Question 1. [25 MARKS]

All parts of this question consider the discrete-time linear time-invariant system with impulse response:

$$h[n] = 2^{-n}u[n]$$

Part (a) [4 MARKS]

Is the LTI system stable? Causal? Give brief explanations.

Stability $\sum_{n=-\infty}^{\infty} |h[n]| < \infty \rightarrow \sum_{n=0}^{\infty} 2^{-n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$ Yes Stable ✓

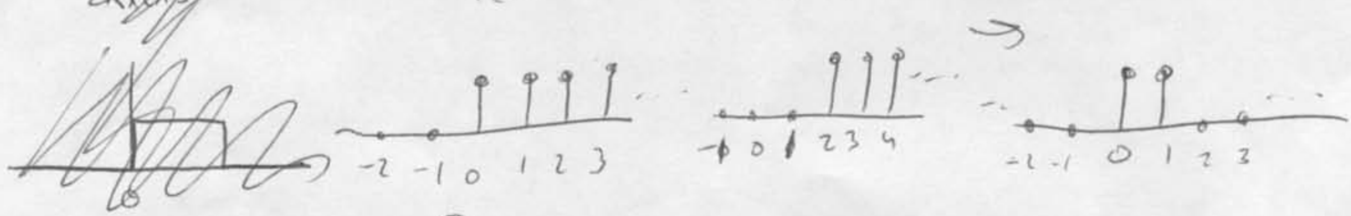
Causal: $h[n] = 0 \quad n < 0$ Here $h[n] = \begin{cases} 2^{-n} & n \geq 0 \\ 0 & n < 0 \end{cases}$ Yes, Causal ✓

Part (b) [6 MARKS]

Suppose that the input to the system is $x_1[n] = u[n] - u[n-2]$. Sketch and label the output $y_1[n]$.

$x_1[n] = \delta[n] + \delta[n-1]$

~~$x_1[n]$~~ $u[n]$ $u[n-2]$

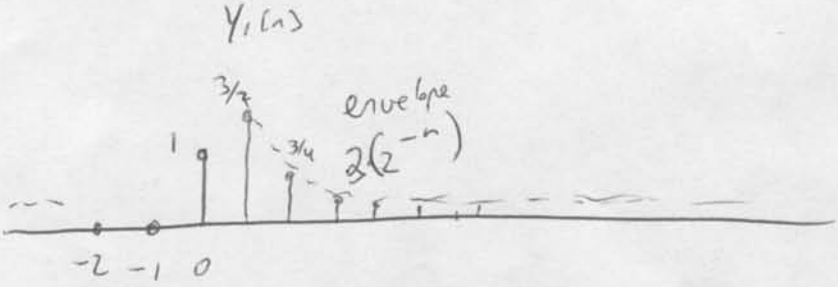


$y_1[n] = x_1[n] * h_1[n] = \sum_{k=-\infty}^{\infty} x_1[k]h_1[n-k]$

$= \sum_{k=-\infty}^{\infty} (\delta[k] + \delta[k-1])h_1[n-k] = h_1[n] + h_1[n-1] = 2^{-n}u[n] + 2^{-(n-1)}u[n-1]$

$2^{-n} + 2^{-(n-1)} = 2^{-n} + 2 \cdot 2^{-n} = 3 \cdot 2^{-n}$

$y_1[n]$



Part (c) [10 MARKS]

Construct an input signal $x_2[n]$ such that the output $y_2[n] = \delta[n]$.

Time domain

To get $\delta(n)$, put in $\delta(n) \rightarrow 2^{-n} u(n)$

1 at $n=0$

Need to cancel $2^{-n-1} u(n-1) \rightarrow$ use $-\frac{1}{2}\delta(n-1)$

$x_2[n] = \delta(n) - \frac{1}{2}\delta(n-1]$

$y_2(n) = \sum_{k=-\infty}^{\infty} (\delta(k) - \frac{1}{2}\delta(k-1))\, 2^{-(n-k)} u(n-k)$

$= 2^{-n} u(n) - \frac{1}{2} 2^{-(n-1)} u(n-1)$

$= \delta(n) + 2^{-n} u(n-1) - 2^{-n} u(n-1) =$

Freq Domain

$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$

$X(e^{j\omega}) = (1 - \frac{1}{2}e^{-j\omega})\, Y(e^{j\omega})$

$\rightarrow$ in inverse system, put in $\delta(n)$, get out IFT of

$1 - \frac{1}{2}e^{-j\omega} = \delta(n) - \frac{1}{2}\delta(n-1]$

$x[n] = y(n) - \frac{1}{2} y(n-1)$

$= \delta(n) - \frac{1}{2}\delta(n-1)$

Part (d) [5 MARKS]

Construct an input signal $x_3[n]$ such that the output $y_3[n] = u[n]$.

