the periods and fundamental frequencies of the following signals:
(a) $x_a(t) = 2\cos(10\pi t + \pi/6)$
(b) $x_b(t) = 5\cos(17\pi t - \pi/4)$
(c) $x_c(t) = 3\sin(19\pi t - \pi/3)$
(d) $x_d(t) = x_a(t) + x_b(t)$
(e) $x_e(t) = x_a(t) + x_c(t)$
(f) $x_f(t) = x_b(t) + x_c(t)$

1-12. (a) Write the signals of Problem 1-11 as the real part of the sum of rotating phasors.
(b) Write the signals of Problem 1-11 as the sum of counterrotating phasors.
(c) Plot the single-sided amplitude and phase spectra for these signals.
(d) Plot the double-sided amplitude and phase spectra for these signals.

1-13. (a) Write the signals given in Problem 1-9 as the real parts of rotating phasors.
(b) Write each of the signals given in Problem 1-9 as one-half the sum of a rotating phasor and its complex conjugate.
(c) Sketch the single-sided amplitude and phase spectra of the signals given in Problem 1-9.
(d) Sketch the double-sided amplitude and phase spectra of the signals given in Problem 1-9.

1-14. (a) Express the signal given below in terms of step functions. Sketch it first.

$$x_a(t) = \Pi[(t-3)/6] + \Pi[(t-4)/2]$$

(b) Express the derivative of the signal given above in terms of unit impulses.

1-15. Suppose that instead of writing a sinusoid as the real part of a rotating phasor, we agree to use the convention

$$\sin(\omega_0 t + \theta) = \text{Im}\exp[j(\omega_0 t + \theta)]\}$$

or

$$\sin(\omega_0 t + \theta) = \exp[j(\omega_0 t + \theta)]/2j - \exp[-j(\omega_0 t + \theta)]/2j$$

(a) What change, if any, will there be to the two-sided amplitude spectrum of a signal from the case where the real-part convention is used?
(b) What change, if any, will there be to the two-sided phase spectrum of a signal from the case where the real-part convention is used?

1-16. Sketch the following signals:
(a) $u[(t-2)/4]$ (d) $\Pi(-3t+1)$
(b) $r[(t+1)/3]$ (e) $\Pi[(t-3)/2]$
(c) $r(-2t+3)$

1-17. Derive expressions for singularity functions $u_i(t)$ for $i = -4$ and $i = -5$. Generalize to arbitrary negative values of i.

1-18. Plot accurately the following signals defined in terms of singularity functions:
(a) $x_a(t) = r(t)u(2-t)$
(b) $x_b(t) = r(t) - r(t-1) - r(t-2) + r(t-3)$
(c) $x_c(t) = \sum_{n=0}^{\infty} x_a(t-2n)$ (Plot for $0 \le t \le 8$, and use three dots to indicate its semi-infinite extent.)
(d) $x_d(t) = \sum_{n=0}^{\infty} x_b(t-3n)$ (Plot for $0 \le t \le 9$ and use three dots to indicate its semi-infinite extent.)

1-19. (a) Sketch the signal $y(t) = \sum_{n=0}^{\infty} u(t-2n)u(1+2n-t)$.
(b) Is it periodic? If so, what is its period? If not, why not?
(c) Repeat parts (a) and (b) for the signal $y(t) = \sum_{n=-\infty}^{\infty} u(t-2n)u(1+2n-t)$.

1-20. Express the signals shown in terms of singularity functions (they are all zero for $t < 0$):

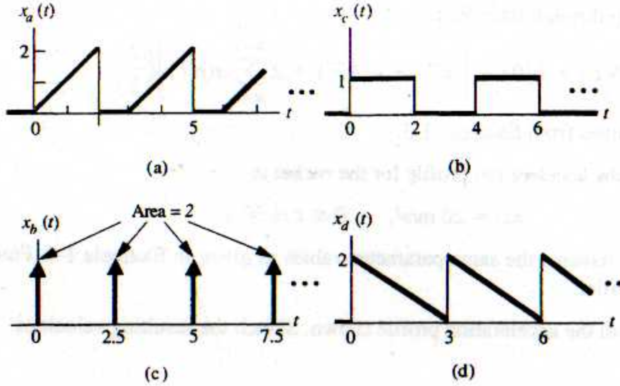


FIGURE P1-20

1-21. Represent the signals shown in terms of singularity functions.

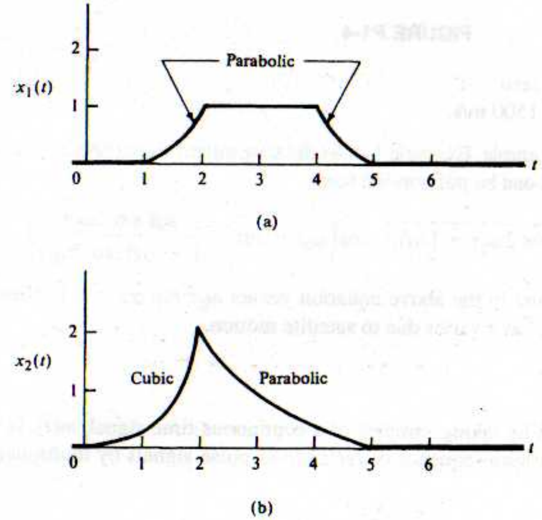


FIGURE P1-21

1-22. Write the signals shown in Figure P1-22 in terms of singularity functions.

1-23. (a) Show that

$$\delta_\epsilon(t) = \frac{e^{-t/\epsilon}}{\epsilon}u(t)$$

has the properties of a delta function in the limit as $\epsilon \to 0$.

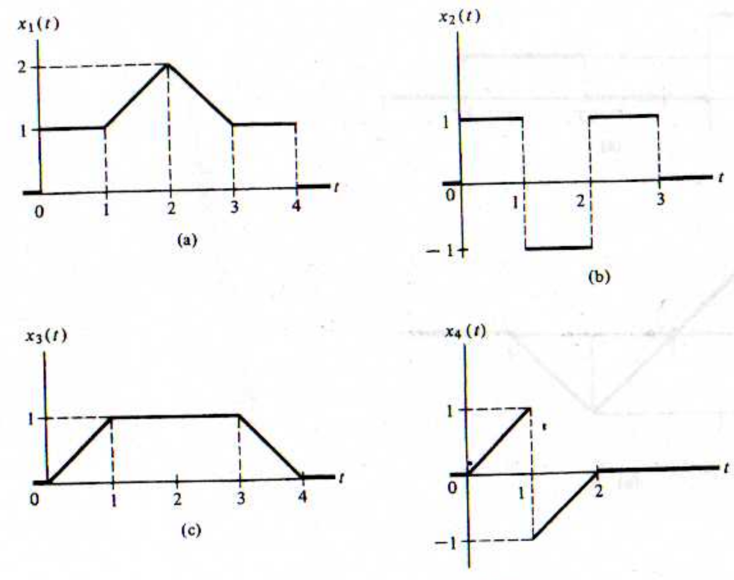


FIGURE P1-22

(b) Show that $\exp[-t^2/2\sigma^2]/\sqrt{2\pi\sigma^2}$ has the properties of a unit impulse function as $\sigma \to 0$. (*Hint:* Look up the integral of $\exp(-\alpha t^2)$ in a table of definite integrals.)

1-24. **(a)** By plotting the derivative of the function given in Problem 1-23(b) for $\sigma = 0.2$ and $\sigma = 0.05$, deduce what a unit doublet must "look like."
(b) By plotting the second derivative of the function given in Problem 1-23(b) for the same values of σ as given in part (a), deduce what a unit triplet must "look like."

1-25. In taking derivatives of product functions, one of which is a singularity function, one must exercise care. Consider the second derivative of

$$h(t) = e^{-\alpha t}u(t)$$

Blindly carrying out the derivative twice yields

$$\frac{d^2h}{dt^2} = \alpha^2 e^{-\alpha t}u(t) - 2\alpha e^{-\alpha t}\delta(t) + e^{-\alpha t}\dot{\delta}(t)$$

Use the fact that

$$\alpha e^{-\alpha t}\delta(t) = \alpha e^{-\alpha 0}\delta(t) = \alpha\delta(t)$$

to obtain the correct result.

1-26. Evaluate the following integrals.

(a) $\int_5^{10} \cos 2\pi t\, \delta(t-2)\, dt$

(b) $\int_0^{5} \cos 2\pi t\, \delta(t-2)\, dt$

(c) $\int_0^{5} \cos 2\pi t\, \delta(t-0.5)\, dt$

(d) $\int_{-\infty}^{\infty} (t-2)^2\delta(t-2)\, dt$

(e) $\int_{-\infty}^{\infty} t^2\delta(t-2)\, dt$

1-27. Evaluate the following integrals (dots over a symbol denote time derivative).

(a) $\int_{-\infty}^{\infty} e^{3t}\ddot{\delta}(t-2)\, dt$

(b) $\int_0^{10} \cos(2\pi t)\dddot{\delta}(t-0.5)\, dt$

(c) $\int_{-\infty}^{\infty} [e^{-3t} + \cos(2\pi t)]\dot{\delta}(t)\, dt$

1-28. Find the unspecified constants, denoted as $C_1, C_2, \ldots$, in the following expressions.
(a) $10\delta(t) + C_1\dot{\delta}(t) + (2 + C_2)\ddot{\delta}(t) = (3 + C_3)\delta(t) + 5\dot{\delta}(t) + 6\ddot{\delta}(t)$
(b) $(3 + C_1)\delta^{(4)}(t) + C_2\ddot{\delta}(t) + C_3\dot{\delta}(t) = C_4\delta^{(3)}(t) + C_5\delta(t)$

1-29. **(a)** Sketch the following signals:
(1) $x_1(t) = r(t+2) - 2r(t) + r(t-2)$
(2) $x_2(t) = u(t)u(10-t)$
(3) $x_3(t) = 2u(t) + \delta(t-2)$
(4) $x_4(t) = 2u(t)\delta(t-2)$
(b) For the signal shown, write an equation in terms of singularity functions.

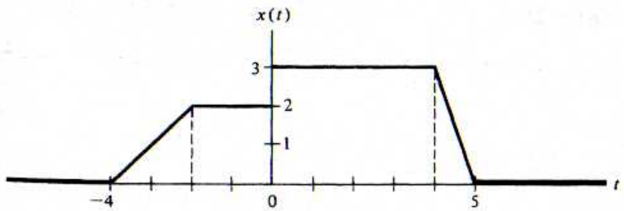


FIGURE P1-29

1-30. For the signal shown, write an equation in terms of singularity functions.

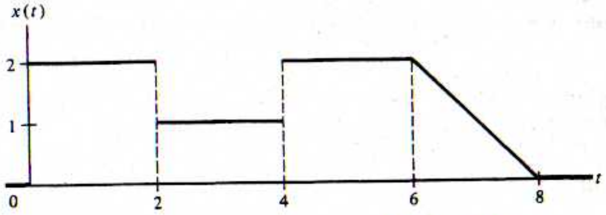


FIGURE P1-30

1-31. Evaluate the integrals given below.

(a) $\int_{-\infty}^{\infty} t^3\delta(t-3)\,dt$

(b) $\int_{-\infty}^{\infty} (3t+\cos 2\pi t)\dot{\delta}(t-5)d$

(c) $\int_{-\infty}^{\infty} (1+t^2)\dot{\delta}(t-1.5)\,dt$

1-32. Write the signals in Figure P1-32 in terms of singularity functions.

Section 1-4

1-33. Sketch the following signals and calculate their energies.

(a) $e^{-10t}u(t)$
(b) $u(t) - u(t-15)$
(c) $\cos 10\pi t\, u(t)u(2-t)$
(d) $r(t) - 2r(t-1) + r(t-2)$

1-34. Obtain the energies of the signals in Problem 1-22.

1-35. Which of the signals given in Problem 1-18 are energy signals? Justify your answers.

1-36. Obtain the average powers of the signals given in Problem 1-9.

1-37. Obtain the average powers of the signals given in Problem 1-11.

1-38. Which of the following signals are power signals and which are energy signals? Which are neither? Justify your answers.

(a) $u(t) + 5u(t-1) - 2u(t-2)$
(b) $u(t) + 5u(t-1) - 6u(t-2)$
(c) $e^{-5t}u(t)$
(d) $(e^{-5t}+1)u(t)$
(e) $(1-e^{-5t})u(t)$
(f) $r(t)$
(g) $r(t) - r(t-1)$
(h) $t^{-1/4}u(t-3)$

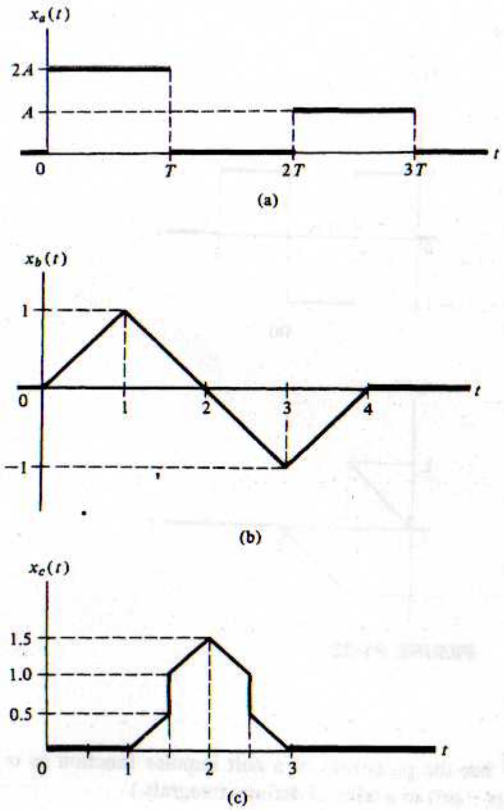


FIGURE P1-32

1-39. Given the signal

$$x(t) = 2\cos(6\pi t - \pi/3) + 4\sin(10\pi t)$$

(a) Is it periodic? If so, find its period.
(b) Sketch its single-sided amplitude and phase spectra.
(c) Write it as the sum of rotating phasors plus their complex conjugates.
(d) Sketch its two-sided amplitude and phase spectra.
(e) Show that it is a power signal.

1-40. Which of the following signals are energy signals? Find the energies of those that are. Sketch each signal.

(a) $u(t) - u(t-1)$
(b) $r(t) - r(t-1) - r(t-2) + r(t-3)$
(c) $t\exp(-2t)u(t)$
(d) $r(t) - r(t-2)$
(e) $u(t) - \frac{1}{3}u(t-10)$

1-41. Given the following signals:

(1) $\cos 5\pi t + \sin 6\pi t$

(2) $\sin 2t + \cos \pi t$

(3) $e^{-10t}u(t)$

(4) $e^{2t}u(t)$

(a) Which are periodic? Give their periods.

(b) Which are power signals? Compute their average powers.

(c) Which are energy signals? Compute their energies.

1-42. Prove Equation (1-84) by starting with (1-76).

1-43. Given the signal

$$x(t) = \sin^2(7\pi t - \pi/6) + \cos(3\pi t - \pi/3)$$

(a) Sketch its single-sided amplitude and phase spectra.

(b) Sketch its double-sided amplitude and phase spectra after writing it as the sum of complex conjugate rotating phasors.

Section 1-5

1-44. Plot the power spectral density of the signal given in Problem 1-43.

1-45. Given the signal

$$x(t) = 16\cos(20\pi t + \pi/4) + 6\cos(30\pi t + \pi/6) + 4\cos(40\pi t + \pi/3)$$

(a) Find and plot its power spectral density.

(b) Compute the power contained in the frequency interval 12 Hz to 22 Hz.

Computer Exercises

1-1. Use the functions given in Section 1-6 for the step and ramp signals to plot the signal shown in Problem 1-30 using MATLAB.

1-2. Use the functions given in Section 1-6 for the step and ramp signals to plot the signals shown in Problem 1-32 using MATLAB.

1-3. Use the elementary function programs given in Section 1-6 to compute and plot the following:

(a) A step of height 3 starting at $t = 3$ and going backwards to $t = -\infty$;

(b) A signal that starts at $t = 1$, increases linearly to a value of 2 at $t = 2$, and is constant thereafter;

(c) A stairstep signal that is 0 for $t < 0$, jumps to a value of 1 at $t = 0$, a value of 2 at $t = 1$, a value of 3 at $t = 2$, a value of 4 at $t = 3$, and stays at 4 thereafter;

(d) A ramp starting at $t = 2$ and going downward with a slope of -3.

1-4. (a) Generate a cosine burst of frequency 2 Hz, lasting for 5 seconds; (b) generate a sine burst of frequency 2 Hz and lasting for five seconds; (c) combine the results of (a) and (b) to produce a sinusoidal burst of frequency 2 Hz, 5 seconds long, and with starting phase at $t = 0$ of $\pi/4$ radians. [Hint: recall the trigonometric identity $\sin(x + y) = 0.5(\sin x \cos y - \cos x \sin y)$. Set $y = \pi/4$.]

1-5. (a) Write a MATLAB function to generate a sequence of impulses spaced by an arbitrary amount `t_rep`, lasting for `t_width` and centered on $t = 0$. Call it `cmb_fn(t,t_rep, t_width,delta)`

(b) Generate an impulse comb with spacing between impulses of 1.25 seconds and containing 5 impulses starting at $t = 0$.

1-6. (a) Write a MATLAB function to generate a unit parabola singularity function.

(b) Write a MATLAB function to generate a unit cubic singularity function.

(c) Use them to generate a plots of the signals shown in Problem 1-21.

CHAPTER 2

System Modeling and Analysis in the Time Domain

2-1 Introduction

Having discussed some continuous-time signal models, we now wish to consider ways of modeling systems and to analyze the effects of systems on signals. The systems analysis problem is: *given a system and an input, what is the output?* System design or synthesis is important also, but generally is more difficult than analysis. In addition, analysis procedures also suggest synthesis techniques.

We will discuss several mathematical characterizations for systems in this chapter, but our main focus throughout the book will be on *linear, time-invariant systems.* The reasons for this are threefold: First, powerful analysis techniques exist for such systems. Second, many real-world systems can be closely approximated as linear, time-invariant systems.† Third, analysis techniques for linear, time-invariant systems suggest approaches for the analysis of nonlinear systems.

There are three basic approaches for finding the response of a fixed, linear system to a given input: (1) Obtain a solution to the modeling equations through standard methods of solving differential equations. (2) Carry out a solution in the time domain, using the superposition integral. (3) Use frequency-domain analysis by means of the Fourier or Laplace transforms. The first method should be familiar to the student from circuits courses. The objective of this chapter is to examine the second method. The third method is dealt with in later chapters.

In passing, we note that the condition of time invariance is not necessary in many of the analysis procedures presented in this chapter, but computational problems are simplified under the assumption of time invariance.

We begin our consideration of systems in the time domain with appropriate classification procedures and modeling techniques. The primary tool for analysis of linear, time-invariant systems will be the *convolution* integral, discussed in Section 2.4.

2-2 System Modeling Concepts

Some Terminology

We now wish to look at ways of representing the effects of systems on signals; that is, we need to be able to construct appropriate *system models* that adequately represent the interaction of signals and systems and the relationship of *causes* and *effects* for that system. Usually, we refer to certain causes of in-

†The terms "linear" and "time invariant" will be given precise mathematical definitions shortly. Although it is possible to analyze systems that are time varying, it is more difficult than for time-invariant systems. Because of this, and because many systems are time invariant, we restrict our attention to this category.

terest as *inputs* and certain effects of interest as *outputs*. For example, in the accelerometer example of Chapter 1, the input of interest could be the position of the weight and the output could be a voltage proportional to velocity.

An obvious choice for input and output quantities may not always be readily apparent, easily isolated, or physically distinct. For instance, in the communications link example, the input, $x(t)$, and output of interest, $s(t)$, appear at the same physical location: the transmitter. The relationship governing their interaction was very simple: that is,

$$s(t) = x(t) + \alpha\beta x(t - 2\tau), \qquad 0 \le t \le T + 2\tau \tag{1-15}$$

Usually, we will not have such simple input-output relationships.

It is convenient to visualize a system schematically by means of a box, as shown in Figure 2-1. (Sometimes symbols other than x and y may be used.) On the left-hand side of the box we represent the inputs (excitations, causes, stimuli) as a series of arrows labeled $x_1(t), x_2(t), \ldots, x_m(t)$. The outputs (responses, effects), $y_1(t), y_2(t), \ldots, y_p(t)$, are represented as arrows emanating from the right-hand side of the box. In general, the inputs and outputs vary with time, which is represented by the independent variable t; sometimes it will not be shown explicitly in order to simplify notation. The number of inputs and outputs need not be equal. Quite often, however, we will be concerned with situations where there is a single input and a single output. Such systems are referred to as *two-port* or *single-input, single-output systems* since they have one input port and one output port.

Representations for Systems

Usually, we are interested in obtaining explicit relationships between the input and output variables of a system, and we discuss systematic procedures for doing so shortly. For now, however, we will express the dependence of the system output on the input symbolically as

$$y(t) = \mathcal{H}[x(t)] \tag{2-1a}$$

Eliminating the explicit use of t, we can write this more compactly as

$$y = \mathcal{H}x \tag{2-1b}$$

which is read

$y(t)$ is the response of $\mathcal{H}$ to $x(t)$

where initial conditions are included if pertinent. The symbol $\mathcal{H}$, which is known as an *operator,* serves the dual role of identifying the system and specifying the operation to be performed on $x(t)$ to produce $y(t)$. (For a multiple-input, multiple-output system, $x(t)$ and $y(t)$ are vectors.)

Several examples of system input-output relationships are given below to illustrate more clearly some of the basic concepts.

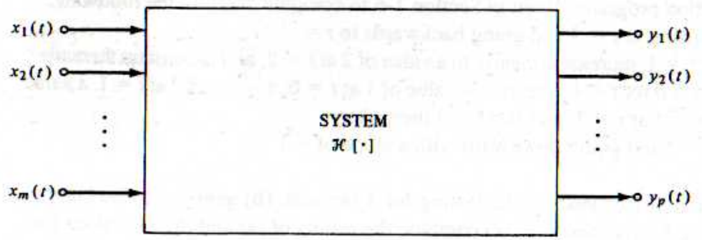


FIGURE 2-1. Block diagram representation of a system.

1. Instantaneous (Nondynamic) Relationships. Many systems are adequately modeled for many situations by relationships such as

$$y(t) = Ax(t) + B \tag{2-2a}$$

or

$$y(t) = Ax(t) + Bx^3(t) \tag{2-2b}$$

where A and B are constants. The former equation could represent an amplifier of gain A with a dc bias of B volts at its output, and the latter equation would be a possible model for an amplifier that introduces *nonlinear distortion*, $Bx^3(t)$, into the amplified output, $Ax(t)$. The concept of nonlinearity for a system will be explicitly defined later.

2. Linear, Constant-Coefficient, Ordinary Differential Equations. The student is assumed to be familiar with obtaining the relationships between currents and voltages in an electrical circuit composed of passive elements (resistors, capacitors, and inductors) in terms of ordinary integrodifferential equations through the application of Kirchhoff's voltage and current laws. If all integrals are removed through repeated differentiation, the general form for such equations may be written as

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}} + \cdots + b_0 x(t) \tag{2-3}$$

where $a_n, a_{n-1}, \ldots, a_0, b_m, b_{m-1}, \ldots, b_0$ are constants that depend on the structure of the system and where nonzero initial conditions may be specified. For systems whose input-output relationships can be represented in this manner, the *order* of the system is defined as the order of the highest derivative of the *dependent variable*, or system output, present (integrals on the input might still be present in some cases).†

Three variations of (2-3) sometimes result from the modeling and analysis of systems. First, time-varying components may be present in the system (e.g., a capacitor whose capacitance varies with time). In such cases one or more of the *coefficients* of (2-3) will be *functions of time*. These are examples of *time-varying systems*, although the class of time-varying systems is broader than that described by ordinary differential equations with nonconstant coefficients.

A second situation that results in a governing differential equation of different form than (2-3) is where one or more nonlinear components are present in the system. Consider a voltage source $v_s(t)$ in series with an inductor and a resistor whose resistance depends on the current through it: for example, $R = R_0 + \alpha i(t)$, where R_0 and α are constants. Then application of Kirchhoff's voltage law results in

$$[R_0 + \alpha i(t)]i(t) + L\frac{di}{dt} = v_s(t) \tag{2-4}$$

and the relationship between the input, $v_s(t)$, and the output, $i(t)$, is no longer linear. This is an example of a *nonlinear* system. Precise definitions for time-varying and nonlinear systems are given later.

The third departure from (2-3) which may result is that of a system described by a *partial differential equation*. Such systems are said to be *distributed*, in contrast to *lumped* systems, for which the cause-effect properties of the system can be ascribed to elements modeled as infinitesimally small in the spatial dimension. This is permissible if their dimensions are small compared with the wavelength of the variable quantities within the system (voltages, currents, etc.), where the wavelength is given by

$$\lambda = \frac{c}{f} \quad \text{meters}$$

†The order of a multiple-input, multiple-output system can be determined through state-variable techniques, which are discussed in Chapter 7.

where $c = 3 \times 10^8$ m/s is the speed of electromagnetic radiation and f is the frequency of the oscillating waveform in hertz. A light bulb, for example, has dimensions that are small compared with $\lambda = 3 \times 10^8/60 = 5 \times 10^6$ m, which is the wavelength of the 60-Hz power-line voltage.

3. Integral Relationships. We discuss an integral representation of a system input-output relation of the form

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\,d\lambda \tag{2-5a}$$

$$= \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda \tag{2-5b}$$

which is known as a *superposition integral*. The function $h(\lambda)$ is called the *impulse response* of the system and is its response to a unit impulse applied at $\lambda = 0$.† All systems that can be represented by ordinary, constant-coefficient, linear differential equations can also be represented by an equation of the form (2-5). However, not all systems that can be represented by a relationship of the form (2-5) can be represented by ordinary, constant-coefficient linear differential equations.‡

For a time-varying system, the generalization of (2-5) is

$$y(t) = \int_{-\infty}^{\infty} h(t, \lambda)x(\lambda)\,d\lambda \tag{2-6}$$

where $h(t, \lambda)$ is the response of the system at time t to a unit impulse applied at time λ.

EXAMPLE 2-1

Consider the RC circuit depicted in Figure 2-2 with input $x(t)$ and output $y(t)$ as shown. We wish to obtain (a) the differential equation relating $y(t)$ to $x(t)$, and (b) convert this differential equation to the integral form (2-5). Assume that $x(t)$ is applied at $t = t_0$ and $y(t_0) = y_0$.

Solution:

(a) The differential equation relating $y(t)$ and $x(t)$ is found by writing Kirchhoff's voltage law (KVL) around the loop indicated by $i(t)$. This results in the equation

$$x(t) = Ri(t) + y(t) \tag{2-7}$$

where $y(t)$, the voltage across the capacitor, is related to $i(t)$, the current through it, by

$$i(t) = C\frac{dy(t)}{dt} \tag{2-8}$$

Eliminating $i(t)$ in the KVL equation, we obtain

$$RC\frac{dy(t)}{dt} + y(t) = x(t) \tag{2-9}$$

†To show this, let $x(t) = \delta(t)$, so that $x(t-\lambda) = \delta(t-\lambda)$. Then $y(t) = h(t)$ by the sifting property of the unit impulse function, and $h(t)$ is seen to be the system's response to $\delta(t)$.

‡An example is an ideal low-pass filter whose impulse response is

$$h(t) = B\frac{\sin(2\pi Bt)}{2\pi Bt}$$

where B is the bandwidth. This type of filter will be discussed in Chapter 4. Ideal filters are convenient mathematical models, but cannot be realized as physical circuits.

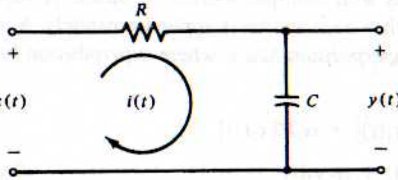


FIGURE 2-2. *RC* circuit to illustrate the derivation of input-output relationships for systems.

(b) To write the input-output relationship as an explicit equation for $y(t)$ in terms of $x(t)$, we first obtain the solution to the homogeneous differential equation

$$RC\frac{dy_h(t)}{dt} + y_h(t) = 0 \tag{2-10}$$

Assuming a solution of the form

$$y_h(t) = Ae^{pt} \tag{2-11}$$

and substituting into the homogeneous equation, we find that $p = -1/RC$, so that

$$y_h(t) = A\exp\left(\frac{-t}{RC}\right) \tag{2-12}$$

The total solution consists of $y_h(t)$ plus a particular solution for a specific $x(t)$. Assume that the input $x(t)$ is applied beginning at $t = t_0$ but is otherwise arbitrary. Also assume that the value of $y(t)$ at $t = t_0$ is y_0. To find the total solution we use the technique of *variation of parameters,* which consists of assuming a solution of the form of $y_h(t)$ but with the undetermined coefficient A replaced by a function of time to be found. Thus we assume that

$$y(t) = A(t)\exp\left(\frac{-t}{RC}\right) \tag{2-13}$$

Differentiating by means of the chain rule, we obtain

$$\frac{dy(t)}{dt} = \left[\frac{dA(t)}{dt} - \frac{A(t)}{RC}\right]\exp\left(\frac{-t}{RC}\right) \tag{2-14}$$

Substituting the assumed solution and its derivative into the nonhomogeneous differential equation, we have

$$RC\exp\left(\frac{-t}{RC}\right)\frac{dA(t)}{dt} = x(t) \tag{2-15}$$

where the term $-A(t)\exp(-t/RC)$ has been canceled by $y(t)$ in the differential equation. Solving for $dA(t)/dt$ and integrating, we get an explicit result for the unknown "varying parameter." It is given by

$$A(t) = \frac{1}{RC}\int_{t_0}^{t} x(\lambda)\exp\left(\frac{\lambda}{RC}\right)d\lambda + A(t_0) \tag{2-16}$$

where λ is a dummy variable of integration and $A(t_0)$ is the initial value of $A(t)$ at the time t_0 that the input $x(t)$ is applied. Since $y(t) = A(t)\exp(-t/RC)$, it follows that $A(t_0) = y_0\exp(t_0/RC)$. Using this result together with the expression for $A(t)$, we obtain

$$y(t) = \left[\int_{t_0}^{t} x(\lambda)\frac{\exp(\lambda/RC)}{RC}\,d\lambda + y_0\exp\left(\frac{t_0}{RC}\right)\right]\exp\left(\frac{-t}{RC}\right)$$

$$= y_0\exp\left(-\frac{t-t_0}{RC}\right) + \int_{t_0}^{t} x(\lambda)\frac{\exp[-(t-\lambda)/RC]}{RC}\,d\lambda \tag{2-17}$$

If we assume that the input $x(t)$ is applied at $t = -\infty$ and that $y_0 \triangleq y(-\infty) = 0$, then

$$y(t) = \int_{-\infty}^{t} x(\lambda)\frac{\exp[-(t-\lambda)/RC]}{RC}\,d\lambda \tag{2-18}$$

If we define

$$h(t) = \frac{\exp(-t/RC)}{RC}u(t) \tag{2-19}$$

so that

$$h(t-\lambda) = \frac{\exp[-(t-\lambda)/RC]}{RC}u(t-\lambda) \tag{2-20}$$

then the solution for $y(t)$, as determined by the variation-of-parameters approach, is exactly of the form given by (2-5b).

Properties of Systems

We now give precise definitions of several system properties, some of which were mentioned in the previous discussion.

Continuous-Time and Discrete-Time Systems. If the signals processed by a system are continuous-time signals, the system itself is referred to as a *continuous-time* system. If, on the other hand, the system processes signals that exist only at discrete times, it is called a *discrete-time* system. The signals in a system may or may not be quantized to a finite number of levels. If they are, the system is referred to as *quantized.* A quantized system may be continuous time or discrete time, although usually a quantized system is also a discrete-time system and is then referred to as a *digital system.* Discrete-time systems are discussed in Chapter 8.

Examples of continuous-time systems are electric networks composed of resistors, capacitors, and inductors that are driven by continuous-time sources. An example of a quantized, discrete-time system is the optical-wand sensing system of a department store cash register.

Fixed and Time-Varying Systems. A system is *time invariant,* or *fixed,* if its input-output relationship does not change with time. Otherwise, it is said to be *time varying.* In terms of the symbolic notation of (2-1a), a system is fixed if and only if

$$\mathcal{H}[x(t-\tau)] = y(t-\tau) \tag{2-21}$$

for *any* $x(t)$ and *any* τ. The system must be at rest[†] prior to the application of $x(t)$. In words, (2-21) states that if $y(t)$ is the response of the system to an input $x(t)$, then its response to a time-shifted version of

[†]A system is at rest, or relaxed, prior to some initial instant, say $t = t_0$, if all energy storage elements have zero initial energy stored. For an nth-order system described by an nth-order ordinary linear differential equation, this means that the dependent variable and all its derivatives up through the $(n-1)$st are zero for $t < t_0$.

this input, $x(t-\tau)$, is the response of the system to $x(t)$ *time-shifted* by the same amount, or $y(t-\tau)$. For example, the system with input-output relationship

$$y(t) = x(t) + Ax(t-T) \tag{2-22}$$

is fixed if A and T are constants, and time varying if either or both are functions of time. Thus the satellite communication system is fixed if $\alpha\beta$ and τ are constants. Similarly, a system described by a differential equation of the form (2-3) is fixed if the coefficients are constant, and time varying if one or more coefficients are functions of time.

Causal and Noncausal Systems. A system is *causal* or *nonanticipatory* if its response to an input does not depend on future values of that input.‡ A precise definition is that a continuous-time system is causal if and only if the condition

$$x_1(t) = x_2(t) \qquad \text{for } t \le t_0 \tag{2-23a}$$

implies the condition

$$\mathcal{H}[x_1(t)] = \mathcal{H}[x_2(t)] \qquad \text{for } t \le t_0 \tag{2-23b}$$

for any t_0, $x_1(t)$, and $x_2(t)$. Stated another way, if the *difference* between two system inputs is zero for $t \le t_0$, the *difference* between the respective outputs must be zero for $t \le t_0$ if the system is causal. Thus the definition applies to systems for which the response to zero input is not zero, such as a system containing independent sources, as well as systems with nonzero inputs.

EXAMPLE 2-2

Consider a system described by (2-2a). Let $x_1(t) = u(t)$ and $x_2(t) = r(t)$. For $t \le t_0 < 0$, both inputs are equal. The outputs for $t \le t_0$ for both inputs are $y_1(t) = y_2(t) = B$. In general, it is seen from (2-2a) that if $x_1(t) = x_2(t)$, $t \le t_0$, then $y_1(t) = y_2(t) = B$ for $t \le t_0$ because $Ax_1(t) = Ax_2(t)$. It is easy to see that this holds for any pair of inputs for which it is true that $x_1(t) = x_2(t)$, $t \le t_0$. Thus this system is *causal.* An easy way to see that the system is causal is to note that $y(t)$ cannot do anything in anticipation of $x(t)$.

Dynamic and Instantaneous Systems. A system for which the output is a function of the input *at the present time only* is said to be *instantaneous* (or *memoryless,* or *zero memory*). A *dynamic* system, or one which is not instantaneous, is one whose output depends on past or future values of the input in addition to the present time. If the system is also causal, the output of a dynamic system depends only on present and past values of the input. Mathematically, the input-output relationship for an instantaneous system must be of the form

$$y(t) = f[x(t), t] \tag{2-24}$$

where $f(\cdot)$ is a function, possibly time dependent, which depends on $x(t)$ *only at the present time, t.*

The systems described by (2-2a) and (2-2b) are instantaneous. Systems described by differential equations are dynamic (nonzero memory). For example, an inductor is a system with memory. This is easy to see if the input is taken as the voltage across its terminals and the output as the current through it so that

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\lambda)\, d\lambda \tag{2-25}$$

It is clear that $i(t)$ depends on past values of $v(t)$ through the integral.

‡Another way of informally defining a causal system is that such a system does not anticipate its input.

Linear and Nonlinear Systems. From your circuits courses, you should already be familiar with the concept of linearity. It allows the analysis of circuits with multiple sources by means of *superposition,* or addition of currents or voltages calculated when each source is applied separately. A precise mathematical definition of a linear system is that *superposition holds,* where superposition for a system with any two inputs $x_1(t)$ and $x_2(t)$ is defined as

$$\begin{aligned}\mathcal{H}[\alpha_1 x_1(t) + \alpha_2 x_2(t)] &= \alpha_1 \mathcal{H}[x_1(t)] + \alpha_2 \mathcal{H}[x_2(t)] \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t)\end{aligned} \tag{2-26}$$

In (2-26), $y_1(t)$ is the response of the system when input $x_1(t)$ is applied alone, and $y_2(t)$ is the response of the system when input $x_2(t)$ is applied alone; α_1 and α_2 are arbitrary constants.

EXAMPLE 2-3

Show that the system described by the differential equation

$$\frac{dy(t)}{dt} + ty(t) = x(t)$$

is linear.

Solution: The response to $x_1(t)$, $y_1(t)$, satisfies

$$\frac{dy_1}{dt} + ty_1 = x_1$$

and the response to $x_2(t)$, $y_2(t)$, satisfies

$$\frac{dy_2}{dt} + ty_2 = x_2$$

where the time dependence of x_1, x_2, y_1, and y_2 is omitted for simplicity. Addition of these equations multiplied, respectively, by α_1 and α_2 results in

$$\alpha_1\frac{dy_1}{dt} + \alpha_2\frac{dy_2}{dt} + \alpha_1 ty_1 + \alpha_2 ty_2 = \alpha_1 x_1 + \alpha_2 x_2$$

or

$$\frac{d}{dt}(\alpha_1 y_1 + \alpha_2 y_2) + t(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 x_1 + \alpha_2 x_2$$

That is, the response to the input $\alpha_1 x_1 + \alpha_2 x_2$ is $\alpha_1 y_1 + \alpha_2 y_2$. Thus superposition holds and the system is linear.

EXAMPLE 2-4

The system described by the differential equation

$$\frac{dy(t)}{dt} + 10y(t) + 5 = x(t)$$

is *nonlinear*. This is demonstrated by attempting to apply the superposition principle. Letting $y_1(t)$ be the response to the input $x_1(t)$ and $y_2(t)$ be the response to the input $x_2(t)$, we write

$$\frac{dy_1}{dt} + 10y_1 + 5 = x_1$$

and

$$\frac{dy_2}{dt} + 10y_2 + 5 = x_2$$

where the time dependence is again dropped for simplicity. Multiplying the first equation by an arbitrary constant α_1, the second equation by the arbitrary constant α_2, adding, and regrouping terms, we obtain.

$$\frac{d}{dt}(\alpha_1 y_1 + \alpha_2 y_2) + 10(\alpha_1 y_1 + \alpha_2 y_2) + 5(\alpha_1 + \alpha_2) = \alpha_1 x_1 + \alpha_2 x_2 \tag{2-27}$$

We *cannot* put this equation into the same form as the original differential equation for an *arbitrary choice* of α_1 and α_2. Thus the system is *nonlinear*.

2-3 The Superposition Integral for Fixed, Linear Systems

In this section we show that (2-5) can be used to obtain the output of a fixed, linear system in response to an input, $x(t)$, where the response of the system to a unit impulse applied at $t = 0$ is $h(t)$ [$h(t)$ is referred to as the *impulse response* of the system]. In this context, (2-5) is called the *superposition integral*. We must emphasize that this integral is more general than for just the computation of the response of a linear, time-invariant system. Some of the other applications of the superposition integral are discussed in Chapter 4.

A short, mathematical derivation of (2-5) is as follows. Represent the input signal to the linear, time-invariant system in terms of the sifting property of the unit impulse as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \tag{2-28}$$

In terms of the system operator, $\mathcal{H}$, the output is

$$y(t) = \mathcal{H}[x(t)]$$

$$= \mathcal{H}\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right] \tag{2-29}$$

Assuming that the order of integration and operation by the system can be interchanged, (2-29) becomes

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\mathcal{H}[\delta(t-\tau)]\,d\tau \tag{2-30}$$

By definition, $\mathcal{H}[\delta(t-\tau)]$ is the response of the system to a unit impulse applied at $t = \tau$. For a *fixed* system, we can write this as $h(t-\tau)$, where $h(t)$ is the system's response to a unit impulse applied at $t = 0$. Substitution of $h(t-\tau)$ for $\mathcal{H}[\delta(t-\tau)]$ in (2-30) results in (2-5).

In addition to being strictly mathematical, an additional difficulty with this derivation is the blind faith required in the interchange of the improper integral and the $\mathcal{H}$-operator. Thus we consider another derivation that is more graphic in nature.

Assume for the time being that the input to the fixed, linear system is a continuous function of time in the time interval $T_1 \le t \le T_2$ and zero elsewhere. Figure 2-3a illustrates an arbitrary input, $x(t)$, in the time interval $T_1 \le t \le T_2$ which has been approximated by a stairstep function $\hat{x}(t)$. That is, $\hat{x}(t)$ can be expressed in terms of the unit pulse function $\Pi(t)$ as the summation

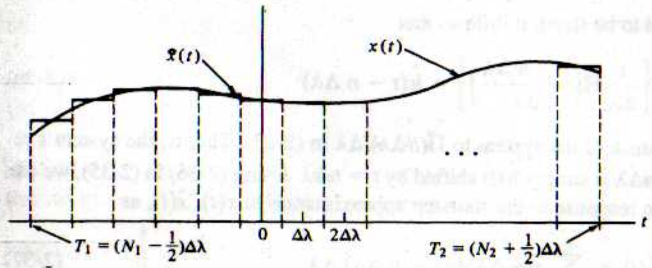
$$\hat{x}(t) = \sum_{n=N_1}^{N_2} x(n\,\Delta\lambda)\Pi\left(\frac{t - n\,\Delta\lambda}{\Delta\lambda}\right)$$

$$= \sum_{n=N_1}^{N_2} x(n\,\Delta\lambda)\frac{1}{\Delta\lambda}\Pi\left(\frac{t - n\,\Delta\lambda}{\Delta\lambda}\right)\Delta\lambda, \qquad T_1 \le t \le T_2 \tag{2-31}$$

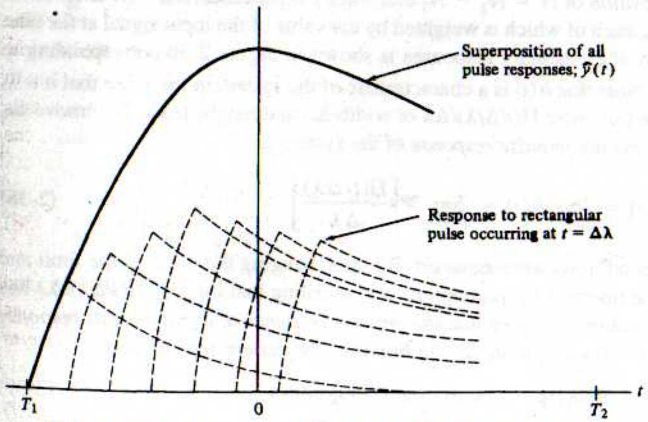
where $T_1 = (N_1 - 1/2)\,\Delta\lambda$ and $T_2 = (N_2 + 1/2)\,\Delta\lambda$. The reason for multiplying and dividing by $\Delta\lambda$ to obtain the second equation will be apparent shortly. Clearly, the approximation of $\hat{x}(t)$ to $x(t)$ improves as $\Delta\lambda$ gets smaller. In fact, in the limit as $\Delta\lambda \to 0$, (2-31) becomes the sifting integral for the unit impulse function since

$$\lim_{\Delta\lambda \to 0} \frac{\Pi[(t - n\,\Delta\lambda)/\Delta\lambda]}{\Delta\lambda} = \delta(t - \lambda), \tag{2-32}$$

where $n\,\Delta\lambda$ has been replaced by the continuous variable λ.



(a) Input to a fixed, linear system and a stairstep approximation



(b) Output of a fixed, linear system due to the stairstep approximate input.

FIGURE 2-3. Input and output signals for a fixed, linear system used to derive the superposition integral.

Now consider the response of a fixed linear system to the summation of pulses (2-31). Let the response of the system to $\Pi(t/\Delta\lambda)\,\Delta\lambda$ be $\hat{h}(t)$. That is, in terms of the operator notation

$$\hat{h}(t) = \mathcal{H}\left[\frac{\Pi(t/\Delta\lambda)}{\Delta\lambda}\right] \tag{2-33}$$

where $\mathcal{H}[\cdot]$ signifies the operation that produces the system output in response to a particular input. We want the output of a fixed, linear system in response to the input (2-31). In terms of the operator $\mathcal{H}[\cdot]$, it is

$$\hat{y}(t) = \mathcal{H}[\hat{x}(t)] = \mathcal{H}\left[\sum_{n=N_1}^{N_2} x(n\,\Delta\lambda)\frac{1}{\Delta\lambda}\Pi\left(\frac{t-n\,\Delta\lambda}{\Delta\lambda}\right)\Delta\lambda\right] \tag{2-34}$$

Because $\mathcal{H}[\cdot]$ represents a linear system, superposition holds and (2-34) can be written as

$$\hat{y}(t) = \sum_{n=N_1}^{N_2} x(n\,\Delta\lambda)\mathcal{H}\left[\frac{1}{\Delta\lambda}\Pi\left(\frac{t-n\,\Delta\lambda}{\Delta\lambda}\right)\right]\Delta\lambda \tag{2-35}$$

where we assume that all initial conditions are zero. More will be said about nonzero initial conditions shortly. Although the superposition property is stated by (2-26) for inputs of the form $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$, we may generalize it easily to the linear combination of any finite number of terms (Problem 2-15).

Because the system is assumed to be fixed, it follows that

$$\mathcal{H}\left[\frac{1}{\Delta\lambda}\Pi\left(\frac{t-n\,\Delta\lambda}{\Delta\lambda}\right)\right] = \hat{h}(t-n\,\Delta\lambda) \tag{2-36}$$

where $\hat{h}_t$ was defined as the response of the system to $\Pi(t/\Delta\lambda)/\Delta\lambda$ in (2-33). That is, the system's response to a pulse centered at $t = n\Delta\lambda$ is simply $\hat{h}(t)$ shifted by $t = n\Delta\lambda$. Using (2-36) in (2-35), we can express the system output, $\hat{y}(t)$, in response to the stairstep approximation to $x(t)$, $\hat{x}(t)$, as

$$\hat{y}(t) = \sum_{n=N_1}^{N_2} x(n\,\Delta\lambda)\hat{h}(t-n\,\Delta\lambda)\,\Delta\lambda \tag{2-37}$$

which shows that y is the *superposition* of $N = N_2 - N_1$ elementary repsponses, $\hat{h}(t - n\,\Delta\lambda)$, of the system to $\Pi[(t - n\,\Delta\lambda)/\Delta\lambda]/\Delta\lambda$, each of which is weighted by the value of the input signal at the time $t = n\,\Delta\lambda$. A typical superposition of elementary responses is shown in Figure 2-3b corresponding to the stairstep input of Figure 2-3a. Note that $\hat{h}(t)$ is a characteristic of the system in the sense that it tells us how the system responds to the test pulse $\Pi(t/\Delta/\lambda)/\Delta\lambda$ of width $\Delta\lambda$ and height $1/\Delta\lambda$. To remove the dependence of $\hat{h}(t)$ on $\Delta\lambda$, we *define* the *impulse response* of the system as

$$h(t) = \lim_{\Delta\lambda\to 0}\hat{h}(t) = \lim_{\Delta\lambda\to 0}\mathcal{H}\left[\frac{\Pi(t/\Delta\lambda)}{\Delta\lambda}\right] \tag{2-38}$$

where we recall that zero initial conditions were assumed. By interchanging the order of the limit and the operator $\mathcal{H}[\cdot]$, which is a linear operator by assumption, and recalling that $\lim_{\Delta\lambda\to 0}\Pi(t/\Delta\lambda)/\Delta\lambda$ has the properties of a unit impulse function, it is seen that *the impulse response of a system is its response to a unit impulse applied at time $t = 0$ with all initial conditions of the system zero.* That is,

$$h(t) \triangleq \mathcal{H}[\delta(t)] \qquad \text{(zero initial conditions)} \tag{2-39}$$

Returning to (2-37) and taking the limit as $\Delta\lambda \to 0$ and $n\,\Delta\lambda \to \lambda$, a continuous variable, we recognize (2-37) as an approximation to the integral

$$y(t) = \int_{T_1}^{T_2} x(\lambda)h(t-\lambda)\,d\lambda, \qquad T_1 \le t \le T_2 \tag{2-40}$$

Assuming that the input may have been present since the infinite past and may last indefinitely into the future, we have, in the limit as $T_1 \to -\infty$ and $T_2 \to \infty$,

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda \tag{2-41}$$

for the response of a system characterized by an impulse response $h(t)$ and having an input $x(t)$.

With the lower limit of $t = -\infty$ on (2-41), we can include the response due to initial conditions by identifying it as the response due to $x(t)$ from $t = -\infty$ to an appropriately chosen starting instant, t_0, usually chosen as $t_0 = 0$. In Chapter 5 we consider another way to handle initial conditions. Making the substitution $\sigma = t - \lambda$, we obtain the equivalent result

$$y(t) = \int_{-\infty}^{\infty} x(t-\sigma)h(\sigma)\,d\sigma \tag{2-42}$$

Because these equations were obtained by superposition of a number of elementary responses due to each individual impulse, they are special cases of what are referred to as *superposition integrals.*† A simplification results if the system under consideration is causal. We defined such a system as being one that did not anticipate its input. For a causal system, $h(t - \tau) = 0$ for $t < \tau$, and the upper limit of (2-41) can be set equal to t. Furthermore, if $x(t) = 0$ for $t < 0$, the lower limit becomes zero, and the resulting integral is given by

$$y(t) = \int_{0}^{t} x(\lambda)h(t-\lambda)\,d\lambda, \qquad t \ge 0 \tag{2-43}$$

for a causal system with input zero for $t < 0$ and zero initial conditions.

EXAMPLE 2-5

A concrete example of the previous discussion will help to make the derivation clearer and lend credence to the stairstep approximation to $x(t)$ being convolved with $h(t)$ actually providing a good approximation to $y(t)$. The signals $x(t)$ and $h(t)$ in this example are given by

$$x(t) = (t+10)e^{-0.4(t+10)}u(t+10)$$

and

$$h(t) = e^{-t}u(t)$$

A MATLAB program (not given here, since the purpose of this example is to illustrate the preceding discussion) provides the plots shown in Figure 2-4. The top figure gives $x(t)$ and the stairstep approximation to it, $\tilde{x}(t)$, where the step length is 0.5. The signal $h(t)$ is shown in the second figure, and the result of convolving $\tilde{x}(t)$ and $h(t)$ is shown in the third figure. Finally, the actual convolution of $x(t)$ and $h(t)$, computed analytically, is shown in the fourth figure. The approximate and actual results are amazingly close for the step length of 0.5. As this step length is decreased, the correspondence will be closer and closer, and it is not hard to believe that (2-37) actually converges to (2-40).

†In this special case (i.e., a linear, time-invariant system) they are of the same form as the convolution integral to be considered in more detail in Section 2-4. A general form for the superposition integral for a linear time-varying system was given by (2-6).

