MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Fall 2003

Final Exam
Tuesday, December 16, 2003

Directions: The exam consists of 7 problems on pages 2 to 33 and additional work space on pages 34 to 37. Please make sure you have all the pages. Tables of Fourier series properties, CT and DT Fourier transform properties and pairs, Laplace transform and z-transform properties and pairs are supplied to you as a separate set of pages. **Enter all your work and your answers directly in the spaces provided on the printed pages of this booklet. Please make sure your name is on all sheets. You may use bluebooks for scratch work, but we will not grade them at all.** All sketches must be adequately labeled. Unless indicated otherwise, **answers must be derived or explained**, not just simply written down. This examination is closed book, but students may use three 8 1/2 $\times$ 11 sheets of paper for reference. Calculators may not be used.

NAME:________________________________

Check your section	Section	Time	Rec. Instr.
☐	1	10-11	Prof. Zue
☐	2	11-12	Prof. Zue
☐	3	1- 2	Prof. Gray
☐	4	11-12	Dr. Rohrs
☐	5	12- 1	Prof. Voldman
☐	6	12- 1	Prof. Gray
☐	7	10-11	Dr. Rohrs
☐	8	11-12	Prof. Voldman

Please leave the rest of this page blank for use by the graders:

Problem	No. of points	Score	Grader
1	30		
2	15		
3	35		
4	30		
5	30		
6	25		
7	35		
Total	200		

PROBLEM 1 (30 pts)

Let $h(t)$ be a right sided impulse response of a system and its Laplace transform is given by

$$H(s) = \frac{10(-s+1)}{(s+10)(s+1)}.$$

Part a. Find the differential equation describing the system.

$$\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 10y(t) = -10\frac{dx(t)}{dt} + 10x(t)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10(-s+1)}{(s+10)(s+1)} = \frac{-10s+10}{s^2+11s+10}$$

$$Y(s)(s^2+11s+10) = X(s)(-10s+10)$$

$$\mathcal{L}^{-1}\{s^2Y(s) + 11sY(s) + 10Y(s) = -10sX(s) + 10X(s)\} =$$

Part b. Is the system causal ?

YES or **NO** $\boxed{\text{Yes}}$

Brief explanation:

Since H(s) is rational, it is causal if h(t) is right-sided or equivalently if the ROC extends rightward from the rightmost pole.

h(t) is right-sided so yes it is causal.

Work Page for Problem 1

 Problem 1 continues on the following page.

Part c. The response of this system to a positive step starts off in a negative direction before turning around. Show this by finding $\lim_{t \to 0^+} \frac{ds(t)}{dt}$. Justify your method.

$$\lim_{t \to 0^+} \frac{ds(t)}{dt} = \underline{\quad -10 \quad}$$

(1) $\frac{ds(t)}{dt} = h(t)$ (2) The Initial Value Theorem says $\lim_{t \to 0^+} h(t) = \lim_{s \to \infty} sH(s)$

if $h(t) = 0$ for $t < 0$ with no impulses or higher order singularities at $t = 0$. Since $h(t)$ is causal and the $Q(N) < Q(D)$ we use the I.V.T.

$$\lim_{t \to 0^+} \frac{ds(t)}{dt} = \lim_{t \to 0^+} h(t) = \lim_{s \to \infty} sH(s) = \lim \frac{10(-s+1)s}{(s+10)(s+1)} = -10$$

Part d. Let $H_I(s)$ be the transfer function of a stable but noncausal inverse system of $H(s)$, i.e., $H_I(s)H(s) = 1$. Find $H_I(s)$ and its region of convergence.

$$H_I(s) = \underline{\quad -\frac{(s+1)(s+10)}{10(s-1)} \quad} \qquad \text{ROC: } \operatorname{Re}\{s\} < 1$$

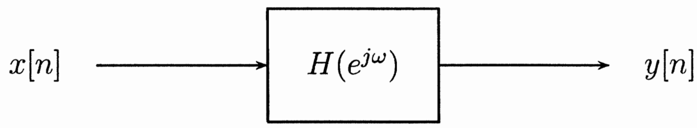
$$H_I(s) = \frac{1}{H(s)}$$

To be stable, the ROC needs to include the $j\omega$ axis. Thus, the ROC extends leftward from the pole at $s = 1$. This also verifies that $h(t)$ is noncausal.

Work Page for Problem 1

PROBLEM 2 (15 pts)

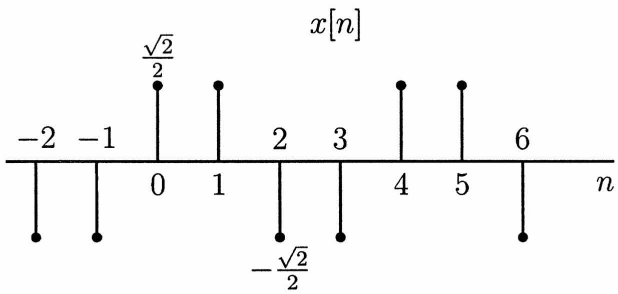
Consider the DT LTI system shown below:



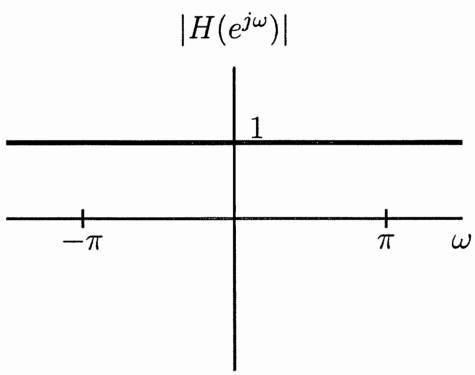
The input sequence is

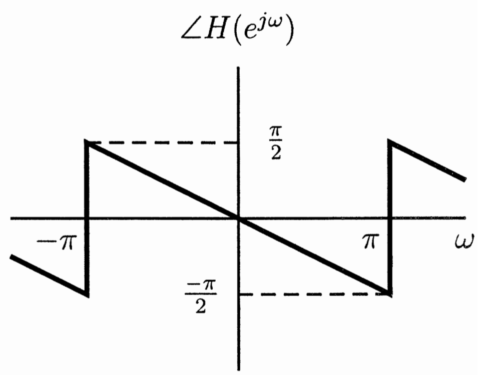
$$x[n] = \cos\left(\frac{5\pi}{2}n - \frac{\pi}{4}\right)$$

as sketched below:

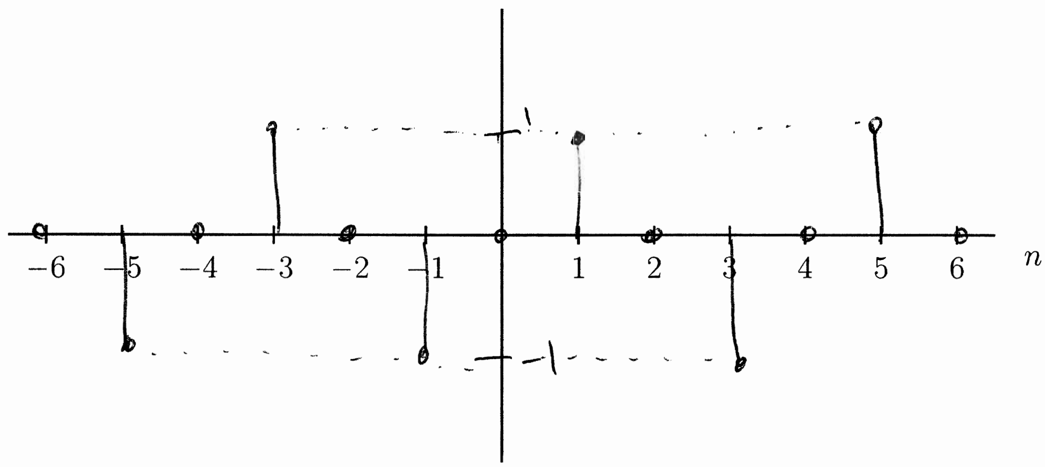


Determine and sketch $y[n]$ if the magnitude and the phase of $H(e^{j\omega})$ are given below:





$y[n] =$ $\sin \frac{\pi}{2} n$



$x[n] = \cos\left(\frac{5\pi}{2}n - \frac{\pi}{4}\right) = \frac{1}{2}e^{j\frac{5\pi}{2}n}e^{-j\frac{\pi}{4}} + \frac{1}{2}e^{-j\frac{5\pi}{2}n}e^{j\frac{\pi}{4}} = \frac{1}{2}e^{-j\pi/4}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{j\pi/4}e^{-j\frac{\pi}{2}n}$

Take F.T. of each term using the tables.

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty}\left(\pi e^{-j\pi/4}\,\delta(\omega - \tfrac{\pi}{2} + 2\pi l) + \pi e^{j\pi/4}\,\delta(\omega + \tfrac{\pi}{2} + 2\pi l)\right)$$

$Y(e^{j\omega}) = |X(e^{j\omega})|\,|H(e^{j\omega})|\ \measuredangle X(e^{j\omega}) + \measuredangle H(e^{j\omega})$

$Y(e^{j\omega})$ only exists at locations of impulses of $X(e^{j\omega})$ so.

$Y(e^{j(\frac{\pi}{2}+2\pi l)}) = \pi\,\delta(\omega - \frac{\pi}{2} + 2\pi l)\, e^{-j\pi/4} e^{-j\pi/4}$ $\quad l = 0, \pm 1, \ldots$

$Y(e^{j(-\frac{\pi}{2}+2\pi l)}) = \pi\,\delta(\omega + \frac{\pi}{2} + 2\pi l)\, e^{j\pi/4} e^{j\pi/4}$ $\quad l = 0, \pm 1, \ldots$

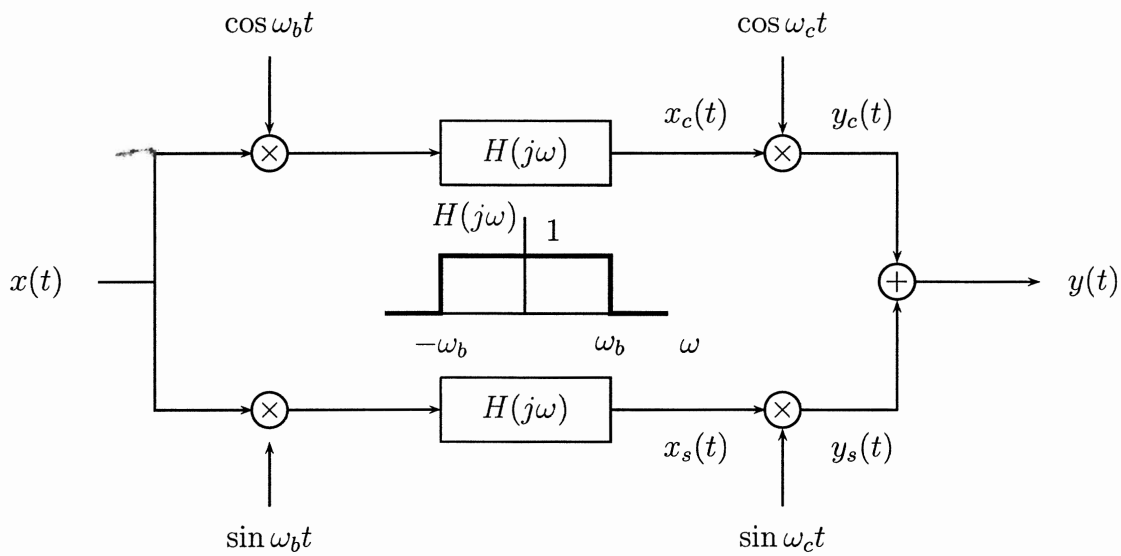
$Y(e^{j\omega}) = 0$ for all other ω.

$$Y(e^{j\omega}) = \sum_{l=\infty}^{\infty} \frac{\pi}{j}\,\delta(\omega - \tfrac{\pi}{2} + 2\pi l) - \frac{\pi}{j}\,\delta(\omega + \tfrac{\pi}{2} + 2\pi l) \Rightarrow y[n] = \sin\frac{\pi}{2}n$$

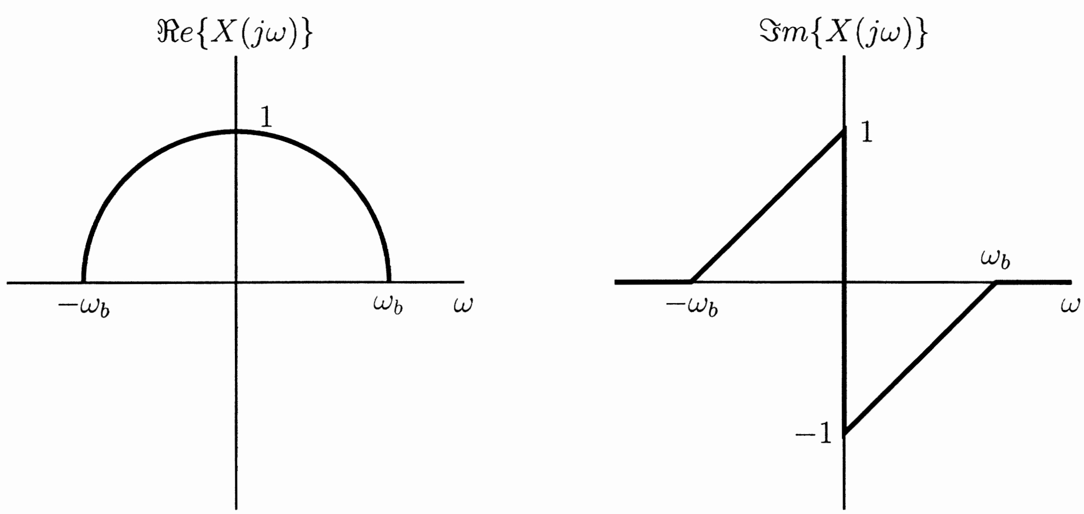
Work Space for Problem 2

PROBLEM 3 (35pts)

Consider the following system:

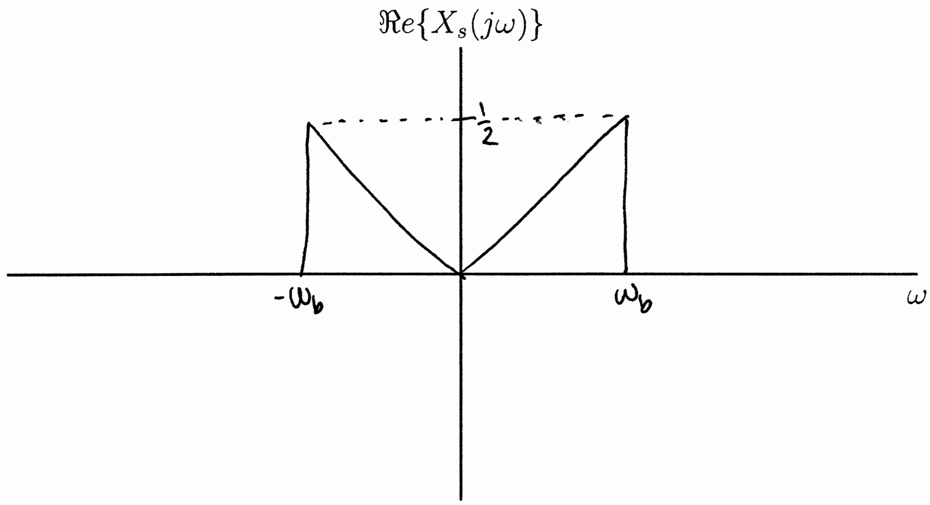


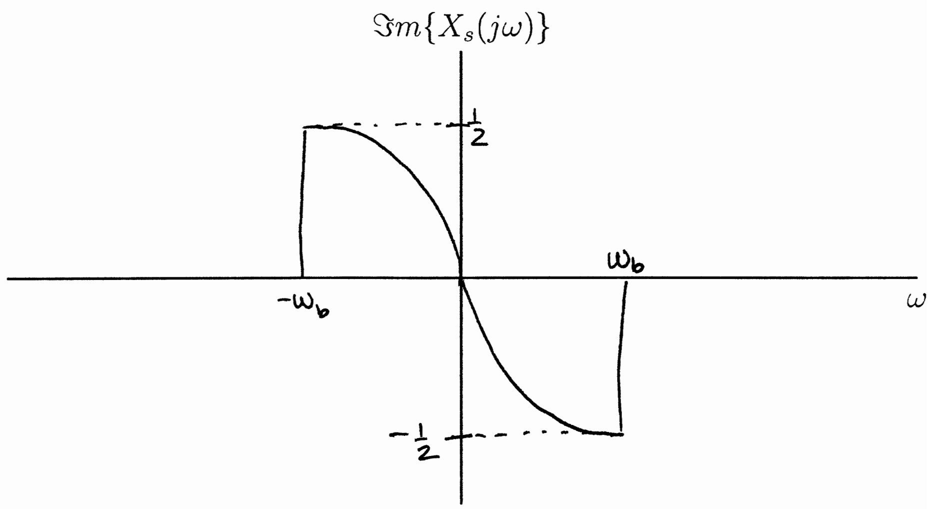
The Fourier transform of $x(t)$, $X(j\omega)$ has real and imaginary parts given below:



For your convenience, the identical figures above are attached along with the transform tables.

Part a. Provide labeled sketches of the real and imaginary parts of $X_s(j\omega)$.



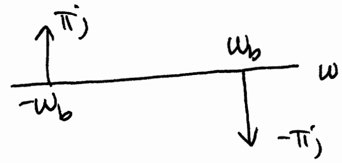


Work Page for Problem 3

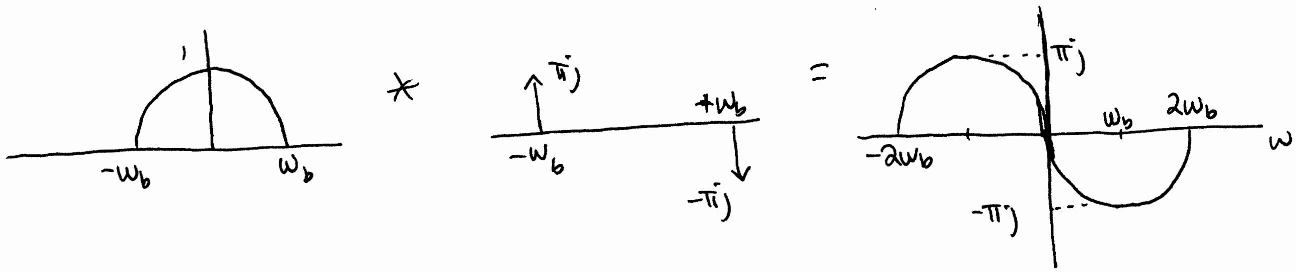
Part a: $X_s(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * \mathcal{F}\{\sin\omega_b t\}\right] \times H(j\omega)$

$$= \frac{1}{2\pi}\left[\mathrm{Re}\{X(j\omega)\} * \mathcal{F}\{\sin\omega_b t\} + j\,\mathrm{Im}\{X(j\omega)\} * \mathcal{F}\{\sin\omega_b t\}\right] \times H(j\omega)$$

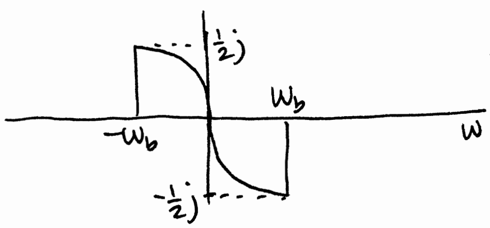
$\mathcal{F}\{\sin\omega_b t\} = \pi j\,\delta(\omega+\omega_b) - \pi j\,\delta(\omega-\omega_b) \rightarrow$



$\mathrm{Re}\{X(j\omega)\} * \mathcal{F}\{\sin\omega_b t\}$

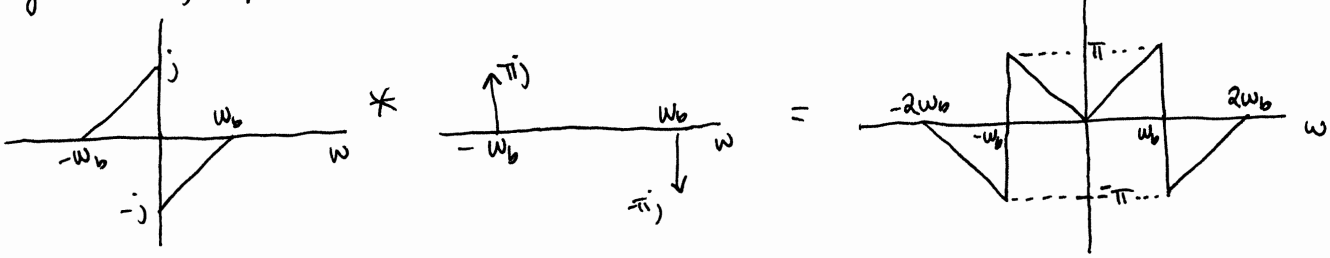


$\frac{1}{2\pi}\left[\mathrm{Re}\{X(j\omega)\} * \mathcal{F}\{\sin\omega_b t\}\right] \times H(j\omega) =$