From part c, we know that $\delta(n) - \frac{1}{2}\delta(n-1)$ gives us $\delta(n)$ as output

Since LTI, $\delta[n-n_0] - \frac{1}{2}\delta(n-1-n_0)$ gives us $\delta(n-n_0)$ as output

But $u[n] = \sum_{k=0}^{\infty} \delta[n-k] \Rightarrow$ Input is $\sum_{k=0}^{\infty} \delta[n-k] - \frac{1}{2}\delta[n-k-1]$

$= \delta(n) - \frac{1}{2}\delta[n-1] + \delta(n-1] - \frac{1}{2}\delta(n-2] + \delta(n-2] + \cdots$

$= \delta[n] + \frac{1}{2}\sum_{k=1}^{\infty} \delta[n-k]$

$x_3(n) = \begin{cases} 1 & n=0 \\ 1/2 & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

Question 2. [25 MARKS]

Note: In some classes, the frequency variable in the discrete time Fourier transform was written as Ω, not ω. Use the protocol you are comfortable with. In this question, it is written as ω.

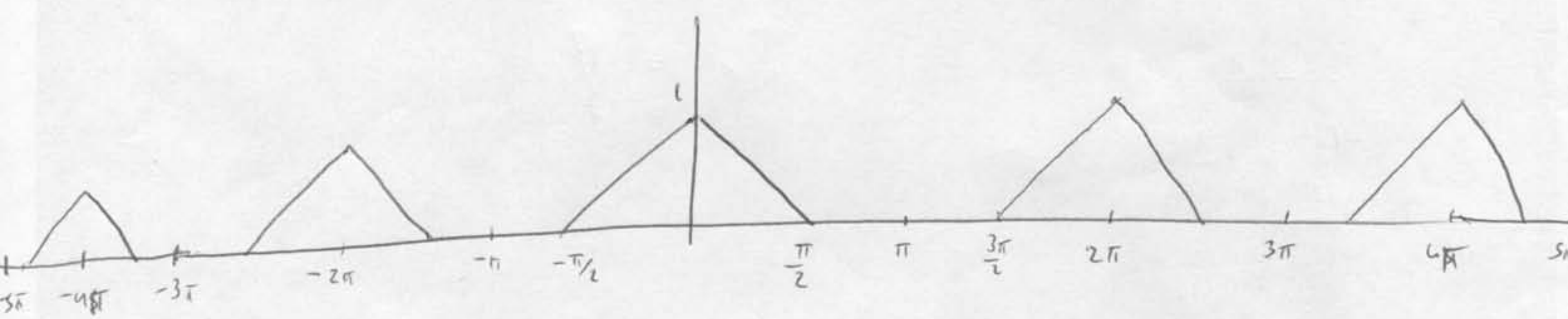
Part (a) [4 MARKS]

A discrete time low pass filter is designed such that, in the region $\omega \in (-\pi, \pi)$, its transfer function (frequency response) $H(e^{j\omega})$ is defined by

$$H(e^{j\omega}) = \begin{cases} 1 - \frac{2|\omega|}{\pi} & \text{if } |\omega| \leq \pi/2 \\ 0 & \text{if } |\omega| > \pi/2 \end{cases}$$

Sketch the transfer function (frequency response) in the region $\omega \in (-5\pi, 5\pi)$.

$H(e^{j\omega})$ — remember periodic with period 2π



Part (b) [3 MARKS]

Is the system described in part (a) causal, i.e., $h[n] = 0$ for $n < 0$? Briefly explain your answer.

Tricky - for $H(e^{j\omega})$ to be real, $h[n]$ must have even symmetry
$\Rightarrow h[n]$ is not causal, since even symmetry gives it $h[n] \neq 0$ for $n < 0$ if $h[n] \neq 0$ for $n > 0$

Part (c) [10 MARKS]

The figure below illustrates a discrete time system composed of two individual systems. The output, $y[n]$, is the sum of the two individual outputs, i.e. $y[n] = y_1[n] + y_2[n]$.