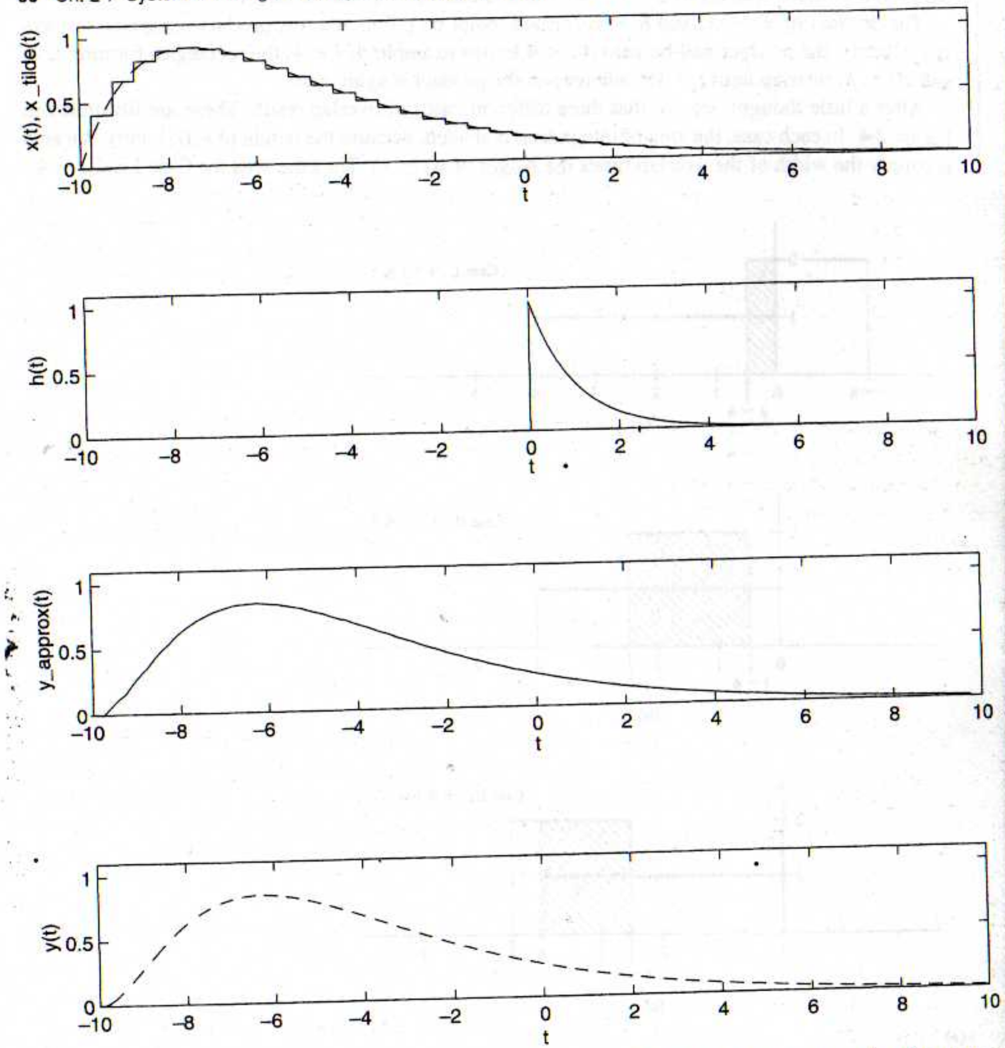


FIGURE 2-4. Plots for Example 2-5 showing a specific example for the stairstep approximation to the derivation of the superposition integral.

EXAMPLE 2-6

Consider the system with impulse response

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t) \tag{2-44}$$

Let the input be a unit step:

$$x(t) = u(t) \tag{2-45}$$

Applying (2-42), we obtain the output

$$a(t) = \int_{-\infty}^{\infty} u(t-\sigma) \frac{1}{RC} e^{-\sigma/RC} u(\sigma)\, d\sigma$$

$$= \int_{0}^{t} \frac{1}{RC} e^{-\sigma/RC} d\sigma, \qquad t \geq 0 \tag{2-46}$$

$$= 1 - e^{-t/RC}, \qquad t \geq 0$$

Since $a(t) = 0$ for $t < 0$, this output, referred to as the *step response* of the system, can be written compactly as

$$a(t) = \left[1 - \exp\left(\frac{-t}{RC}\right)\right] u(t) \tag{2-47}$$

Note that $a(t) = \int_{-\infty}^{t} h(\lambda)\, d\lambda$.

2-4 Examples Illustrating Evaluation of the Convolution Integral

The convolution of two signals, $x(t)$ and $h(t)$, is a new function of time, $y(t)$, which is given by (2-5). A useful symbolic notation often employed to denote (2-5a) and (2-5b), respectively, is

$$y(t) = h(t) * x(t) \tag{2-48}$$

$$y(t) = x(t) * h(t) \tag{2-49}$$

It is possible to prove several properties of the convolution integral. These are listed below, but their proofs are left to the problems.

1. $h(t) * x(t) = x(t) * h(t)$.
2. $h(t) * [\alpha x(t)] = \alpha[h(t) * x(t)]$, where α is a constant.
3. $h(t) * [x_1(t) + x_2(t)] = h(t) * x_1(t) + h(t) * x_2(t)$.
4. $h(t) * [x_1(t) * x_2(t)] = [h(t) * x_1(t)] * x_2(t)$.
5. If $h(t)$ is time-limited to (a,b) and $x(t)$ is time-limited to (c,d), then $h(t) * x(t)$ is time-limited to $(a + c, b + d)$.
6. If A_1 is the area under $h(t)$ and A_2 is the area under $x(t)$, then the area under $h(t) * x(t)$ is $A_1 A_2$.

In these relations, $h(t)$, $x_1(t)$, and $x_2(t)$ are signals. The parentheses and brackets show the order of carrying out the operations.

The integrand of (2-5a) is found by three operations: (1) reversal in time, or folding, to obtain $x(-\lambda)$; (2) shifting, to obtain $x(t - \lambda)$; and (3) multiplication of $h(\lambda)$ and $x(t - \lambda)$, to obtain the integrand. A similar series of operations is required for (2-5b).

Four examples will be given to illustrate these three operations and the subsequent evaluation of the integral to form $y(t)$.

EXAMPLE 2-7

Consider the convolution of the two rectangular-pulse signals

$$x(t) = 2\Pi\left(\frac{t-5}{2}\right) \tag{2-50}$$

and

$$h(t) = \Pi\left(\frac{t-2}{4}\right) \tag{2-51}$$

These signals are sketched in Figures 2-5a and b.

We base our evaluation of the convolution of $x(t)$ and $h(t)$ on (2-5a). Thus $x(t)$ is reversed and the variable changed to λ, which results in $x(-\lambda)$, also shown in Figure 2-5. Finally, $x(-\lambda)$ is shifted so that what was the origin now appears at $\lambda = t$. It is sometimes helpful to think of $x(t - \lambda)$ as $x[-(\lambda - t)]$; that is, t in $x(t)$ is replaced by $\lambda - t$, with λ thought of as the independent variable,

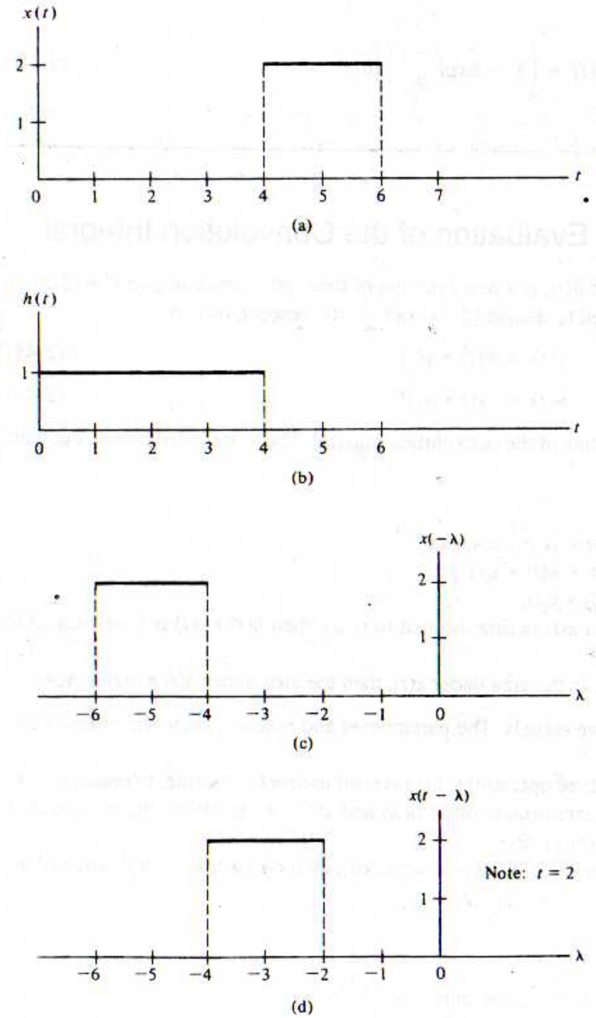


FIGURE 2-5. Signals to be convolved in Example 2-7.

whereupon the function $x(\lambda - t)$ is reversed or folded (its mirror image about the ordinate is taken). The result of these operations is $x(t - \lambda)$, which is shown in Figure 2-5d for $t = 2$.

The product of $x(t - \lambda)$ and $h(\lambda)$ is formed, point by point, and this product integrated to form $y(t)$. Clearly, the product will be zero if $t < 4$ in this example. If $t \geq 4$, the rectangles forming $h(\lambda)$ and $x(t - \lambda)$ overlap until $t > 10$, whereupon the product is again zero.

After a little thought, we see that three different cases of overlap result. These are illustrated in Figure 2-6. In each case, the area of integration is shaded; because the height of $h(t)$ is unity, the area is simply the width of the overlap times the height of $x(t - \lambda)$. Thus the area for Case I is $2(t - 4)$;

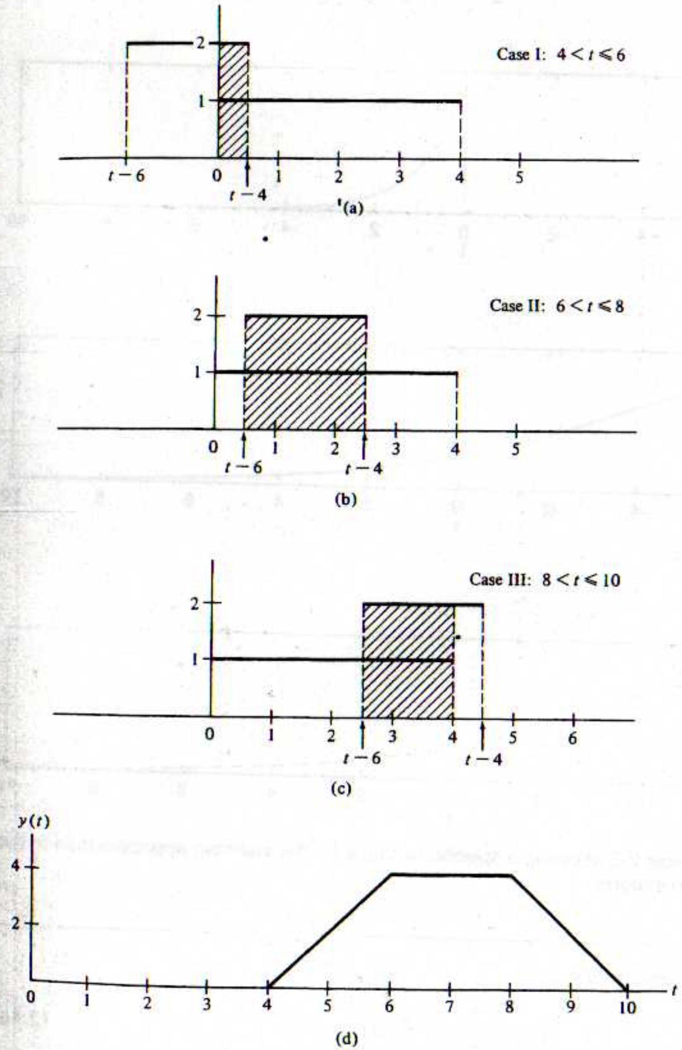


FIGURE 2-6. Steps in the convolution of Example 2-7 and the final result.

that is, it is linearly increasing with t. For Case II, it is $2[(t-4)-(t-6)] = 4$. Finally, for Case III, it is $2[4-(t-6)] = 2(10-t)$; that is, the area linearly decreases with t. The result for $y(t)$ is sketched in Figure 2-6d.

EXAMPLE 2-8

As a second example of the convolution operation, we consider the signals shown in Figure 2-7a. We use the convolution integral (2-5b) in obtaining the result. This example differs from Example

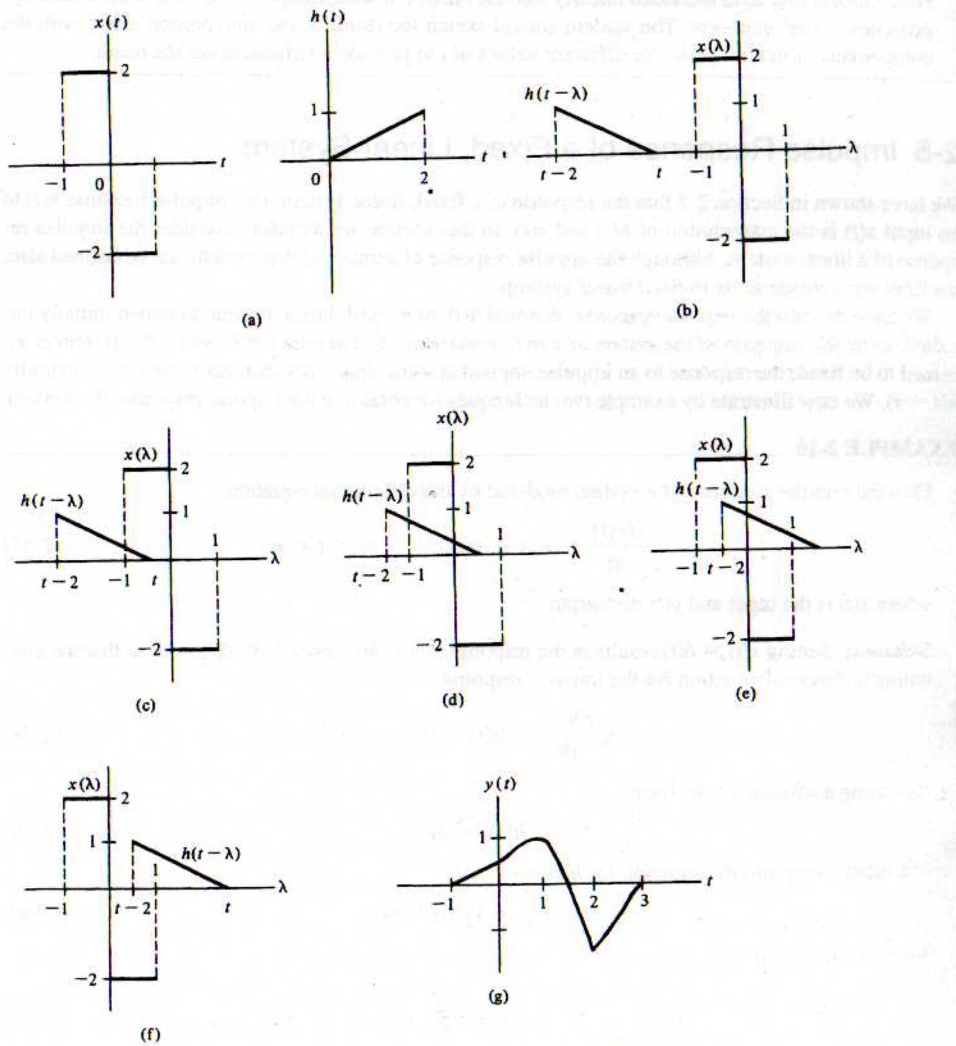


FIGURE 2-7 Waveforms pertinent to the convolution of Example 2-8.

2-7 in that part of the area under the product $x(\lambda)h(t-\lambda)$ is negative. Our first step is to express $h(t-\lambda)$ mathematically. Since

$$h(t) = \begin{cases} \dfrac{t}{2}, & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases} \tag{2-52}$$

we obtain

$$h(t-\lambda) = \begin{cases} \dfrac{t-\lambda}{2}, & 0 \le t-\lambda \le 2 \quad \text{or} \quad t-2 \le \lambda \le t \\ 0, & \text{otherwise} \end{cases} \tag{2-53}$$

The signals $x(\lambda)$ and $h(t-\lambda)$ are sketched in Figure 2-7b. Some thought on the part of the student should result in the four cases of nonzero overlap illustrated in Figures 2-7c through f. The resulting convolutions for these cases are expressed mathematically by the following equations:

$$y(t) = \begin{cases} 0, & t < -1 \quad \text{or} \quad t > 3 \\ \displaystyle\int_{-1}^{t} (t-\lambda)\,d\lambda, & -1 \le t < 0 \\ \displaystyle\int_{-1}^{0} (t-\lambda)\,d\lambda - \int_{0}^{t} (t-\lambda)\,d\lambda, & 0 \le t < 1 \\ \displaystyle\int_{t-2}^{0} (t-\lambda)\,d\lambda - \int_{0}^{1} (t-\lambda)\,d\lambda, & 1 \le t < 2 \\ \displaystyle -\int_{t-2}^{1} (t-\lambda)\,d\lambda, & 2 \le t < 3 \end{cases} \tag{2-54}$$

Integration of these expressions results in the following for $y(t)$:

$$y(t) = \begin{cases} 0, & t < -1 \quad \text{or} \quad t > 3 \\ \frac{1}{2}(t+1)^2, & -1 \le t < 0 \\ \frac{1}{2}(-t^2+2t+1), & 0 \le t < 1 \\ \frac{1}{2}(-t^2-2t+1)+2, & 1 \le t < 2 \\ \frac{1}{2}(t^2-2t+1)-2, & 2 \le t < 3 \end{cases} \tag{2-55}$$

The resulting signal, which is the convolution of $x(t)$ with $h(t)$, is shown in Figure 2-7g.

MATLAB Application

MATLAB can be used to do convolutions with the `conv(x,h)` function. The MATLAB program given below implements the convolution of the signals of Example 2-8 (echo on):

```
EDU» c2ex8
%       MATLAB plot of Example 2-8
%
del_t=.005;
t=-2:del_t:4;
```

```
L=length(t);
tp=[2*t(1):del_t:2*t(L)];
x=2*(pls_fn(t+.5)-pls_fn(t-.5)); % Defined in Chapter 1
h=.5*rmp_fn(t).*stp_fn(2-t);     % Defined in Chapter 1
y=del_t*conv(x,h);               % Multiply by step size to approximate
                                 % rectangular rule integration.
subplot(3,1,1), plot(t,x), xlabel('t'), ylabel ('x(t)'),
axis([t(1) t(L) -3 3])
subplot(3,1,2), plot(t,h), xlabel('t'), ylabel('h(t)'),
axis[(t(1) t(L) 0 2])
subplot(3,1,3), plot(tp,y), xlabel('t'), ylabel('y(t)'),
axis([t(1) t(L) -2 2])
```

The plot output of the program, given in Figure 2-8, is seen to be identical to Figure 2-7g.

EXAMPLE 2-9

In this example, we consider the convolution of a ramp and the decaying exponential signal of the form $x(t) = \exp(-\alpha t)u(t)$. The convolution integral in this case is

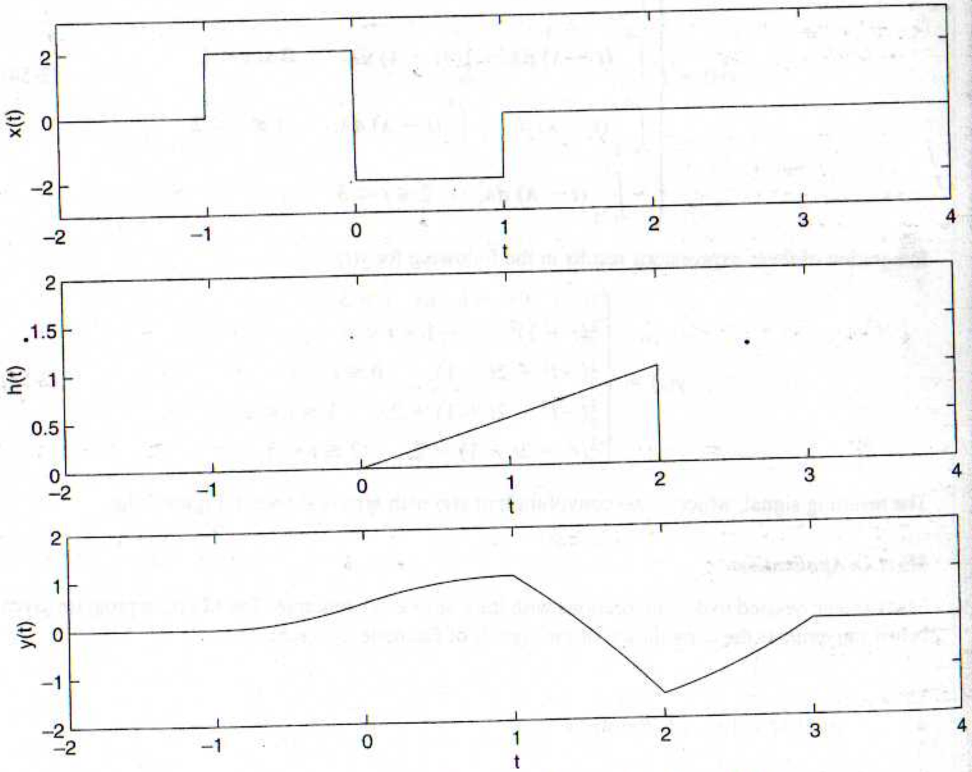


FIGURE 2-8. MATLAB verification of the result of Example 2-8.

$$y(t) = \int_{-\infty}^{\infty} r(\tau)e^{-\alpha(t-\tau)}u(t-\tau)\,d\tau$$

$$= \int_0^t \tau e^{-\alpha(t-\tau)}\,d\tau, \qquad t \geq 0$$

$$= e^{-\alpha t}\int_0^t \tau e^{\alpha\tau}\,d\tau, \qquad t \geq 0$$

$$= \frac{t}{\alpha} - \frac{1}{\alpha^2}(1 - e^{-\alpha t}), \qquad t \geq 0 \tag{2-56}$$

Note that the first term increases linearly like the ramp, but with a slope of $1/\alpha$. The second term approaches $-1/\alpha^2$ as $t \to \infty$. The student should sketch the result of the convolution along with the components of the integrand for different values of t to provide justification for the result.

2-5 Impulse Response of a Fixed, Linear System

We have shown in Section 2-3 that the response of a fixed, linear system with impulse response $h(t)$ to an input $x(t)$ is the convolution of $h(t)$ and $x(t)$. In this section we therefore consider the impulse response of a linear system. Although the impulse response of a time-varying system can be defined also, we limit our consideration to *fixed* linear systems.

We have defined the ***impulse response***, denoted $h(t)$, of a fixed, linear system, assumed initially unexited, to be *the response of the system to a unit impulse applied at time* $t = 0$. Since the system is assumed to be fixed, the response to an impulse applied at some time other than zero, say $t = \tau$, is simply $h(t - \tau)$. We now illustrate by example two techniques for obtaining the impulse response of a system.

EXAMPLE 2-10

Find the impulse response of a system modeled by the differential equation

$$\tau_0 \frac{dy(t)}{dt} + y(t) = x(t), \qquad -\infty < t < \infty \tag{2-57}$$

where $x(t)$ is the input and $y(t)$ the output.

Solution: Setting $x(t) = \delta(t)$ results in the response $y(t) = h(t)$. For $t > 0$, $\delta(t) = 0$, so that the governing differential equation for the impulse response is

$$\tau_0 \frac{dh(t)}{dt} + h(t) = 0, \qquad t > 0 \tag{2-58}$$

Assuming a solution of the form

$$h(t) = Ae^{pt} \tag{2-59}$$

and substituting into the equation for $h(t)$, we obtain

$$(\tau_0 p + 1)Ae^{pt} = 0 \tag{2-60}$$

which is satisfied if

$$p = \frac{-1}{\tau_0} \tag{2-61}$$

Thus

$$h(t) = Ae^{-t/\tau_0}, \qquad t > 0 \tag{2-62}$$

To fix A, we require an initial condition for $h(t)$. The system is unexcited for $t < 0$. Therefore, from the definition of the impulse response,

$$h(t) = 0, \qquad t < 0 \tag{2-63}$$

From (2-57) it follows that the impulse response for $t \geq 0$ obeys the differential equation

$$\tau_0 \frac{dh(t)}{dt} + h(t) = \delta(t), \qquad t \geq 0 \tag{2-64}$$

For the left-hand side to be identically equal to the right-hand side, one of the terms on the left-hand side must contain an impulse at $t = 0$. It cannot be $h(t)$, for then $\tau_0[dh(t)/dt]$ would contain a doublet,† and there is no doublet on the right-hand side [recall (1-68)]. Thus $h(t)$ is discontinuous at $t = 0$, but no impulse is present in $h(t)$. Therefore, its integral through $t = 0$ must be zero,

$$\int_{t=0^-}^{0^+} h(t)\, dt = 0 \tag{2-65}$$

and the integral of (2-64) from $t = 0^-$ to $t = 0^+$ gives

$$\int_{t=0^-}^{0^+} \tau_0 \frac{dh(t)}{dt}\, dt + \int_{t=0^-}^{0^+} h(t)\, dt = \int_{t=0^-}^{0^+} \delta(t)\, dt$$

$$\tau_0[h(0^+) - h(0^-)] + \quad 0 \quad = \quad 1 \tag{2-66}$$

Since $h(0^-) = 0$, it follows that

$$h(0^+) = \frac{1}{\tau_0} \tag{2-67}$$

Setting this result equal to (2-62) with $t = 0^+$ shows that

$$A = \frac{1}{\tau_0} \tag{2-68}$$

Thus

$$h(t) = \begin{cases} 0, & t < 0 \\ \dfrac{1}{\tau_0} e^{-t/\tau_0}, & t \geq 0 \end{cases} \tag{2-69}$$

is the impulse response of the system.

EXAMPLE 2-11

Considering next the RC circuit shown in Figure 2-9, we see that the governing differential equation is (2-57) with $\tau_0 = RC$. We will find the impulse response by considering the physical properties of the resistor and capacitor.

†A doublet was defined as $d\delta(t)/dt$ in Chapter 1.

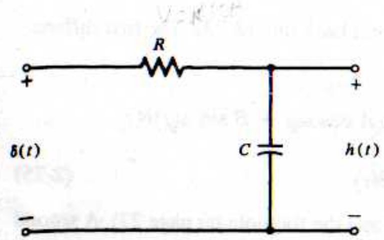


FIGURE 2-9. *RC* circuit to illustrate the calculation of impulse response.

At $t = 0^-$, the capacitor is uncharged and acts as a short circuit (infinite charge sink) to any current flowing through R. With $x(t) = \delta(t)$, the current through R at $t = 0$ is given by

$$i(t) = \frac{\delta(t)}{R}, \qquad 0^- \leq t \leq 0^+ \tag{2-70}$$

The charge stored in C due to this initial current flow is

$$q(0^+) = \int_{0^-}^{0^+} \frac{\delta(t)}{R}\, dt = \frac{1}{R} \tag{2-71}$$

Thus the capacitor voltage is $v_c(0^+) = 1/RC$. For $t > 0$, $x(t) = 0$; that is, the input is a short circuit and C discharges. The voltage across C exponentially decays according to

$$v_c(t) = Ae^{-t/RC}, \qquad t > 0 \tag{2-72}$$

Setting $v_c(0^+) = A = 1/RC$ and noting that $h(t) = v_c(t)$, we obtain the same result for $h(t)$ as in Example 2-10 with $\tau_0 = RC$.

EXAMPLE 2-12

Find the impulse response of the LC circuit shown in Figure 2-10.

Solution: For the unit impulse input, the differential equation governing the response of the circuit is

$$LC\frac{d^2h(t)}{dt^2} + h(t) = \delta(t) \tag{2-73}$$

For $t > 0$ the right-hand side is zero, and the solution to the homogeneous differential equation is

$$h(t) = (A \cos \omega_0 t + B \sin \omega_0 t)u(t) \tag{2-74}$$

where $\omega_0 = 1/LC$ and the fact that $h(t) = 0$ for $t < 0$ has been incorporated into the expression for $h(t)$ by multiplying by a unit step function. To obtain the unknown constants A and B, we

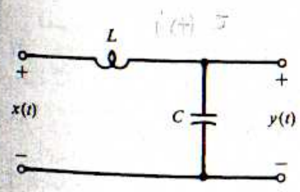


FIGURE 2-10. Circuit for Example 2-12.

differentiate $h(t)$ twice with respect to t and substitute the result back into (2-73). The first differentiation gives

$$\frac{dh(t)}{dt} = (-A\omega_0 \sin \omega_0 t + B\omega_0 \cos \omega_0 t)u(t) + (A \cos \omega_0 t + B \sin \omega_0 t)\delta(t)$$

$$= -\omega_0(A \sin \omega_0 t - B \cos \omega_0 t)u(t) + A\delta(t) \qquad (2\text{-}75)$$

where we used the facts that $0 \cdot \delta(t) = 0$ (see Problem 1-25 and the footnote on page 27). A second differentiation gives

$$\frac{d^2h(t)}{dt^2} = -\omega_0^2(A \cos \omega_0 t + B \sin \omega_0 t)u(t) - \omega_0(A \sin \omega_0 t - B \cos \omega_0 t)\delta(t) + A\frac{d\delta(t)}{dt}$$

$$= -\omega_0^2(A \cos \omega_0 t + B \sin \omega_0 t)u(t) + \omega_0 B\delta(t) + A\frac{d\delta(t)}{dt} \qquad (2\text{-}76)$$

Substitution of this result into (2-73) along with the assumed $h(t)$, (2-74), gives

$$LC\left[-\omega_0^2(A \cos \omega_0 t + B \sin \omega_0 t)u(t) - \omega_0 B\delta(t) + A\frac{d\delta(t)}{dt}\right]$$

$$+ (A \cos \omega_0 t + B \sin \omega_0 t)u(t) = \delta(t) \qquad (2\text{-}77)$$

But $\omega_0^2 = (LC)^{-1}$. Canceling like terms, we have

$$LC\left[\omega_0 B\delta(t) + A\frac{d\delta(t)}{dt}\right] = \delta(t) \qquad (2\text{-}78)$$

By equating coefficients of like derivatives of $\delta(t)$, we obtain $A = 0$ and $B = \omega_0$. Thus the result for the impulse response of this circuit is

$$h(t) = \omega_0 \sin(\omega_0 t)\, u(t) \qquad (2\text{-}79)$$

2-6 Superposition Integrals in Terms of Step Response

We saw in Example 2-6 that the step response of the *RC* circuit shown in Figure 2-11 is simply the integral of the impulse response of the circuit. We now ask: Can the response of a system to an arbitrary input be expressed in terms of its step response? The answer is yes.

To find the appropriate relationship, consider the superposition integral in terms of impulse response. Repeating (2-5a) here for convenience, we have

$$y(t) = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)\, d\lambda \qquad (2\text{-}5)$$

We employ the formula for integration by parts:

$$\int_a^b u\, dv = uv\,\Big|_a^b - \int_a^b v\, du \qquad (2\text{-}80)$$

with $u = x(t - \lambda)$ and $dv = h(\lambda)\, d\lambda$. Thus

$$v(\lambda) = \int_{-\infty}^{\lambda} h(\zeta)\, d\zeta \triangleq a(\lambda) \qquad (2\text{-}81)$$

is simply the step response, and

$$du(\lambda) = -\dot{x}(t-\lambda)\, d\lambda \qquad (2\text{-}82)$$

where the overdot denotes differentiation with respect to the argument. Substituting these expressions into (2-80), we obtain

$$y(t) = a(\lambda)x(t-\lambda)\,\Big|_{\lambda=-\infty}^{\infty} + \int_{-\infty}^{\infty} \dot{x}(t-\lambda)a(\lambda)\, d\lambda \qquad (2\text{-}83)$$

The system is initially unexcited, so that $a(-\infty) = 0$ and $x(t-\lambda)|_{\lambda=\infty} = 0$. Thus the first term is zero when the limits are substituted, and the final result is

$$y(t) = \int_{-\infty}^{\infty} \dot{x}(t-\lambda)a(\lambda)\, d\lambda \qquad (2\text{-}84)$$

The change of variables $\eta = t - \lambda$ results in the alternative form

$$y(t) = \int_{-\infty}^{\infty} \dot{x}(\eta)a(t-\eta)\, d\eta \qquad (2\text{-}85)$$

Thus in terms of the step response $a(t)$, the response of a system to an input $x(t)$ is the *convolution of the derivative of the input with the step response.* Equations (2-84) and (2-85) are known as *Duhamel's integrals.* Note the similarity to (2-5a) and (2-5b), respectively.

EXAMPLE 2-13

Consider a system with a ramp input for which

$$x(t) = tu(t) \qquad (2\text{-}86a)$$

$$\dot{x}(t) = u(t) \qquad (2\text{-}86b)$$

Applying (2-85), we obtain the response to a ramp as

$$y_R(t) = \int_{-\infty}^{\infty} u(t-\lambda)a(\lambda)\, d\lambda$$

$$= \int_{-\infty}^{t} a(\lambda)\, d\lambda \qquad (2\text{-}87)$$

Thus the response of a system to a unit ramp, which is the integral of the unit step, is the integral of the step response.

Generalizing, we conclude that *for a fixed, linear system, any linear operation on the input produces the same linear operation on the output.*

EXAMPLE 2-14

Find the response of the *RC* circuit of Figure 2-9 to the triangular signal

$$x_\Delta(t) = r(t) - 2r(t-1) + r(t-2) \qquad (2\text{-}88)$$

Solution: We first substitute (2-47) into (2-87) to obtain the ramp response. Thus

$$y_R(t) = \int_{-\infty}^{t}\left[1 - \exp\left(\frac{-\lambda}{RC}\right)\right]u(\lambda)\,d\lambda$$

$$= r(t) + RC\left[\exp\left(\frac{-\lambda}{RC}\right)\right]_0^t u(t)$$

$$= r(t) - RC\left[1 - \exp\left(\frac{-t}{RC}\right)\right]u(t) \qquad (2\text{-}89)$$

By superposition, the response to the triangle $x_\Delta(t)$ is $y_\Delta(t) = y_R(t) - 2y_R(t-1) + y_R(t-2)$. This input and output are compared in Figure 2-11. Note that for $RC << 1$, the output closely approximates the input, whereas if $RC = 1$ the output does not resemble the input.

MATLAB *Application*

In this example, we verify the result of Example 2-14, shown in Figure 2-11, by writing a MATLAB program to carry out a Duhamel's integral evaluation. The program listing is given below:

```
EDU>>c2ex14
%    DuHammel's computation of the output for Example 2-14
%
RC=0.1;
t_min=-0.5;                                     % Minimum t-value for
                                                % computation window
t_max=2.5;                                      % Maximum t-value for
                                                % computation window
del_t=0.001;                                    % Step size for 5
t=t_min:del_t:t_max;                            % Vector of t-values
L=length(5);
tp=[2*t(1):del_t:2*t(L)];                       % t-variable defined for
                                                % plotting output
x_del_dot=stp_fn(5)-2*stp_fn(t-1)-stp_fn(t-2);  % Derivative of the input
a=(1 - exp(-t/RC)).*stp_fn(t);                  % Step response of the
                                                % system
y=del_t*conv(x_del_dot,a);                      % The convolution;
                                                % multiply by del_t to
                                                % scale
```

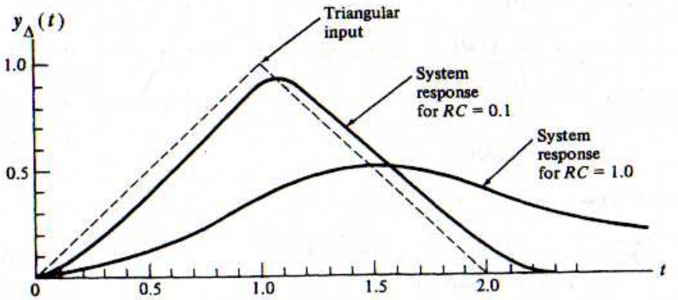


FIGURE 2-11. Response of an *RC* circuit to a triangular input signal.


```
subplot(3,1,1),plot(t,x_del_dot),xlabel('t'),ylabel('der.of input'),
axis([t_min t_max -1.5 1.5])
subplot(3,1,2),plot(t,a'),xlabel('t'),ylabel('step resp.'),axis([t_min
t_max 0 1.5])
subplot(3,1,3),plot(tp,y),xlabel('t'),ylabel('output'),axis([t_min
t_max 0 1])
```

A plot of the output of the system in response to the triangle input is shown in Figure 2-12 for $RC = 0.1$. Note that it is the same as the corresponding response plotted in Figure 2-11.

EXAMPLE 2-15

We again consider the *RC* circuit of Figure 2-9 and compute its impulse response by first finding its step response and then differentiating it. The differential equation relating input and output is

$$RC\frac{dy(t)}{dt} + y(t) = x(t) \qquad (2\text{-}90)$$

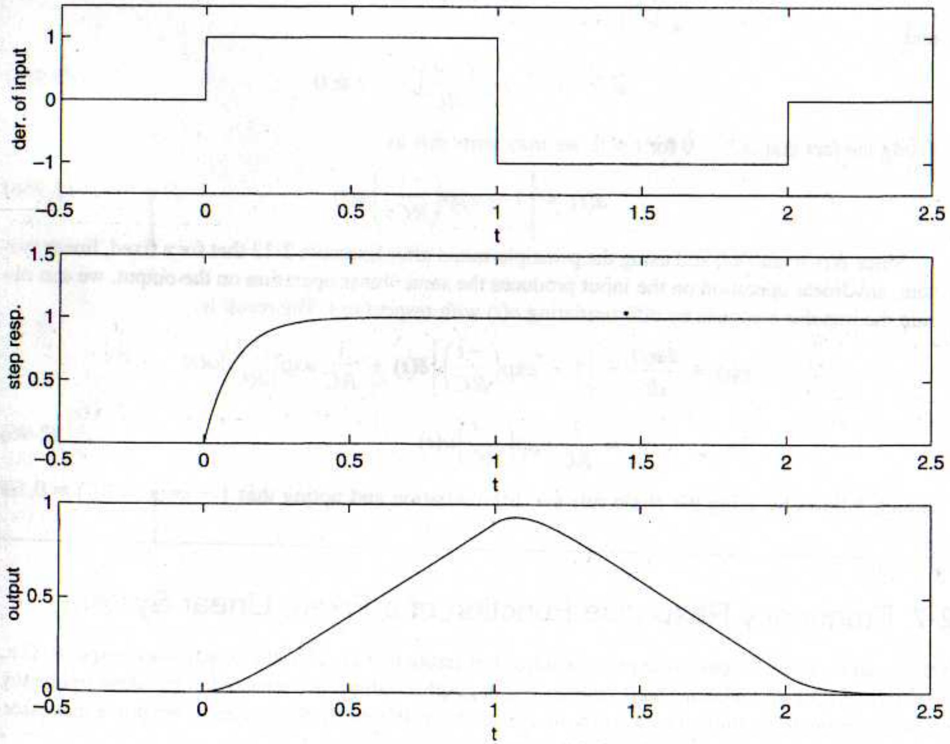


FIGURE 2-12. The response of an RC lowpass filter to a triangular input.

which is (2-57) with $\tau_0 = RC$. With a unit step input, this equation can be written as

$$RC\frac{da(t)}{dt} + a(t) = 1, \qquad t \geq 0 \tag{2-91}$$

where $a(t)$ is the step response. We have already considered the solution of this differential equation in Example 2-1 for an arbitrary input. As mentioned in that example, the solution consists of the sum of a homogeneous solution, which is

$$a_h(t) = A\exp\left(\frac{-t}{RC}\right) \tag{2-92}$$

plus a particular solution for the forcing function $u(t)$. This particular solution is simply $a_p(t) = 1$, which can be seen by direct substitution into the nonhomogeneous differential equation for $a(t)$. Thus

$$a(t) = a_h(t) + a_p(t) = A\exp\left(\frac{-t}{RC}\right) + 1, \qquad t \geq 0 \tag{2-93}$$

The initial charge on the capacitor is zero for $t < 0$. Since the forcing function does not contain impulses (it is a step and therefore cannot change the capacitor charge instantaneously), it follows that the charge on the capacitor at $t = 0^+$ is zero. Therefore, $a(0^+) = 0$ or

$$a(0^+) = A\exp\left(\frac{-t}{RC}\right)_{t=0} + 1 = A + 1 = 0 \tag{2-94}$$

and

$$a(t) = 1 - \exp\left(\frac{-t}{RC}\right), \qquad t \geq 0 \tag{2-95a}$$

Using the fact that $a(t) = 0$ for $t < 0$, we may write this as

$$a(t) = \left[1 - \exp\left(\frac{-t}{RC}\right)\right]u(t) \tag{2-95b}$$

Since $\delta(t) = du(t)/dt$ and using the principle stated after Example 2-13 that for a fixed, linear system, any linear operation on the input produces the same linear operation on the output, we can obtain the impulse response by differentiating $a(t)$ with respect to t. The result is

$$h(t) = \frac{da(t)}{dt} = \left[1 - \exp\left(\frac{-t}{RC}\right)\right]\delta(t) + \frac{1}{RC}\exp\left(\frac{-1}{RC}\right)u(t)$$

$$= \frac{1}{RC}\exp\left(\frac{-t}{RC}\right)u(t) \tag{2-96}$$

which follows by using the chain rule for differentiation and noting that $1 - \exp(-t/RC) = 0$ for $t = 0$.

2-7 Frequency Response Function of a Fixed, Linear System

If the input to a fixed, linear system is a sinusoid of frequency ω rad/s, the steady-state response (i.e., the response after all transients have approached negligible values) is a sinusoid of the same frequency, but with amplitude multiplied by a factor $A(\omega)$ and phase-shifted by $\theta(\omega)$ radians. We prove this assertion in this section and illustrate it with an example.

The complex function of frequency

$$H(\omega) = A(\omega)e^{j\theta(\omega)} \tag{2-97}$$

is called the *frequency-response function* of the system; $A(\omega)$ and $\theta(\omega)$ are referred to as the *amplitude-response* and *phase-response functions* of the system, respectively.

The frequency-response function, (2-97), completely characterizes the steady-state response of a fixed, linear system to a sinusoid or, equivalently, a rotating phasor, $e^{j\omega t}$. That is, the steady-state output of a fixed, linear system is of the same form as its input when its input is $e^{j\omega t}$. In mathematical terms, $e^{j\omega t}$ is said to be an *eigenfunction* of the system. That this is the case can be shown by considering the superposition integral with the input $x(t) = e^{j\omega t}$. Then the output is

$$y(t) = \int_{-\infty}^{\infty} e^{j\omega(t-\lambda)}h(\lambda)\,d\lambda \tag{2-98}$$

where $h(\lambda)$ is the impulse response. Since $e^{j\omega t}$ can be factored out of the integral, the output can be written as

$$y(t) = e^{j\omega t}\int_{-\infty}^{\infty} h(\lambda)e^{-j\omega\lambda}\,d\lambda \triangleq e^{j\omega t}H(\omega) \tag{2-99}$$

We identify the integral multiplying $e^{j\omega t}$ as the frequency response $H(\omega)$. That is,

$$H(\omega) = \int_{-\infty}^{\infty} h(\lambda)e^{-j\omega\lambda}\,d\lambda \tag{2-100}$$

Later we shall see that $H(\omega)$ corresponds to the *Fourier transform* of the impulse response $h(t)$.

EXAMPLE 2-16

To illustrate these remarks, consider the RC filter of Example 2-1 as illustrated in Figure 2-2. Its impulse response was found in Examples 2-10 and 2-11 to be

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t) \tag{2-101}$$

From (2-100), the frequency response function is found to be

$$H(\omega) = \int_{-\infty}^{\infty} \frac{1}{RC}e^{-t/RC}e^{-j\omega t}u(t)\,dt$$

$$= \int_{0}^{\infty} \frac{1}{RC}e^{-(j\omega+1/RC)t}\,dt$$

$$= \left[-\frac{1/RC}{j\omega + 1/RC}e^{-(j\omega+1/RC)t}\right]_0^{\infty}$$

$$= \frac{1}{1 + j\omega RC} \tag{2-102}$$

We write this in polar form (2-97) as

$$H(\omega) = \frac{1}{\sqrt{1 + (\omega RC)^2}}e^{-j\tan^{-1}(\omega RC)} \tag{2-103}$$

Thus, the amplitude and phase responses are, respectively, given by

$$A(\omega) = \frac{1}{\sqrt{1 + (\omega RC)^2}} \tag{2-104a}$$

and

$$\theta(\omega) = -\tan^{-1}(\omega RC) \tag{2-104b}$$

The output, for an input of the form $\alpha e^{j\omega t}$, is

$$y(t) = \frac{\alpha}{\sqrt{1 + (\omega RC)^2}} e^{j[\omega t - \tan^{-1}(\omega RC)]} \tag{2-105}$$

For a cosinusoidal input, we may find the output by writing the cosine in exponential form and using superposition to obtain the output

$$y(t) = \frac{\alpha}{\sqrt{1 + (\omega RC)^2}} \cos[\omega t - \tan^{-1}(\omega RC)] \tag{2-106}$$

Note that the same result could have been obtained by taking the real part of (2-105). This is not surprising, since the cosinuosidal input is obtained by taking the real part of the rotating phasor input, and the operations of taking the real part and convolution are interchangeable since both are linear operations.

Some numerical values will illustrate the behavior of $A(\omega)$ and $\theta(\omega)$ versus ω. For $(2\pi RC)^{-1} =$ 1 kHz and $\alpha = 1$, the values in Table 2-1 result. For example, for $x(t) = \cos(2{,}000\pi t)$, the output is

$$y(t) = 0.707 \cos(2{,}000\pi t - 45°) \tag{2-107}$$

Using the superposition property of the linear RC filter, we obtain the output for an input of the form

$$x(t) = \cos(2{,}000\pi t) + \cos(20{,}000\pi t) \tag{2-108}$$

as

$$y(t) = 0.707 \cos(2{,}000\pi t - 45°) + 0.0995 \cos(20{,}000\pi t - 84.29°) \tag{2-109}$$

Applying the superposition property an indefinite number of times, one could find a series expression for the output due to any periodic input simply by representing the input in terms of its Fourier series, as we will show in the next chapter.

TABLE 2-1
Amplitude and Phase Response Values for an *RC* Filter

ω	$A(\omega)$	$\theta(\omega)$
200π	0.995	−5.71°
$1{,}000\pi$	0.894	−26.57°
$2{,}000\pi$	0.707	−45.00°
$10{,}000\pi$	0.196	−78.69°
$20{,}000\pi$	0.0995	−84.29°

2-8 Stability of Linear Systems

One of the considerations in any system design is the question of stability. Although there are various definitions of stability in common usage, the one we use is referred to as *bounded-input, bounded-output* (BIBO) *stability.* By definition, a *system is BIBO stable if and only if every bounded input results in a bounded output.* Clearly, stability is a desirable property of most systems.

For a fixed, linear system we may obtain a condition on the impulse response which guarantees BIBO stability. To derive this condition, consider (2-41), which relates the input and output of a fixed, linear system:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)\, d\lambda$$

It follows that

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)\, d\lambda \right| \le \int_{-\infty}^{\infty} |x(\lambda)|\, |h(t - \lambda)|\, d\lambda \tag{2-110}$$

If the input is bounded, then

$$|x(\lambda)| \le M < \infty \tag{2-111}$$

where M is a finite constant. Replacing $|x(\lambda)|$ by M in (2-110), we have the inequality

$$|y(t)| \le M \int_{-\infty}^{\infty} |h(t - \lambda)|\, d\lambda \tag{2-112a}$$

or

$$|y(t)| \le M \int_{-\infty}^{\infty} |h(\eta)|\, d\eta \tag{2-112b}$$

which follows by the change of variables $\eta = t - \lambda$. Thus the output is bounded provided that

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \tag{2-113}$$

That is, (2-113) is a sufficient condition for stability.

To show that it is also a necessary[†] condition, consider the input

$$x(\lambda) = \begin{cases} +1 & \text{if } h(t - \lambda) > 0 \\ 0 & \text{if } h(t - \lambda) = 0 \\ -1 & \text{if } h(t - \lambda) < 0 \end{cases} \tag{2-114}$$

For this input it follows that

$$|y(t)| = \left| \int_{-\infty}^{\infty} |h(t - \lambda)|\, d\lambda \right| = \int_{-\infty}^{\infty} |h(\eta)|\, d\eta \tag{2-115}$$

for any fixed value of t since the integrand is always nonnegative. Thus the output will be unbounded if (2-113) is not satisfied. That is, it is also a necessary condition for BIBO stability.

[†]A condition from which a given statement logically follows is said to be a *sufficient condition;* a condition which is a logical consequence of a given statement is said to be a *necessary condition.* A condition may be necessary but not sufficient, and vice versa.

EXAMPLE 2-17

The system of Example 2-11 is BIBO stable since

$$\int_{-\infty}^{\infty} \frac{1}{RC} e^{-t/RC}u(t)\, dt = \int_0^{\infty} \exp(-v)\, dv$$
$$= -\exp(-v)\Big|_0^{\infty}$$
$$= 1 < \infty \tag{2-116}$$

EXAMPLE 2-18

Is the system whose impulse response was derived in Example 2-12 BIBO stable?

Solution: Substituting the impulse response for this system into (2-113), we obtain

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \omega_0 \int_{-\infty}^{\infty} |\sin(\omega_0 t)\, u(t)|\, dt = \omega_0 \int_0^{\infty} |\sin \omega_0 t|\, dt \tag{2-117}$$

The integral does not converge, so the condition for BIBO stability does not give a bounded result. This system is BIBO unstable.

We will return to the idea of stability of a system in Chapter 6, where additional methods for determining the stability of systems describable by constant coefficient, linear differential equations, are discussed.

2-9 System Modeling and Simulation†

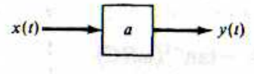
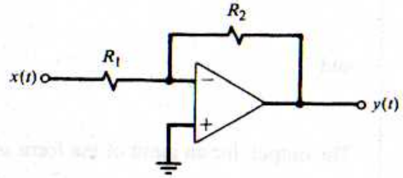
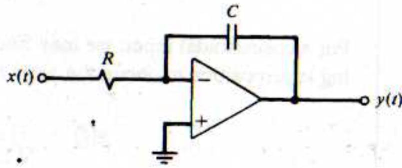
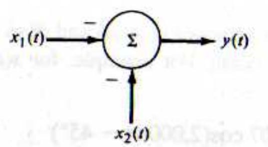
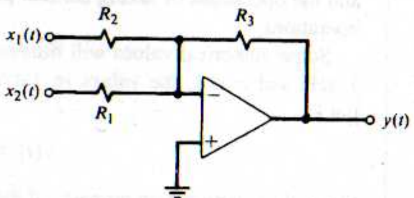
We briefly consider in this section how systems can be modeled and simulated in terms of several basic building blocks. These basic building blocks are given in Table 2-2, and consist of ideal adders or subtractors, constant multipliers, and integrators. Any *n*th-order constant-coefficient differential equation can be simulated by means of these components. It may seem that we are simply replacing one unsolved problem with another in so doing, or that we really don't need such simulations at all because linear, constant-coefficient differential equations are amenable to solution. In answer to the first thought, when a simulation block diagram is available for a system, the realization of such a simulation can take one of two forms: (1) The realization can be accomplished in analog fashion by means of operational amplifier circuits that approximate the ideal summers and integrators needed (such circuits are also shown in Table 2-2); (2) the simulation can be realized by means of a digital computer program where the integration is done numerically, as in Example 1-3. In regard to simulations being unnecessary because the differential equation representing a system is analytically solvable, we simply point out that the system realizations and simulation procedures touched on in this section also apply to cases where the system is nonlinear or time varying as well, and these problems are often not solvable by analytical means.

To see how one might simulate a system, either by means of operational amplifier circuits or by digital computer algorithms, consider the first-order differential equation

$$\frac{dy}{dt} + ay = b_1 \frac{dx}{dt} + b_0 x \tag{2-118}$$

†Systematic modeling procedures for electrical and mechanical systems are discussed in Appendix B.

TABLE 2-2
Building Blocks for System Modeling

Operation	Symbol	Op Amp Realization
Multiplication by a constant	 $y(t) = ax(t)$	 $-a = R_2/R_1$
Integration	 $y(t) = a\int^t x(\lambda)\, d\lambda$	 $-a = 1/RC$
Summation	 $-y(t) = x_1(t) + x_2(t)$	 $R_3/R_2 = 1$ $R_3/R_1 = 1$

It is realized by the block diagram of Figure 2-13. To show that this is the case, let the output of the integrator be $q(t)$. Its input is then dq/dt. Equating dq/dt to the sum of the inputs to the left-hand summer, we obtain

$$\frac{dq}{dt} = -ay + b_0 x \tag{2-119}$$

Considering the output of the right-hand summer, we find that

$$y = q + b_1 x \tag{2-120}$$

Differentiating both sides of (2-120) and substituting (2-119), we obtain (2-118).