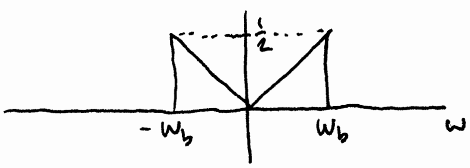


$= \mathrm{Im}\{X_s(j\omega)\}$

$j\,\mathrm{Im}\{X(j\omega)\} * \mathcal{F}\{\sin\omega_b t\}$

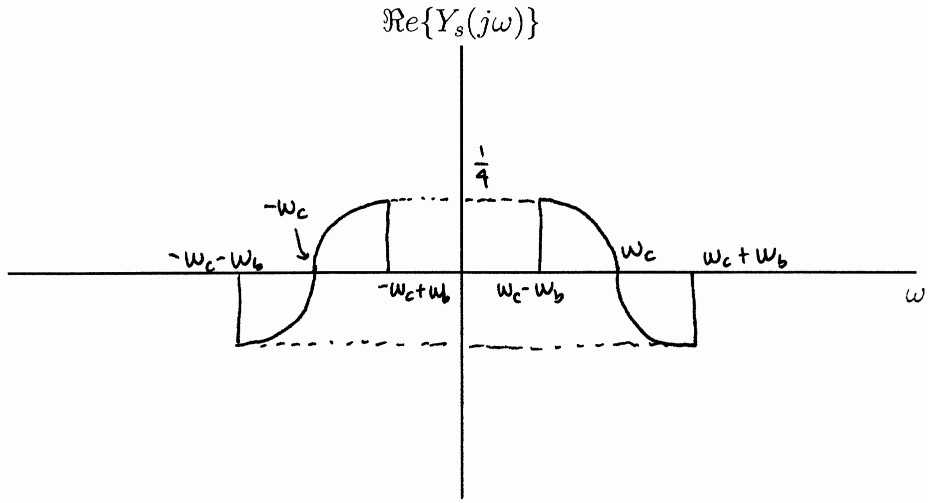


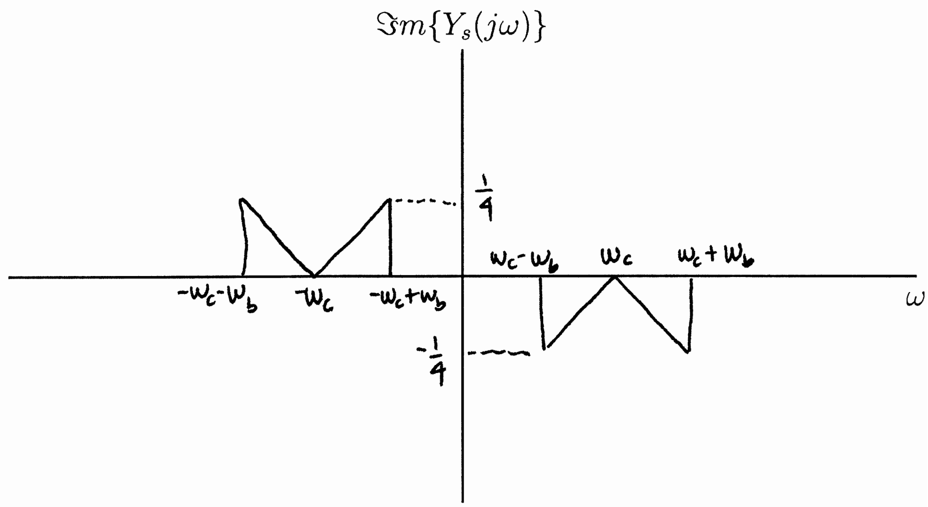
$\frac{1}{2\pi}\left[j\,\mathrm{Im}\{X(j\omega)\} * \mathcal{F}\{\sin\omega_b t\}\right] \times H(j\omega)$



 Problem 3 continues on the following page.

Part b. Provide labeled sketches of the real and imaginary parts of $Y_s(j\omega)$.



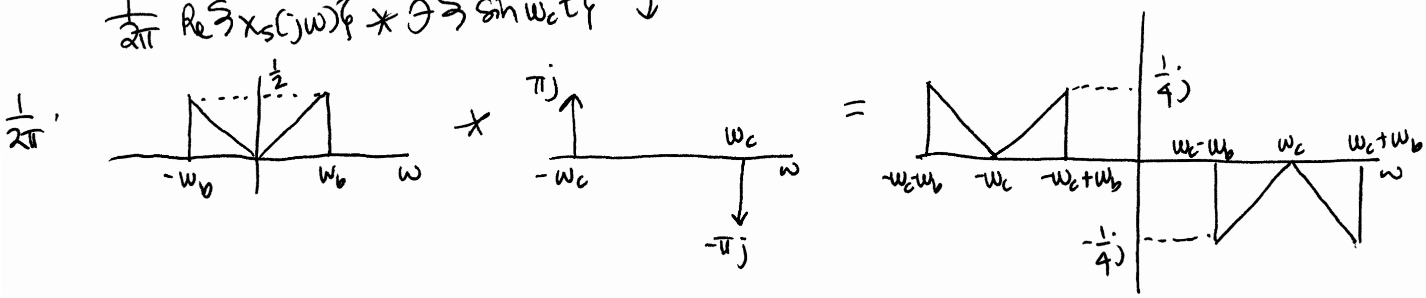


Work Page for Problem 3

Part b: $Y_s(j\omega) = \frac{1}{2\pi} X_s(j\omega) * \mathcal{F}\{\sin\omega_c t\}$

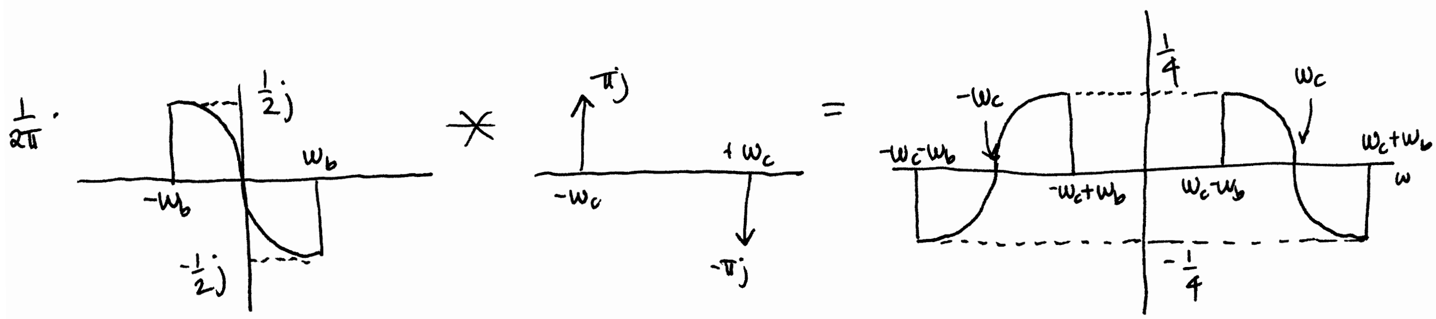
$= \frac{1}{2\pi}\left(\mathrm{Re}\{X_s(j\omega)\} + j\,\mathrm{Im}\{X_s(j\omega)\}\right) * \mathcal{F}\{\sin\omega_c t\}$