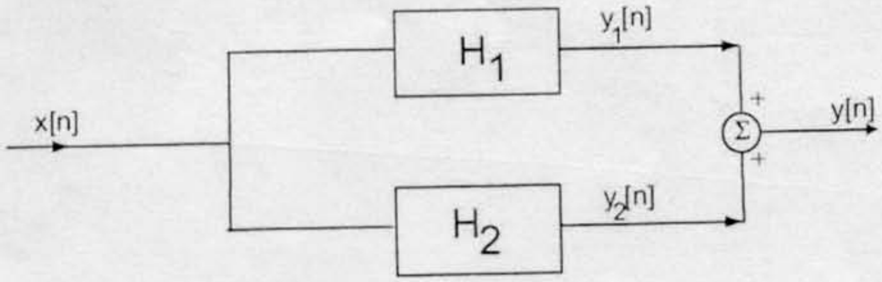


Figure 1: Block diagram for Question 2, Part (c).

The two individual system transfer functions are:

$$H_1(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j(\omega+\pi/3)}}$$
$$H_2(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j(\omega-\pi/3)}}$$

Find the linear constant-coefficient difference equation relating the input $x[n]$ and the <u>*overall*</u> output $y[n]$. *The coefficients are real and must be written as such. Note:* $e^{j\pi/3} = \cos(\pi/3) + j\sin(\pi/3) = 0.5 + j\sqrt{3}/2$.

$$y[n] = y_1[n] + y_2[n] = x[n] * h_1[n] + x[n] * h_2[n]$$
$$= x[n] * [h_1[n] + h_2[n]]$$

$$\Rightarrow Y(e^{j\omega}) = X(e^{j\omega})\left(H_1(e^{j\omega}) + H_2(e^{j\omega})\right)$$
$$= X(e^{j\omega})\left(\frac{1}{1-0.5e^{-j(\omega+\pi/3)}} + \frac{1}{1-0.5e^{-j(\omega-\pi/3)}}\right)$$
$$= X(e^{j\omega})\left(\frac{2 - 0.5e^{-j(\omega+\pi/3)} - 0.5e^{-j(\omega-\pi/3)}}{1 - 0.5e^{-j(\omega+\pi/3)} - 0.5e^{-j(\omega-\pi/3)} + 0.25e^{-2j\omega}}\right)$$

$$Y(e^{j\omega})\left[1 - 0.5e^{-j(\omega-\pi/3)} - 0.5e^{-j(\omega+\pi/2)} + 0.25e^{-2j\omega}\right] = X(e^{j\omega})\left[2 - 0.5e^{-j(\omega+\pi/3)} - 0.5e^{-j(\omega-\pi/3)}\right]$$

or $y[n] - 0.5e^{j\pi/3}y[n-1] - 0.5e^{-j\pi/3}y[n-1] + 0.25y[n-2] = 2x[n] - 0.5e^{-j\pi/3}x[n-1] - 0.5e^{j\pi/3}x[n-1]$

$$y(n) - \frac{1}{2}\underbrace{\left(e^{j\pi/3} + e^{-j\pi/3}\right)}_{\cos\pi/3 = 1/2} y(n-1) + \frac{1}{4}y(n-2) = 2x(n) - \frac{1}{2}\underbrace{\left(e^{j\pi/3} + e^{-j\pi/3}\right)}_{\cos\pi/3 = 1/2} x(n-1)$$

$$y[n] - \frac{1}{2}y(n-1) + \frac{1}{4}y(n-2) = 2x(n) - \frac{1}{2}x[n-1]$$

Part (d) [8 MARKS]

What is the impulse response $h[n]$ for the overall system in part (c)?

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega}) \qquad H_1(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j(\omega+\pi/3)}} \qquad H_2(e^{j\omega}) = \frac{1}{1 - e^{-j(\omega-\pi/3)}}$$

$$h(n) = h_1(n) + h_2(n)$$

$$G(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \longrightarrow g(n) = \left(\frac{1}{2}\right)^n u(n)$$