In a similar manner to the first-order case, we can realize an *n*th-order system by a block diagram similar to Figure 2-12. For the differential equation

$$\frac{d^n y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{n} b_i \frac{d^i x(t)}{dt^i} \tag{2-121}$$

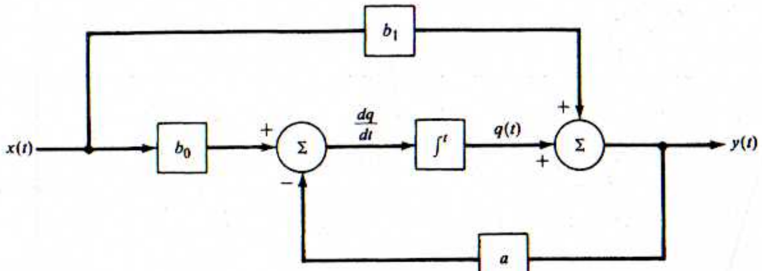


FIGURE 2-13. Integrator realization of a general first-order system.

we have the realization diagram shown in Figure 2-14. That this is the realization of (2-121) can be demonstrated in a similar, although longer, procedure to that used for the first-order case. [Note that this is only one of many ways to realize this system. Also note that the number of derivatives on either side of (2-121) can differ by having some a_i's or b_i's zero.]

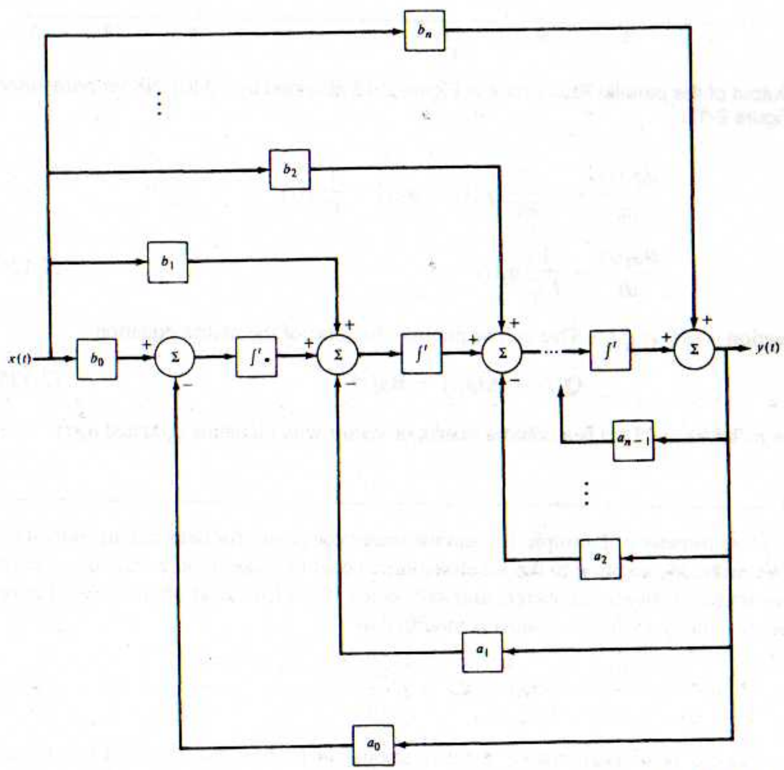


FIGURE 2-14. Integrator realization of a general *n*th-order system.

EXAMPLE 2-19

Consider the *RLC* circuit shown in Figure 2-15. Obtain an integrator realization of this circuit.

Solution: The differential equation describing the voltage response of the circuit is

$$\frac{v(t)}{R} + \frac{1}{L}\int^t v(\lambda)\,d\lambda + C\frac{dv(t)}{dt} = x(t) \tag{2-122}$$

Differentiating once and rearranging, we obtain

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{C}\frac{dx}{dt} \tag{2-123}$$

This is of the form (2-121) with $y = v$, $n = 2$, $a_0 = 1/LC$, $a_1 = 1/RC$, $b_0 = 0$, $b_1 = 1/C$, and $b_2 = 0$. The integrator realization is shown in Figure 2-16.

MATLAB APPLICATION

An integrator (or state variable) simulation of a system can be carried out using SIMULINK, which is a block diagram oriented toolbox. There is a limited version available in *The Student Version of* MATLAB. One invokes SIMULINK by typing "simulink" after the command window prompt. One can then construct a block diagram realization of the desired system using the menus available. Such a SIMULINK block diagram realization is shown in Fig. 2-17 for the RLC circuit of Example 2-19.

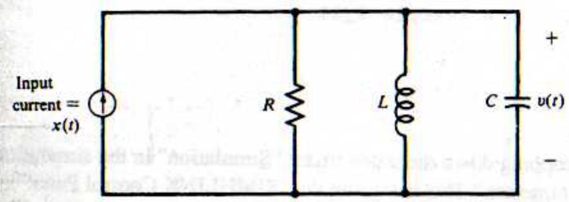


FIGURE 2-15. Circuit for Example 2-19.

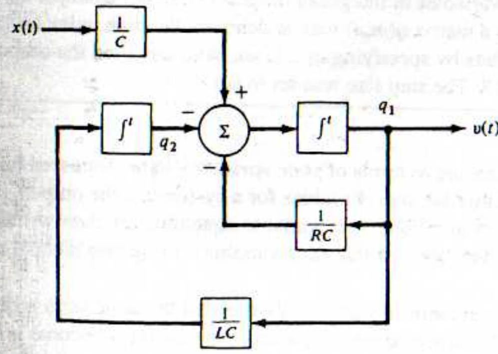


FIGURE 2-16. Integrator realization of a parallel *RLC* circuit.

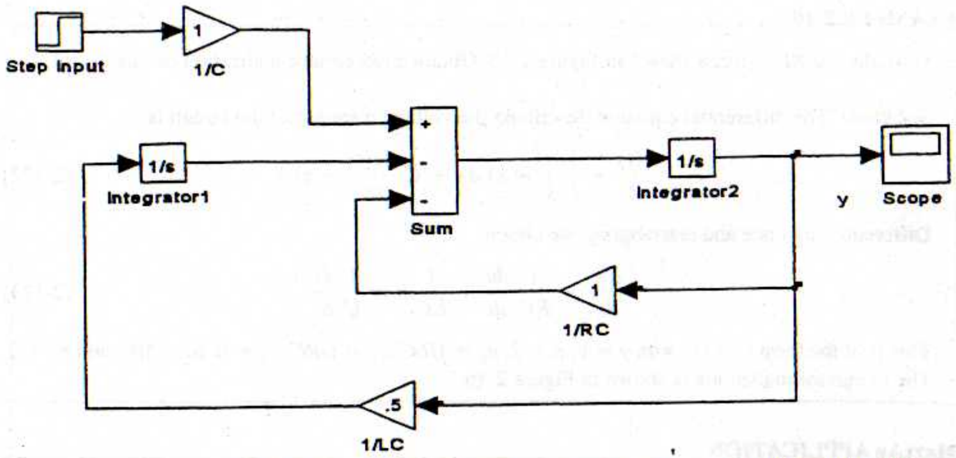


FIGURE 2-17. A SIMULINK simulation block diagram for the RLC circuit of Example 2-19.

The simulation can be carried out directly in the model window or it can be carried out in the MATLAB command window using the function `lsim` as suggested by the script below:

```
EDU>>c2ex19pl
%      Plot of simulation output for Example 2_19
%
[t,q]=lsim('c2ex19',20);
plot(t, q(:,1)), xlabel('t'), ylabel('output')
```

The output variables are defined by dropping down the menu under "Simulation" in the simulation window, and then the menu under "parameters." This brings up the "SIMULINK Control Panel" in which various simulation variables are specified such as start and end time of the simulation. The last blank to be filled out is labeled "Return Variables" which can be left blank if desired and the return variables default to time (t) and the state variables or integrator outputs (x). In our example here we gave the state variables the label q. This is a matrix $q(m,n)$ with m denoting the time index and n indicating the number of the state variable. Thus by specifying $q(:, 1)$ we have specified the output in this example, which is shown in Figure 2-18. The step size was set to 0.1.

A useful way to write the equations for a system are in terms of *state variables,* to be discussed further in Chapter 7. For now, we simply point out that the state variables for a system are the outputs of the integrators of the integrator realization of a system. The state variable equations are then written down by taking the input to each integrator (the derivative of that state variable) and setting it equal to the summer output appearing there.

The advantage of writing the system equations in terms of state variables is that the same form is obtained no matter what the order of the system. For orders greater than one, the equations become matrix equations, as will be pointed out in Chapter 7. For example, the state equations for the system of Example 2-21 are

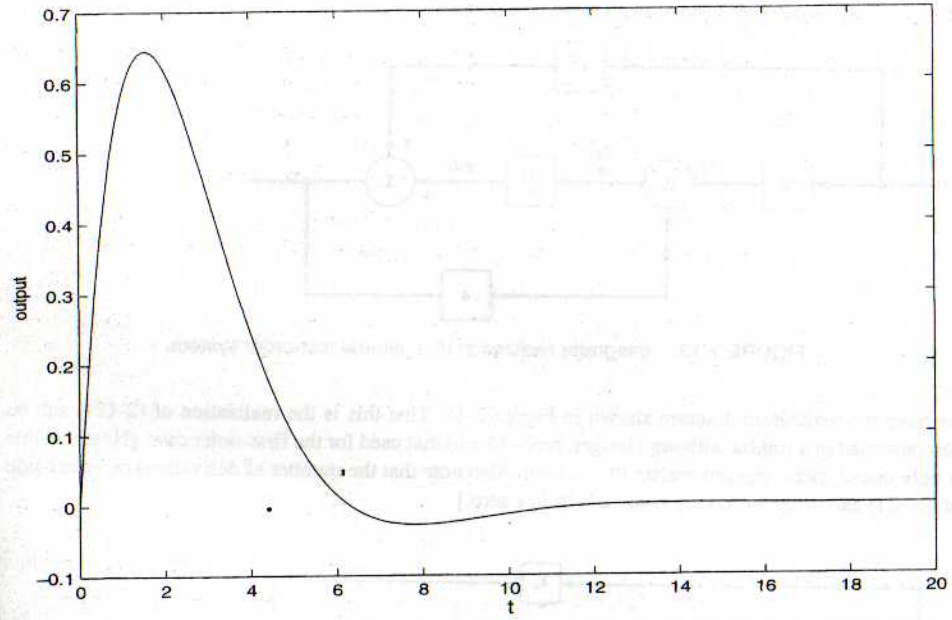


FIGURE 2-18. Output of the parallel RLC circuit of Figure 2-15 obtained by SIMULINK for parameter values shown in Figure 2-17.

$$\frac{dq_1(t)}{dt} = -\frac{1}{RC}q_1(t) - q_2(t) + \frac{1}{C}x(t)$$
$$\frac{dq_2(t)}{dt} = \frac{1}{LC}q_1(t) \qquad (2\text{-}124)$$

and the output equation is $v(t) = q_1(t)$. This can be put into the form of the matrix equation

$$\dot{\mathbf{Q}}(t) = \mathbf{A}\mathbf{Q}(t) + \mathbf{B}\mathbf{x}(t) \qquad (2\text{-}125)$$

where **A** and **B** are matrices and $\mathbf{Q}(t)$ is a column matrix or vector with elements $q_1(t)$ and $q_2(t)$.

EXAMPLE 2-20

Reconsider the accelerometer of Example 1-1 and the integrator circuit for determining velocity of Example 1-2. We make one addition to the accelerometer model to make it more realistic—the addition of viscous friction between the weight and its housing. Such friction is often modeled as being proportional to velocity so that (1-1) now is modified to

$$Ma = Kx + B\frac{dx}{dt} \qquad (2\text{-}126)$$

where B is the constant of proportionality. A small amount of friction will hardly affect the accelerometer action at all, whereas a large amount will make its response sluggish and the indicated velocity at the integrator circuit output will lag that of the actual velocity. To investigate this effect, we model the rest of the system and simulate it with SIMULINK. The remaining equations necessary for constructing the SIMULINK model are (1-8), which is repeated here

$$v_o(t) = -\frac{1}{RC}\int_0^t v_s(\lambda)\,d\lambda \quad \text{or} \quad \frac{dv_o(t)}{dt} = -\frac{1}{RC}x(t) \tag{2-127}$$

and the equation relating v_s to the mass excursion from equilibrium, x. To make the maximum of $v_s = 0.1$ V when the mass is at its maximum excursion of 1 cm = 0.01 m, this equation is simply

$$v_s(t) = 10x(t) \tag{2-128}$$

We define *auxiliary variables* (called state variables) as the outputs of the integrators in the SIMULINK model. Thus $q_1 = x$ and $q_2 = v_o$, and the system governing equations may be written as

$$\frac{dq_1(t)}{dt} = -\frac{K}{B}q_1(t) + \frac{M}{B}a(t)$$

and

$$\frac{dq_2(t)}{dt} = -\frac{1}{RC}v_s(t) = -\frac{10}{RC}x(t) \tag{2-129}$$

The desired output is $q_2(t) = v_o(t)$ which is proportional to the velocity of the rocket (recall that the integrator circuit was designed in Example 1-2 such that when the operational amplifier output voltage is -10 volts, the rocket's velocity is the burnout value of 1500 m/s).

A SIMULINK block diagram modeling the accelerometer and integrator circuit is shown in Figure 2-19. Recall that the following parameter values were used in Examples 1-1 and 1-2:

$$M = 2 \text{ grams} = 0.002 \text{ kg}; \quad K = 8 \text{ kg/s}^2; \quad RC = 0.36 \text{ s}$$

We will take three values for B, in particular 1, 10, and 100 kg/s which corresponds to low, moderate, and high damping. Note that the ramp acceleration input has been modeled as a step integrated.

The SIMULINK simulation can take place either in the SIMULINK model window itself, in which variables of interest can be monitored on scopes strategically placed, or it can be run in the MATLAB command window with the statement below:

```
EDU>>[t, q]=sim('c2ex20',72);
```

Various plots can then be made with the plot command. In this case, three simulations were run for the three different values of B (1, 10, and 100) and the plot held in each case so that the integrator output for each succeeding case could be plotted on top of the previous cases. The resulting graph is shown in Figure 2-20. Note that the largest value of B = 100 gives considerable lag in the output.

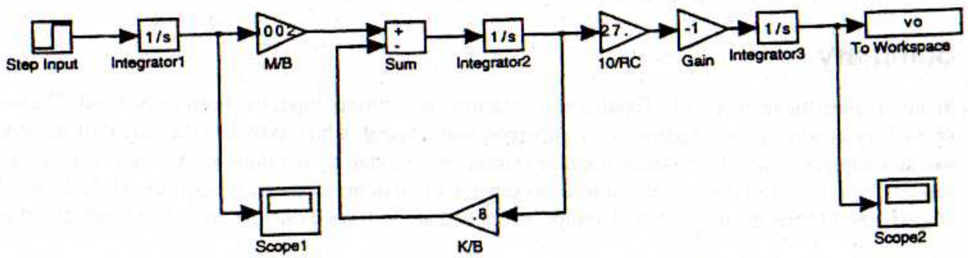


FIGURE 2-19. SIMULINK model for the accelerometer/integrator circuit of Examples 1-1 & 1-2.

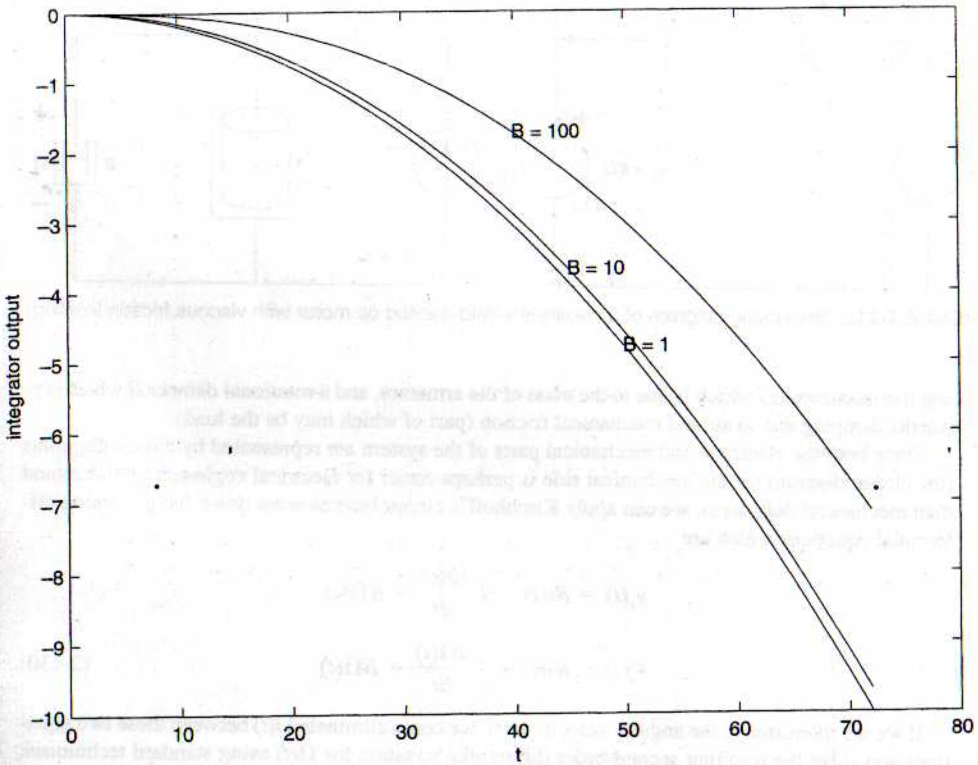


FIGURE 2-20 Output voltages for the accelerometer/integrator system of Examples 1-1 and 1-2 for the three damping parameter values of 1, 10, and 100 kg/s.

EXAMPLE 2-21

As a final example using SIMULINK, we do another system that involves both mechanical and electrical subsystems. Figure 2-21 represents a direct current (dc) motor with separately excited field coils (a separate source supplies the dc current to provide the magnetic field in which the armature rotates). The applied voltage to the armature, $v_s(t)$, is assumed to be a step applied at $t = 0$. The angular velocity (rotational speed), radians/second, of the armature as a function of time is desired.

The left-hand side of the block diagram of Figure 2-21, representing the armature circuit, includes the applied source, a resistor representing the resistance of the armature windings, an inductor representing the inductance of the armature windings, and a controlled source $v_b = K\,d\theta/dt = K\Omega$ representing the back-electromotive force (emf) generated because the armature turning in the magnetic field also acts as a generator. The latter is proportional to the angular velocity of the armature.

The right-hand side of the block diagram of Figure 2-21 represents the rotating armature of the motor. It consists of an ideal torque source $T_s = Ki$ (note that the same constant relates torque to current as relates back emf to angular velocity) which is proportional to the armature current, an

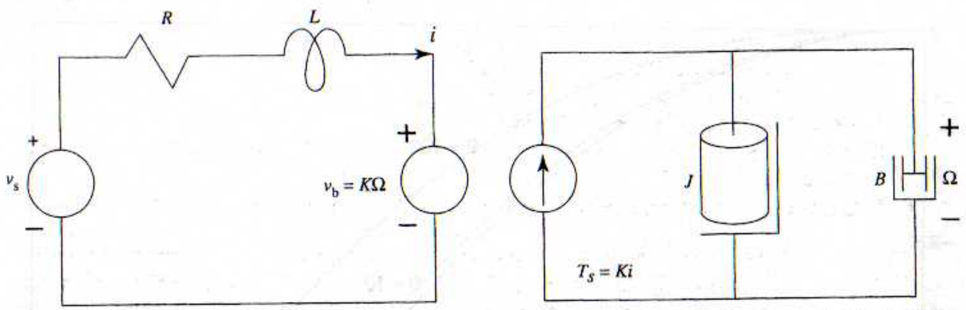


FIGURE 2-21. Schematic diagram of a separately field-excited dc motor with viscous friction loading.

angular momentum J which is due to the mass of the armature, and a rotational damper B which represents damping due to air and mechanical friction (part of which may be the load).

Since both the electrical and mechanical parts of the system are represented by circuit diagrams (the circuit diagram for the mechanical side is perhaps easier for electrical engineers to understand than mechanical diagrams), we can apply Kirchhoff's circuit laws to write down the governing differential equations which are

$$v_s(t) = Ri(t) + L\frac{di(t)}{dt} + K\Omega(t)$$

$$T_s(t) = Ki(t) = J\frac{d\Omega(t)}{dt} + B\Omega(t) \qquad \text{(2-130)}$$

If we are interested in the angular velocity $\Omega(t)$, we could eliminated $i(t)$ between these two equations and solve the resulting second-order differential equation for $\Omega(t)$ using standard techniques. The purpose of this example is to illustrate the application of SIMULINK, however. Easier techniques for solving system equations based on Laplace transforms will be presented in Chapters 5 and 6. In order to easily construct the SIMULINK block diagram, we solve (2-130) for di/dt and $d\Omega/dt$. These will then be the inputs to the two required integrators.

$$\frac{di(t)}{dt} = -\frac{R}{L}i(t) - \frac{K}{L}\Omega(t) + \frac{1}{L}v_s(t)$$

$$\frac{d\Omega(t)}{dt} = -\frac{B}{J}\Omega(t) + \frac{K}{J}i(t) \qquad \text{(2-131)}$$

Using these equations, the SIMULINK diagram shown in Figure 2-22 can be constructed. We have two options for running the simulation—in particular, we can simulate directly in the SIMULINK window, or we can use `lsim` in the MATLAB command window. The latter is convenient for generating plots that can be stored in files or printed. Note that the coupling between the electrical and mechanical parts of the system is controlled by the parameter K. We are interested in studying the effects of the dissipative elements on the system operation—in particular R and B. We intuitively expect the final angular velocity of the motor to be less the higher B, and we expect it to take longer in getting to its final angular velocity the higher B. These outputs for three values of B are illustrated in Figure 2-23 where it is seen that this is indeed the case.

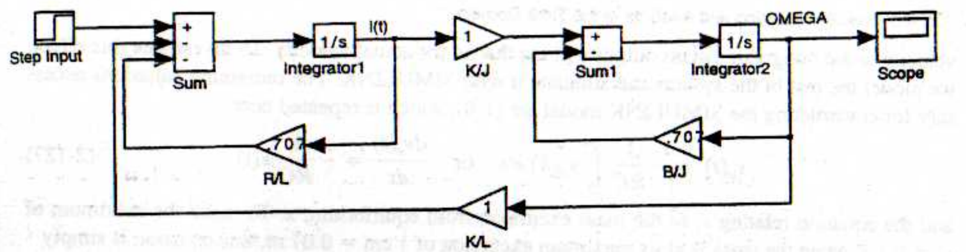


FIGURE 2-22. SIMULINK diagram for the separately excited dc motor system of Figure 2-21 (L = 1).

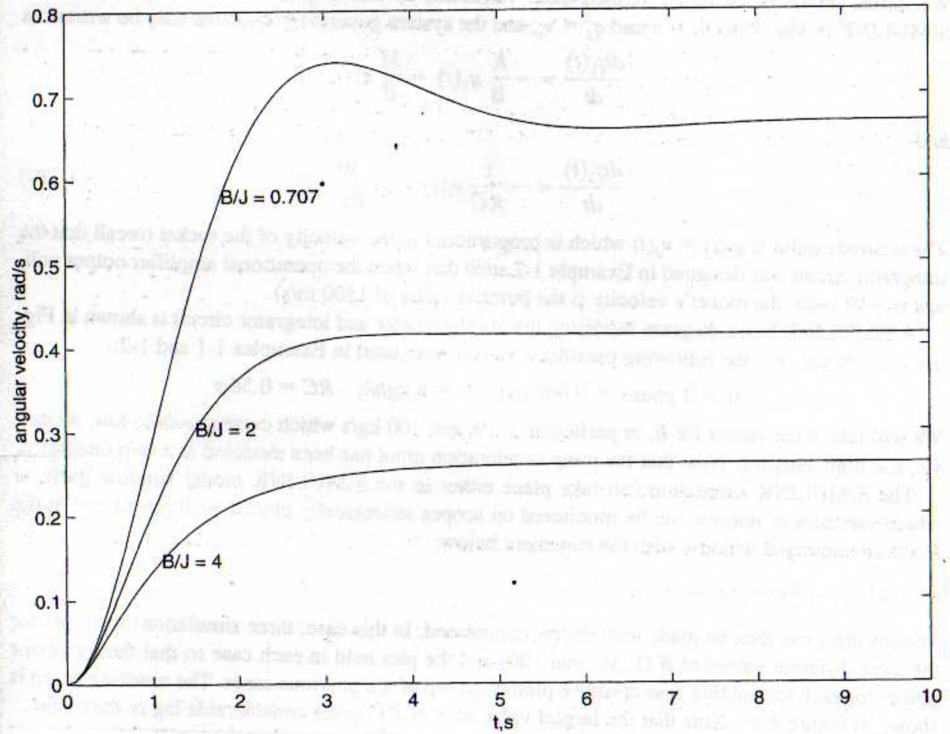


FIGURE 2-23. Angular velocity versus time for motor armature for different coefficients of friction.

Summary

In this chapter, the response of a fixed, linear system to an arbitrary input has been considered. The basic tool for obtaining the response is the superposition integral, which provides the output of the system as a superposition of elemental impulse responses weighted by the input signal values at various delays. The linearity of the system allowed the superposition of the elemental responses, while the fixed property means that the response to an impulse applied at $t = 0$ need only be considered with all other possible delays obtained by delaying the impulse response due to the fixed nature of the system. The following are the main points made in this chapter.

1. A system is said to be *continuous time* if the signals processed by the system are continuous-time signals. It is said to be a *discrete-time system* if it processes discrete-time signals. It is said to be *quantized* if the signals processed by it are quantized signals. If a system is built to process signals that are both discrete time and quantized, it is said to be a *digital system.* Continuous-time systems are dealt with almost exclusively in this chapter. Discrete-time systems will be the subject of Chapters 8-10. Furthermore, the systems considered in this chapter are all single-input, single-output; that is, they have a single input and a single output. Systems with multiple inputs and outputs can be handled by the techniques to be explored in Chapter 7.
2. A system is *time invariant,* or *fixed,* if its input-output relationship does not change with time. Thus, for a time-invariant system, if the input $x(t)$ produces the output $y(t)$, then the delayed input $x(t-\tau)$, where τ is a constant delay, produces the output $y(t-\tau)$.
3. A system is *causal,* or *nonanticipatory,* if its response to an input does not depend on future values of that input.
4. A system is *instantaneous,* or *zero memory,* if its output depends only on present values of the input, and not on past or future values.
5. If a single-input, single-output system is represented symbolically as $y(t) = \mathcal{H}[x(t)]$, where $\mathcal{H}[\cdot]$ is an operator, the system is linear if superposition holds. That is, a system is linear if, for the arbitrary linear combination of two inputs $\alpha_1 x_1(t) + \alpha_2 x_2(t)$, the output can be expressed as

$$\begin{aligned} y(t) &\triangleq \mathcal{H}[\alpha_1 x_1(t) + \alpha_2 x_2(t)] \\ &= \alpha_1 \mathcal{H}[x_1(t)] + \alpha_2 \mathcal{H}[x_2(t)] \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

where $y_1(t)$ is the response to $x_1(t)$ and $y_2(t)$ is the response to $x_2(t)$ and α_1 and α_2 are arbitrary constants.

6. For a linear system that is not fixed, the superposition integral relating the input $x(t)$ to the output $y(t)$ is

$$y(t) = \int_{-\infty}^{\infty} h(t, \lambda)x(\lambda)\, d\lambda$$

where $h(t, \lambda)$ is the response of the system at time t to a unit impulse applied at time λ. If, in addition, the system is fixed, then $h(t, \lambda) = h(t - \lambda)$, and the superposition integral becomes

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)\, d\lambda \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau \end{aligned}$$

where the second form follows from the first by substituting $\tau = t - \lambda$. These integrals are known as convolution integrals.

7. In addition to its use for finding the output of a fixed, linear system, the convolution integral has many other uses. We denote the convolution operation by $y(t) = h(t) * x(t) = x(t) * h(t)$. The following properties of the convolution integral hold:

1. $h(t) * x(t) = x(t) * h(h)$.
2. $h(t) * [\alpha x(t)] = \alpha[h(t) * x(t)]$, where α is a constant.
3. $h(t) * [x_1(t) + x_2(t)] = h(t) * x_1(t) + h(t) * x_2(t)$.
4. $h(t) * [x_1(t) * x_2(t)] = [h(t) * x_1(t)] * x_2(t)$.
5. If $h(t)$ is time-limited to (a, b), and $x(t)$ is time-limited to (c, d), then $h(t) * x(t)$ is time-limited to $a + c, b + d)$.
6. If A_1 is the area under $h(t)$ and A_2 is the area under $x(t)$, then the area under $h(t) * x(t)$ is $A_1 A_2$.

8. The integrand of the convolution operation is found by three operations: (1) reversal in time, or folding, to obtain $x(-\lambda)$; (2) shifting, to obtain $x(t-\lambda)$; (3) multiplication of $h(\lambda)$ and $x(t-\lambda)$, to obtain the integrand. A similar series of operations is required to obtain the second form of the convolution integrand.
9. The impulse response of a fixed, linear system is the response of the system to a unit impulse applied at $t = 0$ with zero initial conditions.
10. For systems described by constant-coefficient, linear differential equations, the time-domain impulse response can be found by solving the differential equation in response to a unit impulse with appropriate initial conditions. Since the unit impulse is zero for $t > 0$, this reduces to solving the homogeneous equation and finding the initial conditions by integrating the differential equation with unit impulse forcing function through $t = 0$.
11. A second method of finding the impulse response of a lumped-element circuit by using time-domain techniques is to look at the properties of the lumped circuit elements in response to impulse forcing functions. To this end, capacitors are infinite charge sinks (short circuits) and inductors are open circuits at the instant of application of the impulse forcing function.
12. The output of a fixed, linear system can also be found in terms of a superposition of its step response. The resulting integrals, called Duhamel's integrals, are given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \dot{x}(t-\lambda)a(\lambda)\, d\lambda \\ &= \int_{-\infty}^{\infty} \dot{x}(\eta)a(t-\eta)\, d\eta \end{aligned}$$

where $a(t)$ is the response of the system to a unit step and the overdot denotes differentiation with respect to time.

13. For a fixed, linear system, *any linear operation on the input produces the same linear operation on the output.* Thus, for example, the step response is the integral of the impulse response, since a unit step is the integral of a unit impulse. The ramp response is the integral of the step response, since a unit ramp is the integral of the unit step.
14. For a fixed, linear system, the steady-state response (i.e., the response after all transients have approached negligible values) to a rotating phasor input signal is a rotating phasor, but usually with different amplitude and different phase. If the rotating phasor input has unit amplitude and zero phase, then the complex proportionality constant, denoted $H(\omega)$, relating output phasor to input phasor is called the *frequency response function* of the system. The magnitude of $H(\omega)$ is called the *amplitude response* of the system, and the argument, or angle, of $H(\omega)$ is called the *phase response* of the system. For example, through superposition, the steady-state output of a fixed, linear system in response to the cosinusoidal input $A\cos(\omega_0 t + \theta)$ is

$$y(t) = A|H(\omega_0)|\cos[\omega_0 t + \theta + \underline{/H(\omega_0)}]$$

Likewise, the steady-state response of the system to any sum of sinusoids could be found by using superposition.

15. Bounded-input, bounded-output (BIBO) stability of a system means that every bounded input produces a bounded output. For a fixed, linear, system a necessary and sufficient condition for BIBO stability is

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

where $h(t)$ is the impulse response of the system.

16. Fixed, linear systems can be modeled by interconnections of operational amplifier circuits configured as integrators, summers, inverters, and scale changers (amplifiers). A standard configuration is given in Figure 2-14 for a system described by the differential equation

$$\frac{d^n y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{n} b_i \frac{d^i x(t)}{dt^i}$$

In Chapter 7, such configurations will be found systematically by state-variable techniques. State-variable techniques are very powerful in that they provide a means to represent multiple-input, multiple-output systems, time-varying systems, and nonlinear systems. A block diagram, such as Figure 2-14, is called an analog computer simulation of the system.

17. Analog computer simulations are very seldom used nowadays. The reason is that digital computers have become very powerful and reasonably priced. Even a desktop computer or workstation is capable of simulating very complex systems. The last section of this chapter introduced the topic of numerical simulation of systems using MATLAB SIMULINK. Several linear systems are employed as examples. The SIMULINK feature of MATLAB makes numerical solution straight forward.

Further Reading

In addition to the references listed in Chapter 1, the following books provide alternative reading to systems modeling in the time domain.

T. KAILATH, *Linear Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1980.
E. W. KAMEN, *Introduction to Signals and Systems.* 2nd ed. New York: Macmillan, 1990.
N. K. SINHA, *Linear Systems.* New York: Wiley, 1991.
H. KWAKERNAAK and R. SIVAN, *Modern Signals and Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1991.

The following is an old book and out of print. Yet, it is referenced because many of the definitions and examples in this chapter are patterned after material found in it.

R. J. SCHWARZ and B. FRIEDLAND, *Linear Systems.* New York: McGraw-Hill, 1965.

The following references provide extensive treatments on modeling physical systems.

W. A. BLACKWELL, *Mathematical Modeling of Physical Networks.* New York: Macmillan, 1968.
C. M. CLOSE and D. K. FREDERICK, *Modeling and Analysis of Dynamic Systems.* Boston: Houghton-Mifflin, 1978.

Material on numerical solution of system equations can be found in the following book:

J. M. SMITH, *Mathematical Modeling and Digital Simulation for Engineers and Scientists,* 2nd ed. New York: Wiley, 1987.

For guidance on the use of SIMULINK, see the following reference:

The Student Edition of SIMULINK, Version 2: User's Guide. Upper Saddle River, NJ: Prentice Hall, 1998.

Problems

Section 2-2

2-1. A system is defined by the set of algebraic equations

$$y_1(t) = 2x_1(t) - x_2(t)$$
$$y_2(t) = 5x_2(t) + 3x_3(t)$$

Represent this system in terms of a matrix equation of the form

$$y(t) = Hx(t)$$

That is, give $y(t)$, H, and $x(t)$ explicitly.

2-2. What is the order of each system defined by the following equations?

(a) $2\frac{dy(t)}{dt} + 3y(t) = \frac{d^2x(t)}{dt^2} + x(t)$

(b) $3y(t) + \int_{-\infty}^{t} y(\lambda)\, d\lambda = x(t)$

(c) $4y(t) + 10 = \frac{dx(t)}{dt} + 5x(t)$

(d) $\frac{dy(t)}{dt} + t^2 y(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda$

(e) $\frac{d^2y(t)}{dt^2} + y(t)\frac{dy(t)}{dt} + y(t) = 5x(t)$

2-3. Which of the systems defined by the equations of Problem 2-2 are fixed? Justify your answers.

2-4. Which of the systems defined by the equations of Problem 2-2 are nonlinear? Justify your answers.

2-5. A system is defined by the input-output relationship

$$y(t) = x(t^{1/2})$$

Is this system causal or noncausal? Justify your answer by choosing a specific pair of inputs that will or will not satisfy (2-23).

2-6. A system is defined by the input-output relationship

$$y(t) = 10x(t + 2) + 5$$

(a) Is this system linear? Prove your answer.
(b) Is it causal or noncausal? Why?

2-7. A system is defined by the input-output relationship

$$y(t) = x(t^2)$$

Is this system:
(a) Linear?
(b) Causal?
(c) Fixed?
Prove your answers.

2-8. An echo system is defined by the input-output relationship $y(t) = x(t) + \alpha x(t - \tau_0)$.
(a) Show that it is a linear system.
(b) Is it zero memory?
(c) Show that it is causal if $\tau_0 \geq 0$.
(d) Sketch its output for $\alpha = 0.5$ and 1.5 if $x(t) = u(t) - u(t - 1)$ and $\tau_0 = 1$.

2-9. An averager is defined by the input-output relationship

$$y(t) = \frac{1}{T_1 + T_2} \int_{t-T_1}^{t+T_2} x(\lambda)\, d\lambda$$

where T_1 and T_2 are positive constants.
(a) Show that this system is linear.
(b) What conditions on T_1 and T_2 make this a causal system?

2-10. (a) Write a differential equation relating $y(t)$ to $x(t)$ for the circuit shown in Figure P2-10.
(b) Show that this system is linear.
(c) Show that this system is fixed.
(d) Convert the differential equation found in part (a) to an integral equation of the form (2-5). Assume that the input $x(t)$ is applied at $t = 0$, and that the current through the inductor is initially zero.

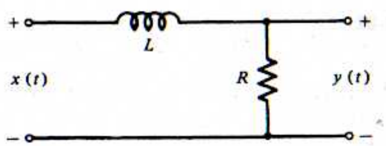


FIGURE P2-10

2-11. Fill in the table at the end of this problem to state whether the systems specified are linear or nonlinear, causal or noncausal, fixed or time-varying, and dynamic or instantaneous. Also give their order.
(a) $\frac{dy}{dt} + 3y + 2\int_{-\infty}^{t} y(\lambda)\, d\lambda = x(t)$
(b) $\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y^2(t) = x(t)$
(c) $\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + y(t) = x(t)$
(d) $y(t)\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + y(t) = x(t)$
(e) $y(t) = x(t^2) + x(t)$
(f) $y(t)\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + y(t) = x(t + 5)$

If true, check the appropriate box. Fill in the bottom row with system order.

	System					
Property	**a**	**b**	**c**	**d**	**e**	**f**
Linear						
Causal						
Fixed						
Dynamic						
Order						

2-12. Classify the following input-output relations for systems as to linearity, order, causality, and time invariance. State the order of each.
(a) $\frac{dy}{dt} + \frac{1}{RC}y(t) = x(t)$; RC = constant
(b) $\frac{dy}{dt} + t^2y(t) = x(t)$
(c) $\frac{d^2y}{dt^2} + y(t)\frac{dy}{dt} + y(t) = x(t)$
(d) $y(t) = x^2(t) + 1$

2-13. Given the system

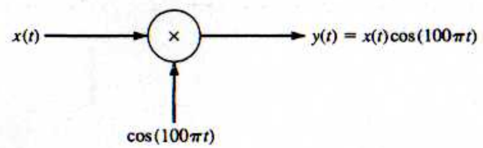


FIGURE P2-13

(a) Show that it is linear.
(b) Is it time varying? Why?
(c) Is it causal? Why?
(d) Is it instantaneous? Why?

2-14. A fixed, linear system responds to the input $x(t) = \Pi[(t - 1)/2]$ with the output $y(t) = (t + 1)\exp[2(t + 1)]u(t + 1)$.
(a) Is it causal? Why or why not?
(b) Write down the response to the input $2\,\Pi[(t - 1)/2] + 3\,\Pi[(t - 3)/2]$. Sketch input and output signals.

Section 2-3

2-15. Assuming the superposition property for the linear combination of two signals, extend this expression to the linear combination of N signals.

2-16. If $x_1(t)$, $x_2(t)$, and $h(t)$ are arbitrary signals, and α is a constant, show the following:
(a) $h(t) * [x_1(t) + x_2(t)] = h(t) * x_1(t) + h(t) * x_2(t)$
(b) $h(t) * [x_1(t) * x_2(t)] = [h(t) * x_1(t)] * x_2(t)$
(c) $h(t) * [\alpha x_1(t)] = \alpha h(t) * x_1(t)$
(d) If $h(t)$ is time-limited to (a, b) and $x(t)$ is time-limited to (c, d), then $h(t) * x(t)$ is time-limited to $(a + c, b + d)$.
(e) If A_1 is the area under $h(t)$ and A_2 is the area under $x(t)$, then the area under $h(t) * x(t)$ is A_1A_2.

Section 2-4

2-17. Find and sketch the signal $y(t)$, which is the convolution of the following pairs of signals.
(a) $x(t) = 2\exp(-10t)u(t)$ and $h(t) = \Pi(t/2)$
(b) $x(t) = \Pi[(t-1)/2]$ and $h(t) = u(t-10)$
(c) $x(t) = 2\exp(-10t)u(t)$ and $h(t) = u(t-2)$
(d) $x(t) = [\exp(-2t) - \exp(-10t)]u(t)$ and $h(t) = \Pi\left(\frac{t-1}{2}\right)$
(e) $x(t) = r(t)$ and $h(t) = 2\exp(-2t)u(t)$

2-18. Given the signals

$$x(t) = \Pi(t - 0.5)$$

and

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - 2n)$$

Write down an expression for their convolution. Sketch the result.

Section 2-5

2-19. Obtain the impulse response of the system shown by:
(a) Using the method of Example 2-10.
(b) Using the method of Example 2-11.

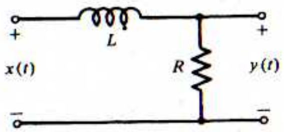


FIGURE P2-19

2-20. Find the impulse response for the circuit

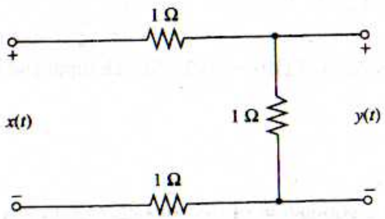


FIGURE P2-20

2-21. Given the circuit

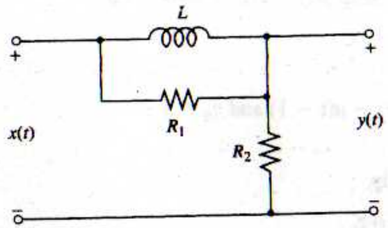


FIGURE P2-21

(a) Show that the differential equation relating the output to the input is

$$\frac{dy(t)}{dt} + \frac{R_2}{R_1 + R_2}\frac{R_1}{L}y(t) = \frac{R_2}{R_1 + R_2}\left[\frac{dx(t)}{dt} + \frac{R_1}{L}x(t)\right]$$

(b) Obtain the impulse response of this circuit.

2-22. Obtain the impulse response of the system shown.

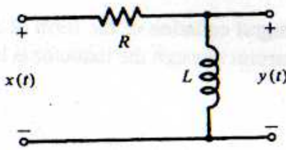


FIGURE P2-22

2-23. Obtain the impulse response of the system shown.

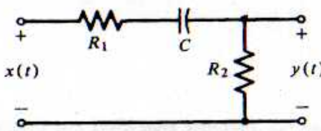


FIGURE P2-23

2-24. **(a)** Show that the impulse response $h(t)$ of the operational amplifier circuit shown is $h(t)$ proportional to $u(t)$.
(b) Express the output of the circuit in terms of the superposition integral for an arbitrary input. Is it the expected result given the function of this circuit?

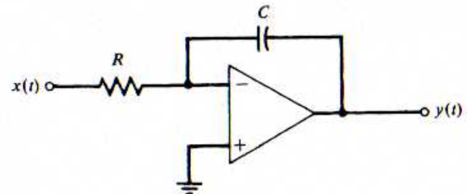


FIGURE P2-24

Section 2-6

2-25. (a) What is the step response of the circuit in Problem 2-24?
(b) Given the input $x(t) = u(t) + u(t-1) - 2u(t-2)$. Find the output for this input. Sketch both the input and the output. Label carefully.

2-26. Obtain the output of the system of Problem 2-19 in response to the input

$$x(t) = \Pi(t - 0.5)$$

Sketch for $R/L = 0.1$ and 1.0. Use (2-41).

2-27. Obtain the output of the system of Problem 2-22 in response to the input

$$x(t) = \Pi(t - 0.5)$$

Sketch for $R/L = 0.1$ and 1.0.

2-28. Using (2-85), obtain the output of the system of Problem 2-19 in response to the input

$$x(t) = \Pi(t - 0.5)$$

Sketch for $R/L = 0.1$ and 1.0.

2-29. Given that the RL filter shown has impulse response

$$h(t) = \delta(t) - (R/L)\, e^{-(R/L)t} u(t)$$

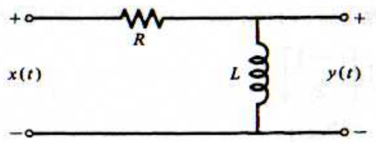


FIGURE P2-29

(a) Find the step response.
(b) Find the ramp response.

2-30. Obtain and sketch the response of the filter of Problem 2-29 to the input shown for:
(a) $R/L = 0.1$
(b) $R/L = 1.0$
(*Hint:* Use the answers obtained for Problem 2-29 together with superposition.)

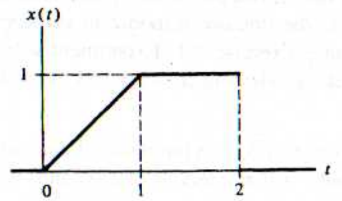


FIGURE P2-30

2-31. (a) Using the superposition property, find the response of the RC circuit of Example 2-11 to the input

$$x(t) = r(t) - 2r(t-1) + r(t-2)$$

Sketch the input and the output for $RC = 0.1$.
(b) Find and sketch the response to $\frac{dx}{dt}$.

2-32. Given the circuit shown

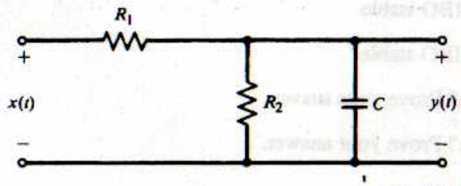


FIGURE P2-32

(a) Show that the input-output relationship is

$$R_1 C \frac{dy}{dt} + \left[1 + \frac{R_1}{R_2}\right] y = x$$

(b) Show that the impulse response is

$$h(t) = (R_1 C)^{-1} \exp\left(-\frac{t}{\tau}\right) u(t)$$

where $\tau = R_1 R_2 C/(R_1 + R_2)$.
(c) Show that the step response is

$$a(t) = \frac{R_2}{R_1 + R_2}\left[1 - \exp\left(-\frac{t}{\tau}\right)\right] u(t)$$

(d) If the input is $x(t) = \Pi[(t-1)/2]$, obtain the output.

Answer: $y(t) = \frac{R_2}{R_1 + R_2}\left\{\left[1 - \exp\left(-\frac{t}{\tau}\right)\right] u(t) - \left[1 - \exp\left(-\frac{t-2}{\tau}\right)\right] u(t-2)\right\}$

(e) If the input is $x(t) = r(t)$, find the output. (*Hint:* Use Duhamel's integral.)

Answer: $y_r(t) = \frac{R_2}{R_1 + R_2}\left\{t - \tau\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]\right\} u(t)$

(f) If the input is $x(t) = r(t) - 2r(t-1) + r(t-2)$, find the output in terms of the answer of part (e).

Section 2-7

2-33. (a) Obtain the frequency-response function for the system of Problem 2-29.
(b) Obtain the amplitude-response function for this system. Put it in terms of the parameter $f_3 = R/2\pi L$ and plot, dimensioning carefully.

(c) Obtain the phase-response function for this system. Put it in terms of the parameter $f_3 = R/2\pi L$ and plot, dimensioning carefully.

2-34. **(a)** Obtain the frequency-response function for the system of Problem 2-32.
(b) Obtain the amplitude- and phase-response functions for this system.
(c) If $R_1 = R_2 = 1\Omega$ and $C = 1$ F, obtain the steady-state response of this system to the input $x(t) = \cos 2\pi t + \sin 5\pi t$.

Section 2-8

2-35. Show that the system of Problem 2-19 is BIBO stable.

2-36. Show that the system of Problem 2-22 is BIBO stable.

2-37. Is the system of Problem 2-24 BIBO stable? Prove your answer.

2-38. Is the system of Problem 2-20 BIBO stable? Prove your answer.

Section 2-9

2-39. Given the system described by the differential equation

$$\frac{d^2y}{dt^2} + \omega_0^2 y = x(t), \qquad y(0^-) = 0 \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{t=0^-} = 0$$

(a) Show that the impulse response of the corresponding system is

$$h(t) = \omega_0^{-1} \sin(\omega_0 t)\, u(t)$$

(b) Show that this second-order differential equation is equivalent to the system of first-order differential equations

$$\frac{dq_1}{dt} = q_2$$

$$\frac{dq_2}{dt} = -\omega_0^2 q_1 + x$$

$$y = q_1$$

(c) Using only integrators, summers, and inverting amplifiers, draw a block diagram that realizes the equations given in part (b).
(d) Using the block diagram of part (c) and noting the impulse response derived in part (a), design an oscillator for the frequency range 100 Hz $\le \omega_0/2\pi \le$ 10,000 Hz.

2-40. Show that the state equations for the circuit of Figure 2-12 are

$$\frac{dq_1(t)}{dt} = -\frac{1}{RC} q_1(t) - q_2(t) + \frac{1}{C} x(t)$$

$$\frac{dq_2(t)}{dt} = \frac{1}{LC} q_1(t)$$

and that the output equation is $y(t) = q_1(t)$.

Problem Extending Text Material

2-41. To illustrate the impulse response for a time-varying system, consider a boat that is spreading chemicals into a lake. Thus its mass changes according to $M(t) = M_0 - kt$, where M_0 is the initial mass and k is a constant. The water exerts a drag force of $\alpha v(t)$ on the boat. Therefore, the equation of motion is

$$\frac{d}{dt}[Mv(t)] + \alpha v(t) = F(t)$$

or

$$(M_0 - kt)\frac{dv(t)}{dt} + (\alpha - k)v(t) = F(t)$$

where $F(t)$ is the applied force. If this force is a unit impulse applied at time $t = \tau$, show that the impulse response is

$$h(t, \tau) = \frac{(M_0 - kt)^{(\alpha-k)/k}}{(M_0 - k\tau)^{\alpha/k}}, \qquad t > \tau > 0$$

(b) If $M = 10$ kg, $\alpha = 2$, and $k = 1$, find the response of the boat to a step applied at time $t = 0$. (See Schwartz and Friedland, pp. 84–85).

Computer Exercises

2-1. **(a)** Following the MATLAB program of Example 2-8, write a program to convolve the signals

$$x(t) = \Pi\left[\frac{t-1}{2}\right]$$

$$\text{and} \quad h(t) = e^{-0.5t}u(t)$$

Plot the two signals $x(t)$ and $h(t)$ along with the result of the convolution, $y(t)$. For this purpose, use the subplot capability of MATLAB. Note that the result of the convolution will have twice as many t-values minus 1 as defined for either input signal, so you will have to use the axis function on the plot for $y(t)$ to make its plot have the same t-range as for $x(t)$ and $h(t)$.
(b) Analytically compute the convolution and compare the exact result with the numerically computed result. Vary the step size for t in the numerical computation and try to make a statement as to what size step size leads to unsatisfactory numerical results.

2-2. Assuming $x(t)$ and $h(t)$ to be the same as Computer Exercise 2-1, where the input and output of a fixed, linear system use Duhamel's integral, (2-85), to find the output of the system after finding the step response from $h(t)$ (recall that you integrate the impulse response to get step response). Compare your result with the result of Computer Exercise 2-1. Experiment with the width of the impulse approximation used for dx/dt to make a judgment on when it is too wide to give satisfactory results.

2-3. Using the MATLAB program developed in Computer Exercise 1-5, find the result of convolving a train of seven unit impulses symmetrically spaced about $t = 0$ at intervals of 2 seconds with a triangle function $\Lambda(t)$. Plot the result and compare with the analytically computed result.

2-4. Use your MATLAB program of Computer Exercise 2-1 to demonstrate the fixed property of a fixed, linear system. You have the result for an input $x(t) = \Pi[(t-1)/2]$. Now delay $x(t)$ some arbitrary amount, τ, and obtain the convolution of this delayed $x(t)$ with $h(t)$. Show that it is the same result as obtained by delaying the original output (that obtained in Computer Exercise 2-1) the same amount τ.

2-5. Use your MATLAB program of Computer Exercise 2-1 to demonstrate the superposition of property of a fixed, linear system. You have the result for an input $x(t) = \Pi[(t-1)/2]$. Now rewrite $x(t)$ as the difference of steps and obtain the convolutions of the two steps with $h(t)$. Show that the difference of the two outputs due to the steps is the same result as obtained in Computer Exercise 2-1.

2-6. Write a SIMULINK program to verify the results of Problem 2-39. Note that you can obtain an impulse source in SIMULINK by using a step function source followed by a differentiator. In part (d) you will want to use lower frequencies than specified there for plotting purposes.

2-7. Write a SIMULINK program to verify the result stated in Problem 2-41.

CHAPTER 3

The Fourier Series

3-1 Introduction

After considering some simple signal models in Chapter 1, we turned to systems characterization and analysis in the time domain in Chapter 2. The principal tool developed there was the superposition integral that resulted from resolving the input signal into a continuum of unit impulses and superimposing the elemental system outputs due to each impulsive input—thus the name "superposition integral." The application of the superposition integral involves first finding the impulse response of the system and then carrying out the actual integration—tasks that often are not simple.