$\frac{1}{2\pi}\mathrm{Re}\{X_s(j\omega)\} * \mathcal{F}\{\sin\omega_c t\}$ ⤵



So this part becomes the imaginary part of $Y_s(j\omega)$

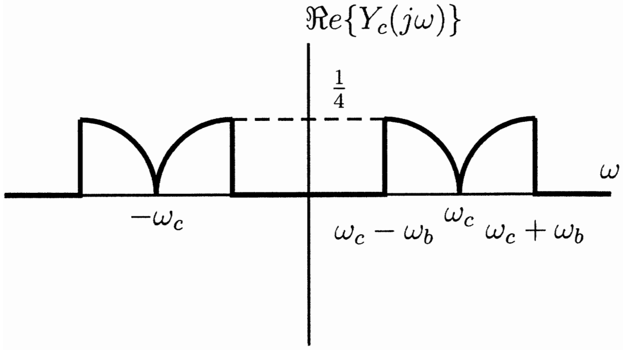
$\frac{1}{2\pi}j\,\mathrm{Im}\{X_s(j\omega)\} * \mathcal{F}\{\sin\omega_c t\}$ ⤵

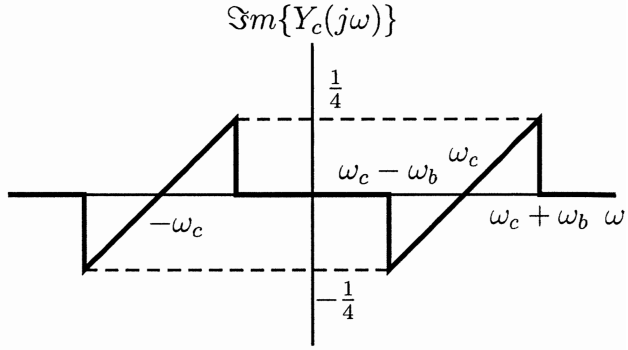


So this part becomes the real part of $Y_s(j\omega)$

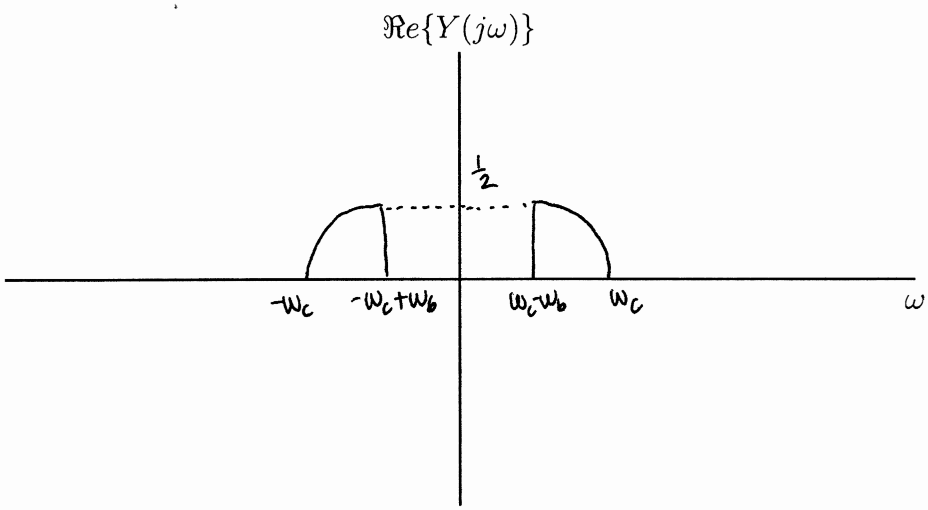
 Problem 3 continues on the following page.

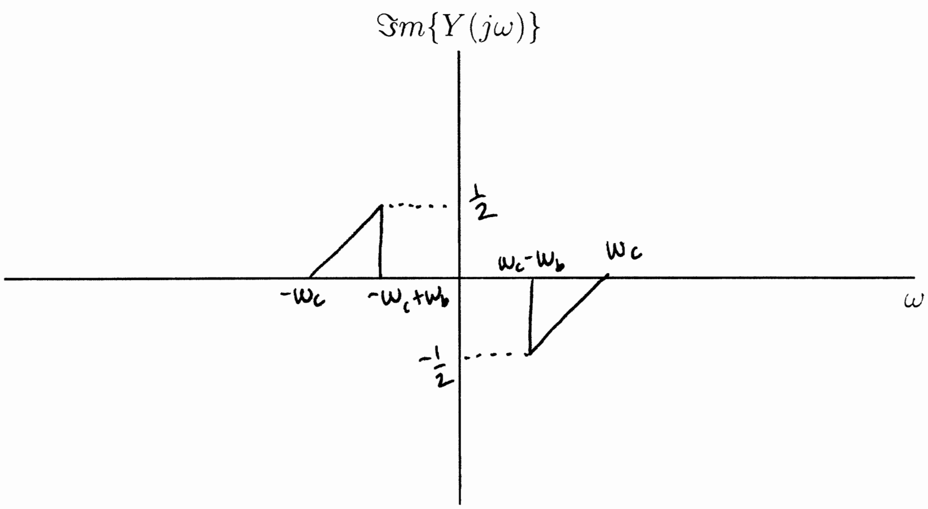
Part c. $Y_c(j\omega)$ has real imaginary parts as shown below





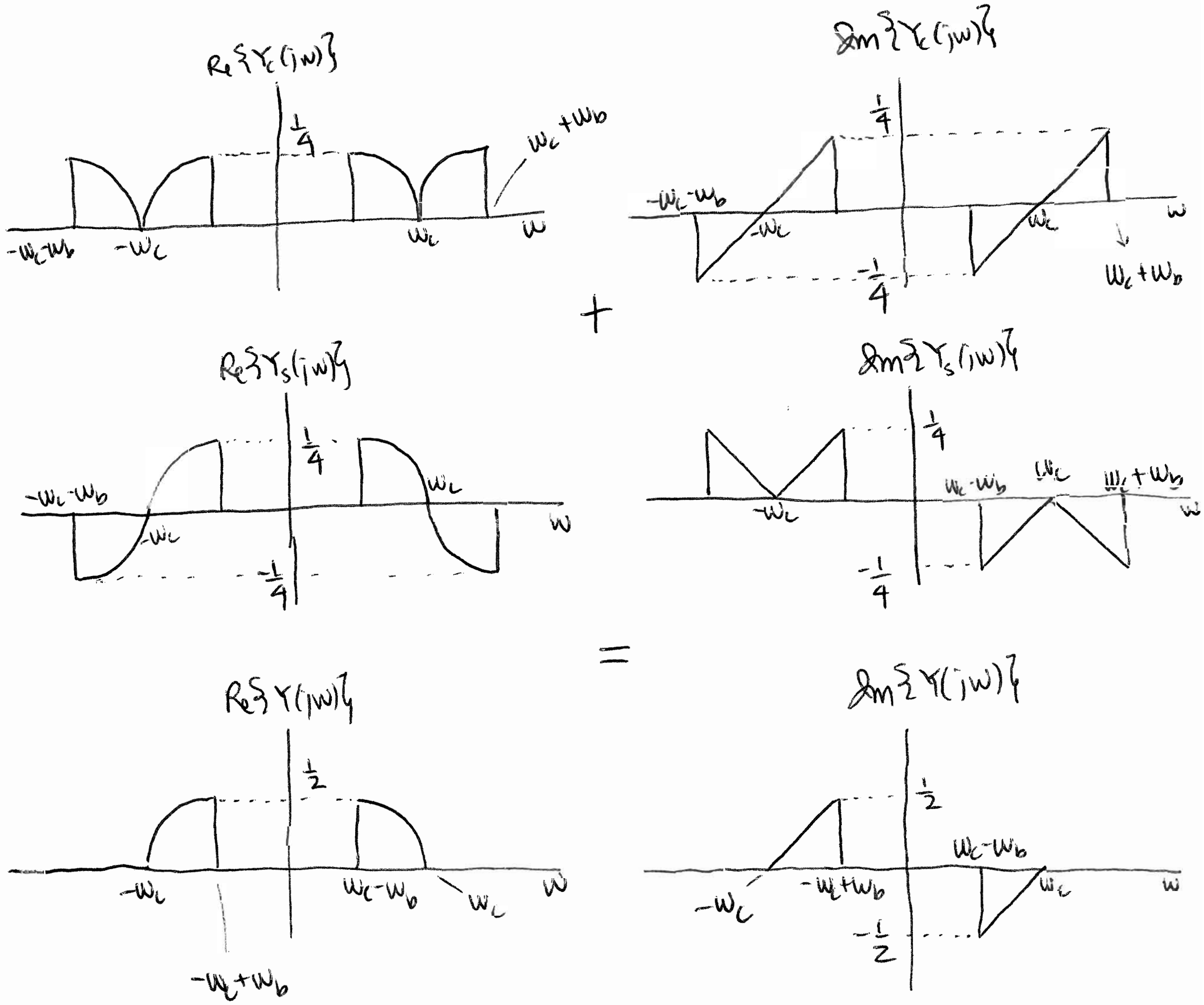
Provide labeled sketches of the real and imaginary parts of $Y(j\omega)$.