[illegible]

$$\rightarrow H_1(e^{j\omega}) = G\left(e^{j(\omega+\pi/3)}\right) \rightarrow h_1(n) = e^{-j\pi/3\,n} g(n)$$

$$= e^{-j\pi/3\,n}\left(\frac{1}{2}\right)^n u(n)$$

Similarly $H_2(e^{j\omega}) \rightarrow h_2(n) = e^{j\pi/3\,n}\left(\frac{1}{2}\right)^n u(n)$

$$h(n) = \left(e^{j\pi/3\,n} + e^{-j\pi/3\,n}\right)\left(\frac{1}{2}\right)^n u(n)$$

$$= 2\cos\left(\frac{\pi n}{3}\right)\left(\frac{1}{2}\right)^n u(n)$$

Student #: ____________

Question 3. [25 MARKS]

As sketched in Figure 2, the signal $x(t)$ has a continuous-time Fourier transform

$$X(j\omega) = \begin{cases} 1 - \frac{|\omega|}{2\omega_M} & \text{if } |\omega| < \omega_M \\ 0 & \text{if } |\omega| \geq \omega_M \end{cases},$$

where $\omega_M = 2\pi \times 10^4$ radians/second.

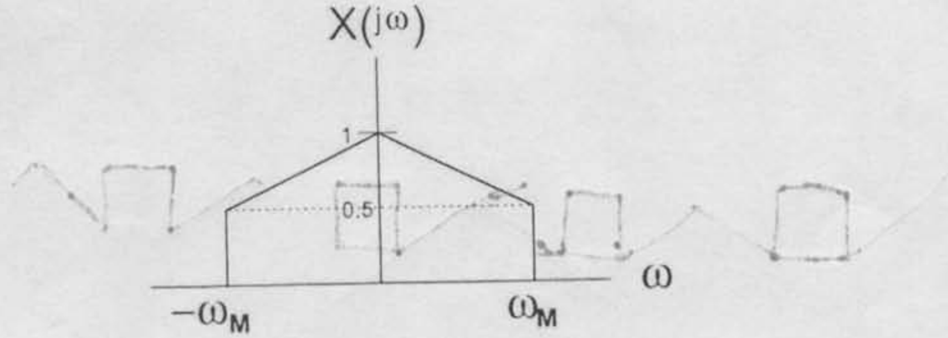


Figure 2: The spectrum of $x(t)$.

As shown in Figure 3, the signal $x(t)$ is sampled by the impulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, producing $x_p(t) = x(t)p(t)$.

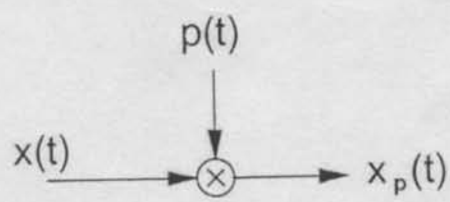


Figure 3: The impulse train sampler producing $x_p(t) = x(t)p(t)$.

Part (a) [4 MARKS]

In terms of seconds, indicate the range of T for which aliasing occurs.

Range of T:

Nyquist Sampling theorem - For baseband signal bandlimited by ω_M, need to sample at $\omega_s \geq 2\omega_m$ for no aliasing

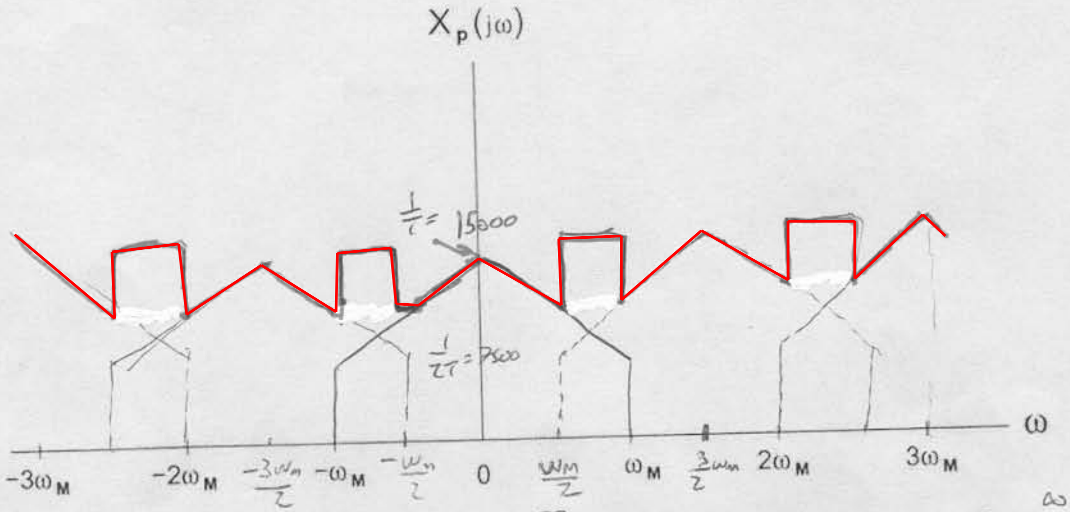
$$T_s = \frac{2\pi}{\omega_s} \leq \frac{2\pi}{2\omega_m} \leq \frac{\pi}{\omega_m} = \frac{\pi}{2\pi \times 10^4 \text{ rad/s}} = \frac{1}{2\times 10^4} \text{ sec.}$$