In this chapter and in Chapter 4 we consider procedures for resolving certain classes of signals into superpositions of sines and cosines or, equivalently, complex exponential signals of the form $\exp(j\omega t)$. For periodic power signals this resolution results in the Fourier series coefficients of the signal and the resulting representation of such signals, known as the Fourier series of the signal, is considered in this chapter. For energy signals of finite or infinite extent, the Fourier integral provides the desired resolution with the signal representation for such signals given by the inverse Fourier transform integral. This resolution technique is examined in Chapter 4. Signals of the *exponential class,* of which energy and power signals are subclasses, may be resolved with complex exponential signals of the form $\exp[(\sigma + j\omega t)]$ by means of the Laplace transform integral. We deal with the Laplace transform in Chapter 5.

Advantages of Fourier series and Fourier transform representations for signals are twofold. First, in the analysis and design of systems it is often useful to characterize signals in terms of *frequency-domain* parameters such as bandwidth or spectral content. Second, the superposition property of linear systems, and the fact that the steady-state response of a fixed, linear system to a sinusoid of a given frequency is itself a sinusoid of the same frequency, provide a means of solving for the response of such systems. Indeed, the Fourier or Laplace transform, as appropriate, allows one to convert the constant-coefficient linear differential equation representation for a lumped system to an algebraic expression that considerably simplifies the solution for the system output. We consider this method of solution in Chapters 4 through 7.

We begin this chapter with some simple examples to convince ourselves that a sine-cosine series is a useful representation for a periodic signal.

3-2 Trigonometric Series†

Recalling Example 1-6, parts (d) and (e), we note that the sum of two sinusoids is periodic provided that their frequencies are commensurable. Stated another way, the sum of two sinusoids is periodic provided that their frequencies are integer multiples of a fundamental frequency.

†An alternative approach to Fourier series is provided in Section 3-10 in terms of generalized vector-space concepts.

To examine the fruitfulness of representing periodic signals by sums of sinusoids whose frequencies are harmonics, or integer multiples, of a fundamental frequency, we consider two such series and plot their partial sums.

EXAMPLE 3-1

As a first example, consider the trigonometric series, assumed to hold for all t, given by

$$x(t) = \sin \omega_0 t + \tfrac{1}{3} \sin 3\omega_0 t + \tfrac{1}{5} \sin 5\omega_0 t + \cdots \tag{3-1}$$

where $2\pi/\omega_0$ is the period. Its partial sums are

$$s_1 = \sin \omega_0 t$$
$$s_2 = \sin \omega_0 t + \tfrac{1}{3} \sin 3\omega_0 t$$

and so on. Several of them are plotted in Figure 3-1 with $\omega_0 t$ as the independent variable. Stretching our imagination, we see that a square wave appears to be taking shape as more terms are included in the partial sum. As each harmonic is added, the ripple in the flat-top approximation of the square wave becomes smaller and possesses a frequency equal to that of the highest harmonic included in the series.

Somewhat disconcerting, however, is the appearance of "ears" (or overshoot) in the plot of the partial sums where the square wave is discontinuous. This phenomenon, referred to as the Gibbs phenomenon, is examined in more detail in Section 4-10.

The convergence of the series at any particular point may be examined by substituting the appropriate value of $\omega_0 t$. For example, setting $\omega_0 t = \pi/2$ in (3-1), we obtain the alternating series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \tag{3-2}$$

which may be checked by consulting a table of sums of series.‡ We can normalize the series (3-1) to have a value of unity at $\omega_0 t = \pi/2$ by multiplying it by $4/\pi$. Later we will show that a unit-amplitude square wave can be represented by the series

$$x_{sq}(t) = \frac{4}{\pi}(\sin \omega_0 t + \tfrac{1}{3} \sin 3\omega_0 t + \tfrac{1}{5} \sin 5\omega_0 t + \cdots), \qquad -\infty < t < \infty \tag{3-3}$$

EXAMPLE 3-2

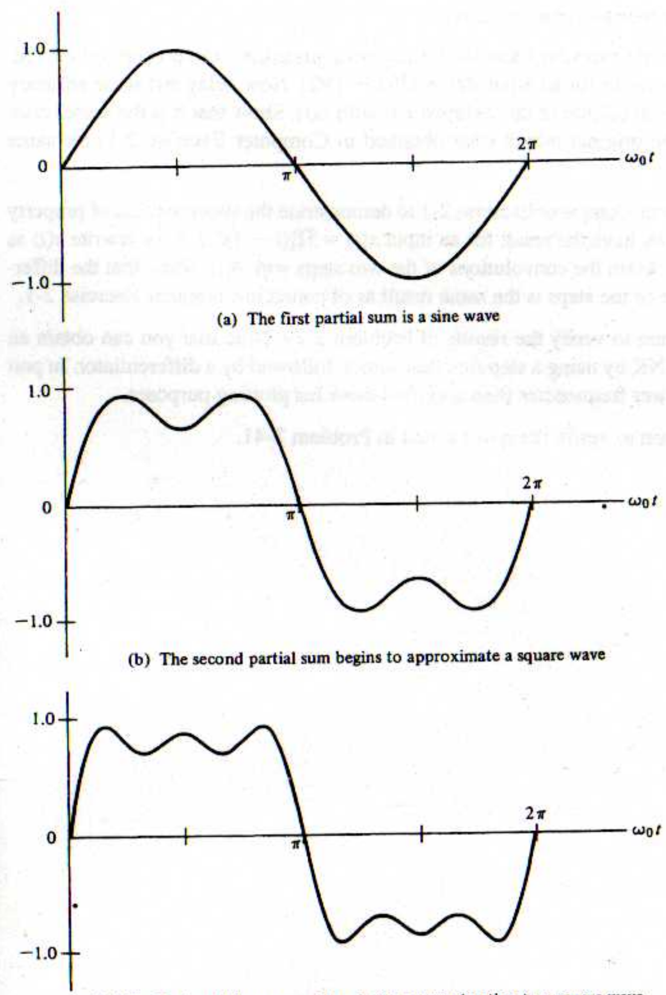
As a second example, consider the partial sums of the trigonometric series

$$y(t) = \sin \omega_0 t - \tfrac{1}{2} \sin 2\omega_0 t + \tfrac{1}{3} \sin 3\omega_0 t \cdots, \qquad -\infty < t < \infty \tag{3-4}$$

Plots showing its partial sums are given in Figure 3-2. Again, with a little imagination, we can see the beginnings of a sawtooth waveform. Each term added decreases the ripple in magnitude. The frequency of the ripple is equal to that of the highest-frequency term included in the series. The overshoot phenomenon, present at the discontinuities of the square wave considered previously, is also present with the sawtooth waveform.

Having convinced ourselves of the possibility of building up arbitrary periodic waveforms from sums of harmonically related sinusoidal terms, we turn to the crux of the matter: *Given a specific periodic waveform, how do we find its trigonometric series representation?* The resulting series, which is unique for each periodic signal, is called the *trigonometric Fourier series* of the signal.

‡See, for example, *Mathematical Tables from Handbook of Chemistry and Physics* (Boca Raton, Fla.: CRC Press, 1962), p. 325.



(a) The first partial sum is a sine wave

(b) The second partial sum begins to approximate a square wave

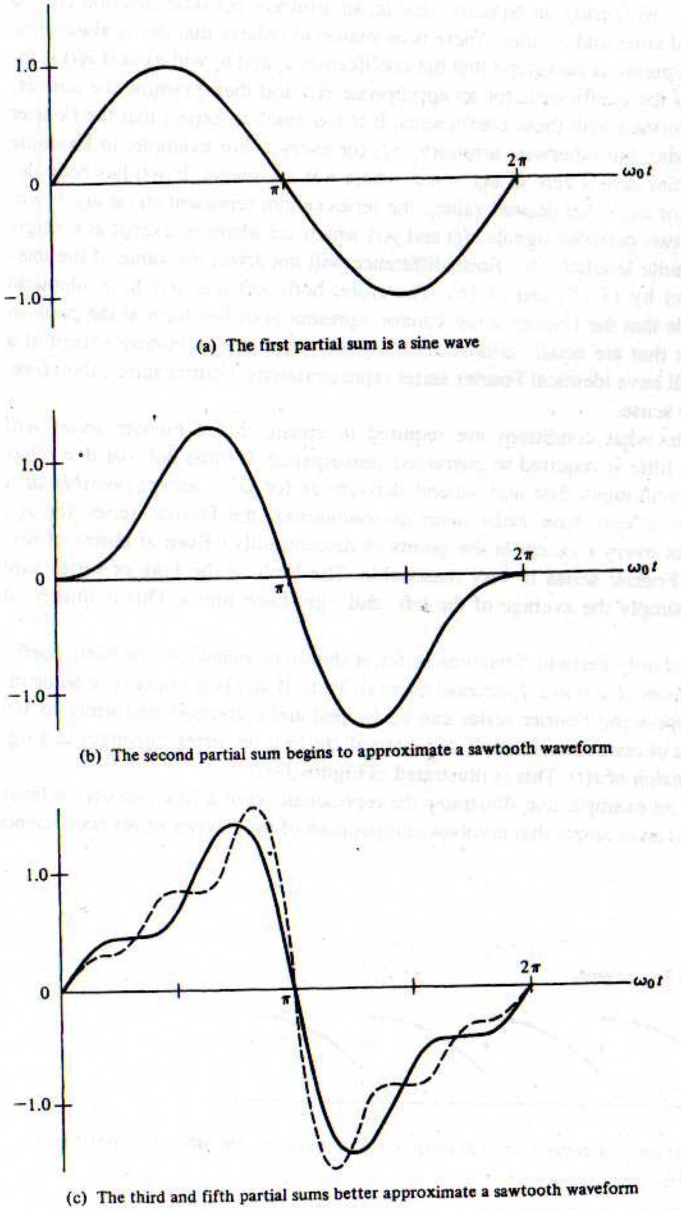
(c) The third partial sum provides a better approximation to a square wave

FIGURE 3-1. The series of Example 3-1 shows how a trigonometric series approximates a square wave (only one period shown).

3-3 Obtaining Trigonometric Fourier Series Representations for Periodic Signals

We begin by writing down the general form that the trigonometric Fourier series representation of a periodic signal may have:

$$\begin{aligned} x(t) = a_0 &+ a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \cdots \\ &+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \cdots, \qquad -\infty < t < \infty \end{aligned} \tag{3-5}$$



(a) The first partial sum is a sine wave

(b) The second partial sum begins to approximate a sawtooth waveform

(c) The third and fifth partial sums better approximate a sawtooth waveform

FIGURE 3-2. The trigonometric series of Example 3-2 illustrates approximation of a sawtooth waveform (only one period shown).

This may be written compactly in terms of summations as

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \tag{3-6}$$

where $-\infty < t < \infty$.† Since the right-hand side is a sum of harmonically related sinusoids, which themselves are periodic functions, the left-hand side is periodic. The problem that faces us is to find $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ for a given $x(t)$.

To obtain relationships for finding the a_n's and b_n's for an arbitrary (periodic) $x(t)$ we begin with a_0. Integrating the series (3-5) term by term over one period of $x(t)$, we obtain

$$\int_{T_0} x(t)\,dt = a_0 \int_{T_0} dt + a_1 \int_{T_0} \cos \omega_0 t\,dt + a_2 \int_{T_0} \cos 2\omega_0 t\,dt + \cdots$$
$$+ b_1 \int_{T_0} \sin \omega_0 t\,dt + b_2 \int_{T_0} \sin 2\omega_0 t\,dt + \cdots \tag{3-7}$$

where $\int_{T_0} (\cdot)\,dt$ denotes integration over any period. All terms except the first involve the integration of a sine or cosine over an integral number of periods and are therefore zero. (For $\sin n\omega_0 t$ or $\cos n\omega_0 t$, as much area appears above the t-axis as below it in one period.) The first term yields $a_0 T_0$. Thus the coefficient a_0 is given by

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\,dt \tag{3-8}$$

which is the average value of the waveform.

Using the series form (3-6) we derive a general expression valid for any of the a_n's. The derivation proceeds as follows for any a_n except a_0. Multiplying both sides of (3-6) by $\cos m\omega_0 t$ (the use of m allows us to choose any a_n coefficient), and integrating over a period of $x(t)$, we obtain

$$\int_{T_0} x(t) \cos m\omega_0 t\,dt = a_0 \int_{T_0} \cos m\omega_0 t\,dt + \int_{T_0} \left(\sum_{n=1}^{\infty} a_n \cos n\omega_0 t \right) \cos m\omega_0 t\,dt$$
$$+ \int_{T_0} \left(\sum_{n=1}^{\infty} b_n \sin n\omega_0 t \right) \cos m\omega_0 t\,dt \tag{3-9}$$

The first term integrates to zero. We multiply each term of the series in parentheses by $\cos m\omega_0 t$ and integrate term-by-term to obtain

$$\int_{T_0} x(t) \cos m\omega_0 t\,dt = \sum_{n=1}^{\infty} a_n \int_{T_0} \cos n\omega_0 t \cos m\omega_0 t\,dt + \sum_{n=1}^{\infty} b_n \int_{T_0} \sin n\omega_0 t \cos m\omega_0 t\,dt \tag{3-10}$$

To continue, we note some rather interesting properties of integrals involving products of sines and cosines. Three possible cases occur. The resulting integrals are

$$I_1 = \int_{T_0} \sin m\omega_0 t \sin n\omega_0 t\,dt = \begin{cases} 0, & m \neq n \\ T_0/2, & m = n \neq 0 \end{cases} \tag{3-11}$$

$$I_2 = \int_{T_0} \cos m\omega_0 t \cos n\omega_0 t\,dt = \begin{cases} 0, & m \neq n \\ T_0/2, & m = n \neq 0 \end{cases} \tag{3-12}$$

†Since most signals in this chapter are defined over the entire t-axis, we dispense with giving the range of definition unless it is other than $-\infty < t < \infty$.

and

$$I_3 = \int_{T_0} \sin m\omega_0 t \cos n\omega_0 t \, dt = 0, \qquad \text{all } m, n \tag{3-13}$$

where $T_0 = 2\pi/\omega_0$ is a period of the fundamental and n and m are integers.[†] We note that with $n = 0$ in (3-12) and (3-13), we have the integral of $\cos m\omega_0 t$ and $\sin m\omega_0 t$, respectively. We also note that the product of two sinusoids with harmonically related frequencies is periodic, so that the integrals could be taken over *any* period.

Applying (3-13), we see that each term of the second series on the right-hand side of (3-10) is zero. Applying (3-12), we see that all terms of the first series on the right-hand side of (3-10) are also zero except for the term for which $n = m$. For $n = m$, the integral gives $T_0/2$, which yields

$$\int_{T_0} x(t) \cos m\omega_0 t \, dt = a_m\left(\frac{T_0}{2}\right) \tag{3-14}$$

Multiplying both sides of (3-14) by $2/T_0$ gives

$$a_m = \frac{2}{T_0}\int_{T_0} x(t) \cos m\omega_0 t \, dt, \qquad m \neq 0 \tag{3-15}$$

In a similar manner, by employing (3-6), (3-11), and (3-13), we may show that

$$b_m = \frac{2}{T_0}\int_{T_0} x(t) \sin m\omega_0 t \, dt \tag{3-16}$$

Without concerning ourselves about the validity of the steps in obtaining (3-8), (3-15), and (3-16), we *define* a Fourier series to be any trigonometric series of the form (3-6), where the coefficients are found according to the formulas just derived. To find the coefficients, we require only that the integrals involved exist.[‡]

[†]To show (3-11) through (3-13), trigonometric identities could be used. However, Euler's theorem gives a more easily remembered derivation. Since, by Euler's theorem, we have

$$e^{\pm jn\omega_0 t} = \cos n\omega_0 t \pm j \sin n\omega_0 t$$

we can add and subtract the expressions for $e^{jn\omega_0 t}$ and $e^{-jn\omega_0 t}$ to obtain

$$\sin n\omega_0 t = \frac{1}{2j}(e^{jn\omega_0 t} - e^{-jn\omega_0 t}) \qquad \text{and} \qquad \cos n\omega_0 t = \frac{1}{2}(e^{jn\omega_0 t} + e^{-jn\omega_0 t})$$

(The student should memorize these expressions as well as Euler's theorem.) Substituting for $\sin n\omega_0 t$ in (3-11), and using the exponential form for sine above, we obtain

$$I_1 = \int_0^{T_0} \frac{1}{2j}(e^{jm\omega_0 t} - e^{-jm\omega_0 t})\frac{1}{2j}(e^{jn\omega_0 t} - e^{-jn\omega_0 t})\, dt$$

$$= -\frac{1}{4}\int_0^{T_0} [e^{j(m+n)\omega_0 t} - e^{j(n-m)\omega_0 t} - e^{j(m-n)\omega_0 t} + e^{-j(n+m)\omega_0 t}]\, dt$$

Now $\int_0^{T_0} e^{jk\omega_0 t}\, dt = 0$ for k an integer not equal to zero by Euler's theorem (the integral of a sine or cosine over an integer number of periods is zero). If $k = 0$, we have $\int_0^{T_0} dt = T_0$. Using this in the expression for I_1, we have

$$I_1 = \begin{cases} 0, & n \neq m \\ T_0/2 & n = m \end{cases}$$

where the nonzero result for $n = m$ results from the second and third terms in the integrand for I_1. A similar series of steps can be used to prove (3-12) and (3-13).

[‡]We give only an informal discussion of convergence here. Conditions concerning the possibility of expanding a function in a Fourier series are given in a theorem proved by Dirichlet in 1829. These sufficient conditions are that $x(t)$ be defined and bounded on the range $(t_0, t_0 + T_0)$ and have only a finite number of maxima and minima and a finite number of discontinuities on this range. These conditions, being *sufficient conditions*, are more restrictive than necessary. Fejér, in 1904, published two theorems giving conditions for the convergence of Fourier series that are less restrictive than the Dirichlet conditions.

In going from (3-9) to (3-10) it was necessary to interchange the order of summation and integration. We did not worry about the validity of this step. Indeed, the whole development so far has been based on the assumption that (3-6) is truly an equality; that is, an arbitrary periodic function can be represented in terms of a sum of sines and cosines. There is no reason to believe that this is always the case. However, we noted in the previous paragraph that the coefficients a_n and b_n will exist if $x(t)$ is integrable. We can therefore find the coefficients for an appropriate $x(t)$ and then examine the convergence properties of the series formed with these coefficients. It is too much to expect that the Fourier series will converge to a periodic, but otherwise arbitrary, $x(t)$ for every t. For example, in Example 3-1 it is apparent that each partial sum is zero at $\omega_0 t = n\pi$, where n is an integer. If $x(t)$ has been defined to be $\pi/4$ at these points (or any other desired value), the series cannot represent $x(t)$ at $\omega_0 t = n\pi$. As another example, consider two periodic signals $x(t)$ and $y(t)$ which are identical except at a single point, where they differ by a finite amount. This finite difference will not affect the value of the integrals for the coefficients given by (3-15) and (3-16). Therefore, both $x(t)$ and $y(t)$ have identical Fourier series, and we conclude that the Fourier series cannot represent both functions at the point in question. In general, functions that are equal "almost everywhere"—that is, everywhere except at a number of isolated points—will have identical Fourier series representations. Fourier series, therefore, do not converge in a *pointwise* sense.

The question remains as to what conditions are required to ensure that a Fourier series *will* converge to $x(t)$. Indeed, very little is required to guarantee convergence. It turns out that if $x(t)$ has period equal to T_0, and has continuous first and second derivatives for all t, *except possibly at a finite number of points where it may have finite jump discontinuities,* the Fourier series for $x(t)$ converges uniformly to $x(t)$ for every t except at the points of discontinuity.[‡] Even at points of discontinuity, the behavior of a Fourier series is very reasonable. The limit of the Fourier series sum at a point of discontinuity is simply the average of the left- and right-hand limits. This is illustrated in Figure 3-3.

Although we have discussed only periodic functions so far, it should be noted that the basic coefficient formulas *use* only the values of $x(t)$ in a T_0-second interval. Thus, if $x(t)$ is given only in some interval of length T_0, the corresponding Fourier series can be formed and converges uniformly to $x(t)$ within this interval at all points of continuity. Outside this interval, the Fourier series converges to a signal which is the *periodic extension of* $x(t)$. This is illustrated in Figure 3-4.

We close this section with an example that illustrates the representation of a function over a finite interval by a Fourier series and an example that involves computation of the Fourier series coefficients for a square wave.

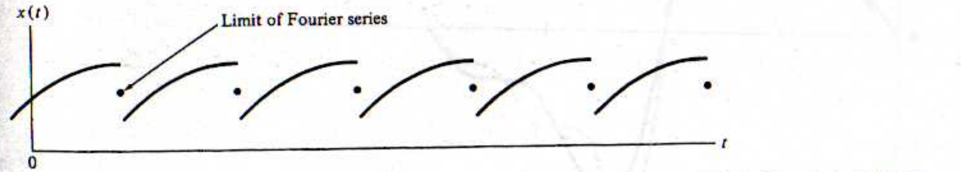


FIGURE 3-3. The Fourier series of a waveform converges to the mean of the left- and right-hand limits at a point of discontinuity.

[‡]By uniform convergence of an infinite series whose terms are functions of a variable, it is meant that the numerical value of the remainder after the first n terms is as small as desired *throughout the given interval* for n greater than a sufficiently large chosen number.

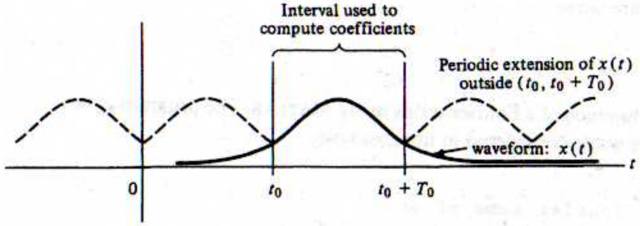


FIGURE 3-4. Expansion of a nonperiodic signal in terms of a Fourier series.

EXAMPLE 3-3

A violin string, to be plucked by a musician, has the initial shape shown in Figure 3-5. (a) Given that the trigonometric Fourier series coefficients of such a triangular wave of amplitude A are given by

$$\begin{aligned} a_0 &= A/2, \\ a_n &= 4A/\pi^2 n^2, \quad n \neq 0 \text{ and odd} \\ a_n &= 0, \quad n \text{ even} \\ b_n &= 0, \quad \text{all } n \end{aligned} \tag{3-17}$$

write out the first four nonzero terms of the Fourier series describing the initial shape of the violin string in the interval $|x| \leq 9$ inches. (b) Sketch the function to which the Fourier series converges for $-\infty < x < \infty$.

Solution:

(a) Note that the independent variable is x, not t. Assume that the $y(x)$ shown is to be represented by one full period of the Fourier series for a triangular waveform. Since the b_n's are zero (even function), the Fourier series is of the form

$$y(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 x \tag{3-18}$$

The fundamental frequency in rad/inch of this series is $\omega_0 = 2\pi/T_0 = 2\pi/18 = \pi/9$. The Fourier series coefficients are

$$\begin{aligned} a_0 &= 0.5/2 = 1/4 \\ a_1 &= 4(0.5)/\pi^2 = 2/\pi^2 \\ a_2 &= 0 \\ a_3 &= 4(0.5)/9\pi^2 = 2/9\pi^2 \\ a_4 &= 0 \\ a_5 &= 4(0.5)/25\pi^2 = 2/25\pi^2 \end{aligned} \tag{3-19}$$

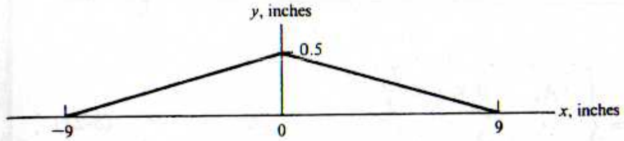


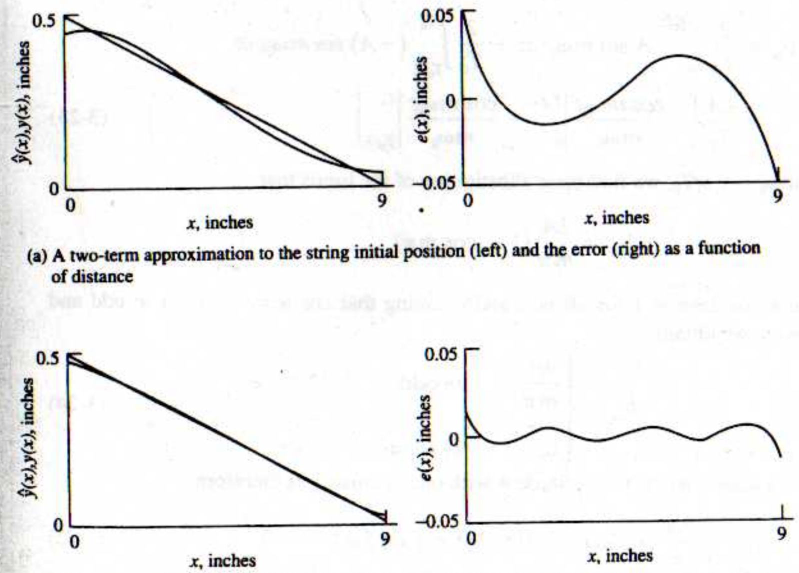
FIGURE 3-5. Initial position of a violin string to be represented by a Fourier series.

Thus, in the interval $|x| \leq 9$ inches, the approximating series is

$$\hat{y}(x) = \frac{1}{4} + \frac{2}{\pi^2}\cos\frac{\pi x}{9} + \frac{2}{9\pi^2}\cos\frac{\pi x}{3} + \frac{2}{25\pi^2}\cos\frac{5\pi x}{9} + \cdots \tag{3-20}$$

where the circumflex indicates that this is an approximation to the true initial shape of the string. Figure 3-6 shows the series approximation to the initial shape of the string for the first two terms only, and all four terms of the approximating series. Only the right half of the string and approximating series are shown, since the left half is the mirror image of the right half. Also shown is the error in the approximation, which is the difference between the actual string position and the approximating series. Note that the error is largest for $x = 9$ inches (i.e., at the end of the string). For a two-term approximation, the error is about 10% of the maximum deflection, while for a four-term approximation, the error is about 3.5% of the maximum deflection.

(b) The actual Fourier series converges to a triangular waveform infinite in extent.



(a) A two-term approximation to the string initial position (left) and the error (right) as a function of distance

(b) A four-term approximation to the string initial position (left) and the error (right) as a function of distance

FIGURE 3-6. Fourier series representations and the error for the initial position of the violin string. (Only the right half is shown due to symmetry.)

EXAMPLE 3-4

Consider the square wave defined by

$$x(t) = \begin{cases} A, & 0 < t < \dfrac{T_0}{2} \\ -A, & \dfrac{T_0}{2} < t < T_0 \end{cases} \tag{3-21}$$

and periodically extended outside this interval. The average value is zero, so $a_0 = 0$. From (3-15), the a_m's are

$$\begin{aligned} a_m &= \frac{2}{T_0}\int_0^{T_0/2} A\cos m\omega_0 t\,dt + \frac{2}{T_0}\int_{T_0/2}^{T_0}(-A)\cos m\omega_0 t\,dt \\ &= \frac{2A}{T_0}\left[\frac{\sin m\omega_0 t}{m\omega_0}\bigg|_0^{T_0/2} - \frac{\sin m\omega_0 t}{m\omega_0}\bigg|_{T_0/2}^{T_0}\right] \end{aligned} \tag{3-22}$$

Recalling that $\omega_0 = 2\pi/T_0$, we see that

$$\sin\frac{m\omega_0 T_0}{2} = \sin m\pi = 0$$

and

$$\sin m\omega_0 T_0 = \sin 2m\pi = 0$$

Thus all the a_m coefficients are zero. The b_m coefficients result by applying (3-16). This gives

$$\begin{aligned} b_m &= \frac{2}{T_0}\int_0^{T_0/2} A\sin m\omega_0 t\,dt + \frac{2}{T_0}\int_{T_0/2}^{T_0}(-A)\sin m\omega_0 t\,dt \\ &= \frac{2A}{T_0}\left[-\frac{\cos m\omega_0 t}{m\omega_0}\bigg|_0^{T_0/2} + \frac{\cos m\omega_0 t}{m\omega_0}\bigg|_{T_0/2}^{T_0}\right] \end{aligned} \tag{3-23}$$

Again, recalling that $\omega_0 = 2\pi/T_0$, we find upon substitution of the limits that

$$b_m = \frac{2A}{m\pi}(1 - \cos m\pi)$$

which results because $\cos 2m\pi = 1$ for all m. Finally, noting that $\cos m\pi = -1$ for m odd and $\cos m\pi = 1$ for m even, we obtain

$$b_m = \begin{cases} \dfrac{4A}{m\pi}, & m \text{ odd} \\ 0, & m \text{ even} \end{cases} \tag{3-24}$$

The Fourier series of a square wave of amplitude A with odd symmetry is therefore

$$x(t) = \frac{4A}{\pi}\left(\sin\omega_0 t + \tfrac{1}{3}\sin 3\omega_0 t + \tfrac{1}{5}\sin 5\omega_0 t + \cdots\right) \tag{3-25}$$

We note that only odd harmonic terms are present. This is a consequence of $x(t)$ having a special type of symmetry, referred to as half-wave odd symmetry. More will be said about special symmetry cases later. If $A = 1$, (3-25) becomes the series first shown in (3-3) and verifies the previous assertion that (3-3) represents a unity amplitude square wave.

MATLAB *Application*

It is interesting to examine the convergence of a Fourier series using MATLAB. The program given below accomplishes this for the square wave considered in this example.

```
%     Program to give partial Fourier sums of an
%     odd square wave of unit amplitude
%
n_max=input('Enter vector of highest harmonic values desired (odd)');
N=length(n_max);
t=0:.002:1;
omega_0=2*pi;
for k=1:N
    n=[];
    n=[1:2:n_max(k)];
    b_n=4./(pi*n);                    % Form vector of Fourier sine-coefficients
    x=b_n*sin(omega_0*n'*t);          % Rows of sine matrix are versus time;
                                      % columns are versus n,
                                      % so matrix multiply sums over n and a
                                      % vector for x(t) results
    subplot(N,1,k),plot(t,x), xlabel('t'), ylabel('partial sum'),...
    axis([0 1 -1.5 1.5]), text(.05,-5, ['max. har.=',
    num2str(n_max(k))])
end
```

Note that matrix multiplication is used to form the partial sums of the Fourier series. This saves considerable computation time over using a for loop because MATLAB is an interpretive language which compiles each statement separately. The for loop simply runs through different cases for the maximum number of harmonics summed. Note that any number of cases for the maximum number of harmonics can be run, but the plots would get rather small for more than three or four.

In this case, the program is not run with echo on because one would see it go through the for loop three times and this would be rather boring and long. The entry in the command window is simply

```
EDU»c3ex4
Enter vector of highest harmonic values desired (odd) [11 21 31]
```

where the second line comes up after the program begins running and the [11 21 31] is entered by the user to specify the three cases to be run for the maximum harmonic. If four cases are desired, this would be a 1 by 4 vector. The resulting plots are shown in Figure 3-7. Note that the number of humps on the flat part of the waveform is determined by the number of harmonics included in the partial sum, and also note the presence of ears at the discontinuity points of the square wave. The latter are known as the Gibbs phenomenon, to be discussed in the Chapter 4, Section 4-10.

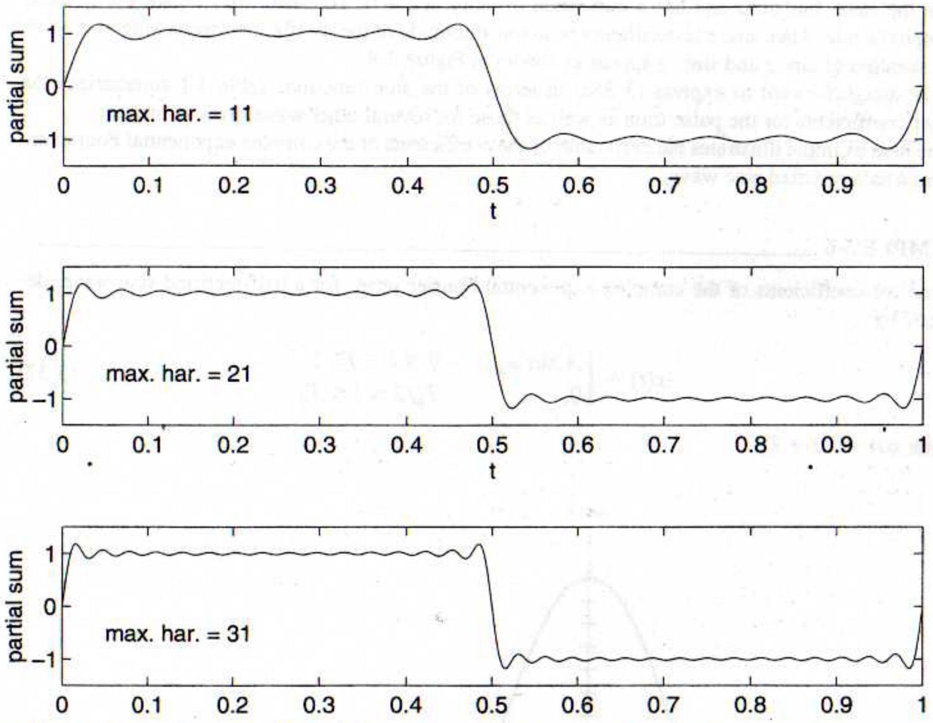


FIGURE 3-7. Partial sums of the Fourier series of an odd square wave for the highest harmonic being 11, 21, and 31, top to bottom.

3-4 The Complex Exponential Fourier Series

Another form of the Fourier series that involves complex exponential functions can be obtained by substituting the complex exponential forms of $\sin \omega_0 t$ and $\cos \omega_0 t$ into (3-6). Thus, letting

$$\sin n\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \tag{3-26a}$$

and

$$\cos n\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \tag{3-26b}$$

we obtain from (3-6) the series

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \\ &= a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - jb_n)e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n + jb_n)e^{-jn\omega_0 t} \end{aligned} \tag{3-27}$$

This is known as the complex exponential form of the Fourier series.

We see that terms involving both $\exp(jn\omega_0 t)$ and $\exp(-jn\omega_0 t)$ are present in the series. This is in keeping with our observation in Chapter 1 that the sum of two complex conjugate rotating phasors is required to produce a real, sinusoidal signal. To generalize the derivation of the coefficients in the complex exponential Fourier series, we write the series in the form

$$\begin{aligned} x(t) &= \cdots X_{-2}e^{-j2\omega_0 t} + X_{-1}e^{-j\omega_0 t} + X_0 + X_1 e^{j\omega_0 t} + X_2 e^{j2\omega_0 t} + \cdots \\ &= \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \end{aligned} \tag{3-28}$$

where the X_n's are, in general, complex constants. To find an expression for computing the X_n's, we multiply both sides of (3-28) by $e^{-jm\omega_0 t}$ and integrate over any period of $x(t)$. This results in

$$\int_{T_0} x(t)e^{-jm\omega_0 t}\,dt = \int_{T_0} \left(\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}\right) e^{-jm\omega_0 t}\,dt \tag{3-29}$$

We next multiply each term in the sum by $e^{-jm\omega_0 t}\,dt$ and integrate the result term by term without worrying about the validity of these steps at this point.† This results in

$$\int_{T_0} x(t)e^{-jm\omega_0 t}\,dt = \sum_{n=-\infty}^{\infty} X_n \int_{T_0} e^{j(n-m)\omega_0 t}\,dt \tag{3-30}$$

If $m \neq n$, the integral

$$\int_{T_0} e^{j(n-m)\omega_0 t}\,dt$$

is zero because

$$e^{j(n-m)\omega_0 t} = \cos(n-m)\omega_0 t + j\sin(n-m)\omega_0 t$$

is a periodic function with period T_0 and is symmetric about the t-axis. For $n = m$, $e^{j(n-m)\omega_0 t} = 1$, and the integral evaluates to T_0. Thus every term in the sum on the right-hand side of (3-30) is zero except the one for $n = m$; (3-30) therefore reduces to

$$\int_{T_0} x(t)e^{-jm\omega_0 t}\,dt = T_0 X_m$$

or, solving for X_m, we obtain

$$X_m = \frac{1}{T_0}\int_{T_0} x(t)e^{-jm\omega_0 t}\,dt \tag{3-31}$$

As with the trigonometric Fourier series, we *define* an exponential Fourier series to be a series of the form (3-28) with the coefficients calculated according to (3-31). The computation of the coefficients is usually considerably simpler than the computation of the trigonometric Fourier series coefficients. In addition, the exponential Fourier series provides us with a direct means of plotting the two-sided amplitude and phase spectra of a signal.

We note, by comparing (3-27) and (3-28), that the coefficients of the trigonometric Fourier series and the complex coefficients (3-31) are related by

$$X_n = \begin{cases} \frac{1}{2}(a_n - jb_n), & n > 0 \\ \frac{1}{2}(a_{-n} + jb_{-n}), & n < 0 \end{cases} \tag{3-32a}$$

†A discussion of convergence for this form of the series would be identical to the one given for the trigonometric form.

and

$$X_0 = a_0 \tag{3-32b}$$

This section will be concluded with two examples to illustrate the calculation of complex Fourier series coefficients.

EXAMPLE 3-5

Consider the periodic sequence of asymmetrical pulses (referred to as a pulse train)

$$x(t) = \sum_{m=-\infty}^{\infty} A\Pi\left(\frac{t - t_0 - mT_0}{\tau}\right), \qquad \tau < T_0 \tag{3-33}$$

whose period is T_0. The exponential Fourier series coefficients are calculated by evaluating the integral

$$X_n = \frac{1}{T_0}\int_{t_0-\tau/2}^{t_0+\tau/2} Ae^{-jn\omega_0 t}\, dt \tag{3-34}$$

where $\omega_0 = 2\pi/T_0$. The evaluation is straightforward. Integration gives

$$\begin{aligned} X_n &= \frac{-A}{jn\omega_0 T_0} e^{-jn\omega_0 t}\Big|_{t_0-\tau/2}^{t_0+\tau/2} \\ &= \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0}\left(\frac{e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}}{2j}\right), \qquad n \neq 0 \\ &= \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0} \sin\frac{n\omega_0\tau}{2}, \qquad n \neq 0 \end{aligned} \tag{3-35a}$$

Letting $n = 0$ in (3-34), we find X_0 to be

$$X_0 = \frac{A\tau}{T_0} \tag{3-35b}$$

Substituting $\omega_0 = 2\pi f_0$ into (3-35a), where f_0 is the fundamental frequency in hertz, we obtain the form

$$X_n = \frac{A\tau}{T_0} e^{-j2\pi n f_0 t_0} \frac{\sin \pi n f_0 \tau}{\pi n f_0 \tau} \tag{3-35c}$$

where the numerator and denominator have been multiplied by τ.

It is convenient to define

$$\text{sinc}\, z = \frac{\sin \pi z}{\pi z} \tag{3-36}$$

which is referred to as the *sinc function.* Values for sinc z and sinc^2 z are tabulated in Appendix F. Examining (3-36), we see that the sinc function is zero whenever

$$\sin \pi z = 0$$

This happens for $z = \pm 1, \pm 2, \pm 3, \ldots$. Furthermore, the sinc function is even (substitute $-z$ for z and obtain the same function) and has a maximum of unity at $z = 0$. The latter may be shown by using L'Hospital's rule. Also, sinc z is oscillatory with the amplitude of the oscillations decreasing as $|z| \to \infty$. Thus sketches of sinc z and sinc^2 z appear as shown in Figure 3-8.

It is straightforward to express (3-35c) in terms of the sinc function. Table 3-1 summarizes the Fourier coefficients for the pulse train as well as those for several other waveforms of interest.

The next example illustrates the derivation of the coefficients of the complex exponential Fourier series for a half-rectified sine wave.

EXAMPLE 3-6

Find the coefficients of the complex exponential Fourier series for a half-rectified sine wave, defined by

$$x(t) = \begin{cases} A \sin \omega_0 t, & 0 \le t \le T_0/2 \\ 0, & T_0/2 \le t \le T_0 \end{cases} \tag{3-37}$$

with $x(t) = x(t + T_0)$.

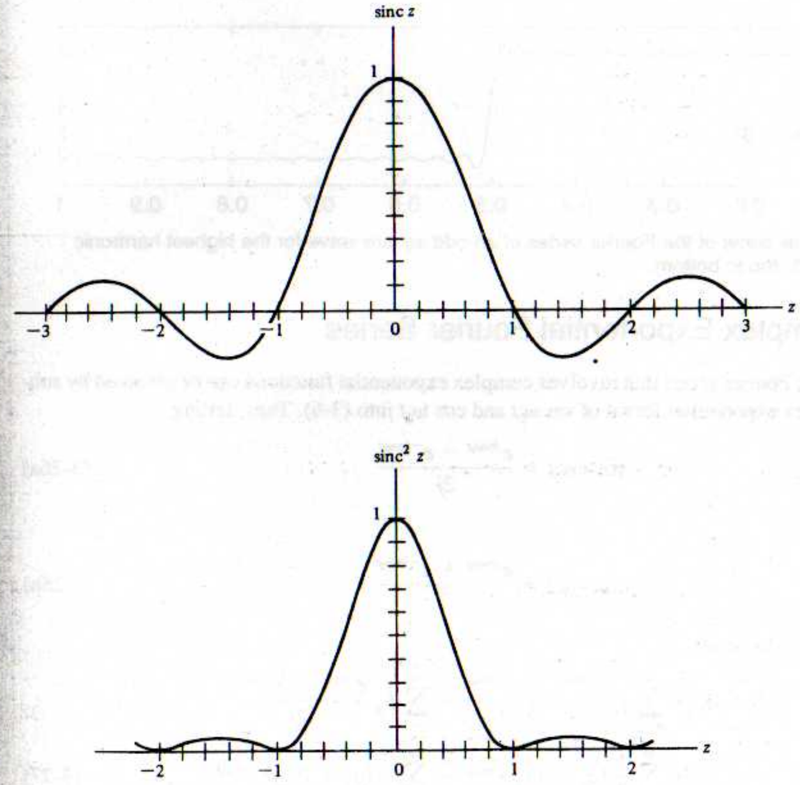
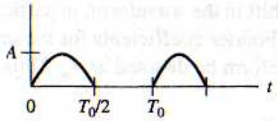
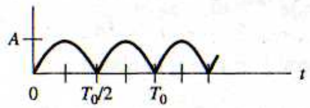
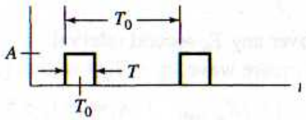
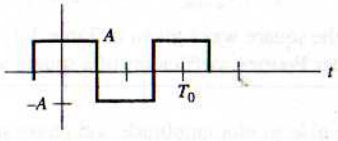
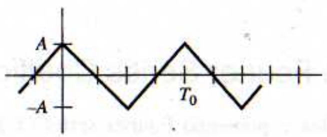


FIGURE 3-8. Sinc z and sinc^2 z.

TABLE 3-1
Coefficients for the Complex Exponential Fourier Series of Several Signals

1. Half-rectified sine wave		$X_n = \begin{cases} \dfrac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n \text{ odd and } \neq \pm 1 \\ -\frac{1}{4}jnA, & n = \pm 1 \end{cases}$
2. Full-rectified sine wave*		$X_n = \begin{cases} \dfrac{2A}{\pi(1-n^2)}, & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$
3. Pulse-train signal		$X_n = \dfrac{A\tau}{T_0} \operatorname{sinc} nf_0\tau\, e^{-j2\pi n f_0 t_0}, \quad f_0 = T_0^{-1}$
4. Square wave		$X_n = \begin{cases} \dfrac{2A}{\lvert n\rvert\pi}, & n = \pm 1, \pm 5, \ldots \\ \dfrac{-2A}{\lvert n\rvert\pi}, & n = \pm 3, \pm 7, \ldots \\ 0, & n \text{ even} \end{cases}$
5. Triangular wave		$X_n = \begin{cases} \dfrac{4A}{\pi^2 n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

*Note that T_0 is *not* the period of the full-wave rectified sinewave. Rather, its period is $T_0/2$.

Solution: From (3-31), the Fourier coefficients are given by the integral

$$X_n = \frac{1}{T_0}\int_0^{T_0/2} A \sin \omega_0 t\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{A}{2jT_0}\left[\int_0^{T_0/2} (e^{j\omega_0 t} - e^{-j\omega_0 t})e^{-jn\omega_0 t}\, dt\right]$$

$$= \frac{A}{2jT_0}\left[\int_0^{T_0/2} e^{j\omega_0(1-n)t}\, dt - \int_0^{T_0/2} e^{-j\omega_0(1+n)t}\, dt\right]$$

$$= -\frac{A}{4\pi}\left[\frac{e^{j(1-n)\pi}-1}{1-n} + \frac{e^{j(1+n)\pi}-1}{1+n}\right], \quad n \neq 1 \text{ or } -1 \qquad (3\text{-}38)$$

where use has been made of the complex exponential form for $\sin \omega_0 t$ and $\omega_0 = 2\pi/T_0$. We note that $e^{j(1\pm n)\pi} = \cos(1 \pm n)\pi + j\sin(1 \pm n)\pi = -(-1)^n$, where n is an integer. Thus, (3-38) simplifies to

$$X_n = 0, \qquad n \text{ odd}, \quad n \neq \pm 1$$

and

$$X_n = \frac{A}{\pi}\frac{1}{1-n^2}, \quad n \text{ even} \qquad (3\text{-}39)$$

The special cases for $n = 1$ and $n = -1$ must be handled separately. For $n = 1$, the calculation is

$$X_1 = \frac{A}{2jT_0}\int_0^{T_0/2} (e^{j\omega_0 t} - e^{-j\omega_0 t})e^{-j\omega_0 t}\, dt$$

$$= \frac{A}{2jT_0}\int_0^{T_0/2} (1 - e^{-j2\omega_0 t})\, dt$$

$$= \frac{A}{4j} \qquad (3\text{-}40)$$

For $n = -1$, a similar integration gives $X_1 = -A/4j$. Putting all these results together, we see that the first entry in Table 3-1 is correct. A similar series of integrations would be carried out to prove the second entry for the full-rectified sine wave.

MATLAB *Application*

To show how fast the partial sums of the Fourier series approaches a half-rectified sinewave, we provide a MATLAB program below for computing the complex exponential Fourier series.

```
%     Program to give partial complex exponential Fourier sums of a
%     half rectified sine wave of unit amplitude
%
n_max=input('Enter vector of highest harmonic values desired (even)');
N=length(n_max);
t=0:.005:1;
omega_0=2*pi;
for k=1:N
    n=[];
    n=[-n_max(k):n_max(k)];
    L_n=length(n);
    X_n=zeros(1, L_n);           % Form vector of Fourier sine-
                                 % coefficients; all
    X_n((L_n+1)/2+1)=-0.25*j;    % odd-order terms are zero except X(1)
                                 % and X(-1)
    X_n((L_n+1)/2-1)=0.25*j;     % so define coefficient array as a zero
                                 % array and
    for i=1:2:L_n                % then fill in nonzero values
        X_n(i)=1/(pi*(1-n(i)^2));
    end
    x=X_n*exp(j*omega_0*n'*t);   % Rows of exponential matrix are versus
                                 % time;
                                 % columns are versus n; matrix multiply
                                 % sums over n
%   Plot real part of x to get rid of small imaginary part due to
%   computational error
```

```
    subplot(N,1,k),plot(t,real(x)), xlabel('t'), ylabel('partial
    sum'),...
    axis([0 1 -0.5 1.5]), text(.05,-.25, ['max. har.=',
    num2str(n_max(k))])
end
```

As with the MATLAB program for Example 3-4, a vector of values can be entered for the highest harmonic present in the sum. The command window appears as follows:

```
EDU»c3ex6
Enter vector of highest harmonic values desired (even) [4 8 12]
```

The resulting plot is shown in Figure 3-9. Note that very few terms are required to make the resulting plot look very much like a half-rectified sine wave. This is because there are no discontinuities present in this waveform.

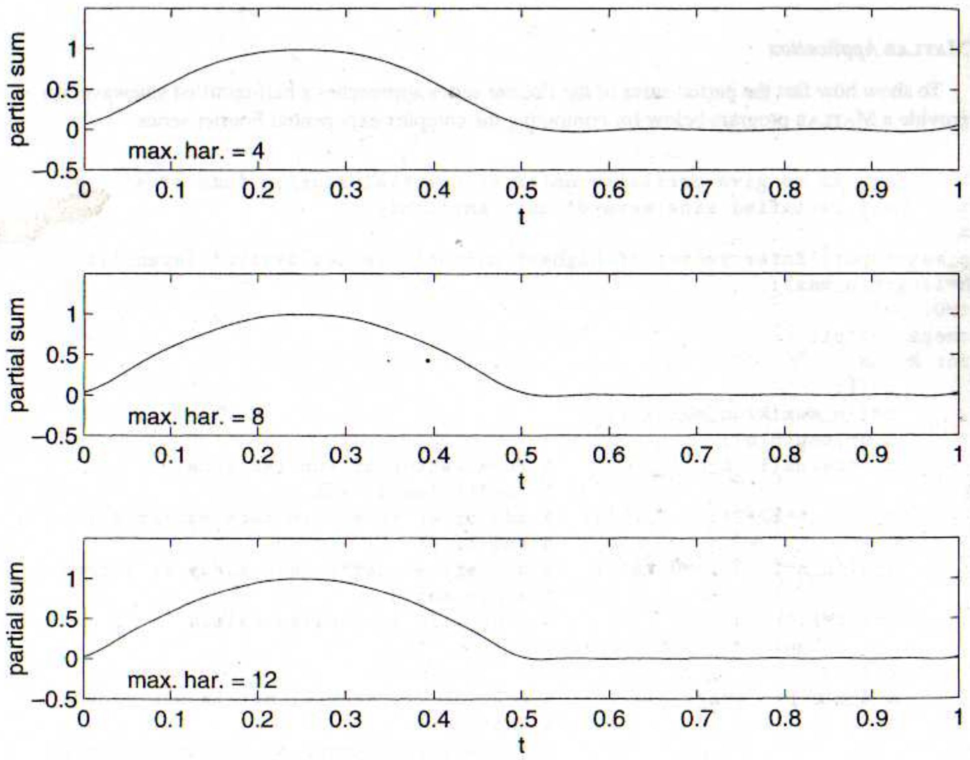


FIGURE 3-9. Partial sums of the complex exponential Fourier series of a half-rectified sine wave. Maximum harmonic is 4, 8, and 12, top to bottom.

EXAMPLE 3-7

Suppose we desire the complex exponential Fourier series for an odd square wave, whereas Table 3-1 has the Fourier series coefficients only for an even square wave. In other words, we want to know the change in the Fourier coefficients due to a time shift in the waveform, in particular, a shift of $T_0/4$ in this case. Consider the general case first. Let the Fourier coefficients for the unshifted waveform be denoted as X_n and those for the time-shifted waveform be denoted as Y_n. Thus

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt$$

and

$$Y_n = \frac{1}{T_0}\int_{T_0} x(t-\tau_0)e^{-jn\omega_0 t}\,dt, \quad \omega_0 = \frac{2\pi}{T_0} \tag{3-41}$$

Make the change of variables in the second integral $t' = t - \tau_0$ to obtain

$$Y_n = \left[\frac{1}{T_0}\int_{T_0} x(t')e^{-jn\omega_0 t'}\,dt'\right]e^{-jn2\pi\tau_0/T_0} = X_n e^{-j2\pi n\tau_0/T_0} \tag{3-42}$$

where we remember that the integration can be over any T_0-second interval.

For the case of going from an even to an odd square wave, $\tau_0 = T_0/4$ so that (3-42) becomes

$$X_{n,\,\text{odd}} = X_{n,\,\text{even}}e^{-j\pi n/2}\ (n\ \text{odd}) = \begin{cases} -jX_{n,\,\text{odd}}, & n = \pm 1, \pm 5, \cdots \\ jX_{n,\,\text{odd}}, & n = \pm 3, \pm 7, \cdots \end{cases} \tag{3-43}$$

where $X_{n,\,\text{odd}}$ are the Fourier coefficients for the square wave given in Table 3-1, item 4, and only n odd need be considered because the even-order Fourier coefficients of a square wave are zero.