Work Page for Problem 3

Part C: We need to add $Y_c(j\omega) + Y_s(j\omega)$



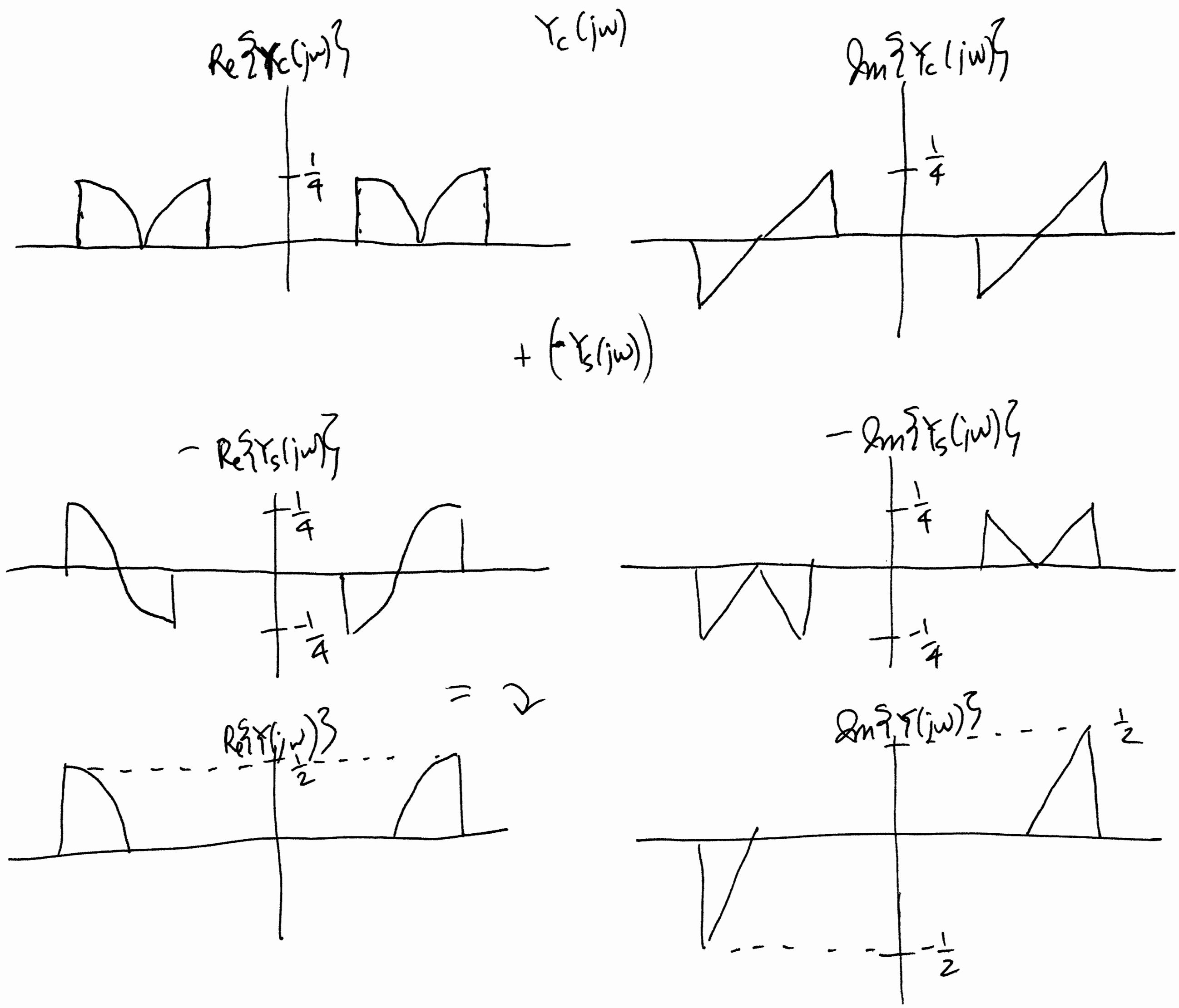
 Problem 3 continues on the following page.

Part d. What small change would you make in this system to create a lower sideband modulation ?

Instead of having $y(t) = y_c(t) + y_s(t)$,

have the system built such that $y(t) = y_c(t) - y_s(t)$.

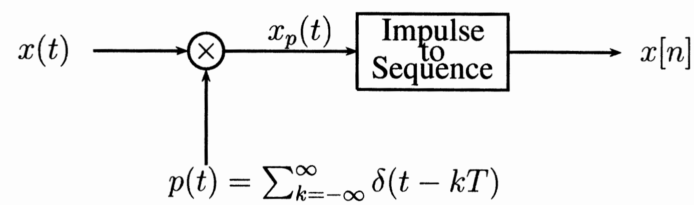
See:



Work Space for Problem 3

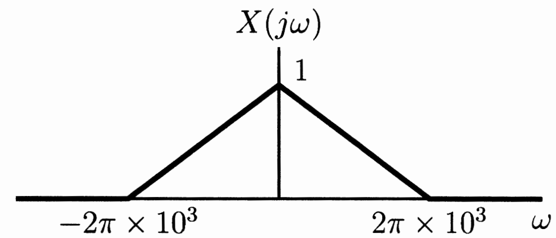
PROBLEM 4 (30 pts)