or 0.05 msec

→ If $T > 0.05$ ms, aliasing occurs

Part (b) [6 MARKS]

Make a detailed and labelled plot of $X_p(j\omega)$ for $-3\omega_M \le \omega \le 3\omega_M$, when $T = (2/3) \times 10^{-4}$ seconds.



$\omega_s = \frac{2\pi}{T}$

$x_p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) \rightarrow \sum_{n=-\infty}^{\infty} \delta(t-nT)$ has FT $\frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$

(even though one $\delta(t)$ has FT of $1 \rightarrow$ why? Since Σ is periodic)

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * \left(\frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) \right) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j(\omega - n\omega_s)) \qquad \frac{1}{T} = 15000$$

Here, $\omega_s = \frac{2\pi}{T} = \frac{2\pi \text{ rad}}{\frac{2}{3} \times 10^{-4} \text{ sec}}$ and $\omega_m = 2\pi \times 10^4$ rad/sec

$\Rightarrow \omega_s = \frac{3}{2}\omega_m < 2\omega_m \rightarrow$ aliasing

Question 4. [25 MARKS]

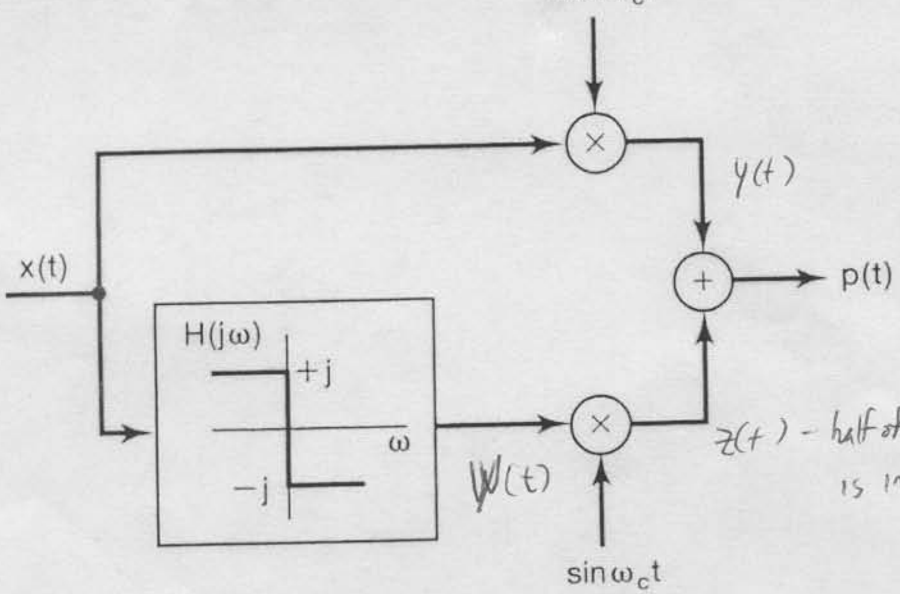
Single side band modulation
- more spectrally efficient than double side band
⇒ more stations in same spectrum

Part (a) [13 MARKS]

The filter,

$$H(j\omega) = \begin{cases} j & \text{if } \omega \leq 0 \\ -j & \text{if } \omega > 0 \end{cases}$$

is used as shown in Figure 5 to create a new signal $p(t)$ from $x(t)$. Assume that $x(t)$ is real and bandlimited to ω_M, which is much less than ω_c. Derive $P(j\omega)$ in terms of $X(j\omega)$. Show your work.