One of our objectives in this chapter is to be able to plot amplitude and phase spectra for periodic signals. Before doing this, however, we discuss symmetry properties of the Fourier coefficients. This will simplify the plotting of spectra.

3-5 Symmetry Properties of the Fourier Series Coefficients

The expression for the coefficients of the complex exponential Fourier series (3-31), through use of Euler's theorem, can be written as

$$X_m = \frac{1}{T_0}\int_{T_0} x(t)\cos m\omega_0 t\,dt - \frac{j}{T_0}\int_{T_0} x(t)\sin m\omega_0 t\,dt \tag{3-44}$$

If $x(t)$ is real, the first term is the real part of X_m and the second term is the imaginary part of X_m. Comparing this with (3-15) and (3-16), we see that

$$X_m = \begin{cases} \frac{1}{2}(a_m - jb_m), & m > 0 \\ \frac{1}{2}(a_{-m} + jb_{-m}), & m < 0 \end{cases} \tag{3-45}$$

which is another way to obtain (3-32a).

Solving (3-45) for a_m and b_m, we obtain

$$a_m = 2\,\text{Re}\,X_m \quad \text{and} \quad b_m = -2\,\text{Im}\,X_m, \qquad m > 0 \tag{3-46}$$

If we replace m by $-m$ in (3-44) or (3-45), it is clear that

$$X_m = X^*_{-m} \tag{3-47}$$

for $x(t)$ real. Writing X_m in polar form as

$$X_m = |X_m| e^{j\theta_m} \tag{3-48}$$

we conclude from (3-44) that

$$|X_m| = |X_{-m}| \quad \text{and} \quad \theta_m = -\theta_{-m} \tag{3-49}$$

That is, for real signals, the magnitude of the Fourier coefficients is an even function of the index, m, and the argument is an odd function of m.

Considering the Fourier coefficients for a real, even signal, that is $x(t) = x(-t)$, we see from (3-44) that the imaginary part will be zero since $x(t) \sin m\omega_0 t$ is an odd function that integrates to zero over an interval symmetrically placed about $t = 0$. That is, the complex exponential Fourier series coefficients X_m of a real, even signal are real (see Example 3-5 with $t_0 = 0$). Furthermore, since the dependence on m is through the function $\cos m\omega_0 t$, they are also even functions of m. From (3-44) and (3-46) it follows that the b_m's are zero for the trigonometric series of real, even signals.

Similar reasoning shows that the X_m's are imaginary and odd functions of m if $x(t)$ is odd, that is, if $x(t) = -x(-t)$. Also $a_m = 0$, all m, if $x(t)$ is odd, so that the trigonometric Fourier series of an odd function consists only of sine terms (see Example 3-4). The proofs of these statements are left to the problems.

Another type of symmetry is *half-wave* (odd) symmetry defined as

$$x\left(t \pm \frac{T_0}{2}\right) = -x(t) \tag{3-50}$$

where T_0 is a period of $x(t)$.† For signals with half-wave symmetry, it turns out that

$$X_n = 0, \qquad n = 0, \pm 2, \pm 4, \ldots \tag{3-51}$$

The proof is left to the problems. Thus the Fourier series of a half-wave (odd) symmetrical signal contains only odd harmonics (see Example 3-4 and Table 3-1, waveforms 4 and 5). We note the half-wave even symmetry, defined by

$$x\left(t \pm \frac{T_0}{2}\right) = x(t) \tag{3-52}$$

simply means that the fundamental period of the waveform is $T_0/2$ rather than T_0.

Table 3-2 summarizes the properties of the two types of Fourier series that we have considered.

3-6 Parseval's Theorem

In Chapter 1 the average normalized power of a periodic waveform $x(t)$ was written as

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \frac{1}{T_0}\int x(t)x^*(t)\, dt \tag{3-53}$$

†To illustrate half-wave symmetry, consider waveform 4 in Table 3-1. Pick an arbitrary point on the t-axis. The value of the signal at this time is equal in magnitude, but opposite in sign, to the value of the signal one-half period earlier or later. The same argument holds for waveform 5 in Table 3-1.

TABLE 3-2
Summary of Fourier Series Properties[a]

Series	Coefficients[b]	Symmetry Properties
1. Trigonometric sine-cosine		
$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$	$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$	a_0 = Average value of $x(t)$
	$a_n = \frac{2}{T_0}\int_{T_0} x(t) \cos n\omega_0 t\, dt$	$a_n = 0$ for $x(t)$ odd, $b_n = 0$ for $x(t)$ even
	$b_n = \frac{2}{T_0}\int_{T_0} x(t) \sin n\omega_0 t\, dt$	$a_n, b_n = 0$, n even, for $x(t)$ odd, half-wave symmetrical
2. Complex exponential		
$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$	$X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt$	X_0 = Average value of $x(t)$
	$X_n = \begin{cases} \frac{1}{2}(a_n - jb_n), & n > 0 \\ \frac{1}{2}(a_{-n} + jb_{-n}), & n < 0 \end{cases}$	X_n real for $x(t)$ even; X_n imaginary for $x(t)$ odd
	$X_n = X^*_{-n}$ for $x(t)$ real	$X_n = 0$, n even, for $x(t)$ odd half-wave symmetrical

[a] $x(t)$ even means that $x(t) = x(-t)$; $x(t)$ odd means that $x(t) = -x(-t)$; $x(t)$ odd half-wave symmetrical means that $x(t) = -x(t \pm T_0/2)$.

[b] $\int_{T_0} (\cdot)\, dt$ means integration over any period T_0 of $x(t)$.

We can express P_{av} in terms of the Fourier coefficients of $x(t)$ by replacing $x^*(t)$ in (3-53) with its Fourier series representation (3-28).‡ The substitution yields

$$P_{av} = \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}\right) dt$$

$$= \sum_{n=-\infty}^{\infty} X_n^*\left[\frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\, dt\right] \tag{3-54}$$

where the order of integration and summation have been interchanged. We recognize the term in brackets as X_n [see (3-31)]. Thus we may write

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |X_n|^2 \tag{3-55a}$$

$$= X_0^2 + 2\sum_{n=1}^{\infty} |X_n|^2 \tag{3-55b}$$

where (3-55b) results by applying (3-47). In words, (3-55a) simply states that the average power of a periodic signal $\tilde{x}(t)$ is the sum of the powers in the phasor components of its Fourier series (recall Example 1-13, in which the power of a phasor was computed). Equivalently, (3-55b) states that the average power in a periodic signal is the sum of the power in its dc component plus the powers in its harmonic components (recall that the average power of a sinusoid is one-half the square of its amplitude).

EXAMPLE 3-8

The average power of the sine wave $x(t) = 4 \sin 50\pi t\, dt$ is

$$P_{av} = 25\int_0^{0.04} 16 \sin^2 50\pi t\, dt = \frac{16}{2} = 8\ W \tag{3-56}$$

computed from (3-53). Its exponential Fourier series coefficients are $X_{-1} = -X_1 = 2j$ and its power, when computed from (3-55a), is $P_{av} = 4 + 4 = 8$ W.

EXAMPLE 3-9

Consider the use of a square wave to test the fidelity of an amplifier. Suppose that the amplifier ideally passes all Fourier components of the input with frequencies less than 51 kHz. By ideal, it is meant that the individual Fourier components of the input with frequencies less than 51 kHz suffer zero phase shift in passing through the amplifier and all are multiplied by the same factor, called the *voltage gain*, G_v, while all Fourier components above 51 kHz do not appear at the output.

Suppose that $G_v = 10$ and the input, $x(t)$, is a 10-kHz square wave with amplitudes $\pm$ 1 V. (a) What is the average power P_x of the input signal? (b) What is the average power P_y of the output signal, $y(t)$? (c) What would the average power of the output signal be if the amplifier passed all frequency components at its input?

Solution: (a) The average power of the input is found by using (1-84) for the average power of a periodic signal. Letting T_0 be the period of the square wave and A its amplitude, we obtain

‡Recall that the conjugate of a sum is obtained by conjugating each term in the sum. Also, the conjugate of a product of factors is the product of the conjugate of each factor.

$$P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^2(t)\, dt$$

$$= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} A^2\, dt \tag{3-57}$$

Equation (3-57) follows by virtue of the square-wave amplitude being either A or $-A$. Thus the squared amplitude of $x(t)$ is A^2, and the average power becomes

$$P_x = A^2 \qquad \text{(square wave of amplitudes } \pm A)$$

$$= 1 \quad \text{W} \tag{3-58}$$

Parseval's theorem could be used to obtain P_x by summing the powers in the spectral components provided that one has the patience to sum an infinite series!

(b) The exponential Fourier series of an even-symmetry square wave, $x(t)$, with amplitude A = 1 V is

$$x(t) = \frac{2}{\pi}\left(\cdots - \tfrac{1}{7}e^{-j7\omega_0 t} + \tfrac{1}{5}e^{-j5\omega_0 t} - \tfrac{1}{3}e^{-j3\omega_0 t} + e^{-j\omega_0 t} + e^{j\omega_0 t} - \tfrac{1}{3}e^{j3\omega_0 t} + \tfrac{1}{5}e^{j5\omega_0 t} - \tfrac{1}{7}e^{j7\omega_0 t} + \cdots\right) \tag{3-59}$$

where $\omega_0 = 2\pi/T_0 = 2\pi f_0$ with $f_0 = 10$ kHz. The amplifier multiplies all spectral components with frequencies less than 51 kHz by the factor $G_v = 10$ and blocks all other spectral components. Thus the Fourier series of the output of the amplifier is

$$y(t) = \frac{20}{\pi}\left(\tfrac{1}{5}e^{-j5\omega_0 t} - \tfrac{1}{3}e^{-j3\omega_0 t} + e^{-j\omega_0 t} + e^{j\omega_0 t} - \tfrac{1}{3}e^{j3\omega_0 t} + \tfrac{1}{5}e^{j5\omega_0 t}\right)$$

$$\equiv \sum_{n=-\infty}^{\infty} Y_n e^{jn\omega_0 t} \tag{3-60}$$

The amplitudes of the Fourier components of the output are therefore

$$Y_n = 0, \qquad |n| > 5$$

$$Y_n = 0, \qquad n \text{ even}$$

$$Y_{-1} = Y_1 = \frac{20}{\pi} \tag{3-61}$$

$$Y_{-3} = Y_3 = -\frac{20}{3\pi}$$

$$Y_{-5} = Y_5 = \frac{20}{5\pi}$$

Using Parseval's theorem, the output power is

$$P_y = Y_0^2 + 2\sum_{n=1}^{\infty} |Y_n|^2 = 2\left[\left(\frac{20}{\pi}\right)^2 + \left(\frac{20}{3\pi}\right)^2 + \left(\frac{20}{5\pi}\right)^2\right]$$

$$= \frac{8 \times 10^2}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25}\right)$$

$$= 93.31 \quad \text{W} \tag{3-62}$$

(c) If the amplifier passes $x(t)$ perfectly, then

$$\hat{y}(t) = G_v x(t) \tag{3-63}$$

and

$$P_{\hat{y}} = G_v^2 P_x$$
$$= 10^2 \quad \text{W} \tag{3-64}$$

Note that $P_y/P_{\hat{y}} = 93.31\%$. This result tells us that only the first through fifth harmonics of the square wave need to be included to obtain 93.3% of the average signal power possible at the amplifier output. The student should compute the percent of average signal power possible at the amplifier output if its cutoff frequency is such that the 7th, 9th, 11th, and so on, harmonics are passed by the amplifier but all others rejected. (*Answers:* 94.96%; 95.96%; 96.63%.)

3-7 Line Spectra

The complex exponential Fourier series (3-28) of a signal consists of a summation of rotating phasors. In Chapter 1 we showed how sums of rotating phasors can be characterized in the frequency domain by two plots, one showing their amplitudes as a function of frequency and one showing their phases. This observation allows a periodic signal to be characterized graphically in the frequency domain by making two plots. The first, showing amplitudes of the separate phasor components versus frequency, is known as the *amplitude spectrum* of the signal. The second, showing the relative phases of each component versus frequency, is called the *phase spectrum* of the signal. Since these plots are obtained from the complex exponential Fourier series, spectral components, or *lines* as they are called, are present at both positive and negative frequencies. Such spectra are referred to as *two-sided.* From (3-49) it follows that, for a real signal, the amplitude spectrum is even and the phase spectrum is odd, which simply is a result of the necessity to add complex conjugate phasors to get a real sinusoidal signal. Figure 3-10 shows the two-sided spectra for a half-rectified sine wave. (See Table 3-1 for the Fourier coefficients.) In order that the phase be odd, X_n is represented as

$$X_n = -\left|\frac{A}{\pi(1-n^2)}\right| = \frac{A}{\pi(n^2-1)} e^{-j\pi} \tag{3-65}$$

for $n = 2, 4, \ldots$, and as

$$X_n = -\left|\frac{A}{\pi(1-n^2)}\right| = \frac{A}{\pi(n^2-1)} e^{j\pi} \tag{3-66}$$

for $n = -2, -4, \ldots$. For $n = \pm 1$, the amplitude spectrum is $A/4$ with a phase shift of $-\pi/2$ for $n = 1$ and a phase shift of $\pi/2$ for $n = -1$, which again ensures that the phase is odd.

The *single-sided line spectra* are obtained by writing the complex exponential form of the Fourier series (3-28) as a cosine series by representing the Fourier coefficients in polar form given by (3-48). For a real signal, the magnitudes and phases of the Fourier coefficients have the symmetry properties given by (3-49). Substituting the polar form of the Fourier coefficients into (3-28) and using the symmetry properties of the magnitudes and phases, we obtain

$$x(t) = \sum_{n=-\infty}^{\infty} |X_n| e^{j\theta_n} e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{-1} |X_n| e^{j(n\omega_0 t + \theta_n)} + X_0 + \sum_{n=1}^{\infty} |X_n| e^{j(n\omega_0 t + \theta_n)}$$
$$= X_0 + \sum_{n=1}^{\infty} \left[|X_n| e^{j(n\omega_0 t + \theta_n)} + |X_{-n}| e^{j(-n\omega_0 t + \theta_{-n})}\right]$$
$$= X_0 + \sum_{n=1}^{\infty} 2|X_n| \frac{e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2}$$
$$= X_0 + \sum_{n=1}^{\infty} 2|X_n| \cos(n\omega_0 t + \theta_n) \tag{3-67}$$

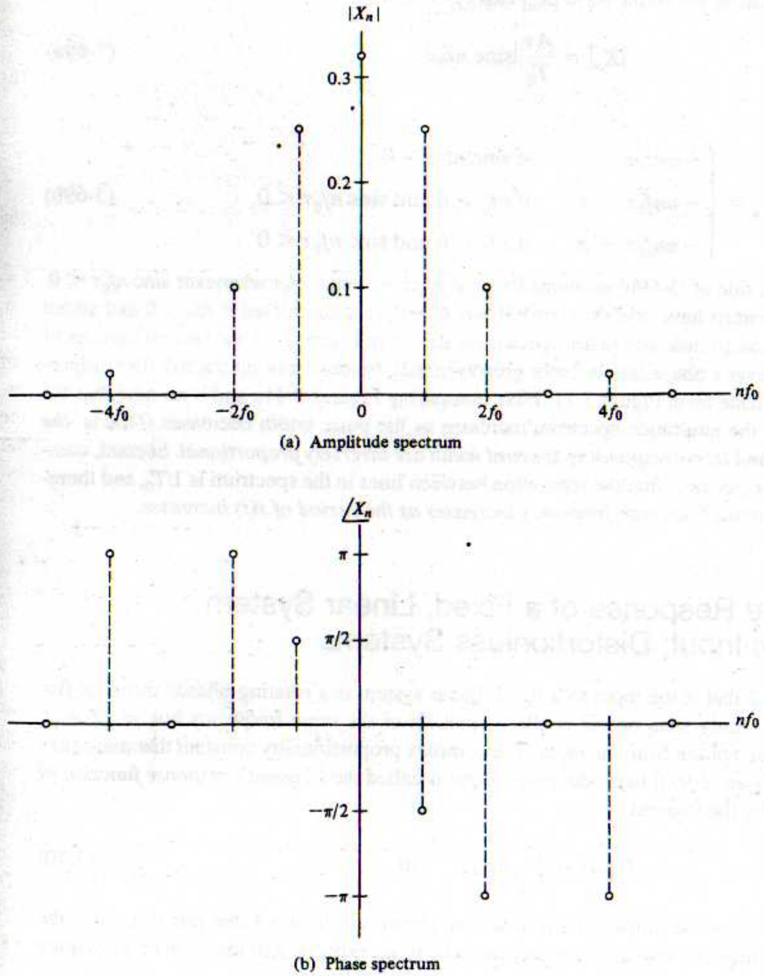


FIGURE 3-10. Two-sided line spectra for a half-rectified sine wave.

From this it is seen that the *single-sided amplitude spectrum* has lines that are twice as high as the lines in the double-sided amplitude spectrum for $f > 0$, with the line at $f = 0$ being the same in both spectra, and that the *single-sided phase spectrum* is identical to the double-sided phase spectrum for $f \geq 0$. We reached the same conclusions in Chapter 1 with regard to spectra of sums of sinusoids. Since the Fourier series of a periodic function is the sum of sinusoids as shown by (3-67), these observations should come as no surprise. The student should plot the single-sided amplitude and phase spectra for the half-rectified sine wave.

As a second example, consider the periodic pulse train shown in Table 3-1 as waveform 3. We consider the special case of $t_0 = \tau/2$. From (3-35c), the Fourier coefficients are

$$X_n = \frac{A\tau}{T_0} \operatorname{sinc} nf_0\tau \, e^{-j\pi n f_0 \tau} \qquad (3\text{-}68)$$

The Fourier coefficients can be put in the form $|X_n| \exp(j\underline{/X_n})$, where

$$|X_n| = \frac{A\tau}{T_0} |\operatorname{sinc} nf_0\tau| \qquad (3\text{-}69a)$$

and

$$\underline{/X_n} = \begin{cases} -\pi n f_0 \tau & \text{if } \operatorname{sinc} nf_0\tau > 0 \\ -\pi n f_0 \tau + \pi & \text{if } nf_0 > 0 \text{ and } \operatorname{sinc} nf_0\tau < 0 \\ -\pi n f_0 \tau - \pi & \text{if } nf_0 < 0 \text{ and } \operatorname{sinc} nf_0\tau < 0 \end{cases} \qquad (3\text{-}69b)$$

The $\pm\pi$ on the right-hand side of (3-69b) accounts for $|\operatorname{sinc} nf_0\tau| = -\operatorname{sinc} nf_0\tau$ whenever $\operatorname{sinc} nf_0\tau < 0$. Since the phase spectrum must have odd symmetry if $x(t)$ is real, π is subtracted if $nf_0 < 0$ and added if $nf_0 > 0$. The two-sided amplitude and phase spectra are shown in Figure 3-11 for several choices of τ and T_0. Note that whenever a phase line is 2π or greater in magnitude, 2π is subtracted. Two important observations can be made from Figure 3-11. First, comparing Figures 3-11a and b we note that the width of the envelope of the amplitude spectrum increases as the pulse width decreases. That is, *the pulse width of the signal and its corresponding spectral width are inversely proportional.* Second, comparing Figures 3-11a and c, we note that the separation between lines in the spectrum is $1/T_0$, and therefore *the density of the spectral lines with frequency increases as the period of $x(t)$ increases.*

3-8 Steady-State Response of a Fixed, Linear System to a Periodic Input; Distortionless Systems

It was shown in Chapter 2 that if the input to a fixed, linear system is a rotating phasor signal of frequency ω rad/s, then the steady-state output is also a sinusoid of the same frequency but, in general, with different amplitude and phase from the input. The complex proportionality constant that multiplies the rotating phasor input [see (2-99)] to produce the output is called the *frequency response function* of the system, and is given by the integral

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \qquad (3\text{-}70)$$

where $h(t)$ is the impulse response of the system. It will be shown in Chapter 4 that this integral is the *Fourier transform* of the impulse response. For any periodic input expressed in the form of a complex

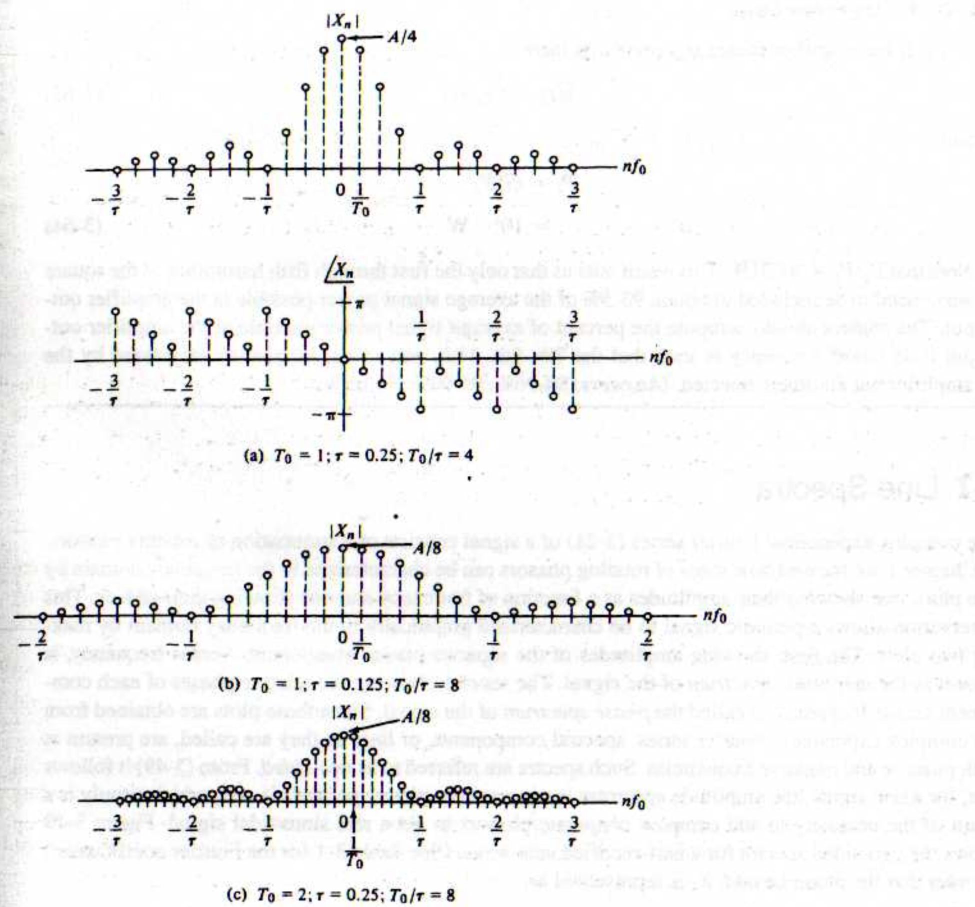


Figure 3-11. Spectra for a periodic pulse-train signal.

exponential Fourier series, and using superposition, we may express the steady-state output of a fixed, linear system as

$$y(t) = \sum_{n=-\infty}^{\infty} X_n H(n\omega_0) e^{jn\omega_0 t} \qquad (3\text{-}71)$$

If $H(\omega)$ is represented in polar form as

$$H(\omega) = A(\omega) e^{j\theta(\omega)} \qquad (3\text{-}72)$$

where $A(\omega)$ is called the amplitude response and $\theta(\omega)$ is called the phase response of the system, it was also shown in Chapter 2 that for a sinusoidal input of the form $\cos \omega_0 t$ the steady-state output is $A(\omega_0)\cos[\omega_0 t + \theta(\omega_0)]$ (see Example 2-16 for a specific case). For any periodic input, it follows from

(3-67) and the superposition property of a fixed, linear system that its steady-state output can be expressed as

$$y(t) = X_0H(0) + \sum_{n=1}^{\infty} 2|X_n|A(n\omega_0)\cos[n\omega_0 t + \underline{/X_n} + \theta(n\omega_0)] \tag{3-73}$$

The same result could have been obtained by expressing the series in (3-71) as a sum of cosinusoids, using the same steps as were used in deriving (3-67). In so doing, it would have been necessary to invoke the following symmetry properties for $A(\omega)$ and $\theta(\omega)$ for $h(t)$ real:

$$A(\omega) = A(-\omega)$$
$$\theta(\omega) = -\theta(-\omega) \tag{3-74}$$

That these relationships hold can be shown from the defining expression (3-70) by substituting $e^{-j\omega t} = \cos \omega t - j \sin \omega t$, writing the integral as the sum of two integrals, and noting that the first integral is the real part of $H(\omega)$ and that the second integral is its imaginary part. By replacing ω by $-\omega$ in these integrals, it is clear that $H(\omega) = H^*(-\omega)$, which is similar to (3-47). The symmetry properties given by (3-74) then follow by expressing $H(\omega)$ in polar form.

EXAMPLE 3-10

The application of (3-71) can be illustrated by finding the output of the RC lowpass filter of Chapter 2 (see Example 2-16) to an odd square wave of unit amplitude. Recall Example 3-7 for finding the complex exponential Fourier coefficients of an odd square wave, and recall (2-102) where the transfer function of the RC lowpass filter was given as

$$H(j\omega) = \frac{1}{1 + j\omega RC} \tag{3-75}$$

This is substituted into (3-71) along with the Fourier coefficients of the odd square wave to obtain the Fourier series expression for steadystate output of the filter. A MATLAB program for computing it is given below. Note that matrix computation is used throughout (with the exception of the for loop to run the different cases for the number of terms in the Fourier series) to keep the program as efficient as possible.

```
%     Plots for Example 3-10
%
RC=input('Enter RC time constant of filter (square wave period=1)');
n_max=input('Enter vector of highest harmonic values desired (odd)');
N=length(n_max);
t=0:.005:1;
omega_0=2*pi;
for k=1:N                              % Loop for various numbers of highest
                                       % harmonics
      n=[-n_max(k):2:n_max(k)];
      L_n=length(n);
      nn=2:L_n/2+1;
      sgn=(-1).^nn;                    % This will make the 1,5, . . . terms + and
                                       % the 3,7, . . . terms -
%     Fourier coefficients for odd square wave; for even square wave,
%     leave exp() off
      X_n=(2./(pi*abs(n))).*[fliplr(sgn)sgn].*exp(-j*0.5*pi*n);
```



```
%     Input to filter
      x=X_n*exp(j*omega_0*n'*t);      % Rows of exponential matrix are
                                       % versus time;
                                       % columns are versus n; matrix
                                       % multiply sums over n
%     Fourier coefficients of output
      Y_n=X_n.*(1./(1+j*n*omega_0*RC));
      y=Y_n*exp(j*omega_0*n'*t);       % The output
%     Plot real part of y to get rid of small imaginary part due to
%     computational error
      subplot(N,1,k),plot(t,real(y)),xlabel('t'),ylabel('approx.
      output'), . . .
      axis([0 1 -1.5 1.5]), text(.1,-.25,['max.har.=',num2str(n_max(k))])
      if k == 1
            title(['Filter output for RC = ',num2str(RC),' seconds'])
      end
end
```

To run the program, the following is entered in the command window, where two prompts are printed in this case—one for RC time constant of the filter and one for the maximum harmonic used in the Fourier series. A plot of the output of the program is shown in Figure 3-11. The plot is most inaccurate at the discontinuity in slope where the square wave input switches sign.

```
EDU»c3ex10
Enter RC time constant of filter (square wave period=1).2
Enter vector of highest harmonic values desired (odd) [7 15 27]
```

EXAMPLE 3-11

The input and output spectra of the RC filter for the previous example will reveal information on why the output time signals look as they do. The MATLAB program below provides plots for the these spectra for the two RC values used to plot Figure 3-12. Note that since we haven't plotted the phase spectrum, the spectral plots will be the same for both the even and odd square wave cases. Figure 3-13 shows the input and output spectra. Since the output spectrum is the product of the input spectrum and the transfer function (magnitude) at each frequency, the plot clearly shows how the filter affects the output spectra.

```
%     Plots for Example 3-11
%
RC=input('Enter vector of RC time constant of filter (sq. wave
period=1)');
n_max=input('Enter highest harmonic value desired (odd)');
N_RC=length(RC);
omega_0=2*pi;
f0=1;
for k=1:N_RC
      n=[-n_max:2:n_max];
      f=[-n_max*f0:.02:n_max*f0];
      H=1./(1+j*2*pi*f*RC(k));
      L_n=length(n);
      nn=2:L_n/2+1;
```

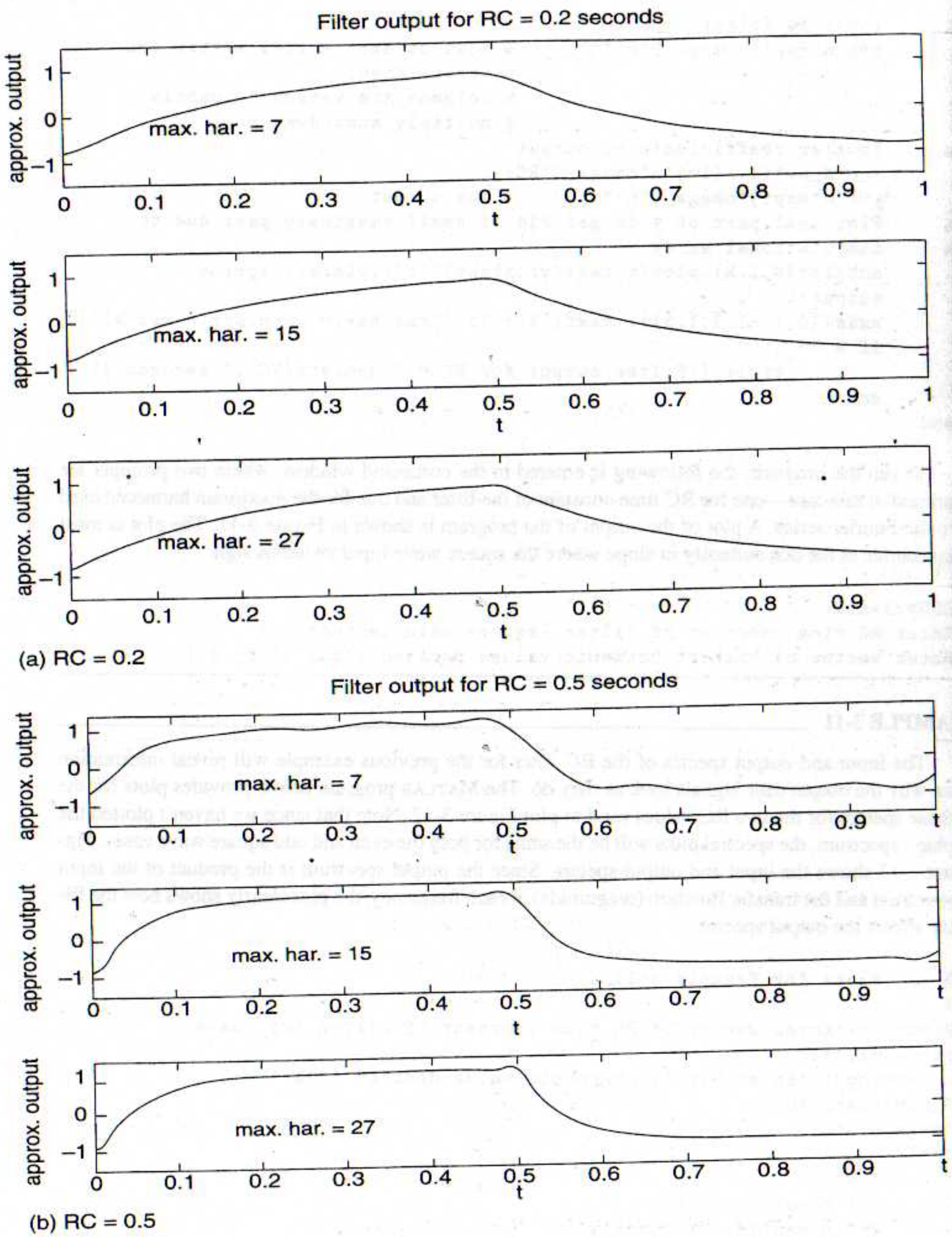


Figure 3-12. Partial sum approximations to the Fourier series for the steady state output of an RC lowpass filter to a square wave.

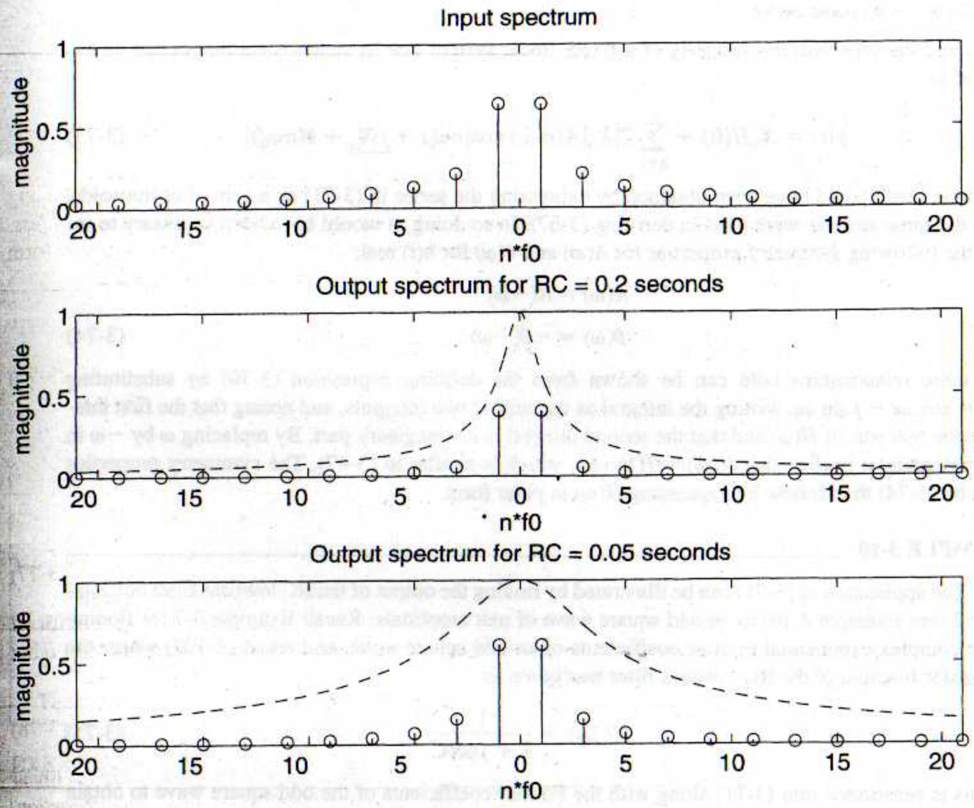


FIGURE 3-13. Input and output amplitude spectra for an RC filter for $RC = 0.2$ and 0.05. The transfer function magnitude is shown superimposed on the spectra.

```
    sgn=(-1).^nn;
%   Fourier coefficients for square wave;
    X_n=(2./(pi*abs(n))).*[fliplr(sgn) sgn];
%   Fourier coefficients of output
    Y_n=X_n.*(1./(1+j*n*omega_0*RC(k)));
%   Plot magnitude of X_n and Y_n
    if k==1
    subplot(N_RC+1,1,1),stem(n*f0,abs(X_n)), xlabel('n*f0'),
    ylabel('magnitude'), . . .
    title('Input spectrum'),axis(-n_max*f0 n_max*f0 01])
    end
    subplot(N_RC+1,1,k+1),stem(n*f0,abs(Y_n)),xlabel('n*f0'),
    ylabel('magnitude'), . . .
    title(['Output spectrum for RC= ', num2str(RC(k)),' seconds']),
    axis([-n_max*f0 n_max*f0 0 1]), hold on
    subplot(N_RC+1,1,k+1), plot(f,abs(H),'-')
end
```

The entries into the command window look as follows:

```
EDU»c3ex11
Enter vector of RC time constant of filter (sq. wave period=1)[.2 .05]
Enter highest harmonic value desired (odd) 21
```

If a fixed, linear system is to have no effect on the input except, possibly, for an amplitude scaling and delay which are the same for all input-frequency components, the system is called *distortionless.* For a distortionless system with periodic input of the form (3-67), the steady-state output is of the form

$$y(t) = K\left\{X_0 + \sum_{n=1}^{\infty} 2|X_n| \cos[n\omega_0(t - \tau_0) + \theta_n]\right\}$$
$$= Kx(t - \tau_0) \qquad (3\text{-}76)$$

where K is the scaling factor and τ_0 is the delay. From (3-73) it follows that

$$A(n\omega_0) = K \quad \text{and} \quad \theta(n\omega_0) = -\tau_0 n\omega_0$$

or

$$A(\omega) = K \quad \text{and} \quad \theta(\omega) = -\tau_0\omega \qquad (3\text{-}77)$$

respectively. That is, the amplitude response of a distortionless system is the same, or *constant,* for all frequency components of the input and the phase response is a *linear* function of frequency. In other words, the frequency response function of a distortionless system is of the form

$$H_d(\omega) = K \exp(-j\tau_0\omega) \qquad (3\text{-}78)$$

In general, three major types of distortion can be introduced by a system. First, if a system is linear but its amplitude response is not constant with frequency, the system introduces *amplitude distortion.* Second, if a system is linear but its phase shift is not a linear function of frequency, it introduces *phase,* or *delay, distortion.* Third, if the system is not linear, *nonlinear* distortion results. These three types of distortion may occur in combination with each other.

EXAMPLE 3-12

To illustrate the ideas of amplitude and phase distortion, consider a system with amplitude response and phase response as shown in Figure 3-14† and the following three inputs:

1. $x_1(t) = 2\cos 10\pi t + \sin 12\pi t$
2. $x_2(t) = 2\cos 10\pi t + \sin 26\pi t$
3. $x_3(t) = 2\cos 26\pi t + \sin 34\pi t$

Although this system has idealized frequency response characteristics, it can be used to illustrate various combinations of amplitude and phase distortion. Using (3-73), we find the outputs for the given inputs to be:

†The filter amplitude- and phase-response functions used here are not realizable, but are employed only to easily illustrate amplitude and phase distortion.

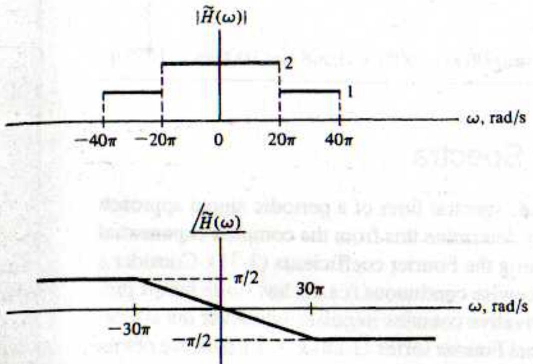


FIGURE 3-14. Amplitude response and phase response of the filter for Example 3-12.

1. $y_1(t) = 4\cos(10\pi t - \pi/6) + 2\sin(12\pi t - \pi/5)$
   $= 4\cos 10\pi(t - 1/60) + 2\sin 12\pi(t - 1/60)$
2. $y_2(t) = 4\cos(10\pi t - \pi/6) + \sin(26\pi t - 13\pi/30)$
   $= 4\cos 10\pi(t - 1/60) + \sin 26\pi(t - 1/60)$
3. $y_3(t) = 2\cos(26\pi t - 13\pi/30) + \sin(34\pi t - \pi/2)$
   $= 2\cos 26\pi(t - 1/60) + \sin 34\pi(t - 1/68)$

Comparison with (3-76) shows that only the input $x_1(t)$ is passed without distortion by the system. The system imposes amplitude distortion on $x_2(t)$ and phase (delay) distortion on $x_3(t)$.

EXERCISE

A second-order low-pass Butterworth filter is one having frequency-response function

$$H(\omega) = \frac{\omega_c^2}{\omega_c^2 - \omega^2 + j\sqrt{2}\,\omega_c\omega} \qquad (3\text{-}79)$$

where ω_c is a filter parameter.

(a) Show that its amplitude- and phase-response functions are

$$A(\omega) = \left[1 + \left(\frac{\omega}{\omega_c}\right)^4\right]^{-1/2} \qquad (3\text{-}80)$$

and

$$\theta(\omega) = \begin{cases} -\tan^{-1}\dfrac{\sqrt{2}\,(\omega/\omega_c)}{1 - \omega^2/\omega_c^2}, & \omega < \omega_c \\ -\pi + \tan^{-1}\dfrac{\sqrt{2}\,(\omega/\omega_c)}{\omega^2/\omega_c^2 - 1}, & \omega > \omega_c \end{cases} \qquad (3\text{-}81)$$

respectively, where the value of the arctangent function is taken as its principal value.

(b) If $\omega_c = 600\pi$ rad/s above and $\omega_0 = 200\pi$ rad/s in (3-25), obtain the attenuations and phase shifts introduced by the filter in the first three terms of the Fourier series of a square-wave input to the filter. (*Answer:* attenuations: 0.994, 0.707, 0.339; phase shifts: −27.9°, −90°, −127°.)

(c) Write out the first three nonzero terms of the Fourier series of the output and compare with the input by plotting.

Answer: $y(t) = 4/\pi[0.994\sin(200\pi t - 27.9°) + 0.236\sin(600\pi t - 90°) + 0.068\sin(1000\pi t - 127°)]$.

3-9 Rate of Convergence of Fourier Spectra

It is handy to be able to predict how rapidly the amplitude spectral lines of a periodic signal approach zero as frequency (or harmonic order) increases. We may determine this from the complex exponential Fourier series expression (3-28) and the formula for finding the Fourier coefficients (3-31). Consider a periodic signal $x(t)$ that has a kth derivative which is piecewise continuous (i.e., is has finite jumps present, but it otherwise continuous). Thus its $(k + 1)$st derivative contains impulses wherever the kth derivative has jumps. Now, by differentiating the exponential Fourier series (3-28) $k + 1$ times, we obtain

$$\frac{d^{k+1}x(t)}{dt^{k+1}} = \sum_{n=-\infty}^{\infty} (jn\omega_0)^{k+1} X_n e^{jn\omega_0 t} \tag{3-82}$$

Therefore the Fourier coefficients of the $(k + 1)$*st derivative of* $x(t)$ *are the Fourier coefficients of* $x(t)$*, multiplied by* $(jn\omega_0)^{k+1}$.

Now consider the computation of the Fourier coefficients of the $(k + 1)$st derivative of $x(t)$. When the expression for the $(k + 1)$st derivative of $x(t)$ is substituted into (3-31), we may evaluate the impulse terms immediately by using the sifting property of the unit impulse function. These terms will give constants in the expression for the Fourier coefficients for the $(k + 1)$st derivative of $x(t)$. This means that, given our observation in relation to (3-82), the Fourier coefficients for $x(t)$ will have terms proportional to $(jn\omega_0)^{-(k+1)}$, since we divide the expression for the Fourier coefficients for the $(k + 1)$st derivative of $x(t)$, which have constant components, by $(jn\omega_0)^{k+1}$ to get X_n.

We conclude, therefore, that a periodic signal $x(t)$ *that has a kth derivative which is piecewise continuous has Fourier coefficients that decrease with frequency as* $(n\omega_0)^{-(k+1)}$. As an example, a square wave is piecewise continuous (zeroth derivative), so its first derivative contains impulses. Therefore, the Fourier coefficients of a square wave decrease inversely with increasing frequency, or order of harmonic.

This is a very useful result for design of waveforms. If we want a signal that has its spectral content concentrated about zero frequency, we think of those waveforms that have many continuous derivatives. An example of a fairly simple waveform of this nature is given in the next example.

EXAMPLE 3-13

Consider a periodic waveform consisting of raised cosine pulses, defined by the equationsand

$$x_p(t) = \begin{cases} \frac{1}{2}(1 + \cos 2\pi t), & |t| \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

and

$$x(t) = \sum_{n=-\infty}^{\infty} x_p(t - 2n) \tag{3-83}$$

The following MATLAB program (note that a function program is used to keep it compact) plots three pulses of this waveform and three of its derivatives. These plots are shown in Figure 3-15. Its third derivative contains impulses. Therefore, its amplitude spectrum approaches zero as $(n\omega_0)^{-3}$.

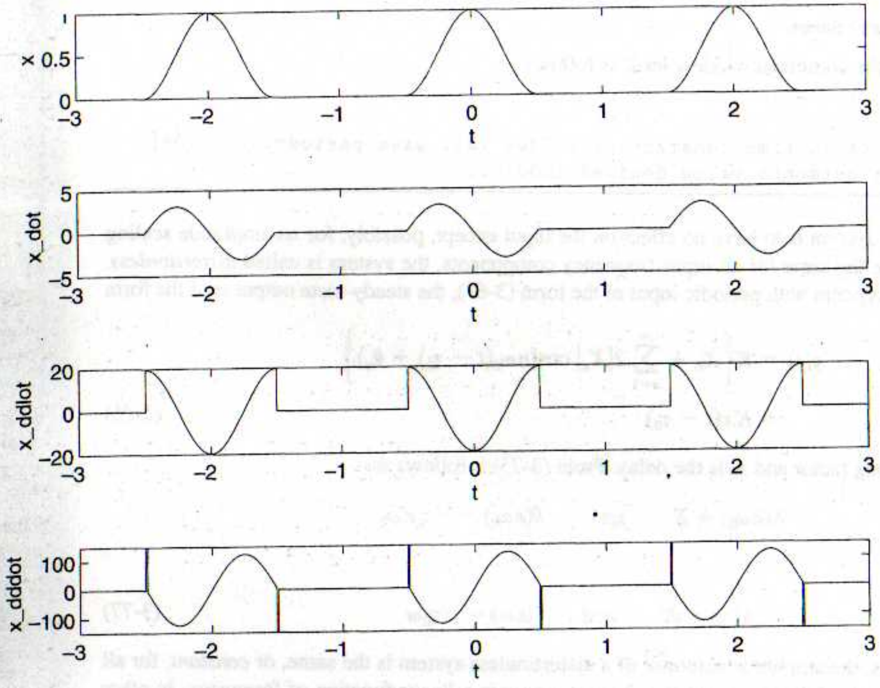


FIGURE 3-15. Plots of a raised cosine periodic waveform and its first three derivatives.

```
EDU»3ex13
%     Example 3-13 plots
%
t=-3:.001:3;
[x1,x1_dot,x1_ddot,x1_dddot]=rsd_cos(t+2);
[x2,x2_dot,x2_ddot,x2_dddot]=rsd_cos(t);
[x3,x3_dot,x3_ddot,x3_dddot]=rsd_cos(t-2);
x=x1+x2+x3;
x_dot=x1_dot+x2_dot+x3_dot;
x_ddot=x1_ddot+x2_ddot+x3_ddot;
x_dddot=x1_dddot+x2_dddot+x3_dddot;
subplot(4,1,1),plot(t,x),xlabel('t'),ylabel('x')
subplot(4,1,2),plot(t,x_dot),xlabel('t'),ylabel('x_dot')
subplot(4,1,3),plot(t,x_ddot),xlabel('t'),ylabel('x_ddlot')
subplot(4,1,4),plot(t,x_dddot),axis([-3 3 -150 150]),...
     xlabel('t'),ylabel('x_dddot')

%    Function to compute raised cosine and derivatives
%
function [x,x_dot,x_ddot,x_dddot]=rsd_cos(t)
x_cos=.5*(1+cos(2*pi*t));
x=x_cos.*pls_fn(t);
x_dot=-pi*sin(2*pi*t).*pls_fn(t);
x_ddot=-2*pi^2*cos(2*pi*t).*pls_fn(t);
```

```
x_dddot=4*pi^3*sin(2*pi*t).*pls_fn(t)+ . . .
     2*pi^2*(impls_fn(t+5,.01)-impls_fn(t-5,.01));
% Function to compute square pulse
%
Function y=pls_fn(t)
y=stp_fn(t+0.5)-stp_fn(t-0.5-eps)
```

3-10 Fourier Series and Signal Spaces

It is often useful to visualize signals as analogous to vectors in a generalized vector space. The Fourier series and the Fourier transform that we have just studied are tools for resolving power and energy signals, respectively, into such generalized vector spaces. We briefly consider this approach to Fourier representations, referred to as *generalized Fourier series.*

We begin our consideration of generalized Fourier series with a review of some concepts about ordinary vectors in three-dimensional space. We were probably first introduced to vectors as physical quantities whose specification involves both magnitude and direction and that obey the parallelogram law of addition. Later, we were perhaps shown that it was convenient to represent a vector, $\mathbf{A}$, in a Cartesian coordinate system in terms of a mutually perpendicular triad of unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ along the x-, y-, and z-axes, respectively, as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{3-84}$$

The components A_x, A_y, and A_z can be expressed as

$$A_x = \mathbf{A}\cdot\hat{\mathbf{i}}, \qquad A_y = \mathbf{A}\cdot\hat{\mathbf{j}}, \qquad A_z = \mathbf{A}\cdot\hat{\mathbf{k}}$$

where the dot denotes the dot product of two vectors, which is the product of their magnitudes and the cosine of the angle between them. Since $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit-length, mutually perpendicular vectors, the xyz components of $\mathbf{A}$ are easily obtained by simply projecting $\mathbf{A}$ onto the x-, y-, and z-axes, respectively.

We now consider representation of a finite-energy signal, $x(t)$, defined on a T-second interval $(t_0, t_0 + T)$ in terms of a set of preselected time functions, $\phi_1(t), \ldots, \phi_n(t)$. It is convenient to choose these functions with properties analogous to $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ of three-dimensional vector space. The mutually perpendicular property, referred to as *orthogonality,* is expressed as

$$\int_{t_0}^{t_0+T} \phi_m(t)\phi_n^*(t)\,dt = 0, \qquad m \neq n \tag{3-85}$$

where the conjugate suggests that complex-valued $\phi_n(t)$'s may be convenient in some cases. We further assume that the $\phi_n(t)$'s have been chosen such that

$$\int_{t_0}^{t_0+T} |\phi_n(t)|^2\,dt = 1 \tag{3-86}$$

In view of (3-86), the $\phi_n(t)$'s are said to be *normalized.* This assumption is invoked to simplify future equations.

We now attempt to approximate $x(t)$ as best we can by a series of the form

$$y(t) = \sum_{n=1}^{N} d_n\phi_n(t) \tag{3-87}$$

where the d_n's are constants to be chosen such that $y(t)$ represents $x(t)$ as closely as possible according to some criterion. It is convenient to measure the error in the integral-square sense, which is defined as

$$\text{integral-square error} = \epsilon_N = \int_{t_0}^{t_0+T} |x(t) - y(t)|^2\,dt \tag{3-88}$$

We wish to find $d_1, d_2, \ldots, d_N$ such that ϵ_N as expressed by (3-88) is a minimum. Substituting (3-87) into (3-88), we obtain

$$\epsilon_N = \int_T \left[x(t) - \sum_{n=1}^{N} d_n\phi_n(t)\right]\left[x^*(t) - \sum_{n=1}^{N} d_n^*\phi_n^*(t)\right] dt \tag{3-89}$$

where

$$\int_T (\cdot)\,dt \triangleq \int_{t_0}^{t_0+T} (\cdot)\,dt$$

This can be expanded to yield

$$\epsilon_N = \int_T |x(t)|^2\,dt - \sum_{n=1}^{N}\left[d_n^*\int_T x(t)\phi_n^*(t)\,dt + d_n\int_T x^*(t)\phi_n(t)\,dt\right] + \sum_{n=1}^{N}|d_n|^2 \tag{3-90}$$

which was obtained by making use of (3-85) and (3-86) after interchanging the orders of summation and integration.

It is convenient to add and subtract the quantity

$$\sum_{n=1}^{N}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2 \tag{3-91}$$

to (3-90) which, after rearrangement of terms, yields

$$\epsilon_N = \int_T |x(t)|^2\,dt - \sum_{n=1}^{N}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2 + \sum_{n=1}^{N}\left|d_n - \int_T x(t)\phi_n^*(t)\,dt\right|^2 \tag{3-92}$$

To show the equivalence of (3-92) and (3-90), it is easiest to work backward from (3-92).