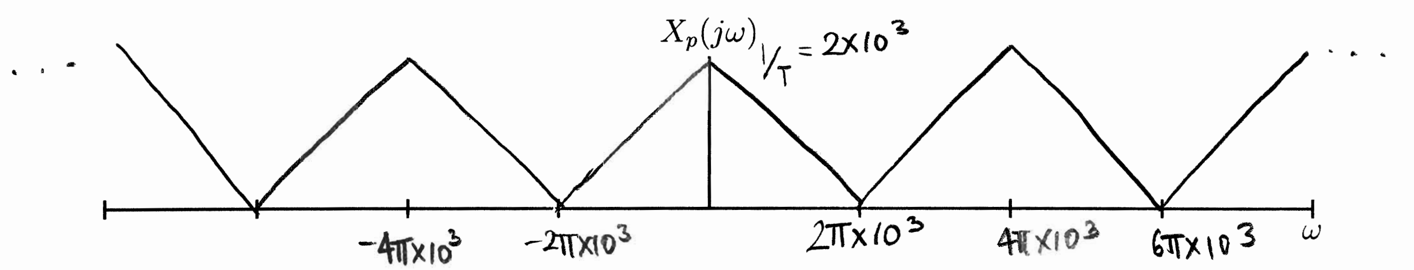
Consider the following system:

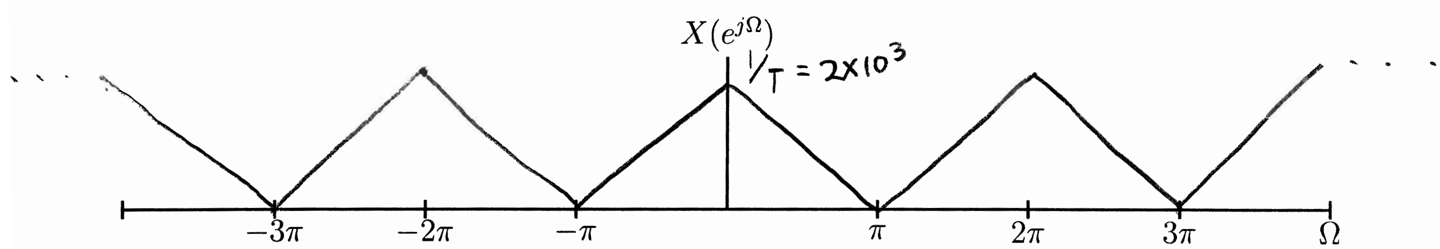


The Fourier transforms of $x(t)$, $x_p(t)$, and $x[n]$ are denoted respectively by $X(j\omega)$, $X_p(j\omega)$ and $X(e^{j\Omega})$.

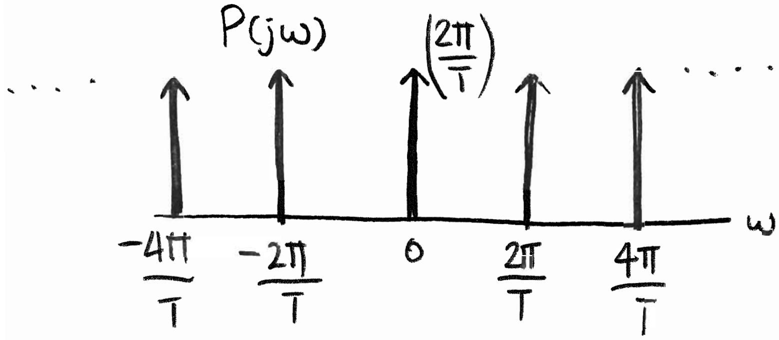
Part a. If $X(j\omega)$ is as shown below and $T = 0.5 \times 10^{-3}$ sec, provide labeled sketches of $X_p(j\omega)$ and $X(e^{j\Omega})$.







Work Page for Problem 4



$$X_p(j\omega) = \frac{1}{2\pi} P(j\omega) * X(j\omega)$$

And $\frac{2\pi}{T} = 2\pi \cdot \frac{1}{0.5\times10^{-3}} = 4\pi\times10^{3}$

$X(e^{j\Omega})$ is just $X_p(j\omega)$ with a scaling on the frequency axis by $T = 0.5\times10^{-3}$

Problem 4 continues on the following page.

Part b. Using the same $X(j\omega)$ and T as in **Part a**, determine

(i) $\int_{-\infty}^{\infty} x(t)dt$

(ii) $\sum_{n=-\infty}^{\infty} x[n]$.

$\int_{-\infty}^{\infty} x(t)dt =$ 1, $\sum_{n=-\infty}^{\infty} x[n] =$ 2×10^3

Part c. Now, assume only that $x(t)$ is bandlimited, i.e., $X(j\omega) = 0$ for $|\omega| \geq W$ and is otherwise arbitrary.

It has been claimed that, for suitable values of T, i.e., $T < A$ for some value A, the total area under the continuous time input signal $x(t)$ is T times the sum of the $x[n]$. Do you believe the claim, i.e., is there any constraint between T and W which will **guarantee** that

$$T \sum_{n=-\infty}^{\infty} x[n] = \int_{-\infty}^{\infty} x(t)dt\,?$$

If your answer is yes, specify in terms of W, the largest value of A for which the claim is true. If your answer is no, explain clearly.

YES **NO**

$A =$ $2\pi/W$

Explanation:

No overlap at

$\Omega = 0$ pt when

$2\pi - WT > 0$

$\therefore\ T < \frac{2\pi}{W}$

Work Space for Problem 4

(b)

$$\int_{-\infty}^{\infty} x(t)\,dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \Big|_{\omega=0}$$

$$= X(j\omega)\Big|_{\omega=0}$$

$$= 1$$

similarly; $$\sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\Big|_{\Omega=0}$$

$$= X(e^{j\Omega n})\Big|_{\Omega=0}$$

$$= 2\times 10^3$$

(c) We know that $$X(e^{j\Omega})\Big|_{\Omega=0} = \frac{1}{T}X(j\omega)\Big|_{\omega=0}$$

$$TX(e^{j\Omega})\Big|_{\Omega=0} = X(j\omega)\Big|_{\omega=0}$$

From (c), we know that

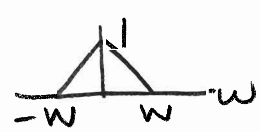
$$X(e^{j\Omega})\Big|_{\Omega=0} = \sum_{n=-\infty}^{\infty} x[n]$$

$$X(j\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} x(t)\,dt$$

So; We need

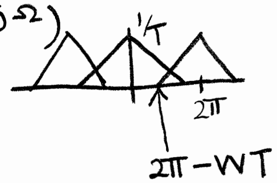
$$TX(e^{j\Omega})\Big|_{\Omega=0} = X(j\omega)\Big|_{\omega=0}$$

Assuming $X(j\omega)$ is



At sub-Nyquist sampling, we have

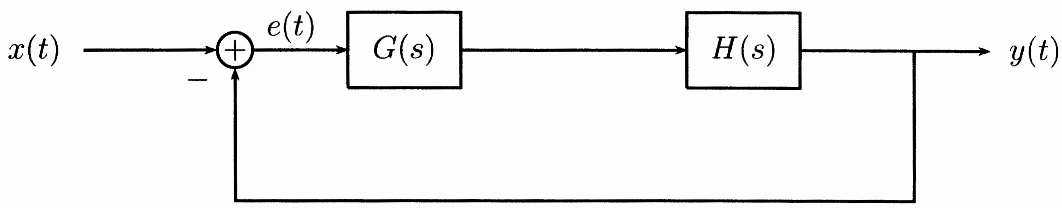
$X(e^{j\Omega})$



As long as there is no overlap between the images at $\Omega=0$, then the property holds. $\therefore\ 2\pi - WT > 0$

PROBLEM 5 (30 pts)

Consider the following feedback system:



where $H(s) = \frac{1}{s^2}$ is the plant, $x(t)$ is the reference input, $e(t) = x(t) - y(t)$ is the error signal, and $y(t)$ is the output of the plant $H(s)$.