$z(t)$ - half of each branch is inverted

Figure 5: Block diagram for Question 4, Part (a).

$$W(j\omega) = X(j\omega)H(j\omega) = \begin{cases} jX(j\omega) & \omega \leq 0 \\ -jX(j\omega) & \omega > 0 \end{cases}$$

$$y(t) = x(t)\cos(\omega_c(t))$$

$$\Rightarrow Y(j\omega) = \frac{1}{2\pi} X(j\omega) * \left(\pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)\right)$$

$$= \frac{1}{2} X(j(\omega-\omega_c)) + \frac{1}{2} X(j(\omega+\omega_c))$$

$$z(t) = w(t)\sin\omega_c(t)$$

$$Z(j\omega) = \frac{1}{2\pi} W(j\omega) * \left(\frac{\pi}{j}\delta(\omega-\omega_c) - \frac{\pi}{j}\delta(\omega+\omega_c)\right) = \frac{1}{2j} W(j(\omega-\omega_c)) - \frac{1}{2j} W(j(\omega-\omega_c))$$

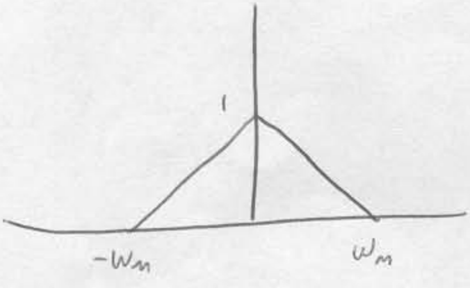
$$= \frac{1}{2} V(j(\omega-\omega_c)) - \frac{1}{2} V(j(\omega+\omega_c))$$

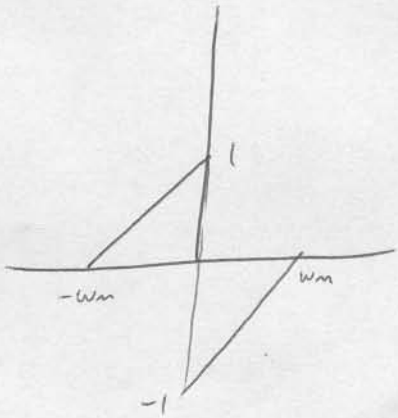
Let $V(j\omega) = \begin{cases} X(j\omega) & \omega \leq 0 \\ -X(j\omega) & \omega > 0 \end{cases}$

Response to 4a (cont)

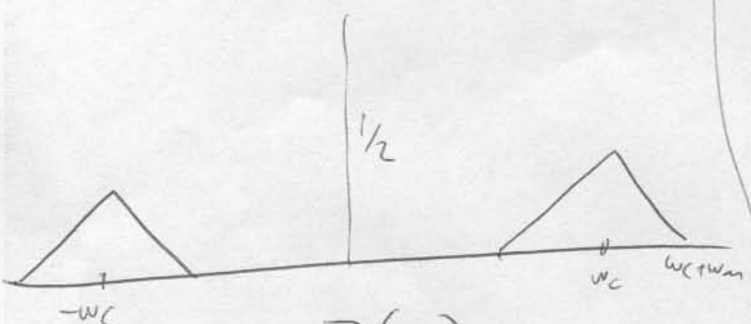
$$P(j\omega) = Y(j\omega) + Z(j\omega)$$
$$= \frac{1}{2}\left[X(j(\omega-\omega_c)) + X(j(\omega+\omega_c)) + V(j(\omega-\omega_c)) - V(j(\omega+\omega_c))\right]$$