Now the first two terms on the right-hand side of (3-92) are independent of the coefficients d_n. The last summation of terms on the right-hand side is nonnegative and is added to the first two terms. Therefore, to minimize ϵ_N through choice of the d_n's, the best we can do is make each term of the last sum zero. That is, we choose the nth coefficient, d_n, such that

$$d_n = \int_{t_0}^{t_0+T} x(t)\phi_n^*(t)dt, \qquad n = 1, 2, \ldots, N \tag{3-93}$$

This choice for $d_1, d_2, \ldots, d_N$ minimizes the integral-square error, ϵ_N. The resulting coefficients are called the *generalized Fourier coefficients.*

The minimum value for ϵ_N is

$$\epsilon_{N,\min} = \int_{t_0}^{t_0+T} |x(t)|^2\,dt - \sum_{n=1}^{N}|d_n|^2 \tag{3-94}$$

It is natural to inquire as to the possibility of $\epsilon_{N,\min}$ being zero. For special choices of the set of functions $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$, referred to as *complete sets* in the space of all integrable-square functions, it will be true that

$$\lim_{N\to\infty} \epsilon_{N,\min} = 0 \tag{3-95}$$

for any signal that is square integrable, that is, for any signal for which

$$\int_{t_0}^{t_0+T} |x(t)|^2\,dt < \infty \tag{3-96}$$

In the sense that the integral-square error is zero, we may then write

$$x(t) = \sum_{n=1}^{\infty} d_n \phi_n(t) \tag{3-97}$$

For a complete set of orthogonal functions, $\phi_1(t), \phi_2(t), \ldots$, we obtain from (3-94) that

$$\int_{t_0}^{t_0+T} |x(t)|^2\, dt = \sum_{n=1}^{\infty} |d_n|^2 \tag{3-98}$$

This is a generalized version of Parseval's theorem.

EXAMPLE 3-14

Given the set of functions shown in Figure 3-16. (a) Show that this is an orthogonal set and that each member of the set is normalized. (b) If $x(t) = \cos 2\pi t,\ 0 \le t \le 1$, find $y(t)$ as expressed by (3-87) such that the integral-square error is minimized. (c) Same question as in part (b) but $x(t) = \sin 2\pi t,\ 0 \le t \le 1$.

Solution:

(a) It is clear that each function is normalized according to (3-86). To show orthogonality, we calculate

$$\int_0^1 \phi_1(t)\phi_2(t)\, dt = \int_0^{0.5} (1)(1)\, dt + \int_{0.5}^1 (1)(-1)\, dt = 0 \tag{3-99}$$

Similarly, it can be shown that $\int_0^1 \phi_1(t)\phi_3(t)\, dt = 0$ and $\int_0^1 \phi_2(t)\phi_3(t)\, dt = 0$.

(b) The generalized Fourier coefficients are found according to (3-93). The coefficients are

$$\begin{aligned} d_1 &= \int_0^1 (1) \cos 2\pi t\, dt \\ &= \frac{\sin 2\pi t}{2\pi}\bigg|_0^1 \\ &= 0 \end{aligned} \tag{3-100}$$

$$\begin{aligned} d_2 &= \int_0^{0.5} (1) \cos 2\pi t\, dt + \int_{0.5}^1 (-1) \cos 2\pi t\, dt \\ &= \frac{1}{2\pi}\left[\sin 2\pi t\Big|_0^{0.5} - \sin 2\pi t\Big|_{0.5}^1\right] \\ &= 0 \end{aligned} \tag{3-101}$$

$$\begin{aligned} d_3 &= \int_0^{0.25} (1) \cos 2\pi t\, dt + \int_{0.25}^{0.75} (-1) \cos 2\pi t\, dt + \int_{0.75}^1 (1) \cos 2\pi t\, dt \\ &= \frac{1}{2\pi}\left[\sin 2\pi t\Big|_0^{0.25} - \sin 2\pi t\Big|_{0.25}^{0.75} + \sin 2\pi t\Big|_{0.75}^1\right] = \frac{2}{\pi} \end{aligned} \tag{3-102}$$

Therefore,

$$y(t) = \frac{2}{\pi}\phi_3(t) \tag{3-103}$$

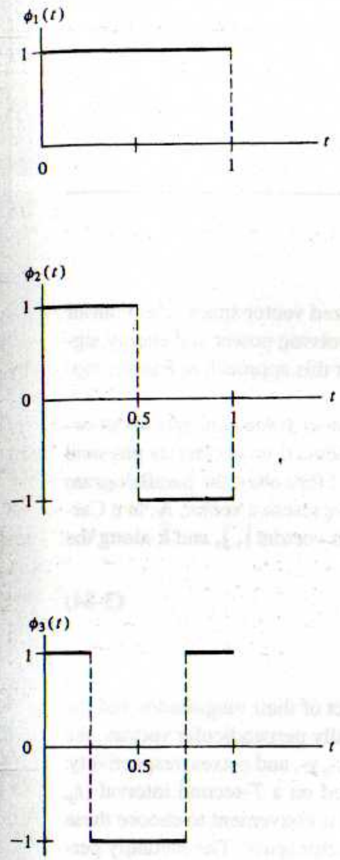


FIGURE 3-16. Orthogonal set of functions for Example 3-11.

The minimum integral-square error is

$$\epsilon_3 = \int_0^1 \cos^2 2\pi t\, dt - \left(\frac{2}{\pi}\right)^2 = \frac{1}{2} - \left(\frac{2}{\pi}\right)^2 \tag{3-104}$$

(c) In a similar manner to the calculations carried out for part (b), it follows that for $x(t) = \sin 2\pi t$.

$$\begin{aligned} d_1 &= 0 \\ d_2 &= \frac{2}{\pi} \\ d_3 &= 0 \end{aligned} \tag{3-105}$$

The minimum integral-square error is the same as for part (b). The student is advised to sketch $y(t)$ for each case and compare it with $x(t)$.

Note that the exponential Fourier series could have been derived as a generalized Fourier series with

$$\phi_n(t) = e^{jn\omega_0 t}, \qquad n = 0, \pm 1, \ldots \tag{3-106}$$

where the interval under consideration is $(t_0, t_0 + T_0)$ and

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} \tag{3-107}$$

Thus, *partial sums of exponential (and trigonometric) Fourier series minimize the integral-square error between the series and the signal under consideration.*

Summary

In this chapter we have considered the representation of periodic signals by trigonometric sums and by rotating phasor sums whose terms have frequencies which are harmonically related. The former representation is called the trigonometric Fourier series, and the latter is referred to as an exponential Fourier series. Fourier series representations facilitate the plotting of spectra for periodic signals, which consist of discrete lines spaced at integer multiples of the fundamental frequency. Although we naturally think of the Fourier series as being a useful representation for periodic signals, an aperiodic signal may be represented by a Fourier series over a finite interval. We also returned to the idea of a frequency response function of a fixed, linear system in this chapter which is a complex function of frequency. A distortionless system has an output which is a scaled, delayed replica of the input signal; for such systems, the magnitude of the transfer function is a constant, and its argument is a linear function of frequency. The following are the major points covered in this chapter:

1. A *trigonometric Fourier series* has the form

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

where

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\, dt, \qquad n \neq 0$$

and

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0\, dt$$

where $\omega_0 = 2\pi/T_0$, with T_0 being the period of the periodic signal or the interval of expansion if the series is used to approximate an aperiodic function over a finite interval.

2. At points of continuity of x, a Fourier series converges pointwise under fairly lenient restrictions. At points of jump discontinuity of x,, say at $t = t_0$, a Fourier series converges to the average of the left- and right-hand limits of $x(t)$ at t_0.
3. The Gibbs phenomenon of a Fourier series, to be treated more precisely in Chapter 4, refers to the tendency of the partial sums of a Fourier series to overshoot a jump discontinuity of a signal, no matter how many terms are included in the series. This is illustrated by Example 3-4 for a square wave.
4. A *complex exponential Fourier series* has the form

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

where

$$X_n = \frac{2}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\, dt$$

5. The coefficients for the trigonometric and exponential forms of the Fourier series are related by

$$X_0 = a_0$$
$$X_n = \tfrac{1}{2}(a_n - jb_n), \qquad n > 0$$

and

$$X_n = \tfrac{1}{2}(a_{-n} + jb_{-n}), \qquad n < 0$$

or

$$a_n = 2\,\mathrm{Re}\, X_n, \qquad n > 0$$

and

$$b_n = -2\,\mathrm{Im}\, X_n, \qquad n > 0$$

6. The following *symmetry properties* hold for the Fourier coefficients of a real signal:

$$X_n = X_{-n}^*$$
$$|X_n| = |X_{-n}|$$
$$\underline{/X_n} = -\underline{/X_{-n}}$$

If a signal is even, then
   a. The X_n's are real.
   b. All b_n's are zero.
On the other hand, if a signal is odd, then
   a. The X_n's are imaginary.
   b. All a_n's are zero.
If a signal is half-wave (odd) symmetrical, i.e., $x(t) = -x(t - T_0/2)$, then all even-indexed coefficients are zero. If a signal has zero average value, then $X_0 = a_0 = 0$.

7. *Parseval's theorem* relates to the average power contained in a periodic signal and states that

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |X_n|^2 = X_0^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

That is, the power can be found by averaging the square of the time-domain signal over a period, or by summing the powers in its rotating phasor components, or by summing the powers in its sinusoidal components plus the power in the dc component.

8. Line spectra for a signal are plotted from its Fourier series representation. The *two-sided amplitude spectrum* of a signal is obtained by plotting the magnitude of its complex exponential Fourier series coefficients versus frequency. Its *two-sided phase spectrum* is obtained by plotting the arguments of its complex exponential Fourier series coefficients versus frequency. The *single-sided amplitude spectrum* is obtained by doubling each line in the double-sided amplitude spectrum for $f > 0$ and leaving the line at $f = 0$ as it is. The *single-sided phase spectrum* is

obtained by keeping each line in the double-sided amplitude spectrum for $f \geq 0$ as it is. (Single-sided line spectra are zero for $f < 0$, as their name implies.)

9. The *frequency response function* of a system is given by the integral

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

where $h(t)$ is its impulse response.

10. For any periodic input expressed in the form of a complex exponential Fourier series, the *steady-state output* of a fixed, linear system may be expressed as

$$y(t) = \sum_{n=-\infty}^{\infty} X_n H(n\omega_0)e^{jn\omega_0 t}$$

11. If $H(\omega)$ is represented in polar form as

$$H(\omega) = A(\omega)e^{j\theta(\omega)}$$

where $A(\omega)$ is called the *amplitude response* and $\theta(\omega)$ is called the *phase response* of the system, it follows that the steady-state output of a fixed, linear system can be expressed as

$$y(t) = X_0H(0) + \sum_{n=1}^{\infty} 2|X_n|A(n\omega_0)\cos[n\omega_0 t + \theta_n + \theta(n\omega_0)]$$

12. For $h(t)$ real,

$$A(\omega) = A(-\omega)$$

and

$$\theta(\omega) = -\theta(-\omega)$$

13. A fixed, linear system that has no effect on its input except, possibly, for an amplitude scaling and delay which are the same for all input-frequency components is called *distortionless.* Such a system is defined by the input-output relationship $y(t) = Kx(t - \tau_0)$, where K is the amplitude scaling factor and τ_0 is the delay. Its amplitude and phase responses are given by

$$A(\omega) = K \quad \text{and} \quad \theta(\omega) = -\tau_0\omega$$

14. Fixed, linear systems with nonconstant amplitude responses are said to introduce *amplitude distortion* into the input signal. Systems with phase responses that are not linear with frequency are said to introduce *phase distortion.*

15. If the kth derivative of a waveform is piecewise continuous such that its $(k + 1)$st derivative contains impulses, its Fourier spectrum decreases with frequency as the inverse $(k + 1)$st power.

16. An alternative way to develop the Fourier expansion of a signal is as a special case of its minimum integral-squared-error expansion over an interval of interest using a complete orthonormal set of functions over the interval. Such a development shows that when a Fourier series is truncated at any finite number of terms, the best representation of the signal of any equivalent length series is obtained in terms of minimum integral-squared error.

Further Reading

Fourier series and the Fourier transform are introduced in almost all beginning circuits texts, but their use as systems analysis tools are not fully exploited until the student becomes involved in more systems-oriented courses, such as communications and signal processing. Nevertheless, rather than cite references in these specialized areas, we give two that deal with these topics from a more mathematical viewpoint.

E. Kreyszig, *Advanced Engineering Mathematics,* 6th ed. New York: Wiley, 1988 (Chapter 10).
A. Papoulis, *Signal Analysis.* New York: McGraw-Hill, 1977 (Chapter 3).

An older reference that provides a very readable treatment of Fourier analysis is long out of print. However, it is well worth exposure to if the student can obtain a copy. It is

E. A. Guillemin, *The Mathematics of Circuit Analysis.* New York: Wiley, 1949 (Chapter 10).

Problems

Section 3-2

3-1. Plot the first through third partial sums of the series

$$x(t) = \frac{4}{\pi}\left(\cos\omega_0 t - \tfrac{1}{3}\cos 3\omega_0 t + \tfrac{1}{5}\cos 5\omega_0 t - \cdots\right)$$

Comment on its similarity to a square wave. Is it even or odd? Why is this series an alternating series, whereas the series of Example 3-1 was not? (*Hint:* In calculating the data for making the plot, note that symmetry can be used to save work.)

Section 3-3

3-2. Prove the relationships (3-12) and (3-13), and show all steps in deriving (3-16).

3-3. The uniqueness of the Fourier series means that if we can somehow find the Fourier series of a waveform, we are assured that there is no other waveform with that Fourier series, except for waveforms differing from the waveform under consideration only over an inconsequential set of values of the independent variable (this is referred to in mathematics as a set of measure zero). With this assistance, find the following trigonometric Fourier series of the signals without doing any integration:

(a) $x_1(t) = \cos^2(100\pi t)$;
(b) $x_2(t) = \exp(j200\pi t)$;
(c) $x_3(t) = \sin(2\pi t)\cos^2(10\pi t)$;
(d) $x_4(t) = \cos^3(20\pi t)\,[1 - \sin^2(10\pi t)]$

3-4. Obtain the trigonometric Fourier series of the square wave

$$x(t) = \begin{cases} A, & \dfrac{-T_0}{4} < t \leq \dfrac{T_0}{4} \\ -A, & \dfrac{-T_0}{2} < t \leq \dfrac{-T_0}{4} \quad \text{and} \quad \dfrac{T_0}{4} < t \leq \dfrac{T_0}{2} \end{cases}$$

with $x(t) = x(t + T_0)$, all t. Why is it composed of only cosine terms?

3-5. (a) In Example 3-3, the violin string was viewed as one full period of a triagular waveform. Find the Fourier series representation if it is viewed as one-half of a period.
(b) In Figure 3-6, it is seen that the two points of maximum error occur at the ends of the string. With the representation found in part (a), what is the error at the ends of the string now?

Sections 3-4 to 3-6

3-6. Obtain the results for the exponential Fourier series examples given in Table 3-1 with the exception of number 3, which was worked in Example 3-5.

3-7. Suppose that differentiation of the periodic signal $x(t)$ results in a signal that has a Fourier series.
(a) Integrate by parts the expression

$$X_n' = \frac{1}{T_0}\int_{T_0} \frac{dx}{dt} e^{-jn\omega_0 t}\, dt$$

to show that

$$X_n' = (jn\omega_0)X_n$$

That is, the Fourier series coefficients of the signal dx/dt are related to the Fourier series coefficients of $x(t)$ through multiplication by $jn\omega_0$.
(b) From Table 3-1, waveform 5, obtain the Fourier series coefficients of an odd-symmetry square wave.

3-8. **(a)** Use Parseval's theorem expressed by (3-55a) and (3-55b) to find P_{av} for

$$x(t) = 2\sin^2(2500\pi t)\cos(2 \times 10^4 \pi t)$$

(b) If $x(t)$ is a signal that is transmitted through a telephone system which blocks dc and frequencies above 12 kHz, compute the ratio of received to transmitted power.

3-9. Consider the periodic pulse-train signal of Example 3-5. Find the ratio of the power in frequency components for which $|nf_0| \le \tau^{-1}$ to total power if:
(a) $T_0/\tau = 2$
(b) $T_0/\tau = 4$
(c) $T_0/\tau = 10$

3-10. Obtain the exponential Fourier series of the sawtooth waveform defined by

$$x(t) = At, \quad -\frac{T_0}{2} \le t < \frac{T_0}{2}$$

$$x(t) = x(t + T_0), \quad \text{all } t$$

3-11. The complex exponential Fourier series of a signal over an interval $0 \le t \le T_0$ is

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{1 + j\pi n} e^{j(3\pi n t/2)}$$

(a) Determine the numerical value of T_0.
(b) What is the average value of $x(t)$ over the interval $(0, T_0)$?
(c) Determine the amplitude of the third-harmonic component.
(d) Determine the phase of the third-harmonic component.
(e) Write down an expression for the third harmonic term in the Fourier series in terms of a cosine.

3-12. Represent the signal

$$x(t) = e^{-|t|}$$

over the interval $(-1, 1)$ using:
(a) The exponential Fourier series
(b) The trigonometric Fourier series
(c) Sketch the function to which either Fourier series converges

3-13. Obtain complex exponential Fourier series expansions for the following signals without doing any integration. Give the fundamental frequency for each case.
(a) $x_a(t) = \cos^2 20\pi t \sin 10\pi t$
(b) $x_b(t) = \sin^3 30\pi t + 2\cos 25\pi t$
(c) $x_c(t) = \sin^2 40\pi t \cos^2 20\pi t + \sin 10\pi t \cos 5\pi t$
(*Hint:* Use trigonometric identities and Euler's theorem.)

3-14. Given that the complex exponential Fourier series coefficients of a signal $x(t)$ are $\ldots, X_{-1}, X_0, X_1, \ldots$. Find the Fourier series coefficients of $y(t)$ in terms of those for $x(t)$ if the two signals are related by:
(a) $y(t) = x(t - \tau)$, where τ is a constant;
(b) $y(t) = \exp(j2\pi f_0 t)x(t)$, where f_0 is a constant.

3-15. **(a)** Prove the relationships (3-32a) and (3-32b), which relate the coefficients of the exponential Fourier series to the trigonometric Fourier series.
(b) Show that the trigonometric Fourier series of a real, even signal consists only of cosine terms, and that of an odd signal consists only of sine terms.

3-16. Show the following symmetry properties for the complex exponential Fourier coefficients:
(a) X_n is real and an even function of n if $x(t)$ is real and even.
(b) X_n is imaginary and odd if $x(t)$ is real and odd.
(c) $X_n = 0$ for even n if $x(t) = -x(t \pm T_0/2)$ (i.e., has odd half-wave symmetry).
(*Hint:* (1) One way to do this is to write down the Fourier series for $x(t)$. Substitute $t + T_0/2$ for t to obtain a series for $x(t + T_0/2)$. What must be true of the coefficients in order for this new series to equal the original series for $x(t)$? (2) A second way to prove $X_n = 0$, n even, is to write down the integral for X_n and break the integral into one from $-T_0/2$ to 0 and 0 to $T_0/2$. Change variables in the one from $-T_0/2$ to 0 to make its limits 0 to $T_0/2$. Use the fact that $x(t - T_0/2) = -x(t)$ in the integrand. Consider n even and n odd separately.

3-17. Consider the waveforms in Figure P3-17 and their complex exponential Fourier series. For which ones are the following true: real coefficients; purely imaginary coefficients; even-indexed coefficients zero; $X_0 = 0$?
If true, check the appropriate box in the table.

3-18. Given the periodic waveform shown in Figure P3-18.
(a) What is the value of a_0 in the sine-cosine Fourier series? Why?
(b) What are the values of the b_m coefficients in the sine-cosine Fourier series? Why?
(c) Are the coefficients in the complex exponential Fourier series real, imaginary, or complex? Why?
(d) Does this waveform have half-wave odd symmetry?

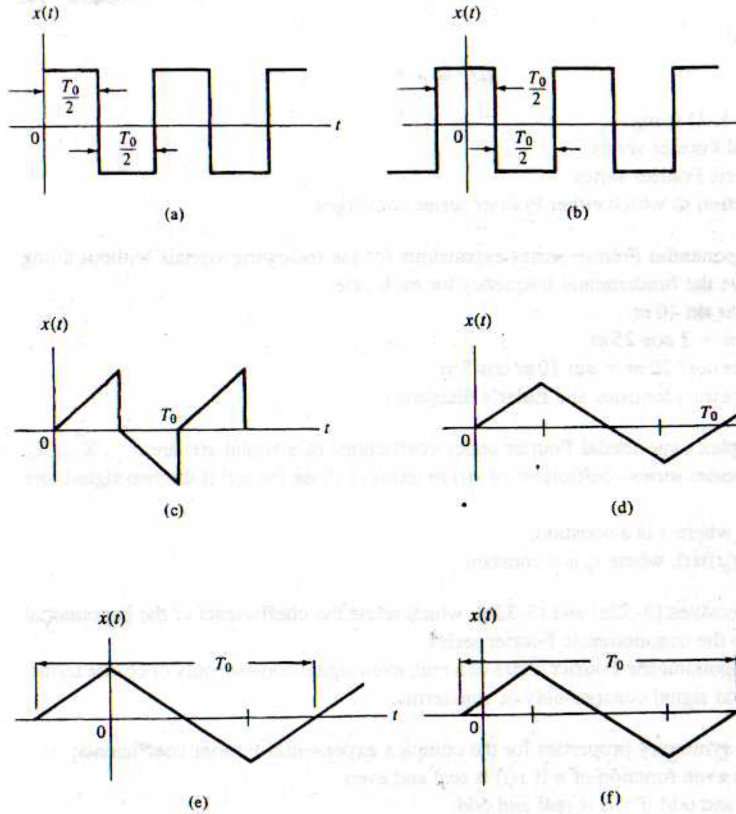


FIGURE P3-17

Property	Waveform a	b	c	d	e	f
Purely real coefficients						
Purely imaginary coefficients						
Complex coefficients						
Even-indexed coefficients zero						
$X_0 = 0$						

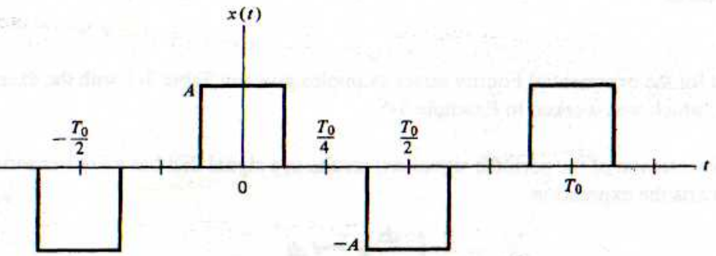


FIGURE P3-18

3-19. The phenomenon of "beats" when two slightly out of tune band instruments play the same nominal note at the same time may be seen by considering the signal

$$x(t) = \cos(\omega_0 t) + \cos[(\omega_0 + \Delta\omega)t]$$

Use trigonometric identies to rewrite this in the form

$$x(t) = A(t)\cos[\omega_0 t + \theta(t)]$$

That is, find explicit expressions for $A(t)$ and $\theta(t)$ in terms of $\Delta\omega$. Plot the waveform for the case of $\omega_0/2\pi = 1000$ hertz and $\Delta\omega/2\pi = 5$ hertz. Discuss how this would sound to the ear.

Section 3-7

3-20. Plot the two-sided amplitude and phase spectra for the full-wave rectified sinewave (waveform 2 of Table 3-1). Convert the two-sided plots to single-sided plots.

3-21. Plot and compare amplitude spectra for the triangular and square waves (waveforms 4 and 5 of Table 3-1). Which requires the most bandwidth for a given fidelity of reproduction?

3-22. Plot and label accurately Figure 3-11 for specific values of τ and T_0 as follows:
(a) $\tau = 2$ ms; $T_0 = 8$ ms
(b) $\tau = 1$ ms; $T_0 = 8$ ms
(c) $\tau = 2$ ms; $T_0 = 16$ ms

Section 3-8

3-23. Show all the steps in the derivation of (3-73).

3-24. For the *RL* filter shown in Figure P3-24, obtain the steady-state output to a triangular waveform with zero average value if $R/L = \omega_0$, where ω_0 is the fundamental frequency of the waveform in rad/s.

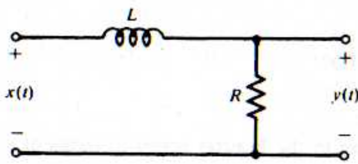


FIGURE P3.24

3-25. If the input to a system is $\cos 10\pi t + 2\cos 20\pi t$, tell what kinds of distortion, if any, a system introduces if its output is:

(a) $\cos(10\pi t - \pi/4) + 5\cos(20\pi t - \pi/2)$
(b) $\cos(10\pi t - \pi/4) + 2\cos(20\pi t - \pi/4)$
(c) $\cos(10\pi t - \pi/4) + 2\cos(20\pi t - \pi/2)$
(d) $2\cos(10\pi t - \pi/4) + 4\,(20\pi t - \pi/8)$

3-26. A function generator generates approximations of the following types of waveforms: (1) square wave; (2) triangular wave; (3) sine wave.

The square wave is generated by a multivibrator circuit that has an asymmetry in switching between positive and negative values so that the waveform is actually given by

$$x(t) = \begin{cases} A, & |t| \le T_0/4 + \epsilon/2 \\ -A, & T_0/4 + \epsilon/2 < |t| \le T_0/2 \end{cases}$$

The triangular waveform is generated by integrating the square wave with an operational amplifier integrator as described in Example 1-2.

The sine wave is obtained by filtering the square wave with a second-order filter with transfer function

$$H(\omega) = \frac{1}{1 + jQ[\omega/\omega_0 - \omega_0/\omega]}$$

where $\omega_0 = 2\pi/T_0$ and Q is a parameter called the filter quality factor.

(a) Sketch the actual and ideal square-wave output. Determine the integral-square error between actual and ideal outputs for the square-wave mode. Divide the integral-square error by T_0 to determine the mean-square error. Plot it as a function of ϵ/T_0 for $A = 1$.

(b) Obtain and sketch the triangular-mode output for an integrator constant $RC = 1$. Other than being inverted due to the operational amplifier circuit, how might one characterize its error from the ideal case?

(c) For the sine-wave mode, define the percent harmonic distortion for the nth harmonic as

$$(HC)_n = \frac{P_{n\text{th har}}}{P_{\text{fund}}} \times 100$$

(i) Find $(HC)_n$ in terms of ϵ and Q. If $\epsilon/T_0 = 0.05$, what must Q be to make $(HC)_2 = 0.1\%$?
(ii) Find $(HC)_3$ in terms of ϵ and Q. What must Q be to make $(HC)_3 = 0.05\%$? ($\epsilon/T_0 = 0.05$)
(iii) Which is the most stringent condition, the one for $(HC)_2$ or the one for $(HC)_3$?

Section 3-9

3-27. Give the rate of convergence of the amplitude spectrum for the signals shown in Problem 3-17(c) and (d) to zero as frequency increases. Justify your answer.

3-28. Give the rate of convergence of the amplitude spectrum for a half-rectified sine wave to zero as frequency increases. Justify your answer.

Section 3-10

3-29. Show that (3-92) and (3-90) are equivalent.

3-30. Given the set of functions

$$f_1(t) = A_1 e^{-t}$$
$$f_2(t) = A_2 e^{-2t}$$
$$\vdots$$

defined on the interval $(0, \infty)$.

(a) Find A_1 such that $f_1(t)$ is normalized to unity on $(0, \infty)$. Call this function $\phi_1(t)$.
(b) Find B such that $\phi_1(t)$ and $f_2(t) - B\,\phi_1(t)$ are orthogonal on $(0, \infty)$. Normalize this new function and call it $\phi_2(t)$.
(c) Do this for the third function, e^{-3t}. That is, choose C and D such that $f_3 - C\phi_2 - D\phi_1$ is orthogonal to both $\phi_1(t)$ and $\phi_2(t)$. Normalize it and call it $\phi_3(t)$. Comment on the feasibility of continuing this procedure.

3-31. What is the integral-square error in representing a square wave with zero average value by:
(a) Its fundamental
(b) Its fundamental plus third harmonic

Computer Exercises

3-1. Using the MATLAB application of Example 3-6 as a model, write a program to compute and plot the partial sums of each waveform given in Table 3-1. You might consider using MATLAB function programs to keep your program well organized.

3-2. Consider the use of a half-wave rectifier followed by an RC lowpass filter to implement a direct current (dc) power supply as shown in the following figure. Write a program to display the output waveform from the RC lowpass filter with a half-wave rectified sine wave at its input. Assume an operational amplifier isolation stage between rectifier and lowpass filter so that you can ignore loading of the rectifier by the filter.

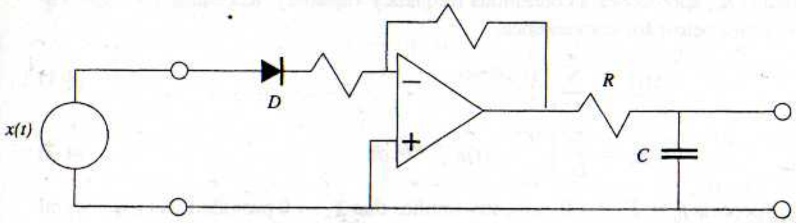


FIGURE P3-01

3-3. Rework Computer Exercise 3-2 for a full-wave rectified sine wave at the input.

3-4. Write a ***simulink*** program to rework Computer Exercise 3-3 without the isolation amplifier between the diode and the RC lowpass filter. Assume an ideal diode with zero forward resistance and infinite back resistance.

3-5. Rework Examples 3-10 and 3-11 for a triangular waveform at the RC lowpass filter input. Comment on the comparison of your results with those of Examples 3-10 and 3-11.

CHAPTER 4

The Fourier Transform and Its Applications

4-1 Introduction

The Fourier integral may be viewed as a limiting form of the Fourier series of a signal as the period goes to infinity. Thus, it is useful for pulse-type signals in providing spectral characterization and a system analysis tool. Since it converges for a relatively restricted class of signals, it finds limited use in the latter application. However, it also provides a bridge to a more general transform for system analysis, namely, the Laplace transform, which we take up in the next chapter. In this chapter, we provide a derivation of the Fourier transform and prove several theorems useful for deriving Fourier transform pairs. Several applications of the Fourier transform are also illustrated.

4-2 The Fourier Integral

Figure 3-11 showed that the spectral components of a pulse train became closer together as the period of the pulse train increased, while the shape of the spectrum remained unchanged if the pulse width stayed constant. We now use the insight gained in that exercise to develop a spectral representation for nonperiodic signals. Specifically, in the complex Fourier series sum and the integral for the coefficients of the Fourier series, we let the frequency spacing $f_0 = T_0^{-1}$ approach zero and the index n approach infinity such that the product nf_0 approaches a continuous frequency variable f. Repeating the exponential form of the Fourier series below for convenience,

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \tag{4-1}$$

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi n f_0 t}\, dt \tag{4-2}$$

we note that the limiting process $f_0 = T_0^{-1} \to 0$ evidently implies that $X_n \to 0$ provided that the integral

$$\left|\int_{-\infty}^{\infty} x(t)e^{-j2\pi n f_0 t}\, dt\right| \le \int_{-\infty}^{\infty} |x(t)|\, dt \tag{4-3}$$

exists. Thus, before carrying out the limiting process, it is useful to write (4-1) and (4-2) in the forms

$$x(t) = \sum_{nf_0=-\infty}^{\infty} \frac{X_n}{f_0} e^{j2\pi n f_0 t}\, \Delta(nf_0) \tag{4-4}$$

$$\tilde{X}(nf_0) \triangleq \frac{X_n}{f_0} = \int_{-1/2f_0}^{1/2f_0} x(t)e^{-j2\pi n f_0 t}\, dt \tag{4-5}$$

respectively, where the symbol $\Delta(nf_0)$ means the increment in the variable nf_0 which is f_0. The limiting process then formally becomes

$$T_0 \to \infty$$

$$f_0 \to 0 \quad \text{and} \quad n \to \infty \quad \text{such that } nf_0 \to f$$

$$\frac{1}{T_0} = \Delta(nf_0) \to df$$

$$\frac{X_n}{f_0} \to \tilde{X}(f)$$

which results in the sum in (4-4) becoming an integral over the continuous variable f, and the integral in (4-5) extending over the entire t-axis, $-\infty < t < \infty$. The resulting expressions are

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df \tag{4-6}$$

and

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt \tag{4-7}$$

where the tilde has been dropped from $X(f)$ to simplify notation.

These expressions define a *Fourier transform pair* for the signal $x(t)$, and are sometimes denoted by the notation $x(t) \leftrightarrow X(f)$, where the double-headed arrow means that $X(f)$ is obtained from $x(t)$ by applying (4-7) and $x(t)$ is obtained from $X(f)$ by applying (4-6). The integral (4-7) is referred to as the *Fourier transform* of $x(t)$, and is frequently denoted symbolically as $X(f) = \mathcal{F}[x(t)]$. Conversion back to the time domain by means of (4-6) is referred to as the *inverse Fourier transform* and is often denoted symbolically as $x(t) = \mathcal{F}^{-1}[X(f)]$.†

Several symmetry properties of the Fourier transform may be shown by writing $X(f)$ in terms of magnitude and phase as

$$X(f) = |X(f)|e^{j\theta(f)} \tag{4-8}$$

For example, assuming that $x(t)$ is real, it can be shown that‡

$$|X(f)| = |X(-f)| \quad \text{and} \quad \theta(f) = -\theta(-f) \tag{4-9}$$

That is, the magnitude is an even function of frequency and the argument, or phase, is an odd function of frequency. Plots of $|X(f)|$ and $\theta(f) = \underline{/X(f)}$ with f chosen as the abscissa are referred to as the amplitude and phase spectra of $x(t)$, respectively.* Thus the amplitude spectrum of a real, Fourier-transformable signal is even, and its phase spectrum is odd.

†Sufficient conditions for the convergence of the Fourier transform integral are that (1) the integral of $|x(t)|$ from $-\infty$ to ∞ exists, and (2) any discontinuities in $x(t)$ be finite. These being sufficient conditions means that signals exist which violate at least one of them and yet have Fourier transforms. An example is $(\sin \alpha t)/\alpha t$. For a discussion of other criteria that ensure convergence of the Fourier transform integral, see Bracewell (1978).

Later, we will include functions that do not possess Fourier transforms in the ordinary sense by generalization to Fourier transforms in the limit. Examples are the signals $x(t) =$ constant, all t, and $x(t) = \delta(t)$, the former violating condition (1) and the latter, condition (2).

‡For complex signals $|X(f)|$ need not be even, and $\theta(f)$ need not be odd.

*More properly, the amplitude spectrum should be referred to as the magnitude spectrum, but the former term is customary. Note that $|X(f)|$ is an amplitude, or magnitude, *density*. Its units are volts (or whatever the dimensions of $x(t)$ are) per hertz. This differs from the Fourier coefficients, which have the same dimensions as the original signal, $x(t)$.

Further properties which can be derived for $X(f)$ if $x(t)$ is real is that the *real part* of $X(f)$ is an *even* function of frequency and the *imaginary part* of $X(f)$ is an *odd* function of frequency.

The foregoing properties are proved by expanding $\exp(-j2\pi ft)$ in (4-7) as $\cos 2\pi ft - j \sin 2\pi ft$ by employing Euler's theorem and writing $X(f)$ as the sum of two integrals which are the real and imaginary parts of $X(f)$. Since the frequency dependence of the real part of $X(f)$ is through $\cos 2\pi ft$, it is therefore an even function of f. Similarly, since the frequency dependence of the imaginary part of $X(f)$ is through $\sin 2\pi ft$, we conclude that it is odd.

Further properties that can be proved (see Problem 4-4) about $X(f)$ are:

1. If $x(t)$ is even [i.e., $x(t) = x(-t)$], then $X(f)$ is a *real, even function of* f.
2. If $x(t)$ is odd [i.e., $x(t) = -x(-t)$], then $X(f)$ is an *imaginary, odd function of* f.

To illustrate the computation of Fourier transforms as well as their symmetry properties discussed above, we consider the following example.

EXAMPLE 4-1

The Fourier transform of $x_a(t) = \Pi(t/2)$, which is sketched in Figure 4-1a, is real and even. On the other hand, the Fourier transform of $x_b(t) = \Pi(t + \frac{1}{2}) - \Pi(t - \frac{1}{2})$, which is the odd function sketched in Figure 4-1b, is imaginary and odd. To prove these statements, we derive their Fourier transforms in this example.

Since $\Pi(t/2) = 1$, $-1 \le t \le 1$, and is zero otherwise, the Fourier transform of $x_a(t)$ is given by

$$\begin{aligned} X_a(f) &= \int_{-1}^{1} e^{-j2\pi ft}\, dt \\ &= -\frac{e^{-j2\pi ft}}{j2\pi f}\bigg|_{-1}^{1} \\ &= 2\,\frac{\sin 2\pi f}{2\pi f} \\ &= 2 \operatorname{sinc} 2f \end{aligned} \tag{4-10}$$

which is clearly a real, even function of f.

From the sketch of $x_b(t)$ shown in Figure 4-1 we may write the integral for its Fourier transform as

$$\begin{aligned} X_b(f) &= \int_{-1}^{0} e^{-j2\pi ft}\, dt - \int_{0}^{1} e^{-j2\pi ft}\, dt \\ &= -\frac{1}{j2\pi f}\left[e^{-j2\pi ft}\Big|_{-1}^{0} - e^{-j2\pi ft}\Big|_{0}^{1} \right] \\ &= \frac{j}{2\pi f}(2 - 2\cos 2\pi f) \\ &= j2\,\frac{\sin^2 \pi f}{\pi f} \\ &= j2\pi f \operatorname{sinc}^2 f \end{aligned} \tag{4-11}$$

This is clearly an imaginary, odd function of f.

The amplitude and phase spectra of $x_a(t)$ and $x_b(t)$ are shown in Figure 4-2. The student should be convinced that these are correct, particularly the phase spectrum for $x_b(t)$.

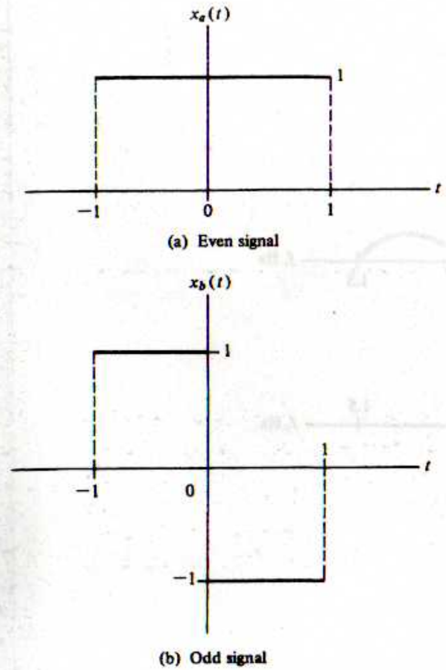


FIGURE 4-1. Even and odd pulse-type signals for illustration of the symmetry properties of Fourier transforms.

4-3 Energy Spectral Density

The energy of a signal can be expressed in the frequency domain by proceeding as follows. By definition of the normalized energy for a signal, given by (1-75), we write

$$\begin{aligned} E &\triangleq \int_{-\infty}^{\infty} |x(t)|^2\, dt \\ &= \int_{-\infty}^{\infty} x^*(t)\left[\int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df\right] dt \end{aligned} \tag{4-12}$$

where $x(t)$ has been written in terms of its Fourier transform. Reversal of the orders of integration results in

$$\begin{aligned} E &= \int_{-\infty}^{\infty} X(f)\left[\int_{-\infty}^{\infty} x^*(t) e^{j2\pi ft}\, dt\right] df \\ &= \int_{-\infty}^{\infty} X(f)\left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt\right]^* df \\ &= \int_{-\infty}^{\infty} X(f) X^*(f)\, df \end{aligned} \tag{4-13}$$

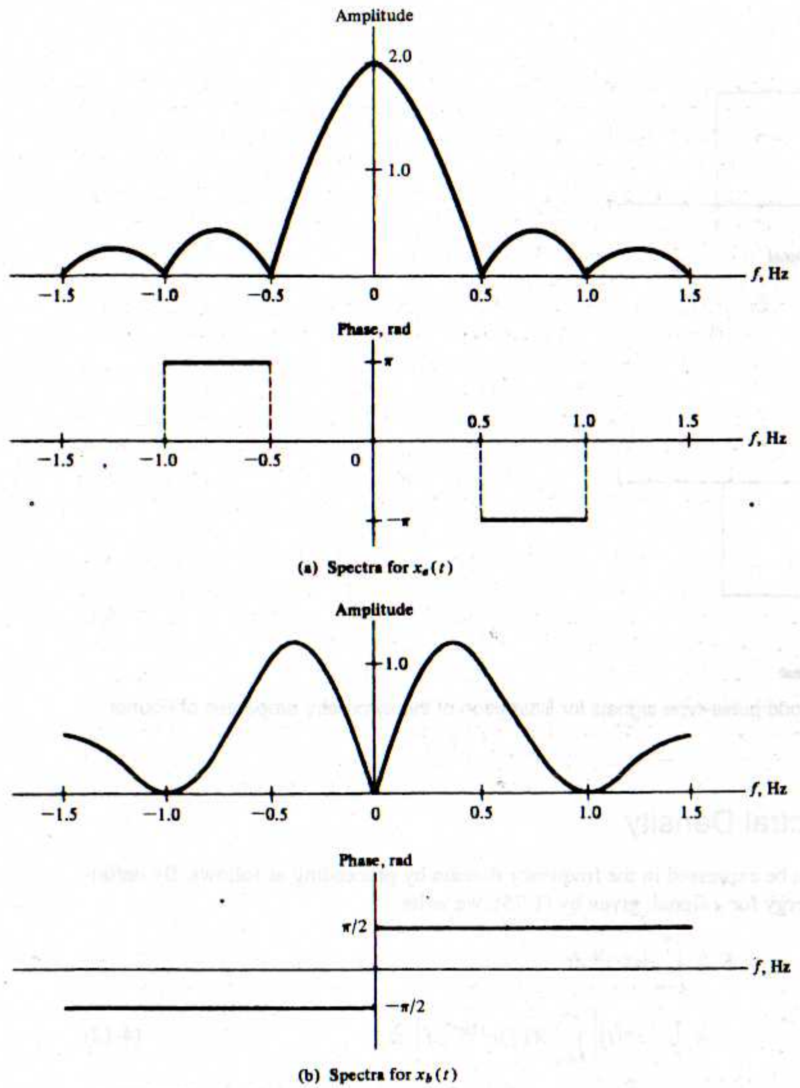


FIGURE 4-2. Amplitude and phase spectra for the signals of Example 4-1.

or

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df \qquad (4\text{-}14)$$

This is referred to as *Parseval's theorem for Fourier transforms.*

The units of $|X(f)|^2$, assuming that $x(t)$ is a voltage, are (V-s)^2 or, since E represents energy on a per ohm basis, (W-s)/Hz = J/Hz. Thus $|X(f)|^2$ has the units of energy density with frequency, and we define the *energy spectral density,* first referred to in connection with (1-87), of a signal as

$$G(f) \triangleq |X(f)|^2 \qquad (4\text{-}15)$$

Integration of $G(f)$ over all frequencies from $-\infty$ to ∞ yields the total (normalized) energy contained in a signal. Similarly, integration of $G(f)$ over a finite range of frequencies gives the energy contained in the signal within the range of frequencies represented by the limits of integration.

EXAMPLE 4-2

The energy in the signal $x(t) = \exp(-\alpha t)u(t)$, $\alpha > 0$, has been shown to be $1/2\alpha$ in Example 1-10. The Fourier transform of this signal is (Problem 4-1)

$$X(f) = \frac{1}{\alpha + j2\pi f} \qquad (4\text{-}16)$$

and its energy spectral density is

$$G(f) = \frac{1}{\alpha^2 + (2\pi f)^2} \qquad (4\text{-}17)$$

The energy contained in this signal in the frequency range $-B < f < B$ is

$$\begin{aligned} E_B &= \int_{-B}^{B} \frac{df}{\alpha^2 + (2\pi f)^2} \\ &= \frac{1}{\pi\alpha}\int_0^{2\pi B/\alpha} \frac{dv}{1 + v^2} \\ &= \frac{1}{\pi\alpha}\tan^{-1}\frac{2\pi B}{\alpha} \end{aligned} \qquad (4\text{-}18)$$

where the fact that the integral is even has been used together with the substitution $v = 2\pi f/\alpha$. The student should plot E_B as a function of B and show that $\lim_{B\to\infty} E_B = 1/2\alpha$, in agreement with the result calculated in Example 1-10.

MATLAB *Application*

Student MATLAB has the capability to manipulate quantities symbolically. The symbolic manipulation capability is illustrated by the following program:

```
%     c4ex2 MATLAB application
%
x=sym('exp(-2*t)*Heaviside(t)') % Symbolically define signal
X=fourier(x)                    % Symbolically take the Fourier transform
Xf=subs(X,'2*pi*f,'w')          % Change independent variable to f in
                                  hertz
Xf_conj=subs(Xf,'-i','i')       % Conjugate Fourier transform
GF=Xf*Xf_conj                   % Multiply the Fourier transform and its
                                  conjugate
ezplot(Gf)                      % Use the ezplot routine to plot energy
                                  spectrum
```

The result of running the program (each line above could be run directly from the command window) produces

```
EDU»c4ex2
x=
exp(-2*t)*Heaviside(t)
X=
1/(2+i*w)
Xf=
1/(2+2*i*pi*f)
Xf_conj=
1/(2-2*i*pi*f)
Gf=
1/(2+2*i*pi*f)/(2-2*i*pi*f)
```

The result of ezplot is shown in Figure 4-3.

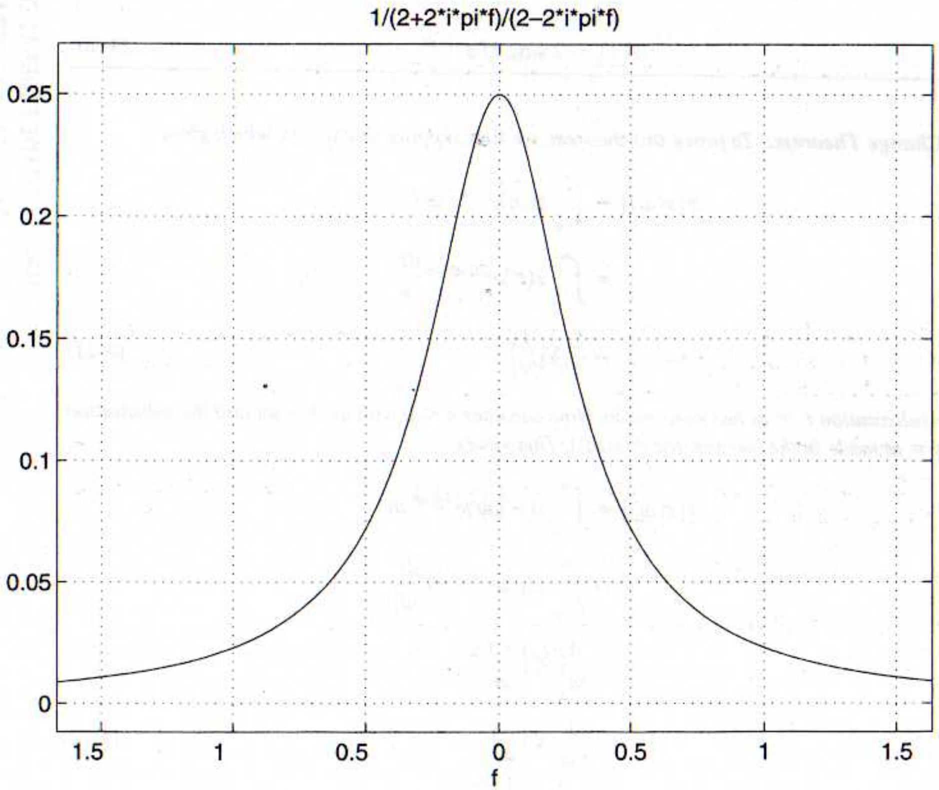


FIGURE 4-3. Energy spectral density for decaying exponential signal.

4-4 Fourier Transforms in the Limit

We now wish to add to our repertoire of Fourier transform pairs. In doing so, we will soon be faced with transforms of signals that are not absolutely integrable, as expressed by (4-3), or which have infinite discontinuities. An example of a signal that is not absolutely integrable is a sinusoid extending over all time from $t = -\infty$ to $t = \infty$. This signal we know has a discrete spectrum and yet cannot be Fourier-transformed through direct application of the defining integral (4-7). An example of a signal that has infinite discontinuities is the impulse function, which in the ordinary sense does not possess a Fourier transform. Rather than exclude such signals, which we know to be useful models, we broaden the definition of the Fourier transform to include Fourier transforms in the limit.

To define the Fourier transform of $\cos \omega_0 t$, $-\infty < t < \infty$, for example, we may first obtain the Fourier transform of the signal $\exp(-\alpha|t|) \cos \omega_0 t$, $\alpha > 0$, which is absolutely integrable.† The Fourier transform of $\cos \omega_0 t$ then is defined as the limit of the Fourier transform of $\exp(-\alpha|t|) \cos\omega_0 t$ as $\alpha \to 0$ since $\lim_{\alpha\to 0} \exp(-\alpha|t|) = 1$, $-\infty < t < \infty$. The Fourier transform of $\cos \omega_0 t$ will be given later.

As a second example of obtaining a Fourier transform in the limit, consider the approximation for the unit impulse function $\delta_\epsilon(t) = \Pi(t/2\epsilon)/2\epsilon$ discussed in Chapter 1. The Fourier transform of $\delta_\epsilon(t)$ exists for each finite value of ϵ and is, in fact, given by sinc $2\epsilon f$. In the limit as $\epsilon \to 0$, $\delta_\epsilon(t)$ approaches a unit impulse and its Fourier transform approaches 1 and is then defined to be the Fourier transform of the unit impulse function.

As a final, somewhat more subtle, example of a Fourier transform in the limit, consider $x(t) = A$, a constant. It should be clear that the Fourier integral in this case does not exist. (Write it down and try to evaluate the limits after integration!) On the other hand, the Fourier transform of $A\Pi(t/T)$ does exist and is similar to $X_a(f)$ derived in Example 4-1. It is, in fact, $\mathcal{F}[A\Pi(t/T)] = AT \operatorname{sinc} fT$. If we now let $T \to \infty$ in order to obtain the Fourier transform of a constant, it is not clear what the limit of AT sinc fT is. You should sketch this Fourier transform to show that it is a damped oscillatory function with a main lobe centered at $f = 0$ which becomes very narrow and high for T large. Furthermore, $\int_{-\infty}^{\infty} AT \operatorname{sinc} fT\, df = A$ for any T. Thus we formally write that $\mathcal{F}[A] = A\delta(f)$.

Easier ways of deriving these transforms will be shown after some basic transforms and theorems are considered.

4-5 Fourier Transform Theorems

Several theorems involving operations on signals and the corresponding operations on their Fourier transforms are summarized in Table 4-1. These theorems are useful for obtaining additional Fourier transform pairs as well as in systems analysis. We now discuss these theorems, outlining proofs in some cases, and working examples to illustrate their application. To denote a Fourier transform pair we will sometimes use the notation $x(t) \leftrightarrow X(f)$, which means that $\mathcal{F}[x(t)] = X(f)$ and $\mathcal{F}^{-1}[X(f)] = x(t)$.

Linearity (Superposition) Theorem: *Because the Fourier transform is an integral operation on a signal $x(t)$, with $x(t)$ appearing in the defining integral linearly, superposition is an obvious but extremely important property of the Fourier transform.*

EXAMPLE 4-3

Find the Fourier transform of $x(t) = \frac{1}{2}[x_a(t) - x_b(t)]$, where $x_a(t)$ and $x_b(t)$ are shown in Figure 4-1.

†Without the use of some Fourier transform theorems to be given later, the derivation of this Fourier transform is too lengthy.

TABLE 4-1
Fourier Transform Theorems[a]

Name of Theorem		
1. Superposition (a_1 and a_2 arbitrary constants)	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
2. Time delay	$x(t - t_0)$	$X(f)e^{-j2\pi f t_0}$
3a. Scale change	$x(at)$	$\lvert a\rvert^{-1}X\left(\frac{f}{a}\right)$
b. Time reversal	$x(-t)$	$X(-f) = X^*(f)$
4. Duality	$X(t)$	$x(-f)$
5a. Frequency translation	$x(t)e^{j\omega_0 t}$	$X(f - f_0)$
b. Modulation	$x(t)\cos\omega_0 t$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
6. Differentiation	$\dfrac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
7. Integration	$\int_{-\infty}^{t} x(t')\,dt'$	$(j2\pi f)^{-1}X(f) + \frac{1}{2}X(0)\delta(f)$
8. Convolution	$\int_{-\infty}^{\infty} x_1(t - t')x_2(t')\,dt'$ $= \int_{-\infty}^{\infty} x_1(t')x_2(t - t')\,dt'$	$X_1(f)X_2(f)$
9. Multiplication	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(f - f')X_2(f')\,df'$ $= \int_{-\infty}^{\infty} X_1(f')X_2(f - f')\,df'$

[a] $\omega_0 = 2\pi f_0$; $x(t)$ is assumed to be real in 3b.

Solution: This signal is a unit-high square pulse starting at $t = 0$ and ending at $t = 1$. Employing the superposition theorem and using the results of Example 4-1, we have

$$
\begin{aligned}
X(f) = X_a(f) - X_b(f) &= \frac{1}{2}\left[2\operatorname{sinc}(2f) - j2\pi f \operatorname{sinc}^2(f)\right] \\
&= \frac{1}{2}\left[2\frac{\sin 2\pi f}{2\pi f} - j2\pi f\frac{\sin^2 \pi f}{(\pi f)^2}\right] \\
&= \frac{1}{2}\left[2\frac{\sin 2\pi f}{2\pi f} - j\frac{1 - \cos 2\pi f}{2\pi f}\right] \\
&= \frac{1}{j2\pi f}\left[1 - \cos 2\pi f + j\sin 2\pi f\right] \\
&= \frac{1}{j2\pi f}\left[1 - e^{-j2\pi f}\right] \\
&= \frac{e^{j\pi f}}{\pi f}\,\frac{e^{j\pi f} - e^{-j\pi f}}{2j} \\
&= \frac{\sin \pi f}{\pi f}e^{-j\pi f} = \operatorname{sinc}(f)e^{-j\pi f}
\end{aligned}
\tag{4-19}
$$

where Euler's theorem has been used to good advantage. Note that the final result is sinc(f) which, as will be seen later, is the Fourier transform of a unit-high square pulse centered on $t = 0$ and the factor $e^{-j\pi f}$, which is due to the delay of the pulse by $t = \frac{1}{2}$ to start it at $t = 0$ as will be shown in the next theorem.

Time-Delay Theorem: *The time-delay theorem follows by replacing $x(t)$ in (4-7) with $x(t - t_0)$ and making the change in variables $t = t' + t_0$, whereupon the factor $\exp(-j2\pi f t_0)$ is taken outside the integral. The remaining integral is then the Fourier transform of $x(t)$ (note that what we call the variable of integration is immaterial).*

EXAMPLE 4-4

Obtain the Fourier transform of a unit-high square pulse 2 units wide starting at $t = 0$ and ending at $t = 2$.

Solution: This signal can be obtained from $x_a(t)$ of Example 4-1 by delaying it by $t_0 = 1$ unit of time. Thus

$$X(f) = 2\operatorname{sinc} 2f\, e^{-j2\pi f} \tag{4-20}$$

Scale Change Theorem: *To prove this theorem, we first suppose that $a > 0$, which gives*

$$
\begin{aligned}
\mathcal{F}[x(at)] &= \int_{-\infty}^{\infty} x(at)e^{-2\pi f t}\,dt \\
&= \int_{-\infty}^{\infty} x(t')e^{-j2\pi f t'/a}\,\frac{dt'}{a} \\
&= \frac{1}{a}X\left(\frac{f}{a}\right)
\end{aligned}
\tag{4-21}
$$

where the substitution $t' = at$ has been made. Now consider $a < 0$ with $at = -\lvert a\rvert t$ and the substitution $t' = -\lvert a\rvert t = at$ made in the integral for $\mathcal{F}[x(at)]$. This yields

$$
\begin{aligned}
\mathcal{F}[x(at)] &= \int_{-\infty}^{\infty} x(-\lvert a\rvert t)e^{-j2\pi f t}\,dt \\
&= \int_{-\infty}^{\infty} x(t')e^{j2\pi f t'/\lvert a\rvert}\,\frac{dt'}{\lvert a\rvert} \\
&= \frac{1}{\lvert a\rvert}X\left(\frac{-f}{\lvert a\rvert}\right) \\
&= \frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)
\end{aligned}
\tag{4-22}
$$

where the last step results because $-\lvert a\rvert = a$ for $a < 0$.

As a special case, we let $a = -1$, which results in the *time-reversal theorem.*

EXAMPLE 4-5

An application of the scale change theorem is provided by considering the recording on magnetic tape of a signal at, say, 15 inches per second (ips) and playing it back at $7\frac{1}{2}$ ips. Assuming a flat frequency response for the recorder which is independent of playback speed and an original signal spectrum of

$$X_r(f) = \frac{1}{1 + (f/1000)^2} \tag{4-23}$$

we find the spectrum of the played back signal to be

$$X_{pb}(f) = \frac{2}{1 + (f/500)^2} \tag{4-24}$$

where $a = \frac{1}{2}$ in the scale change theorem. To see that $a = \frac{1}{2}$, consider an easily visualized signal such as a pulse 1 s in duration recorded at 15 ips. When played back at $7\frac{1}{2}$ ips it will be 2 s in duration. Thus $x_{pb}(t) = x_r(at)$ or $x_{pb}(2) = x_r(2a)$, where $t = 2$ corresponds to the trailing edge of the pulse on playback, which must correspond to the trailing edge on record. This requires that $2a = 1$ or $a = \frac{1}{2}$. The spectrum of the played-back signal must be narrower than that of the recorded signal (time is stretched out on playback, thus lowering the frequency content). The student should sketch $X_r(f)$ and $X_{pb}(f)$ and be convinced that they are indeed reasonable.

EXAMPLE 4-6

As another application of the scale change theorem, consider the signal $x_a(t)$ of Example 4-1, for which we obtained the transform pair

$$x_a(t) = \Pi\left(\frac{t}{2}\right) \leftrightarrow 2 \operatorname{sinc} 2f \tag{4-25}$$

Instead of the Fourier transforms of a pulse of width 2, we wish to generalize this to a pulse of arbitrary width, say τ. Thus we let $a = 2/\tau$ to obtain the transform pair

$$x_a(at) = \Pi\left(\frac{t}{\tau}\right) \leftrightarrow \tau \operatorname{sinc} \tau f \tag{4-26}$$

Plots of this signal and the magnitude of its Fourier transform are shown in Figure 4-3a. Note that the signal duration and spectral width (or *bandwidth*) are inversely proportional. Even though the spectrum of $x(t)$ is infinite in extent in this case, we take some convenient measure as the bandwidth; in this case the width of the main lobe of the sinc function is taken as the bandwidth, which is $2/\tau$.

Duality Theorem: *The duality theorem follows by virtue of the similarity of the direct and inverse Fourier transform relationships, the only difference in addition to the variable of integration being the sign in the exponent. The use of this theorem is illustrated in the following example.*

EXAMPLE 4-7

The duality theorem can be used to derive the Fourier transform pair

$$2W \operatorname{sinc} 2Wt \leftrightarrow \Pi\left(\frac{f}{2W}\right) \tag{4-27}$$

given the transform pair

$$x(t) = \Pi\left(\frac{t}{\tau}\right) \leftrightarrow \tau \operatorname{sinc} \tau f = X(f) \tag{4-28}$$

We do this by replacing f in $X(f)$ by t to get a new function of time, which is

$$X(t) = \tau \operatorname{sinc} \tau t$$

According to the duality theorem, this new time function has the Fourier transform

$$x(f) = \Pi\left(\frac{t}{\tau}\right)\Big|_{t \to -f} = \Pi\left(\frac{-f}{\tau}\right) \tag{4-29}$$

Defining the new parameter $W = \tau/2$ to make the notation clearer, and noting that $\Pi(-f/\tau) = \Pi(f/\tau)$, we obtain the desired transform pair.

Plots of this signal and its spectrum as given in Figure 4-4 again illustrate the inverse relationship between signal duration and bandwidth. Since this signal is actually infinite in duration, we take a convenient measure for its practical duration, such as the width of its main lobe, which in this case is $1/W$.

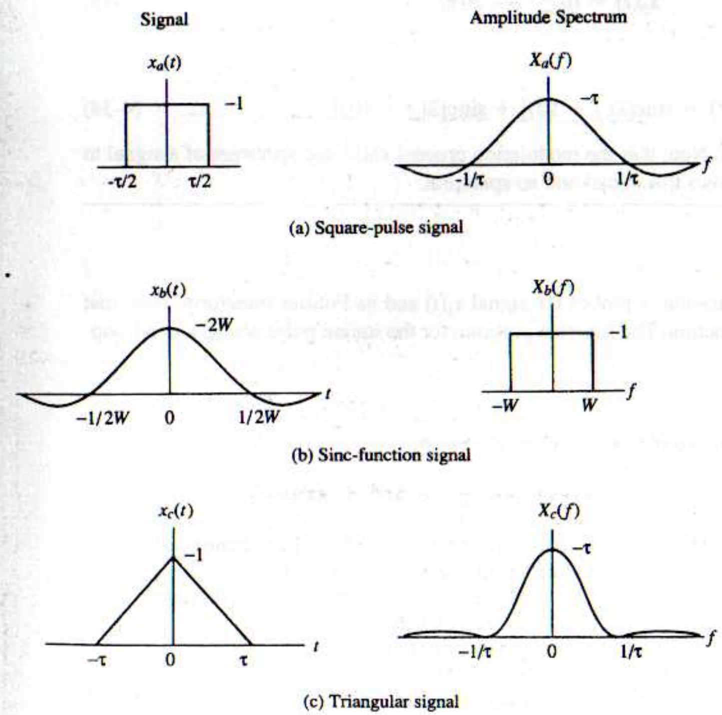


FIGURE 4-4. Various signals and their spectra.

Frequency Translation Theorem: *This theorem is the dual of the time-delay theorem. It is proved by writing down the expression for the Fourier transform of* $x(t)e^{j2\pi f_0 t}$, *which is*

$$\int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_0)t}\,dt \quad (4\text{-}30)$$

We recognize the right-hand side as the Fourier transform of $x(t)$, *with* f *replaced by* $f - f_0$ *and the theorem is proved. The modulation theorem follows from the frequency translation theorem by using Euler's theorem to write* $\cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$ *and then applying superposition.*

EXAMPLE 4-8

The Fourier transform of the signal

$$x_1(t) = \Pi\left(\frac{t}{2}\right)\exp(\pm j20\pi t) \quad (4\text{-}31)$$

is

$$X_1(f) = 2\,\mathrm{sinc}[2(f \mp 10)] \quad (4\text{-}32)$$

where the transform pair obtained in Example 4-1 for $x_a(t)$ has been used. From the modulation theorem, the Fourier transform of

$$x_2(t) = \Pi\left(\frac{t}{2}\right)\cos 20\pi t \quad (4\text{-}33)$$

is

$$X_2(f) = \mathrm{sinc}[2(f - 10)] + \mathrm{sinc}[2(f + 10)] \quad (4\text{-}34)$$

The student should sketch $X_2(f)$. Note that the modulation process shifts the spectrum of a signal to a new frequency. Figure 4-5 shows this signal and its spectrum.

MATLAB *Application*

The MATLAB program below provides a plot of the signal $x_1(t)$ and its Fourier transform. Note that sinc( ) is a MATLAB-defined function. The function program for the square pulse was given in Chapter 1.

```
EDU» c4ex8
%     Plot of signal and spectra of Example 4-8
%
t=-2:.005:2;                       % Define the t and f axes
f=-20:.005:20;
x=pls_fn(t/2).*cos(20*pi*t);       % Use the predefined pulse function to
                                   % build up x(t) (page 136)
X=sinc(2*(f - 10))+sinc(2*(f+10));      % sinc() is a built-in MATLAB
                                        % function
subplot(2,1,1),plot(t,x),xlabel('t'),ylabel('x2(t)')
subplot(2,1,2),plot(f,X),xlabel('f'),ylabel('X2(f)')
```

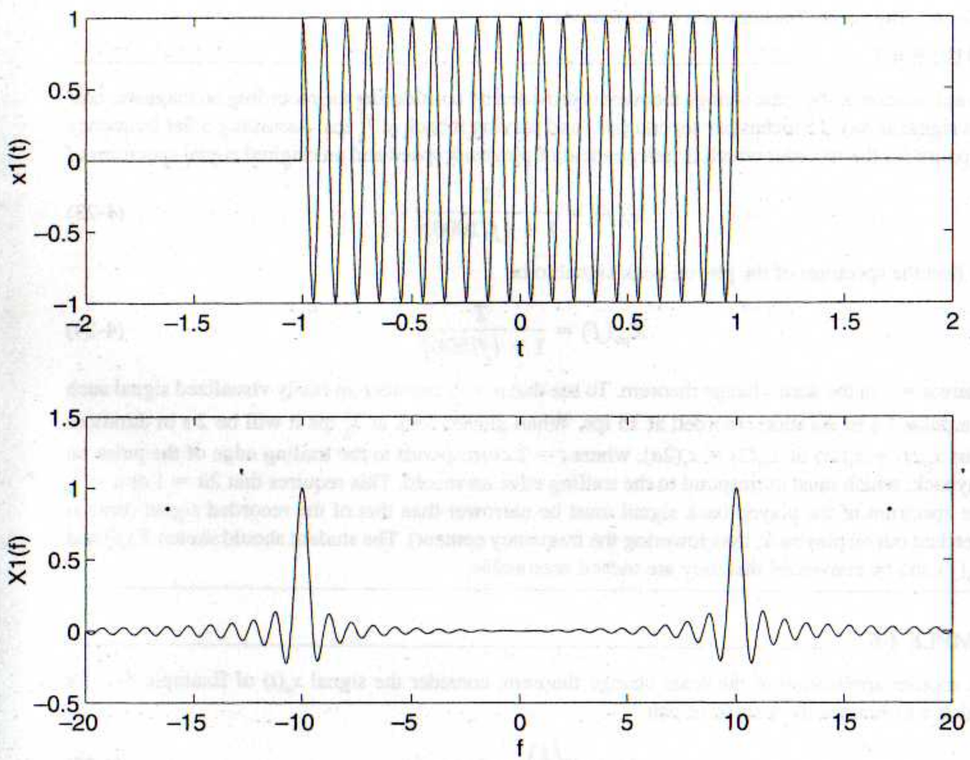



FIGURE 4-5. MATLAB plot of time-domain and frequency domain signals illustrating the modulation theorem. Top: Time-domain signal. Bottom: Fourier transform

EXAMPLE 4-9

Obtain the following transform pairs:

1. $A\delta(t) \leftrightarrow A$
2. $A\delta(t - t_0) \leftrightarrow Ae^{-j2\pi f t_0}$
3. $A \leftrightarrow A\delta(f)$
4. $Ae^{j2\pi f_0 t} \leftrightarrow A\delta(f - f_0)$
5. $A\cos 2\pi f_0 t \leftrightarrow A/2[\delta(f - f_0) + \delta(f + f_0)]$

The impulse function is not properly Fourier transformable. However, its transform is nevertheless useful. The same holds true for a constant. We could find their transforms using limiting operations, as discussed in Section 4-3. It is easier to prove the first transform pair by using the sifting property of the unit impulse:

$$\mathcal{F}[A\delta(t)] = A\int_{-\infty}^{\infty}\delta(t)e^{-j2\pi ft}\,dt = A \quad (4\text{-}35)$$

The second pair is obtained by applying the time-delay theorem to pair 1.

The third pair can be shown by using duality in conjunction with the first transform pair. Thus

$$X(t) = A \leftrightarrow A\delta(-f) = x(-f) = A\delta(f) \tag{4-36}$$

where we recall that the impulse function was defined as even in Chapter 1.

Transform pair 4 follows by applying the frequency translation theorem to pair 3, and pair 5 is obtained by applying the modulation theorem to pair 3.

EXAMPLE 4-10

A convenient signal for use in obtaining transforms of periodic signals is the signal

$$y_s(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_s) \tag{4-37}$$

It is a periodic train of impulses referred to as the *ideal sampling waveform.*

The Fourier transform of $y_s(t)$ can be obtained by noting that it is periodic and, in a formal sense, can be represented by the Fourier series

$$y_s(t) = \sum_{n=-\infty}^{\infty} Y_n e^{j2\pi n f_s t}, \qquad f_s = \frac{1}{T_s} \tag{4-38}$$

The Fourier coefficients are given by

$$Y_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t) e^{-j2\pi n f_s t}\, dt = f_s \tag{4-39}$$

where the sifting property of the unit impulse function has been used. Therefore, $y_s(t)$ can be represented by the Fourier series

$$y_s(t) = f_s \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t} \tag{4-40}$$

Using the Fourier transform pair $e^{j2\pi f_0 t} \leftrightarrow \delta(f - f_0)$, we may take the Fourier transform of this Fourier series term-by-term, to obtain

$$\begin{aligned} Y_s(f) &= f_s \sum_{n=-\infty}^{\infty} \mathcal{F}[e^{j2\pi n f_s t}] \\ &= f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \end{aligned} \tag{4-41}$$

Summarizing, we have obtained the transform pair

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_s) \leftrightarrow f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \tag{4-42}$$

The transform pair (4-42) is useful in spectral representation of periodic signals by the Fourier transform, to be considered in Sections 4-7 and 4-11.

Differentiation and Integration Theorems: *To prove the differentiation theorem, consider*

$$\mathcal{F}\left[\frac{dx}{dt}\right] = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j2\pi f t}\, dt \tag{4-43}$$

Using integration by parts with $u = e^{-j2\pi f t}$ and $dv = (dx/dt)\,dt$, we obtain

$$\mathcal{F}\left[\frac{dx}{dt}\right] = x(t)e^{-j2\pi f t}\Big|_{-\infty}^{\infty} + j2\pi f \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}\, dt \tag{4-44}$$

But if $x(t)$ is absolutely integrable, $\lim_{t\to\pm\infty}|x(t)| = 0$. Therefore, $dx/dt \leftrightarrow j2\pi f X(f)$. Repeated application of integration by parts can be used to prove the theorem for n differentiations.

The integration theorem is proved in a manner similar to that for the differentiation theorem if $X(0) = \int_{-\infty}^{\infty} x(t)\, dt = 0$. If $X(0) \neq 0$, the proof of the integration theorem must be carried out using a transform-in-the-limit approach. The second term of the transform of pair 7, $\frac{1}{2}X(0)\delta(f)$, simply represents the spectrum of the constant $\int_{-\infty}^{\infty} x(t)\, dt$, which is the net area of $x(t)$.

Note that differentiation enhances the high-frequency content of a signal, which is reflected by the factor $j2\pi f$ in the transform of a derivative, while integration smooths out time fluctuations or suppresses the high-frequency content of a signal, as indicated by the $(j2\pi f)^{-1}$ in the transform of an integral.

EXAMPLE 4-11

A useful application of the differentiation theorem is that of obtaining Fourier transforms of piecewise linear signals such as the triangle signal. We generalize the triangle signal in this example to that of a trapezoidal signal and find its Fourier transform through application of the differentiation theorem. The trapezoidal signal is shown in Figure 4-6 along with its first two derivatives. The second derivative consists entirely of unit impulses, and may be written analytically as

$$\frac{d^2x(t)}{dt^2} = K[\delta(t + b) - \delta(t + a) - \delta(t - a) + \delta(t - b)] \tag{4-45}$$

where $K = A/(b - a)$. Applying item 2 in Example 4-9, we find the Fourier transform of the second derivative to be

$$\mathcal{F}\left[\frac{d^2x(t)}{dt^2}\right] = k[e^{j2\pi f b} - e^{j2\pi f a} - e^{-j2\pi f a} + e^{-j2\pi f b}] = 2K[\cos 2\pi f b - \cos 2\pi f a] \tag{4-46}$$

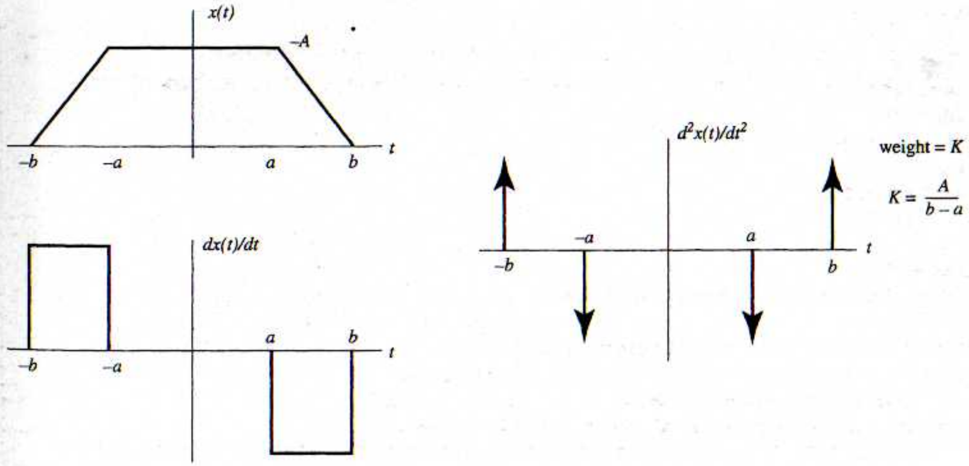


FIGURE 4-6. A trapezoidal pulse and its first two derivatives.

By the differentiation theorem, this transform equals $(j2\pi f)^2X(f)$. Thus

$$X(f) = 2K\left[\frac{\cos 2\pi fb - \cos 2\pi fa}{(j2\pi f)^2}\right]$$

$$= K\left[b^2\frac{\sin^2 \pi fb}{(\pi fb)^2} - a^2\frac{\sin^2 \pi fa}{(\pi fa)^2}\right]$$

$$= K[b^2 \operatorname{sinc}^2(fb) - a^2 \operatorname{sinc}^2(fa)] \quad (4\text{-}47)$$

Another way to find the Fourier transform of a trapezoidal pulse is to write it as the difference of two triangle signals as

$$x(t) = B\Lambda(t/b) - (B - A)\Lambda(t/a) \quad (4\text{-}48)$$

where $B = Ab/(b - a)$. The unit-high triangle signal, $\Lambda(t/\tau)$, centered on $t = 0$ going from $t = -\tau$ to $t = \tau$ will be shown later to have the Fourier transform $\tau \operatorname{sinc}^2 f\tau$ [in fact this may be seen from (4-47) by setting $a = 0$ and $b = \tau$]. Applying this to (4-48) together with superposition gives

$$X(f) = B\tau b \operatorname{sinc}^2(fb) - (B - A)\tau a \operatorname{sinc}^2(fa) \quad (4\text{-}49)$$

which may be demonstrated to be the same result as (4-47). The student should also work out the special cases $a = 0$ and $a = b$.

MATLAB *Application*

We examine the effect of the pitch of the sides of the trapezoid on its spectrum. The following MATLAB program implements a plot of the trapezoid and the corresponding spectrum for several values of a with b fixed.

```
EDU»c4ex11
%     Spectrum of a trapezoidal pulse-Example 4-11
%
t_max=2;
f_max=2;
f=-f_max:.001:f_max;
t=-t_max:.001:t_max;
A=1;
b=1.5;
for k=1:4
k_odd=2*k-1;
k_even=2*k;
a=.5*(k-1)+.0000001;
aa=a;
if aa <=.000001
      aa=0;
end
B=A*b/(b-a);
x=B*trgl_fn(t/b)-(B-A)*trgl_fn(t/a);
K=A/(b-a);
X=K*(b^2*(sinc(f*b)).^2-a^2*(sinc(f*a)).^2);
subplot(4,2,k_even),plot(f,X),axis([-f_max f_max -.8 3.5]),...
      xlabel('f'),ylabel('X(f)')
subplot(4,2,k_odd),plot(t,x),axis([-t_max t_max 0 2]),...
      text(-1.9, 1.6,['a=',num2str(aa)]),xlabel('t'),ylabel('x(t)')
end
```


A plot of the pulse and its spectrum for four values of a are shown in Figure 4-7. Note that as the pulse approaches a triangle its spectrum is more concentrated about $f = 0$. For the limiting case of a square pulse, the spectrum has large lobes along side the main lobe. We infer from this that the smoother transitions of the trapezoid give better spectral containment within the main lobe of the spectrum.

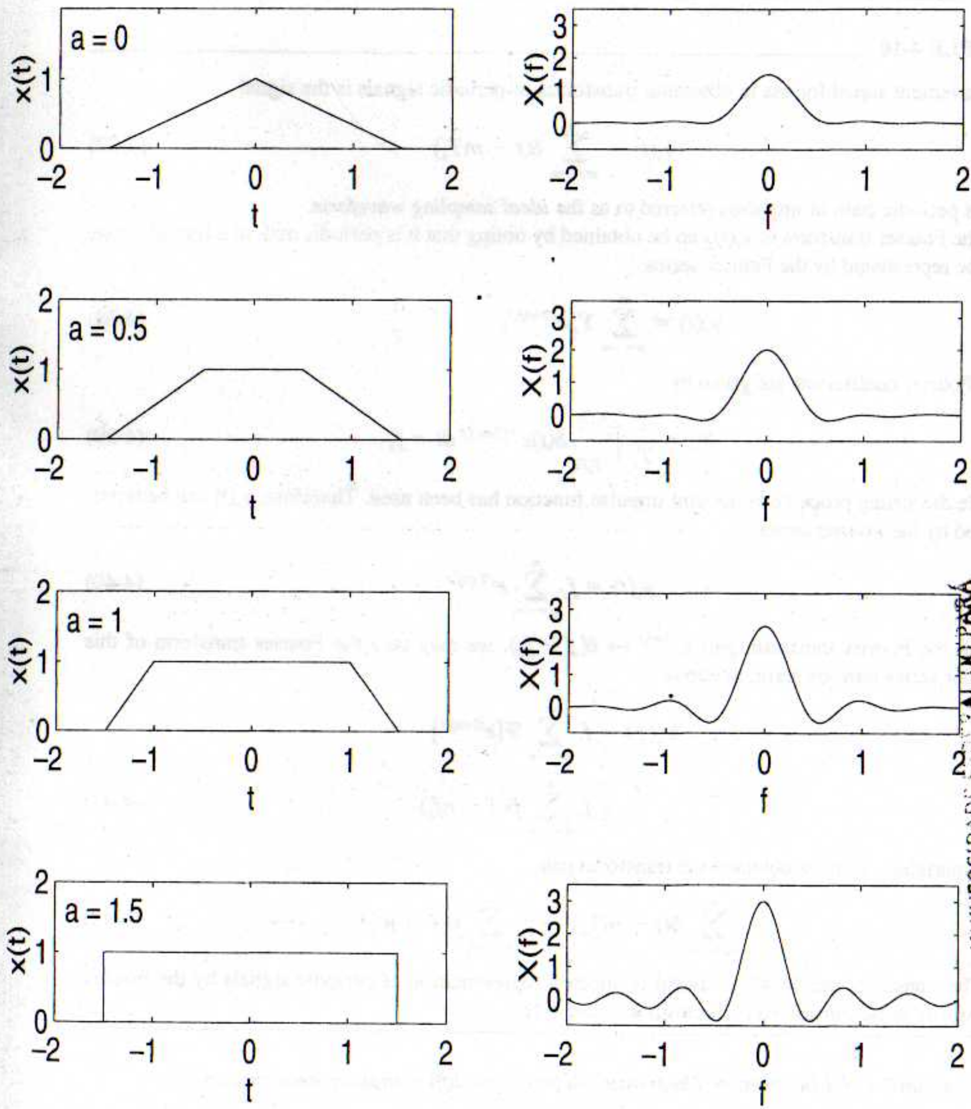


FIGURE 4-7. Trapezoidal pulses and their spectra for four different shapes.

EXAMPLE 4-12

As an application of the integration theorem, consider the Fourier transform of a unit step. Since $u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda$ and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$, we obtain

$$\mathcal{F}[u(t)] = (j2\pi f)^{-1} + \tfrac{1}{2}\delta(f) \tag{4-50}$$

where $x(t)$ in the integration theorem of Table 4-1 has been taken as $\delta(t)$. A signal that is related to the unit step is the signum function, defined as

$$\operatorname{sgn} t = 2u(t) - 1 = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases} \tag{4-51}$$

Using the result obtained above for the Fourier transform of a step, we find the Fourier transform of sgn t to be

$$\mathcal{F}[\operatorname{sgn} t] = (j\pi f)^{-1} \tag{4-52}$$

By the duality theorem, it follows that

$$\mathcal{F}\left[\frac{1}{j\pi t}\right] = \operatorname{sgn}(-f) = -\operatorname{sgn} f \tag{4-53}$$

or

$$\mathcal{F}\left[\frac{1}{\pi t}\right] = -j \operatorname{sgn} f \tag{4-54}$$

which results because sgn f is odd. The last transform pair is useful in defining the Hilbert transform of a signal, which will be discussed in Example 4-14.

The Convolution Theorem: *The proof of the convolution theorem follows by representing $x_2(t - t')$ in the second integral under Theorem 8 of Table 4-1 in terms of its inverse Fourier transform, which is*

$$x_2(t - t') = \int_{-\infty}^{\infty} X_2(f) e^{j2\pi f(t-t')}\, df \tag{4-55}$$

Substitution into the convolution integral gives

$$\begin{aligned} x_1 * x_2 &= \int_{-\infty}^{\infty} x_1(t') \left[\int_{-\infty}^{\infty} X_2(f) e^{j2\pi f(t-t')}\, df\right] dt' \\ &= \int_{-\infty}^{\infty} X_2(f) \left[\int_{-\infty}^{\infty} x_1(t') e^{-j2\pi f t'}\, dt'\right] e^{j2\pi f t}\, df \end{aligned} \tag{4-56}$$

The last step results from reversing the order of the two integrations. Recognizing the bracketed term inside the integral as $X_1(f)$, the Fourier transform of $x_1(t)$, we have obtained

$$x_1 * x_2 = \int_{-\infty}^{\infty} X_1(f) X_2(f) e^{j2\pi f t}\, df \tag{4-57}$$

which is the inverse Fourier transform of $X_1(f)X_2(f)$.

This theorem is useful in systems analysis applications as well as for deriving transform pairs. The latter application is illustrated by the following examples.

EXAMPLE 4-13

Consider the transform of the signal that is the convolution of two rectangular pulses of equal width. The student can verify that

$$\Pi\left(\frac{t}{\tau}\right) * \Pi\left(\frac{t}{\tau}\right) = \tau\Lambda\left(\frac{t}{\tau}\right) \tag{4-58}$$

where $\Lambda(t/\tau)$ is the triangular signal. Therefore, according to the convolution theorem,

$$\begin{aligned} \mathcal{F}\left[\tau\Lambda\left(\frac{t}{\tau}\right)\right] &= \left\{\mathcal{F}\left[\Pi\left(\frac{t}{\tau}\right)\right]\right\}^2 \\ &= \tau^2 \operatorname{sinc}^2 f\tau \end{aligned} \tag{4-59}$$

EXAMPLE 4-14

The Hilbert transform, $\hat{x}(t)$, of a signal, $x(t)$, is obtained by convolving $x(t)$ with $1/\pi t$. That is,

$$\hat{x}(t) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\lambda)}{t - \lambda}\, d\lambda \tag{4-60}$$

From Example 4-12 and the convolution theorem it follows that

$$\mathcal{F}[\hat{x}(t)] = \mathcal{F}\left[\frac{1}{\pi t}\right] \mathcal{F}[x(t)] = -j \operatorname{sgn}(f) X(f) \tag{4-61}$$

A Hilbert transform operation therefore multiplies all positive-frequency spectral components of a signal by $-j$, and all negative-frequency spectral components by j. The amplitude spectrum of the signal is left unchanged by the Hilbert transform operation and the phase is shifted by $\pi/2$ rad.

The Multiplication Theorem: *The proof of the multiplication theorem proceeds in a manner analogous to the proof of the convolution theorem. It is left to the student as a problem. Its application will be illustrated by an example.*

EXAMPLE 4-15

We will use the multiplication theorem to obtain the Fourier transform of the cosinusoidal pulse given by

$$x(t) = A\Pi\left(\frac{t}{\tau}\right) \cos 2\pi f_0 t \tag{4-62}$$

The makeup of this waveform is illustrated in Figure 4-8c on the left-hand side; that is, the product of the waveforms shown in Figure 4-8a and b is the oscillatory pulse shown in Figure 4-8c. Shown on the right-hand side in each part of the figure are the Fourier transforms of the corresponding waveforms on the left. From the multiplication theorem, we know that the convolution of the spectra shown in Figure 4-8a and b is the spectrum of the cosinusoidal pulse of Figure 4-8c. In terms of equations, we have the transform pair

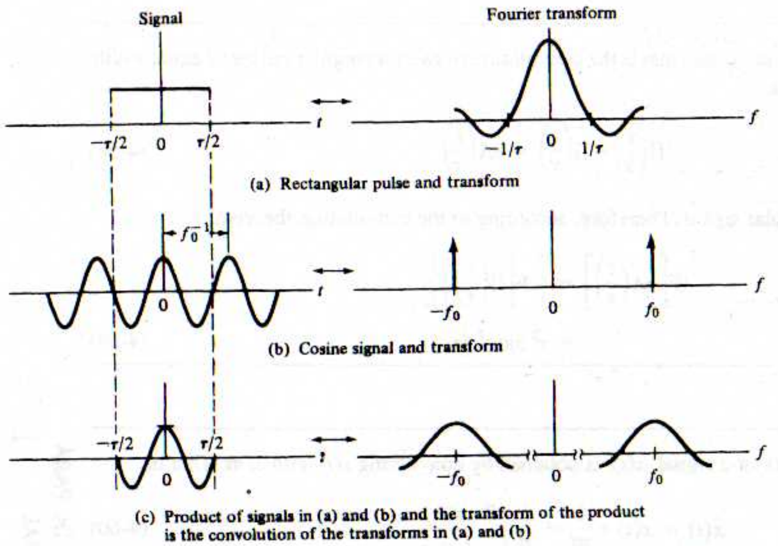


FIGURE 4-8. Application of the multiplication theorem in obtaining the Fourier transform of a finite-duration cosinusoidal signal.

$$A\Pi\left(\frac{t}{\tau}\right) \leftrightarrow A\tau \operatorname{sinc} f\tau \tag{4-63}$$

which was obtained previously. The Fourier transform of $\cos 2\pi f_0 t$ follows by using Euler's theorem to write

$$\begin{aligned}\mathcal{F}[\cos 2\pi f_0 t] &= \mathcal{F}[\tfrac{1}{2}e^{j2\pi f_0 t} + \tfrac{1}{2}e^{-j2\pi f_0 t}] \\ &= \tfrac{1}{2}\delta(f-f_0) + \tfrac{1}{2}\delta(f+f_0)\end{aligned} \tag{4-64}$$

where the superposition theorem has been used together with the transform pair $e^{j2\pi f_0 t} \leftrightarrow \delta(f-f_0)$. The multiplication theorem for this example then states that

$$\begin{aligned}X(f) &= \mathcal{F}\left[A\Pi\left(\frac{t}{\tau}\right)\cos 2\pi f_0 t\right] \\ &= (A\tau \operatorname{sinc} f\tau) * [\tfrac{1}{2}\delta(f-f_0) + \tfrac{1}{2}\delta(f+f_0)] \\ &= \frac{A\tau}{2}[\operatorname{sinc}(f-f_0)\tau + \operatorname{sinc}(f+f_0)\tau]\end{aligned} \tag{4-65}$$

where use has been made of

$$\begin{aligned}\operatorname{sinc} f\tau * \delta(f \pm f_0) &= \int_{-\infty}^{\infty} \operatorname{sinc} \lambda\tau\, \delta(\lambda - f \pm f_0)\, d\lambda \\ &= \operatorname{sinc}(f \pm f_0)\tau\end{aligned} \tag{4-66}$$

Note that the modulation theorem could have equally well been used to obtain this result.

TABLE 4-2
Fourier Transform Pairs

Pair Number	$x(t)$	$X(f)$	Comments on Derivation
1.	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc} \tau f$	Direct evaluation
2.	$2W \operatorname{sinc} 2Wt$	$\Pi\left(\frac{f}{2W}\right)$	Duality with pair 1, Example 4-7
3.	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2 \tau f$	Convolution using pair 1
4.	$\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$	Direct evaluation
5.	$t\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$	Differentiation of pair 4 with respect to α
6.	$\exp(-\alpha\|t\|),\ \alpha > 0$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$	Direct evaluation
7.	$e^{-\pi(t/\tau)^2}$	$\tau e^{-\pi(f\tau)^2}$	Direct evaluation
8.	$\delta(t)$	1	Example 4-9
9.	1	$\delta(f)$	Duality with pair 7
10.	$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$	Shift and pair 7
11.	$\exp(j2\pi f_0 t)$	$\delta(f - f_0)$	Duality with pair 9
12.	$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$	Exponential representation of cos and sin and pair 10
13.	$\sin 2\pi f_0 t$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$	
14.	$u(t)$	$(j2\pi f)^{-1} + \frac{1}{2}\delta(f)$	Integration and pair 7
15.	$\operatorname{sgn} t$	$(j\pi f)^{-1}$	Pair 8 and pair 13 with superposition
16.	$\frac{1}{\pi t}$	$-j\operatorname{sgn}(f)$	Duality with pair 14
17.	$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t - \lambda}\, d\lambda$	$-j\operatorname{sgn}(f)X(f)$	Convolution and pair 15
18.	$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$	$f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s),\ f_s = T_s^{-1}$	Example 4-10

The Fourier transform pairs derived in the preceding section are collected in Table 4-2 together with other pairs that are often useful. A column is provided that gives suggestions on the derivation of each pair.

4-6 System Analysis with the Fourier Transform

In Chapter 2 the superposition integral was developed as a systems analysis tool. Application of the convolution theorem of Fourier transforms, pair 8 of Table 4-1, to the superposition integral gives the result

$$Y(f) = H(f)X(f) \tag{4-67}$$

where $X(f) = \mathcal{F}[x(t)]$, $Y(f) = \mathcal{F}[y(t)]$, and $H(f) = \mathcal{F}[h(t)]$. The latter is referred to as the *transfer function* of the system and is identical to the frequency-response function $H(\omega)$ defined by (3-70) but with

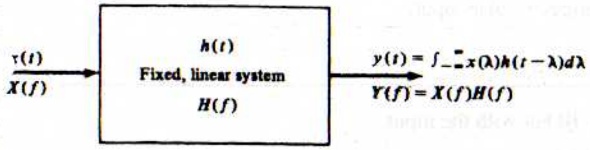


FIGURE 4-9. Representation of a fixed, linear system in terms of its impulse response or, alternatively, its transfer function.

$= \omega/2\pi$ as the independent variable. Either $h(t)$ or $H(f)$ are equally good characterizations of the system, since its output can be found in terms of either one as illustrated in Figure 4-9. In the frequency domain, the output of the system is the inverse Fourier transform of (4-67), which is

$$y(t) = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}\,df \tag{4-68}$$

Since $H(f)$ is, in general, a complex quantity, we write it as

$$H(f) = |H(f)|e^{j\underline{/H(f)}} \tag{4-69}$$

where $|H(f)|$ is the *amplitude-response* function and $\underline{/H(f)}$ the *phase-response* function of the network. If $H(f)$ is the Fourier transform of a real time function $h(t)$, which it is in all cases being considered in this book, it follows that†

$$|H(f)| = |H(-f)| \tag{4-70}$$

and

$$\underline{/H(f)} = -\underline{/H(-f)} \tag{4-71}$$

Since $H(f)$ is such an important system function, we illustrate its computation by the following example.

EXAMPLE 4-16

To obtain the transfer function of the *RC* filter shown in Figure 4-10, any of the methods illustrated in Figure 4-10 may be used. The first method (Fig. 4-10a) involves Fourier transformation of the governing differential equation (integrodifferential equation, in general), which for this system is

$$RC\frac{dy}{dt} + y(t) = x(t), \qquad -\infty < t < \infty \tag{4-72}$$

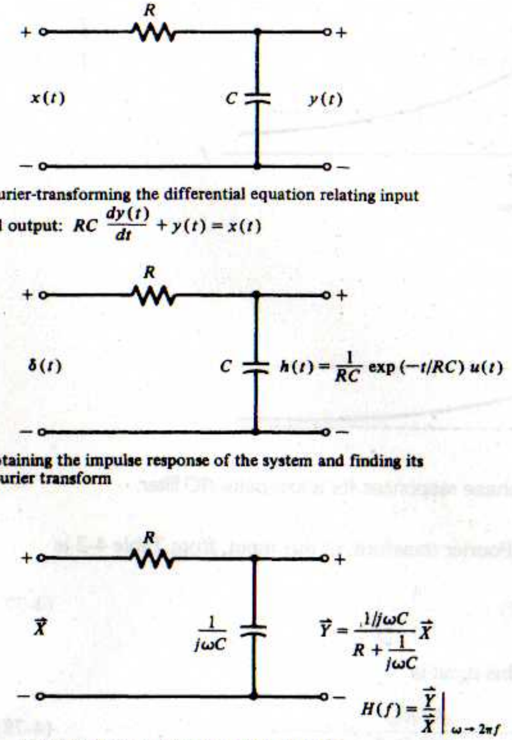
Assuming that the Fourier transform of each term exists, we may apply the superposition and differentiation theorems to obtain

$$(j2\pi fRC + 1)Y(f) = X(f) \tag{4-73}$$

or

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1 + j(f/f_3)} = \frac{1}{\sqrt{1 + (f/f_3)^2}}\,e^{-j\tan^{-1}(f/f_3)} \tag{4-74}$$

†It is useful to model some fixed, linear systems as having complex impulse responses. In such cases (4-70) and (4-71) do not hold.



(a) Fourier-transforming the differential equation relating input and output: $RC\frac{dy(t)}{dt} + y(t) = x(t)$

(b) Obtaining the impulse response of the system and finding its Fourier transform

(c) Using ac sinusoidal steady-state analysis

FIGURE 4-10. Illustration of methods that can be used to obtain the transfer function of a system.

The frequency $f_3 = 1/2\pi RC$ is called the 3-dB or half-power frequency.† Equivalently, we can use Laplace transform theory as discussed in Chapter 5 and replace s by $j2\pi f$.

A second method is to obtain the impulse response and Fourier transform it to obtain $H(f)$. This method is illustrated in Figure 4-10b.

A third alternative is to use ac sinusoidal steady-state analysis as shown in Figure 4-10c and find the ratio of output to input phasors, $\bar{Y}/\bar{X}$.

The amplitude and phase responses of this system are obtained from (4-74). They are

$$|H(f)| = \left[1 + \left(\frac{f}{f_3}\right)^2\right]^{-1/2} \quad \text{and} \quad \underline{/H(f)} = -\tan^{-1}\frac{f}{f_3} \tag{4-75}$$

respectively. They are illustrated in Figure 4-11.

To end this example, we find the response of the system to the input

$$x(t) = Ae^{-\alpha t}u(t), \qquad \alpha > 0 \tag{4-76}$$

†The half-power frequency of a two-port system is defined as the frequency of an input sine wave which results in a steady-state sinusoidal output of amplitude $1/\sqrt{2}$ of the maximum possible amplitude of the output sinusoid. The power of the output sinusoid is then one-half of the maximum possible output power. The ratio of actual to maximum powers in decibels (dB) is therefore (P_{out}/P_{max}) dB $= 10\log_{10}\frac{1}{2} = -3$ dB.

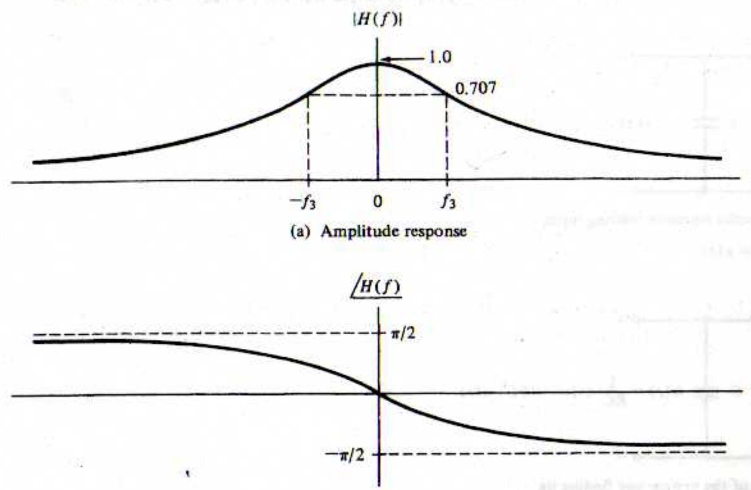


FIGURE 4-11. Amplitude and phase responses for a low-pass *RC* filter.

by using Fourier transform techniques. The Fourier transform of this input, from Table 4-2 is

$$X(f) = \frac{A}{\alpha + j2\pi f} \tag{4-77}$$

The Fourier transform of the output due to this input is

$$Y(f) = \frac{A/RC}{(\alpha + j2\pi f)(1/RC + j2\pi f)} \tag{4-78}$$

For $\alpha RC \neq 1$, this can be written as[†]

$$Y(f) = \frac{A}{\alpha RC - 1}\left[\frac{1}{1/RC + j2\pi f} - \frac{1}{\alpha + j2\pi f}\right] \tag{4-79}$$

The inverse Fourier transform of $Y(f)$ can be found with the aid of Table 4-2 and the superposition theorem. It is given by

$$y(t) = \frac{A}{\alpha RC - 1}\left[\exp\left(\frac{-t}{RC}\right) - \exp(-\alpha t)\right]u(t) \tag{4-80}$$

We can find the response for $\alpha RC = 1$ by using pair 5 of Table 4-2 or by taking the limit of the foregoing result as $\alpha RC \to 1$. In either case, the result is

$$y(t) = A\left(\frac{t}{RC}\right)\exp\left(\frac{-t}{RC}\right)u(t), \qquad \alpha = \frac{1}{RC} \tag{4-81}$$

The student should plot $y(t)$ for various combinations of α and RC and consider what their relationship must be for the output to resemble the input except for a scale change.

[†]Ways of obtaining partial-fraction expansions such as this one will be reviewed in Chapter 5.

As an example of how system characteristics influence the response of the system to a given input, we consider the *RC*-circuit response to a rectangular-pulse input.

EXAMPLE 4-17

Again, we consider the system of Figure 4-10 but with the input

$$x(t) = A\Pi\left(\frac{t - T/2}{T}\right) = A[u(t) - u(t - T)] \tag{4-82}$$

Because the inverse Fourier transform relationship (4-68) is difficult to evaluate in this case, the output is found using time-domain techniques. From Example 2-16 the step response is

$$a_s(t) = (1 - e^{-t/RC})u(t) \tag{4-83}$$

Noting that the $x(t)$ consists of the difference of two steps and using superposition, we find the output to be

$$y(t) = \begin{cases} 0, & t < 0 \\ A(1 - e^{-t/RC}), & 0 \leq t \leq T \\ A[e^{-(t-T)/RC} - e^{-t/RC}], & t > T \end{cases} \tag{4-84}$$

This result is plotted in Figure 4-12 for several values of T/RC together with $|X(f)|$ and $|Y(f)|$. The parameter $T/RC = 2\pi f_3/T^{-1}$ is proportional to the ratio of the 3-dB frequency of the filter to the spectral width (T^{-1}) of the pulse. When the filter bandwidth is large compared with the spectral width of the input pulse, the input is essentially passed undistorted by the system. On the other hand, for $2\pi f_3/T^{-1} << 1$, the system distorts the input signal spectrum and the output does not resemble the input.

Since the energy spectral density of a signal is proportional to the magnitude of its Fourier transform squared, it follows that

$$G_y(f) = |H(f)|^2 G_x(f) \tag{4-85}$$

where $G_x(f)$ and $G_y(f)$ are the energy spectral densities of the system input and output, respectively. Note that the energy spectral density is an even function of frequency. The total energy in the output can be obtained by integrating (4-85) over all frequency. Since $G_y(f)$ is an even function of frequency, the integral can be carried out from $f = 0$ to $f = \infty$ and then doubling this result to get total energy. The energy within any bandwidth of frequencies can be obtained by integrating over that bandwidth. Once again, if the integration is over only positive frequencies, the result of the integration must be doubled.

4-7 Steady-State System Response to Sinusoidal Inputs by Means of the Fourier Transform

In Section 2-7 the frequency response function of a fixed, linear system was defined in terms of the attenuation and phase shift suffered by a sinusoid in passing through the system. We may use (4-67) to

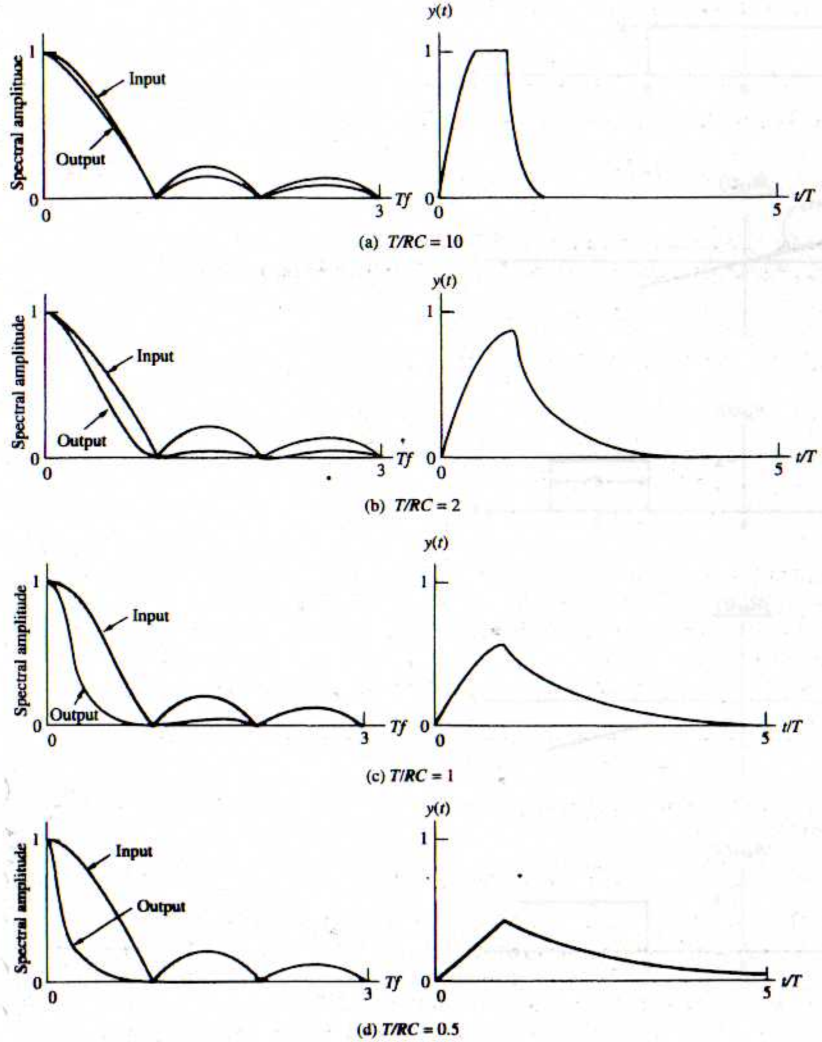


FIGURE 4-12. Input and output signals and spectra for a low-pass *RC* filter (only right side of spectra shown due to evenness).

relate the Fourier transforms of the input and output of a fixed, linear system in response to a periodic waveform. In particular, a periodic input signal $x(t)$ can be represented by its Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n\omega_0 t} \tag{4-86a}$$

Using the transform pair $e^{j2\pi n f_0 t} \leftrightarrow \delta(f - nf_0)$, this can be Fourier-transformed term-by-term to yield

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0) \tag{4-86b}$$

When (4-86b) is multiplied by the system transfer function $H(f)$, and use is made of the relationship $H(f)\delta(f - nf_0) = H(nf_0)\delta(f - nf_0)$, we may express the Fourier transform of the output as

$$Y(f) = \sum_{n=-\infty}^{\infty} X_n H(nf_0)\delta(f - nf_0) \tag{4-87}$$

Writing X_n and $H(nf_0)$ in terms of magnitude and phase as $|X_n| \exp j\underline{/X_n}$ and $|H(nf_0)| \exp j\underline{/H(nf_0)}$, respectively, (4-87) may be inverse-Fourier-transformed term-by-term to give the output

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n|\,|H(nf_0)| \exp\left[j\left\{2\pi n f_0 t + \underline{/X_n} + \underline{/H(nf_0)}\right\}\right] \tag{4-88}$$

To emphasize the significance of (4-88), note that the nth spectral component of the input, X_n, appears at the output with amplitude attenuated (or amplified) by the amplitude-response function $|H(nf_0)|$, and a phase which is the input phase-shifted by the system phase response $\underline{/H(nf_0)}$, both of which are evaluated *at the frequency of the particular spectral component* under consideration. Note that (4-88) is equivalent to (3-71) or (3-73).