ie $X(j\omega) \longrightarrow W(j\omega)$

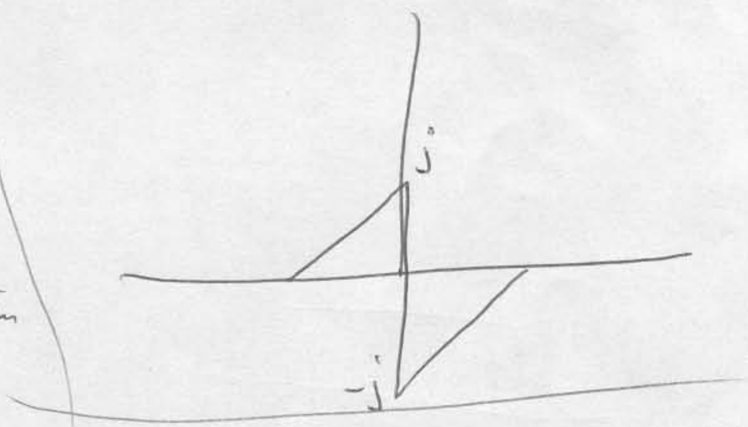




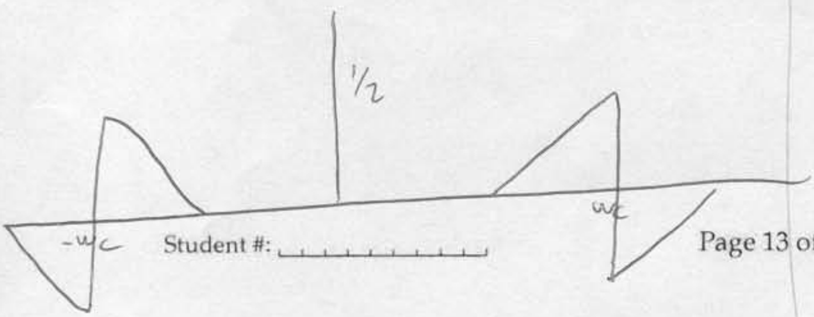
$Y(j\omega)$



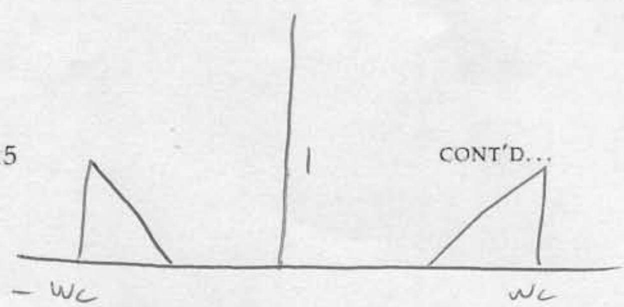
$W(j\omega)$



$Z(j\omega)$

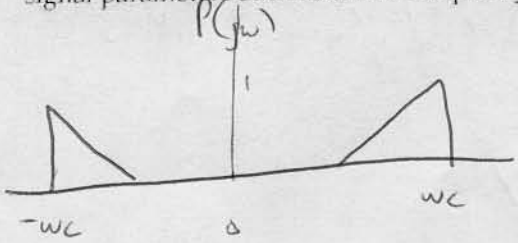


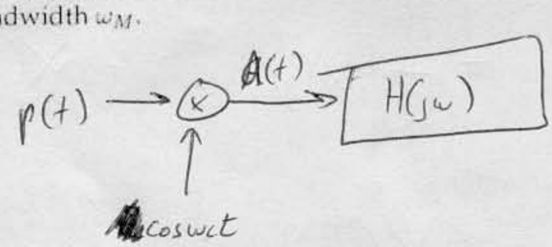
$P(j\omega)$

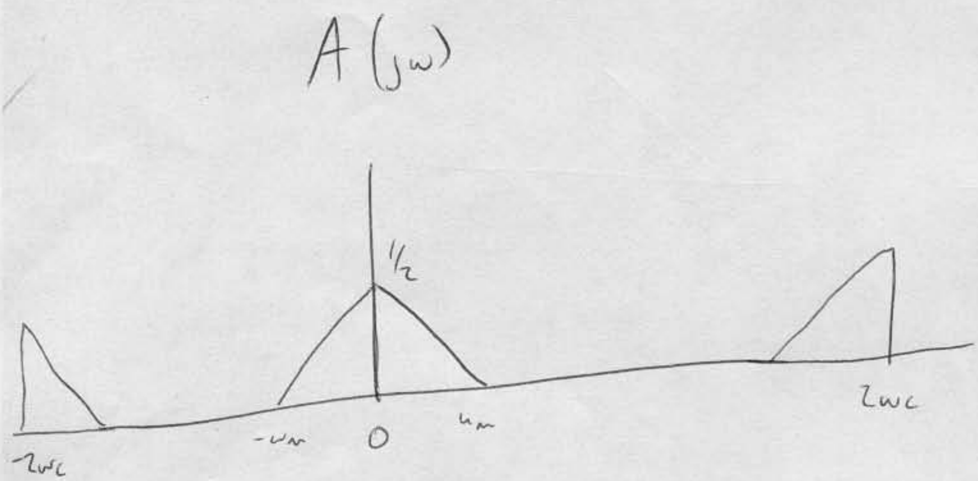


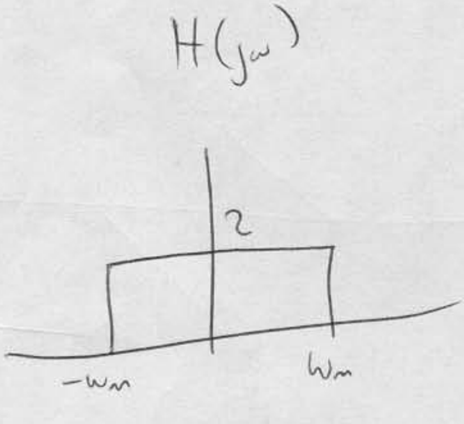
Part (b) [8 MARKS]