EXAMPLE 4-18

Consider a system with amplitude- and phase-response functions given by

$$|H(f)| = K\Pi\left(\frac{f}{2B}\right) = \begin{cases} K, & |f| \le B \\ 0, & \text{otherwise} \end{cases} \tag{4-89a}$$

and

$$\underline{/H(f)} = -2\pi t_0 f \tag{4-89b}$$

respectively. A filter with this transfer function is referred to as an *ideal low-pass filter.*

Its output in response to $x(t) = A\cos(2\pi f_0 t + \theta_0)$ is obtained from (4-88) by noting that

$$X_1 = \tfrac{1}{2}Ae^{j\theta_0} = X_{-1}^* \tag{4-90}$$

with all other X_n's = 0. Thus the output is

$$y(t) = \begin{cases} 0, & f_0 > B \\ KA\cos[2\pi f_0(t - t_0) + \theta_0], & f_0 \le B \end{cases} \tag{4-91}$$

Thus an ideal low-pass filter completely rejects all spectral components with frequencies greater than some cutoff frequency B, and passes all input spectral components below this cutoff frequency except that their amplitudes are multiplied by a constant K and they are phase-shifted by an amount $-2\pi t_0 f_0$ or delayed in time by t_0.

Because ideal filters are used quite often in systems analysis, we consider the characteristics of three types of ideal filters in more detail in the following section.

4-8 Ideal Filters

It is often convenient to work with idealized filters having amplitude-response functions which are constant within the passband† and zero elsewhere. In general, three types of ideal filters, referred to as low-pass, high-pass, and bandpass filters, can be defined. A phase-shift function that is a linear function of frequency throughout the passband is assumed in each case. Figure 4-13 illustrates the frequency-response functions for each type of ideal filter.

The impulse response of any filter can be found by obtaining the inverse Fourier transform of its transfer function. For example, for the ideal low-pass filter, the impulse response can be found from‡

$$h_{LP}(t) = \int_{-\infty}^{\infty} H_{LP}(f)e^{j2\pi ft}\,df$$
$$= \int_{-B}^{B} Ke^{-j2\pi ft_0}e^{j2\pi ft}\,df$$
$$= \int_{-B}^{B} Ke^{j2\pi f(t-t_0)}\,df$$
$$= 2BK\,\mathrm{sinc}[2B(t-t_0)] \qquad (4\text{-}92)$$

Since $\mathrm{sinc}[2B(t-t_0)]$ is nonzero for all t, except when $2B(t-t_0)$ takes on an integer value, it follows that $h_{LP}(t)$ is nonzero for $t < 0$. In other words, an ideal low-pass filter is *noncausal*, as are all types of ideal filters.

In fact, it can be shown that the ideal bandpass filter has impulse response

$$h_{BP}(t) = 2KB\,\mathrm{sinc}[B(t-t_0)]\cos[2\pi f_0(t-t_0)] \qquad (4\text{-}93)$$

and that the ideal high-pass filter has impulse response

$$h_{HP}(t) = K\delta(t-t_0) - 2BK\,\mathrm{sinc}[2B(t-t_0)] \qquad (4\text{-}94)$$

Both of these impulse responses are nonzero for $t < 0$ if t_0 is finite. Figure 4-14 illustrates ideal filter impulse responses. The noncausal nature of ideal filters is a consequence of their ideal attenuation characteristics in going from the passband to the stop band.

Several methods of approximating ideal filter frequency-response characteristics by means of causal filters are discussed in Appendix E. In general, the more closely a causal filter approximates a corresponding ideal filter, the more delay a signal suffers in passing through it.

4-9 Bandwidth and Rise Time

The rise time of a pulse is the amount of time that it takes in going from a prespecified minimum value, say 10% of the final value of the pulse, to a prespecified maximum value, say 90% of the final value of the pulse. For example, whereas the square pulse input to a low-pass RC filter has zero rise time, the

†The passband of a filter is defined as the frequency range where its amplitude response is greater than some arbitrarily chosen minimum value. Quite often, this minimum value is chosen as $1/\sqrt{2}$ of the maximum amplitude response.

‡B denotes filter bandwidth, and W signal bandwidth.

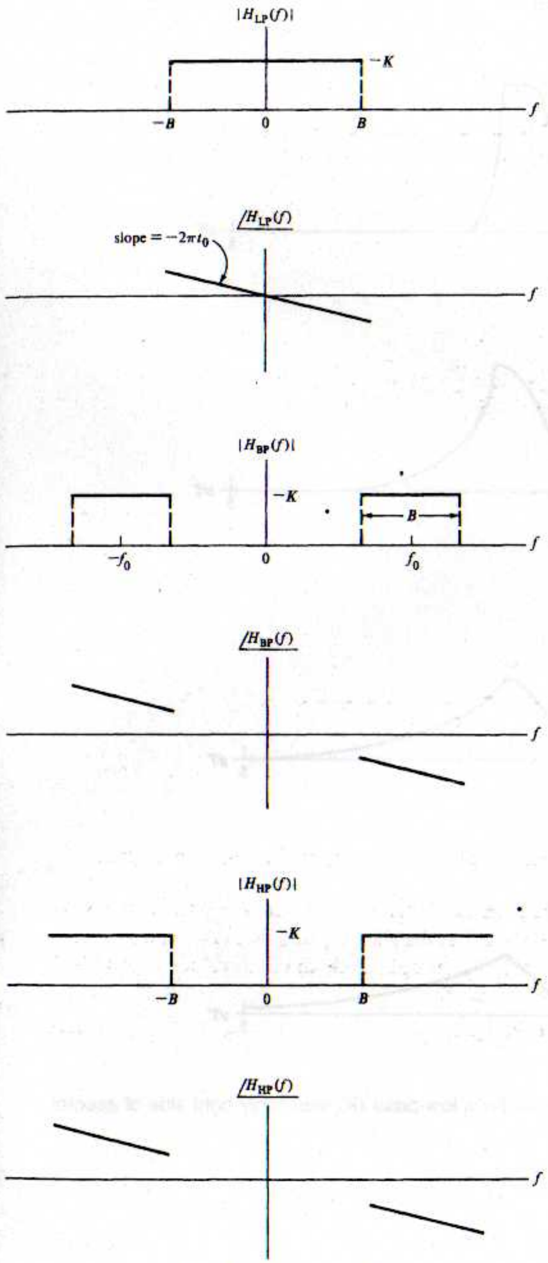


FIGURE 4-13. Ideal filter amplitude- and phase-response functions.

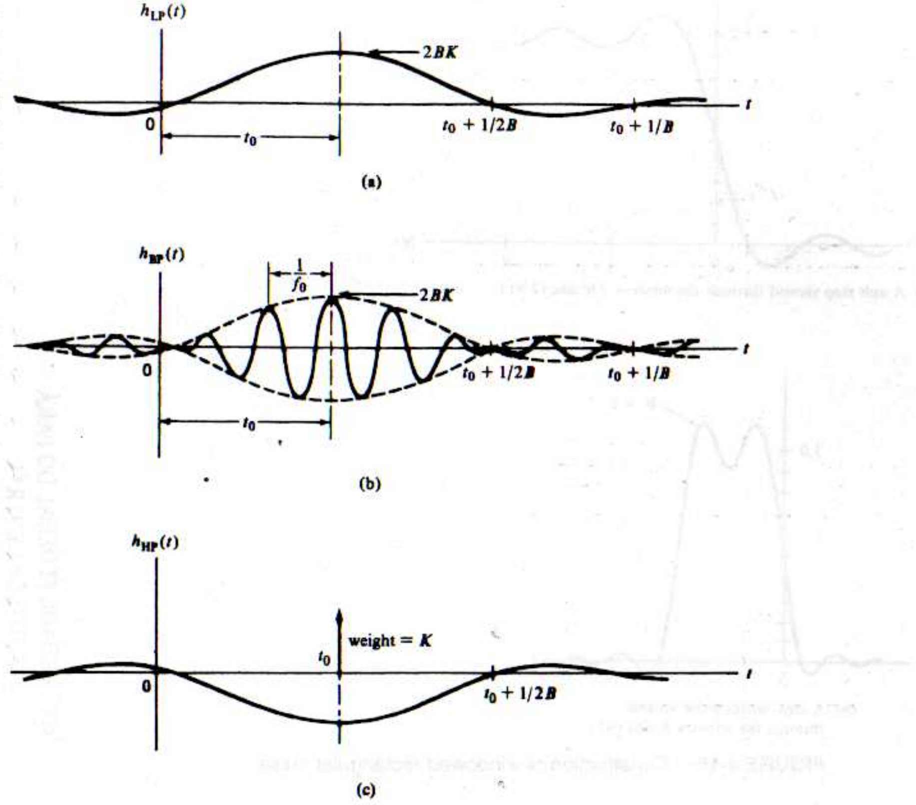


FIGURE 4-14. Impulse responses for ideal filters: (a) Low-pass; (b) bandpass; (c) high-pass.

filter output has a finite rise time as shown by the plots in Figure 4-12. The square-pulse response of this filter is given analytically by (4-84), where the width of the square-pulse input is T and its amplitude is A. The final value of the output pulse is also A. The prespecified minimum value of $0.1A$ (10% of the final value of the output pulse) occurs at time t_m, where

$$A(1 - e^{-t_m/RC}) = 0.1A \tag{4-95a}$$

or

$$\frac{t_m}{RC} = -\ln(0.9)$$

The prespecified maximum value of $0.9A$ (90% of the final value) occurs at time t_M, given by

$$A(1 - e^{-t_M/RC}) = 0.9A \tag{4-95b}$$

or

$$\frac{t_M}{RC} = -\ln(0.1)$$

The rise time is

$$T_R = t_M - t_m = RC[-\ln(0.1) + \ln(0.9)] \tag{4-95c}$$

In terms of the 3-dB bandwidth of the RC filter, which is $f_3 = (2\pi RC)^{-1}$, this becomes

$$T_R = \frac{-\ln(0.1) + \ln(0.9)}{2\pi f_3} = \frac{0.35}{f_3} \tag{4-96}$$

That is, for the lowpass filter the rise time is inversely proportional to the bandwidth. This inverse relationship is illustrated by Example 4-17 and, in particular, Figure 4-12. Although shown only for the particular case of a low-pass RC filter, this inverse relationship between rise time and bandwidth is true in general. In closing this section, note that the 10%–90% definition of rise time used here is only one of the many possible. No matter what the definition of rise time, the inverse relationship between rise time and bandwidth still holds.

*4-10 Window Functions and the Gibbs Phenomenon

In our consideration of trigonometric series at the beginning of Chapter 3, it was noted that the sum of such a series tended to overshoot the signal being approximated at a discontinuity. We can easily examine the reason for this behavior, referred to as the *Gibbs phenomenon,* with the aid of the Fourier transform. Consider a signal $x(t)$ with Fourier transform $X(f)$. We consider the effect of reconstructing $x(t)$ from only the low-pass part of its frequency spectrum. That is, we approximate the signal by

$$\tilde{x}(t) = \mathcal{F}^{-1}\left[X(f)\Pi\left(\frac{f}{2W}\right)\right] \tag{4-97}$$

where

$$\Pi\left(\frac{f}{2W}\right) = \begin{cases} 1, & |f| \le W \\ 0, & \text{otherwise} \end{cases} \tag{4-98}$$

According to the convolution theorem of Fourier transform theory,

$$\begin{aligned}\tilde{x}(t) &= x(t) * \mathcal{F}^{-1}\left[\Pi\left(\frac{f}{2W}\right)\right] \\ &= x(t) * (2W \operatorname{sinc} 2Wt)\end{aligned} \tag{4-99}$$

where the last step follows by virtue of the transform pair developed in Example 4-7. Recalling that convolution is a folding-product, sliding-integration process, it is now apparent why the overshoot phenomenon occurs at a discontinuity. Figure 4-15 illustrates (4-99) for $W = 2$ assuming that $x(t)$ is a unit-square pulse. In accordance with (4-99), as W increases, more and more of the frequency content of the rectangular pulse is used to obtain the approximation $\tilde{x}(t)$. Nevertheless, we see that a finite value of W means that the signal $x(t)$ is viewed through the *window function* $2W \operatorname{sinc} 2Wt$ in the time domain.† As a result, anticipatory wiggles are present in $\tilde{x}(t)$ before the onset of the main pulse, and echoing wiggles remain after the discontinuity of the leading edge of $x(t)$ passes.

To put this discussion on a mathematical basis, we represent the square-pulse signal as

$$x(t) = u(t) - u(t-1) \tag{4-100}$$

†Actually, it is often customary to refer to the multiplying function $\Pi(f/2W)$ as the window function.

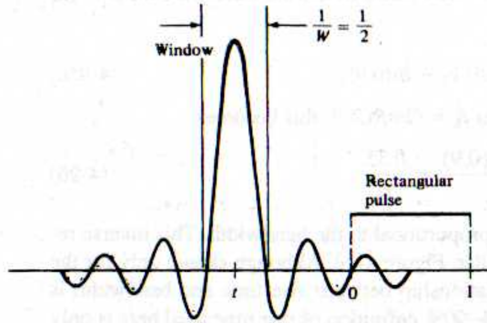


FIGURE 4-15. Convolution of the window function $2W \operatorname{sinc} 2Wt$ with a rectangular pulse.

which results in

$$\tilde{x}(t) = [u(t) - u(t-1)] * 2W \operatorname{sinc} 2Wt \tag{4-101}$$

Considering, for the moment, only the first term of (4-101), we write it as

$$\begin{aligned}\tilde{u}(t) &= u(t) * 2W \operatorname{sinc} 2Wt = 2W \operatorname{sinc} 2Wt * u(t)\\ &= 2W \int_{-\infty}^{\infty} \operatorname{sinc} 2W\lambda\, u(t-\lambda)\, d\lambda\\ &= 2W \int_{-\infty}^{t} \operatorname{sinc} 2W\lambda\, d\lambda\end{aligned} \tag{4-102}$$

This integral cannot be evaluated in closed form but can be expressed in terms of the sine-integral function, $\operatorname{Si}(x)$, defined as

$$\operatorname{Si}(x) = \int_0^x \frac{\sin u}{u}\, du \tag{4-103}$$

which is available in tabular form.† It is useful to note that $\operatorname{Si}(x)$ is an even function and that $\operatorname{Si}(\infty) = \pi/2$. In terms of $\operatorname{Si}(x)$, $\tilde{u}(t)$ can be expressed as

$$\tilde{u}(t) = \begin{cases} \frac{1}{2} + \frac{1}{\pi}\operatorname{Si}(2\pi Wt), & t > 0 \\ \frac{1}{2} - \frac{1}{\pi}\operatorname{Si}(2\pi Wt), & t < 0 \end{cases} \tag{4-104}$$

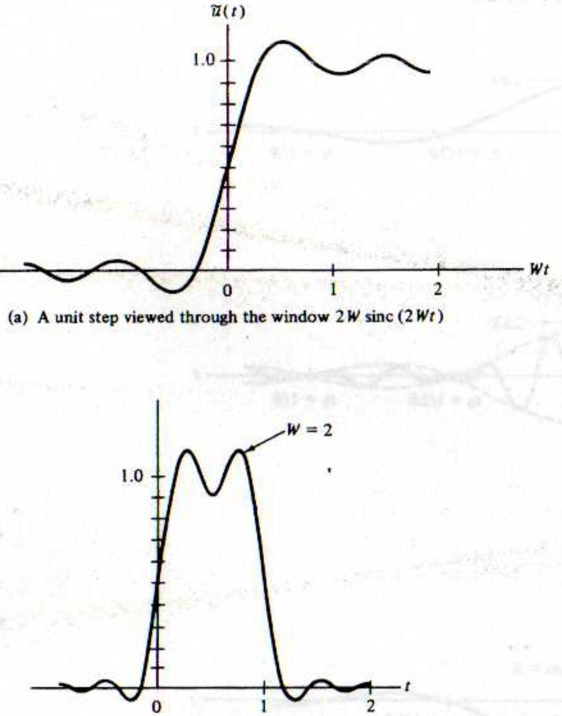
A plot of $\tilde{u}(t)$ versus Wt is shown in Figure 4-16a.

From (4-101) it follows that we may obtain $\tilde{x}(t)$ from $\tilde{x}(t)$ as

$$\tilde{x}(t) = \tilde{u}(t) - \tilde{u}(t-1) \tag{4-105}$$

an operation that is easily carried out graphically; $\tilde{x}(t)$ is shown in Figures 4-16b and c for $W = 2$ and $W = 5$. The approximation of $\tilde{x}(t)$ to $x(t)$ becomes better as W increases relative to the pulse width,

†M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions with Formulas, Tables, and Graphs* (New York: Dover, 1972), p. 243.



(a) A unit step viewed through the window $2W \operatorname{sinc}(2Wt)$

(b) A unit-width pulse viewed through the window $4 \operatorname{sinc}(4t)$

FIGURE 4-16. Construction of windowed rectangular pulse.

except for the overshoot at the discontinuities at $t = 0$ and $t = 1$, which eventually approaches a value of 9% of the pulse height as W becomes large.

The question of what can be done to combat this behavior is answered by considering the possibility of reducing the sidelobes of the $2W \operatorname{sinc} 2Wt$ window function, which was the inverse Fourier transform of $\Pi(f/2W)$. The function

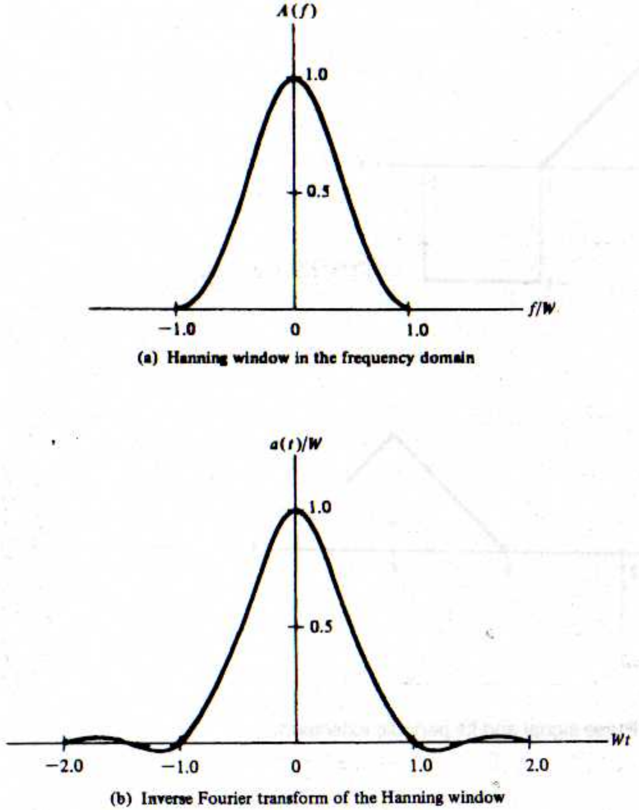
$$A(f) = \begin{cases} 0.5 + 0.5\cos\frac{\pi f}{W}, & |f| \le W \\ 0, & \text{otherwise} \end{cases} \tag{4-106}$$

provides a smooth transition from zero at $f = \pm W$ to its maximum value of unity at $f = 0$.† Its inverse Fourier transform is

$$a(t) = W[\operatorname{sinc} 2Wt + \tfrac{1}{2}\operatorname{sinc}(2Wt - 1) + \tfrac{1}{2}\operatorname{sinc}(2Wt + 1)] \tag{4-107}$$

This function and its transform are shown in Figure 4-17, where it is seen that its sidelobes are much lower than the function $2W \operatorname{sinc} 2Wt$.

†This function is called a *Hanning window.*



(a) Hanning window in the frequency domain

(b) Inverse Fourier transform of the Hanning window

FIGURE 4-17. Window with smoother transitions than the rectangular window.

Note that the inverse relationship between rise time and bandwidth discussed in the previous section is illustrated by Figure 4-16a. The waveform can be viewed as the response of an ideal low-pass filter to a unit step input. From this figure it is seen that the 10%–90% rise time is approximately $T_R = 0.5/W$. In this case W appears as the filter bandwidth rather than B because of the notation used in (4-97), from which we considered building up a square-pulse signal from a finite range of its frequency spectrum.

EXAMPLE 4-19

To study the effect of window half width, W, the following MATLAB program cycles through four windows widths for both rectangular and Hanning windows. It produces the plots shown in Figure 4-18.

```
%    c4ex19
%
t_max=2;
t=-t_max+1:.01:t_max;
L=length(t);
for k=1:4                                  % Loop to do 4 values of window width
k_even=2*k;
```



```
k_odd=2*k-1;
W=2+(k-1);
tp=[2*t(1):.01:2*t(L)];
w_r=2*W*sinc(2*W*t);                       % Inverse transform of rectangular
                                           % window
w_h=W*(sinc(2*W*t)+.5*sinc(2*W*t-1)+.5%sinc(2*W*t+1));  % Hanning window
x=pls_fn(t-.5);
x_tilde_r=.01*conv(w_r,x);                 % Convolve rectangular window with
                                           % pulse
x_tilde_h=.01*conv(w_h,x);                 % Convolve Hanning window with pulse
subplot(4,2,k_odd),plot(tp, x_tilde_r),xlabel('t'),
ylabel('wind.x(t)'),...
text(-.9,1,'rectan.'),text(-.9,.5,['W = ',num2str(W)]),
axis([-t_max+1 t_max -.2 1.5])
subplot(4,2,k_even),plot(tp, x_tilde_h),xlabel('t'),
ylabel('wind. x(t)'),...
text(-.9,1,'Hann.'),text(-.9,.5,['W = ',num2str(W)]),
axis([-t_max+1 t_max -.2 1.5])
end
```

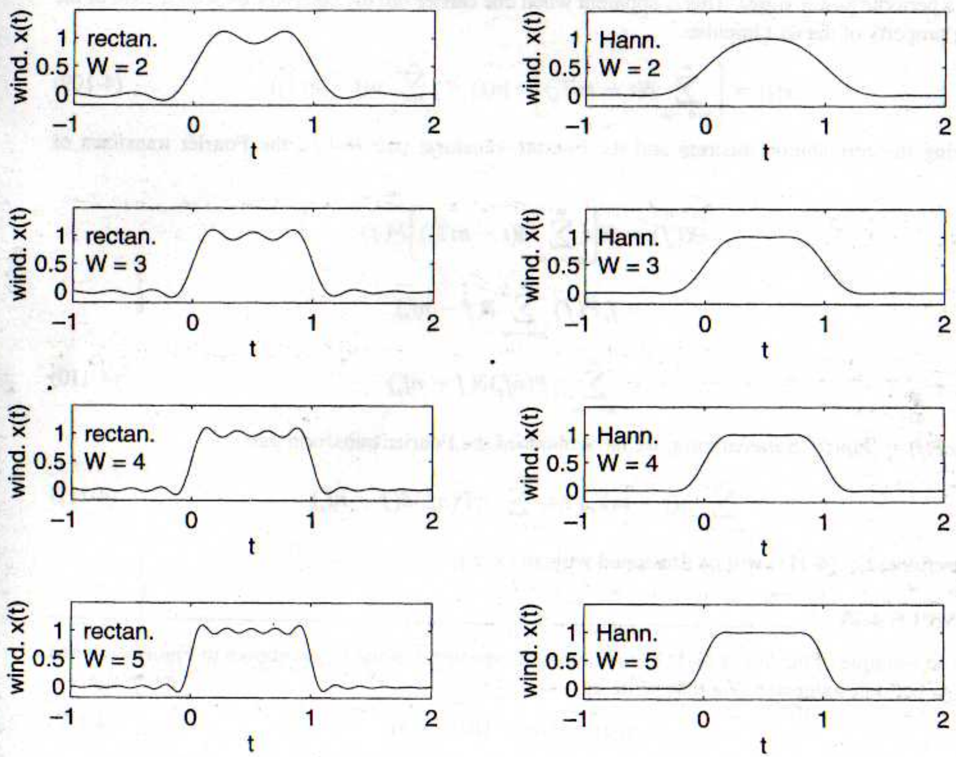


FIGURE 4-18. Effect of windowing the spectrum of a rectangular pulse with rectangular (left) and Hanning (right) frequency windows for four windows half widths, W.

*4-11 Fourier Transforms of Periodic Signals

The Fourier transform of a periodic signal, in a strict mathematical sense, does not exist since periodic signals are not Fourier transformable. However, using the transform pairs derived in Example 4-9 for a constant and a phasor signal we could, in a formal sense, write down the Fourier transform of a periodic signal by Fourier-transforming its complex Fourier series term-by-term. Thus, for a periodic signal $x(t)$ with Fourier series $\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$, where $T_0 = 2\pi/\omega_0 = 1/f_0$ is the period, we have the transform pair

$$\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \leftrightarrow \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0) \qquad (4\text{-}108)$$

Either representation in (4-108) contains the same information about $x(t)$, and either result can be used to plot the two-sided spectra of a signal. If the Fourier transform representation is used [right-hand side of (4-108)], the amplitude spectrum consists of impulses rather than lines.

A somewhat more useful form for the Fourier transform of a periodic signal than (4-108) is obtained by applying the convolution theorem and pair 17 of Table 4-2 for the ideal sampling wave. To obtain it, consider the result of convolving the ideal sampling waveform with a pulse-type signal $p(t)$ to obtain a new signal $x(t)$. If $p(t)$ is an energy signal of limited time extent such that $p(t) = 0$, $|t| \geq T/2 \leq T_s/2$, then $x(t)$ is a periodic power signal. This is apparent when one carries out the convolution with the aid of the sifting property of the unit impulse:

$$x(t) = \left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right] * p(t) = \sum_{m=-\infty}^{\infty} p(t - mT_s) \qquad (4\text{-}109)$$

Applying the convolution theorem and the Fourier transform pair (4-42), the Fourier transform of $x(t)$ is

$$X(f) = \mathcal{F}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right] P(f)$$

$$= f_s P(f) \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$

$$= \sum_{n=-\infty}^{\infty} f_s P(nf_s) \delta(f - nf_s) \qquad (4\text{-}110)$$

where $P(f) = \mathcal{F}[p(t)]$. Summarizing, we have obtained the Fourier transform pair

$$\sum_{m=-\infty}^{\infty} p(t - mT_s) \leftrightarrow \sum_{n=-\infty}^{\infty} f_s P(nf_s) \delta(f - nf_s) \qquad (4\text{-}111)$$

The usefulness of (4-111) will be illustrated with an example.

EXAMPLE 4-20

As an example of the use of (4-111), we obtain the spectrum of the signal shown in Figure 4-19 and of its periodic extension. We may write $x_1(t)$ as

$$x_1(t) = \Lambda(t) - \Pi(t - 1.5) \qquad (4\text{-}112)$$

Using previously derived Fourier transform pairs for the triangle and the pulse, we obtain

$$X_1(f) = \text{sinc}^2 f - \text{sinc}\, f\, e^{-j3\pi f} \qquad (4\text{-}113)$$

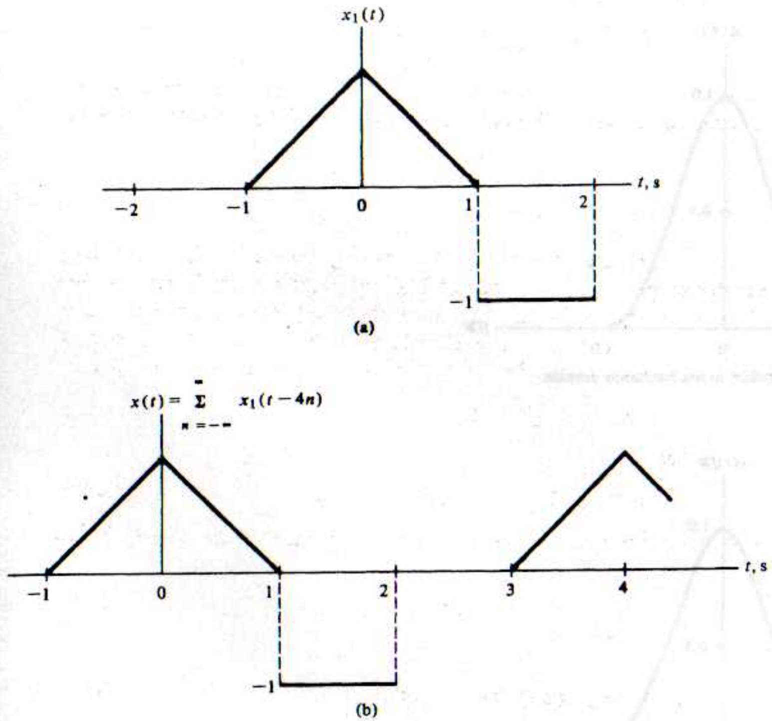


FIGURE 4-19. Pulse signal and its periodic extension.

Recalling that sinc $u = (\sin \pi u)/\pi u$, we see that the spectrum of $x_1(t)$ goes to zero as $1/f$. Checking Figure 4-16a, we see that one derivative is required to produce impulses from the square-pulse portion of $x_1(t)$, which implies the decrease as f^{-1} (see Section 3-11).

We proceed to find the Fourier series of the periodic waveform $x(t)$ by using (4-111). Since $x_1(t)$ is repeated every 4 s to produce $x(t)$, $f_s = 0.25$ in (4-111), and the Fourier transform of $x(t)$, using (4-113) in (4-111), is

$$X(f) = 0.25 \sum_{n=-\infty}^{\infty} [\text{sinc}^2\, 0.25n - \text{sinc}\, 0.25n\, e^{-j0.75\pi n}]\, \delta(f - 0.25n) \qquad (4\text{-}114)$$

Recalling that $e^{j2\pi f_0 t} \leftrightarrow \delta(f - f_0)$, we may inverse-Fourier-transform (4-114) to obtain the exponential Fourier series of $x(t)$. The result is

$$x(t) = 0.25 \sum_{n=-\infty}^{\infty} (\text{sinc}^2\, 0.25n - \text{sinc}\, 0.25n\, e^{-j0.75\pi n}) e^{j0.5\pi n t} \qquad (4\text{-}115)$$

Again, it is seen that the predominant term as $n \to \infty$ is sinc $0.25n \exp(-j0.75\pi n)$, which approaches zero as n^{-1}. The student should plot the amplitude and phase spectra of this signal.

4-12 Applications of the Hilbert Transform

The Hilbert transform, $\hat{x}(t)$ of a signal $x(t)$, was introduced in Example 4-14, and its Fourier transform in terms of $x(t)$ was given as pair 16 of Table 4-2. We discuss two uses of the Hilbert transform in this section.

Analytic Signals

An analytic signal $z(t)$ is a complex-valued signal whose spectrum is single-sided (i.e., is nonzero only for $f > 0$ or $f < 0$). Because of this property of its spectrum, it follows that the real and imaginary parts of an analytic signal cannot be specified independently. It turns out that one is the Hilbert transform of the other. To show this, suppose that $x(t)$ and $y(t)$ are the real and imaginary parts of $z(t)$, respectively. Let $Z(f)$, $X(f)$, and $Y(f)$ be the Fourier transforms of $z(t)$, $x(t)$, and $y(t)$, respectively. It follows that

$$Z(f) = X(f) + jY(f) \tag{4-116}$$

being zero for $f < 0$, say, requires that

$$Y(f) = jX(f), \quad f < 0 \tag{4-117}$$

[note that $X(f)$ and $Y(f)$ may be complex]. If we double the positive-frequency portion of $Z(f)$, it then follows that

$$Y(f) = -jX(f), \qquad f > 0 \tag{4-118}$$

so that

$$Y(f) = -j\,\mathrm{sgn}(f)\,X(f), \qquad \text{all } f \tag{4-119}$$

and

$$Z(f) = \begin{cases} 2X(f), & f > 0 \\ 0, & f < 0 \end{cases} \tag{4-120}$$

From (4-119) and pair 16 of Table 4-2, we see that

$$y(t) = \hat{x}(t) \qquad [Z(f) = 0, \quad f < 0] \tag{4-121}$$

where $\hat{x}(t)$ is the Hilbert transform of $x(t)$. Had we required that $Z(f)$ be zero for $f > 0$ and nonzero for $f < 0$, it follows that the signs in front of the j in (4-117) and (4-118) would have been reversed, so that

$$y(t) = -\hat{x}(t) \qquad [Z(f) = 0, \quad f > 0] \tag{4-122}$$

Analytic signals are of use in modulation theory applications, and in particular to describe single-side-band modulated signals mathematically.

Causality

A causal system is defined as one whose output does not anticipate the input. If a causal system is also linear *and* fixed, so that it can be characterized by an impulse response, $h(t)$, the causality of the system requires that

$$h(t) = 0, \qquad t < 0 \tag{4-123}$$

The Fourier transform of $h(t)$ is the transfer function $H(f)$ of the system and, from the discussion under analytic signals, it should be apparent that the real and imaginary parts of $H(f)$ cannot be independently specified, but are in fact Hilbert transforms of each other. To show this, consider $h(t)$ written in terms of its even and odd parts as

$$h(t) = h_e(t) + h_o(t) \tag{4-124}$$

where

$$h_e(t) = \tfrac{1}{2}[h(t) + h(-t)] \tag{4-125}$$

and

$$h_o(t) = \tfrac{1}{2}[h(t) - h(-t)] \tag{4-126}$$

In order that $h(t)$ be zero for $t < 0$, it follows that

$$h_o(t) = \begin{cases} h_e(t), & t > 0 \\ -h_e(t), & t < 0 \end{cases} = \mathrm{sgn}(t)\,h_e(t) \tag{4-127}$$

Substitution of (4-127) into (4-124) results in

$$h(t) = h_e(t) + \mathrm{sgn}(t)\,h_e(t) \tag{4-128}$$

which, when Fourier-transformed with the aid of pair 14 of Table 4-2 and the multiplication theorem, allows the transfer function to be written as

$$H(f) = H_e(f) + \frac{1}{j\pi f} * H_e(f) = H_e(f) - j\hat{H}_e(f) \tag{4-129}$$

where $\hat{H}_e(f)$ is the Hilbert transform of $H_e(f) = \mathcal{F}[h_e(t)]$.

4-13 The Discrete Fourier Transform

The Fourier transform of a continuous-time signal, $x(t)$, as developed in Chapter 4, is

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt \tag{4-7}$$

In this section we consider the approximation of (4-7) by a discrete sum, and illustrate fast MATLAB programs, called fast Fourier transform (FFT) algorithms to compute this discrete sum efficiently on a computer. In Chapter 10, several FFT algorithms will be developed mathematically. To this end, we assume that $x(t)$ is nonzero only over the finite time interval $[0, T]$ and is bandlimited to W hertz. We saw earlier by means of several examples that it is not possible for a signal to be both strictly time limited and bandlimited. However, from a practical standpoint, we can define a bandwidth beyond which any strictly time limited signal has negligible energy, and vice versa. It is in this context that we take $x(t)$ to be time limited to T seconds and bandlimited to W hertz. Since $x(t)$ is effectively both time limited and bandlimited, we may represent it, to a good approximation, in terms of samples of $x(t)$.* This yields

$$x_s(t) \approx \sum_{n=0}^{N-1} x(n\Delta t)\delta(t - n\Delta t) \tag{4-130}$$

*See Chapter 7 for a derivation of the Sampling Theorem.

where $\Delta t = T/N$, with N being the total number of samples taken in $[0, T]$. In keeping with the band-limited nature of $x(t)$, it follows that $\Delta t \le 1/2W$ in order to avoid significant aliasing. Substituting (4-130) into (4-7) and simplifying by interchanging orders of summation and integration, we find that

$$X_s(f) = \sum_{n=0}^{N-1} x(n\Delta t)e^{-j2\pi fn\,\Delta t} \tag{4-131}$$

This summation is the Fourier transform of the discrete-time signal represented by the sample values $\{x(n\Delta t)\}$ and is often expressed as a function of the variable $\omega = 2\pi f\,\Delta t = 2\pi r$, where r is the normalized frequency, f/f_s.

Since we are interested in digital computation of (4-131), we restrict f to the discrete set of values $\{0, 1/T, 2/T, \ldots, (N-1)/T\}$. Thus setting $f = k/T = k/(N\,\Delta t)$ in (4-131), where k takes on the discrete set of values $\{0, 1, 2, \ldots, N-1\}$, we obtain

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1 \tag{4-132}$$

The explicit dependence of $x(n\,\Delta t)$ on Δt has been dropped, and both $X(f)$ and $x(t)$ are now replaced by the sequences $\{X_k\}$ and $\{x_n\}$, respectively. This is defined to be the *discrete Fourier transform* of the sequence $\{x_0 = x(0), x_1 = x(\Delta t), \ldots, x_{N-1} = x[(N-1)\,\Delta t]\}$. Because it was derived using a sampling approach, it should be clear that the sequence $\{X_k\}$ is periodic with period N. By replacing k by $k+N$ in (4-132), we see that this is indeed the case.

The original time-domain sequence $\{X_n\}$ is obtained from the sequence of frequency-domain samples $\{X_k\}$ by the inverse relationship

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}, \qquad n = 0, 1, \ldots, N-1 \tag{4-133}$$

Thus (4-133) defines the *inverse DFT* operation. We must show that (4-133) with (4-132) substituted produces an identity. Making this substitution, using l for the summation index in (4-132), and summing over k first, we obtain

$$x_n \overset{?}{=} \sum_{l=0}^{N-1} \frac{1}{N} x_l \sum_{k=0}^{N-1} e^{j(2\pi k/N)(n-l)} \tag{4-134}$$

where the $\overset{?}{=}$ means that the equality is being tested.

The inside sum over k can be written as

$$\sum_k \triangleq \sum_{k=0}^{N-1} [e^{j2\pi(n-l)/N}]^k \tag{4-135}$$

Now the sum of a geometric series is

$$S_N \triangleq \sum_{k=0}^{N-1} x^k = \frac{1-x^N}{1-x} \tag{4-136}$$

Letting $x = \exp[j2\pi(n-l)/N]$, we find that

$$\sum_N = \frac{1 - \exp[j2\pi(n-l)]}{1 - \exp[j2\pi(n-l)/N]} = 0, \qquad n \ne l \tag{4-137}$$

which follows because $\exp[j2\pi(n-l)] = 1$. If $n = l$, the result reduces to the indeterminate form 0/0, so we must sum (4-135) directly for this special case. With $k = l$, the exponential in (4-135) is unity and we obtain

$$\sum_N = N, \qquad n = l \tag{4-138}$$

Summarizing, we have shown that

$$\sum_N = \sum_{k=0}^{N-1} e^{j(2\pi k/N)(n-l)} = \begin{Bmatrix} N, & n = l \\ 0, & n \ne l \end{Bmatrix} \triangleq N\delta_{nl} \tag{4-139}$$

where $\delta_{nl} = 1, n = 1$, and $\delta_{kl} = 0, n \ne 1$, is called the *Kronecker delta function*. Thus (4-134) becomes

$$x_n \overset{?}{=} \frac{1}{N}\sum_{l=0}^{N-1} Nx_l\delta_{nl} = x_n \tag{4-140}$$

which shows that (4-132) and (4-133) are indeed inverse operations and therefore constitute a valid transform pair.

Before finding a fast computational technique for evaluating (4-132) and (4-133), we note that they are really equivalent operations. From (4-133) it follows that the complex conjugates of the time samples, x_n^*, are given by

$$x_n^* = \frac{1}{N}\sum_{k=0}^{N-1} X_k^* e^{-j(2\pi/N)nk}, \qquad n = 0, 1, \ldots, N-1 \tag{4-141}$$

so that

$$Nx_n^* = \sum_{k=0}^{N-1} X_k^* e^{-j(2\pi/N)nk}, \qquad n = 0, 1, \ldots, N-1 \tag{4-142}$$

Comparison of (4-142) and (4-132) show that algorithms used to calculate the forward transform can also be used for the calculation of the inverse transform if the frequency samples X_k are conjugated prior to performing the computation. Since the result yields Nx_n^* rather than x_n, it must be conjugated and divided by N. Often, however, the time samples are real and the conjugation of the result can be omitted. Thus once we have developed a computationally efficient algorithm for calculating the sum given by (4-132), we can use the same algorithm for computing the inverse DFT.

EXAMPLE 4-21

To illustrate the FFT program in MATLAB consider the Fourier transformation of square pulses of various widths. The following MATLAB program does this for three different pulse widths. After FFT of a pulse, the inverse FFT is taken and this is also plotted to show that the FFT and inverse FFT are indeed inverse operations. The output of the program is given in Figure 4-20. Note that the shorter pulses have the higher frequency content and more overlap takes place in the middle of the spectral plot (high frequencies). Also note that the FFTs are not symmetric about 0, but the normally negative frequency portion is on the right of the plot.

```
%   Example 4-21 - illustration of the FFT and inverse FFT
%   by taking the FFT of a square pulse of various widths
%
T=4;
del_t=.2;
t=0:del_t:T;
L_t=length(t);
```

```
del_f=1/T:
f_max=(L_t-1)*del_f:
f=0:del_f:f_max;
for k=1:3
width=*0.6;
k_1=3*(k-1)+1;
k_2=3*k-1:
k_3=3*k;
x=pls_fn((t - width/2)/width):
X=fft(x):
X_inv=ifft(X):
subplot(3.3.k_1).stem(t.x).axis([0 T 0 1])....
      xlabel('t').ylabel('x(t)')
subplot(3.3.k_2).stem(f.abs(X)).axis([0 f_max 0 10])....
      xlabel('f').ylabel('X(f)')
subplot(3.3.k_3).stem(t.abs(X_inv)).axis([0 T 0 1])....
      xlabel('t').ylabel('X_inv(t)')
end
```

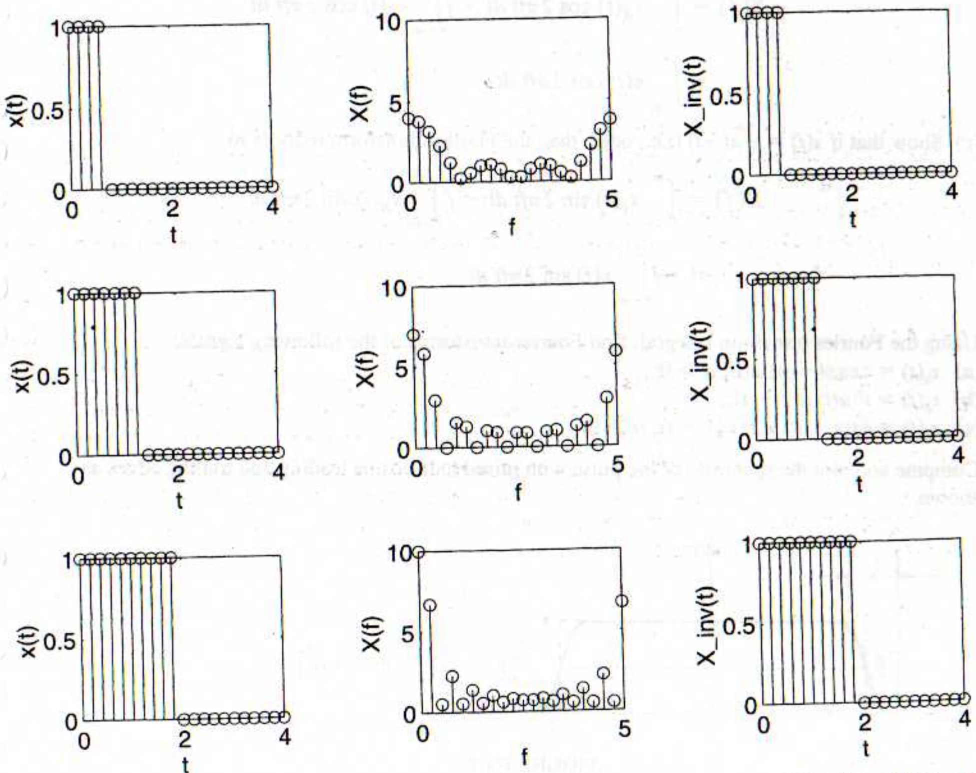


FIGURE 4-20. FFTs (middle column) of square pulses (left column) and inverse FFT of the FFT (right column).

Summary

In this chapter we have considered the representation of signals in terms of the *Fourier transform.* One class of signals to which the Fourier transform representation applies directly is energy signals. Power signals can also be represented in terms of Fourier transforms, but their spectra contain impulses representing finite power (infinite energy) at discrete frequencies. Fourier transforms of energy signals contain no impulsive components. The following are the major points made in this chapter.

1. A signal $x(t)$ and its *Fourier transform* $X(f)$ are related by

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

The former is the inverse relationship, and the latter is the direct relationship.

2. It is useful to represent the Fourier transform of a signal in terms of magnitude and argument or phase as

$$X(f) = |X(f)|e^{j\theta(f)}$$

where $|X(f)|$ plotted versus frequency is known as the *amplitude spectrum,* and $\theta(f)$ plotted versus frequency is called the *phase spectrum.*

3. If a signal is *real,* it follows that its *amplitude spectrum* is an *even* function of frequency and its *phase spectrum* is an *odd* function of frequency. Furthermore, *Fourier transforms of real, even signals are real, even functions of frequency,* while *Fourier transforms of real, odd signals are imaginary, odd functions of frequency.*
4. The *energy spectral density* of a signal is its magnitude-squared Fourier transform.
5. Several useful theorems pertaining to Fourier transforms of signals are summarized in Table 4-1.
6. Several useful Fourier transform pairs are given in Table 4-2. An important conclusion can be drawn from these transform pairs, namely that the *duration and bandwidth of pulse-type signals are inversely proportional.*
7. The Fourier transform, $Y(f)$, of the *output of a fixed, linear system* is related to its transfer function, $H(f)$, and the Fourier transform of the system input, $X(f)$, by

$$Y(f) = H(f)X(f)$$

By taking the magnitude squared of both sides of this equation, it follows that the *energy spectral density of input and output* are related by

$$G_y(f) = |H(f)|^2 G_x(f)$$

where

$$G_x(f) = |X(f)|^2 \quad \text{and} \quad G_y(f) = |Y(f)|^2$$

The *energy contained in the output within a band of frequencies* can be obtained by integrating $G_y(f)$ over the frequency band of interest.

8. The *transfer function* of a fixed, linear system can be obtained by applying any one of the following techniques:

(a) Fourier-transforming the differential equation relating output to input;
(b) Fourier-transforming the impulse response of the system; and

(c) Representing the lumped elements of the system in terms of their ac sinusoidal steady-state impedances and using ac circuit analysis.

These three methods are illustrated pictorially in Figure 4-10.

9. *Ideal filters* are filters passing all frequency components of the input within the passband with the same gain and the same time delay, and rejecting completely all frequency components of the input outside the passband. Usually, three types of ideal filters are defined; *low-pass, high-pass,* and *bandpass. Ideal filters are noncausal* since their impulse responses exist before the impulsive input is applied.
10. The *rise time* of a pulse can be defined in various ways. One common method is to define rise time as the time it takes for a pulse to go from 10% to 90% of its final value. It is true in general that a *pulse's bandwidth and rise time are inversely proportional.*
11. The *Gibb's phenomenon* refers to the property that the Fourier approximation to a pulse that undergoes an abrupt change which requires infinite frequency content tends to overshoot the abrupt change by about 9%. This overshoot phenomenon was shown to be directly attributable to the bandlimiting of its spectrum, which can be viewed in the time domain as a convolution with a sinc-function envelope.
12. An *analytic signal* is one whose spectrum is single sided (i.e., is nonzero only for $f \leq 0$ or for $f \geq 0$). Such a signal has real and imaginary parts which are Hilbert transforms of each other. The *Hilbert transform* of a signal is obtained by phase-shifting its positive-frequency spectral components by $-\pi/2$ rad and its negative-frequency components by $\pi/2$ rad.
13. It can be shown that a *causal system* has a transfer function whose real and imaginary parts are Hilbert transforms of each other.
14. The fast Fourier transform is introduced as a means for computing signal spectra efficiently.

Further Reading

In addition to the references given in Chapter 3, the first reference will provide a refreshing approach to Fourier transforms and their applications. The second reference gives extensive information on the FFT.

R. N. Bracewell, *The Fourier Transform and Its Applications,* 2nd ed. New York: McGraw-Hill, 1978.
W. W. Smith and J. M. Smith, *Handbook of Real-Time Fast Fourier Transforms,* Piscataway, NJ: IEEE Press, 1995.

Problems

Section 4-1

4-1. Obtain the Fourier transforms of the following signals ($\alpha > 0$):
(a) $x_a(t) = Ae^{-\alpha t}u(t)$;
(b) $x_b(t) = Ae^{\alpha t}u(-t)$;
(c) $x_c(t) = Ae^{-\alpha|t|}$;
(d) $x_d(t) = Ae^{-\alpha t}u(t) - Ae^{\alpha t}u(-t)$.

4-2. Plot and compare the amplitude and phase spectra of $x_a(t)$ and $x_b(t)$ in Problem 4-1.

4-3. Plot and compare the amplitude and phase spectra of $x_c(t)$ and $x_d(t)$ in Problem 4-1.

4-4. (a) Show that if $x(t)$ is real and even [i.e., $x(t) = x(-t)$], then the Fourier transform of $x(t)$ may be reduced to

$$X(f) = 2\int_0^{\infty} x(t)\cos 2\pi ft\, dt$$

Therefore, conclude that $X(f)$ is real and even in this case.

(b) Show that for $x(t)$ real and odd [i.e, $x(t) = -x(-t)$], then

$$X(f) = -2j\int_0^{\infty} x(t)\sin 2\pi ft\, dt$$

Therefore, conclude that $X(f)$ is imaginary and odd if $x(t)$ is real and odd.

4-5. (a) Suppose we want the Fourier transform of a complex signal $x(t) = x_R(t) + jx_I(t)$, where $x_R(t)$ is the real part and $x_I(t)$ is the imaginary part. Show that $X(f)$ can be written as

$$X(f) = \int_{-\infty}^{\infty} x_R(t)\cos 2\pi ft\, dt + \int_{-\infty}^{\infty} x_I(t)\sin 2\pi ft\, dt + j\left[\int_{-\infty}^{\infty} x_I(t)\cos 2\pi ft\, dt - \int_{-\infty}^{\infty} x_R(t)\sin 2\pi ft\, dt\right]$$

(b) Provide an argument for the fact that if $x(t) = x(-t)$ (i.e., even), then the Fourier transform reduces to

$$X(f) = \int_{-\infty}^{\infty} x_R(t)\cos 2\pi ft\, dt + j\int_{-\infty}^{\infty} x_I(t)\cos 2\pi ft\, dt = \int_{-\infty}^{\infty} x(t)\cos 2\pi ft\, dt$$

(c) Show that if $x(t) = -x(-t)$ (i.e., odd), then the Fourier transform reduces to

$$X(f) = \int_{-\infty}^{\infty} x_I(t)\sin 2\pi ft\, dt - j\int_{-\infty}^{\infty} x_R(t)\sin 2\pi ft\, dt = -j\int_{-\infty}^{\infty} x(t)\sin 2\pi ft\, dt$$

4-6. Using the Fourier transform integral, find Fourier transforms of the following signals:
(a) $x_a(t) = t\exp(-\alpha t)\,u(t)$, $\alpha > 0$;
(b) $x_b(t) = t^2\,u(t)\,u(1-t)$;
(c) $x_c(t) = \exp(-\alpha t)\,u(t)\,u(1-t)$, $\alpha > 0$.

4-7. Compute and plot the spectrum of the pulse with raised half-cosine leading and trailing edges as shown.

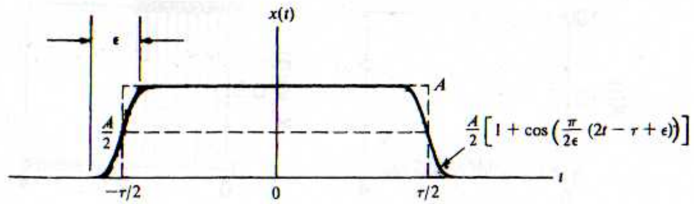


FIGURE P4-7