Devise a receiver to recover $X(j\omega)$ from $P(j\omega)$, i.e., a system that takes $p(t)$ as input and outputs $x(t)$. Draw a block diagram of your receiver. Clearly label key parameters of the receiver and relate them to signal parameters such as carrier frequency ω_c and bandwidth ω_M.









Part (c) [4 MARKS]

Assume that a real band-limited audio signal whose Fourier transform extends from -20 kHz to 20 kHz is to be transmitted by a radio station. Further, let that station share the radio spectrum from 10 Mhz to 20 Mhz with other stations transmitting different audio signals that are also real and band-limited with Fourier transforms extending from -20 kHz to 20 kHz. Each station employs a transmitting scheme as shown in Figure 5, but each uses a different carrier frequency ω_c. What is the maximum number of stations that you can fit into the assigned spectrum without their spectra overlapping? Justify your answer.

$\omega_m = 20$ kHz $\Rightarrow$ uses spectrum of 10 MHz $(20-10)$

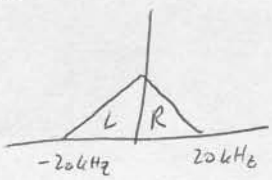
$\Rightarrow$ Can Fit in $\dfrac{10\times10^6}{20\times10^3} = 500$ Channels

As opposed to DSB – each would need 40 kHz $\Rightarrow$ only 250 Chanels

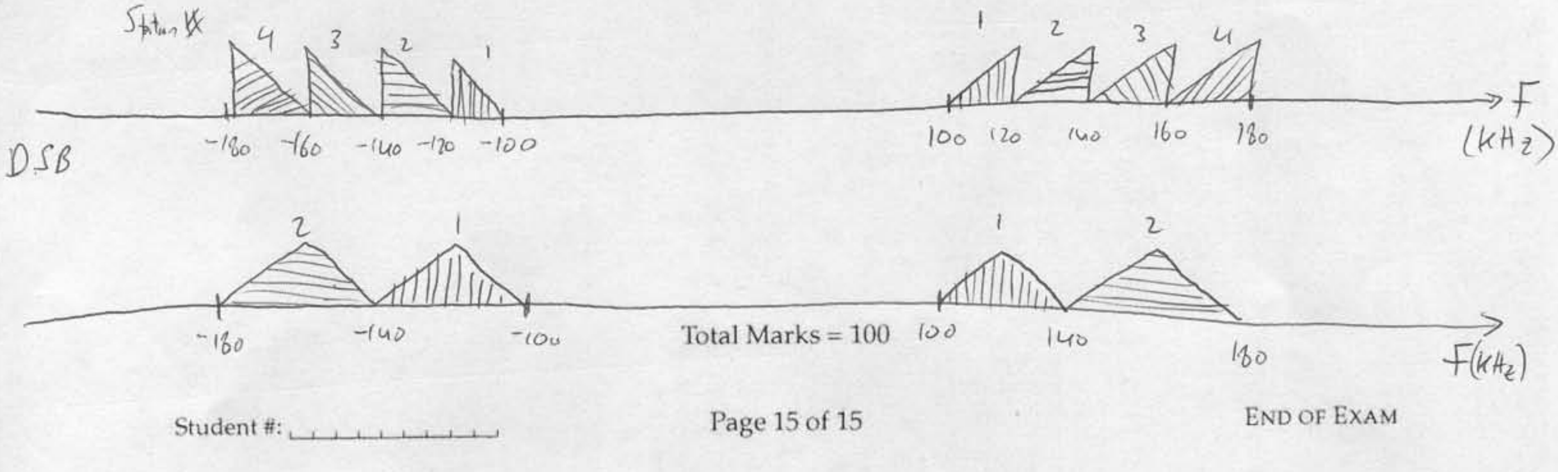
↑

Double sideband

Eg IF each station has spectrum $X(j\omega)$, and 80 kHz available bandwidth From 100 – 180 KHz



SSB



Total Marks = 100