Part a. Is $H(s)$ stable ?

YES or **NO** $\boxed{\text{No}}$

Brief explanation:

Poles on $j\omega$-axis $\therefore$ ROC cannot include $j\omega$-axis.

Part b. Find system functions $\frac{Y(s)}{X(s)}$ and $\frac{E(s)}{X(s)}$. Express your answers in terms of powers of s and $G(s)$.

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{s^2 + G(s)}$$

$$\frac{E(s)}{X(s)} = \frac{s^2}{s^2 + G(s)}$$

Work Page for Problem 5

(b) Using Black's Formula

$$\frac{Y(s)}{X(s)} = \frac{G(s)\,H(s)}{1 + G(s)\,H(s)}$$

Substituting $H(s) = \frac{1}{s^2}$;

$$\frac{Y(s)}{X(s)} = \frac{\frac{G(s)}{s^2}}{1 + \frac{G(s)}{s^2}} = \frac{G(s)}{s^2 + G(s)}$$

$$\frac{E(s)}{X(s)} = \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + \frac{G(s)}{s^2}} = \frac{s^2}{s^2 + G(s)}$$

 Problem 5 continues on the following page.

Part c. Suppose $G(s) = K_d s + K_p$ where K_d and K_p are real numbers. Find the values of K_d and K_p such that the closed loop system is critically damped with undamped natural frequency of 10 rad/s.

$K_d =$ 20 , $K_p =$ 100

$$\frac{Y(s)}{X(s)} = \frac{K_d s + K_p}{s^2 + K_d s + K_p}$$

Comparing with our standard form on the denominator polynomial of $s^2 + 2\zeta\omega_n s + \omega_n^2$

we have $K_p = \omega_n^2$

and $K_d = 2\zeta\omega_n$

Natural frequency $= 10 = \omega_n$

$\therefore \; K_p = 10^2 = 100$

For critical damping $\zeta = 1$

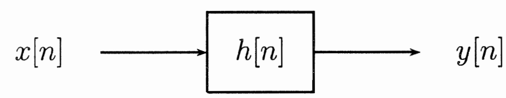
$\therefore \; K_d = 2(1)(10)$

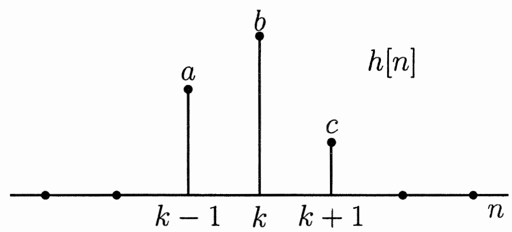
$= 20$

Work Page for Problem 5

PROBLEM 6 (25 pts)

Consider the DT LTI system whose unit sample response, $h[n]$ is shown below:

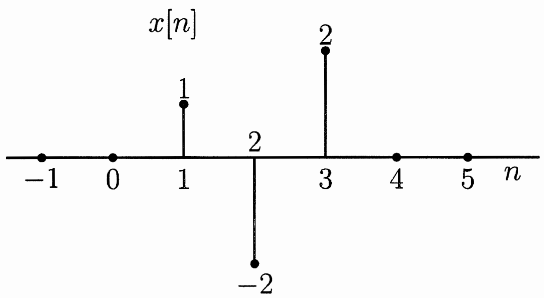


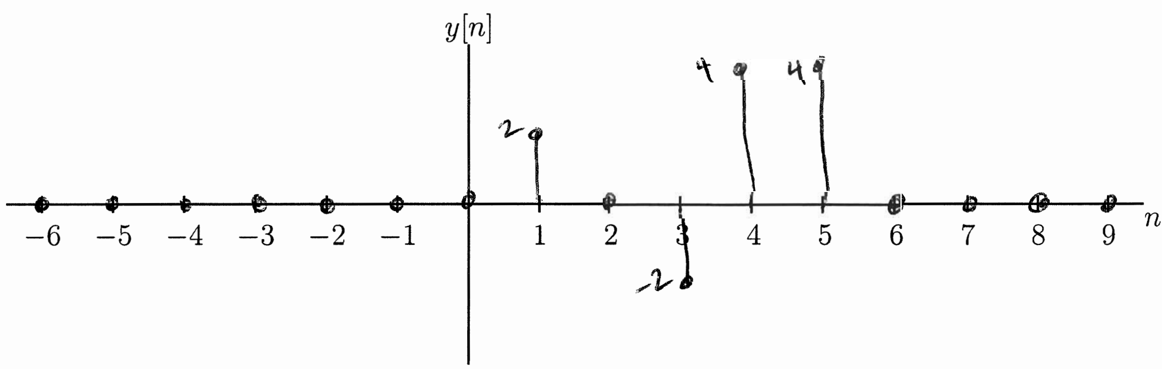


where k is an unknown integer and a, b, and c are unknown real numbers.
It is known that $h[n]$ satisfies the following conditions:

(i) Let $H(e^{j\omega})$ be the Fourier transform of $h[n]$. $H(e^{j\omega})e^{j\omega}$ is real and even.
(ii) If $x[n] = (-1)^n$ for all n, then $y[n] = 0$.
(iii) If $x[n] = \left(\frac{1}{2}\right)^n u[n]$ for all n, then $y[2] = \frac{9}{2}$.

Provide a labeled sketch of the output $y[n]$ when the input $x[n]$ is shown below. Your answer should not include a, b, c, nor k.



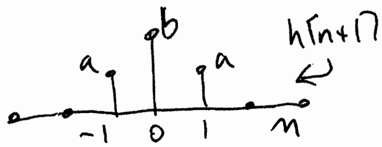


Work Space for Problem 6

(1) $H(z) = az^{-(k-1)} + bz^{-k} + cz^{-(k+1)}$

(2) Since $H(e^{j\omega})e^{j\omega}$ is real and even, and since

$h[n-n_0] \xrightarrow{\mathcal{F}} H(e^{j\omega})e^{-j\omega n_0}$ then $h[n+1]$ is real & even.



$\Rightarrow a = c$ and $k = 1$

(3) $H(z) = a + bz^{-1} + az^{-2} = h[0] + h[1]z^{-1} + h[2]z^{-2}$

(4) if $x[n] = (-1)^n$ $y[n] = 0$ $\Rightarrow$ $y[n] = (-1)^n H(-1) = 0$ $\Rightarrow$ $H(-1) = 0$

($h[n]$ is a finite signal so $H(-1)$ exists)

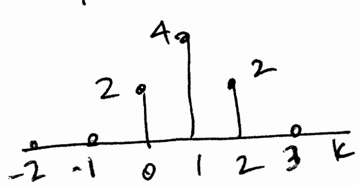
$H(-1) = a - b + a = 0$ $\qquad 2a = b$

(5) if $x[n] = \frac{1}{2}^n u[n]$ $y[2] = \frac{9}{2}$

$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ $\Rightarrow$ $y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0]$

$y[2] = \frac{9}{2} = 1 \cdot a + \frac{1}{2}b + \frac{1}{4}a = \frac{5}{4}a + \frac{1}{2}(2a) = \frac{9}{4}a$

$a = 2$ $\quad b = 4$ $\quad c = 2$



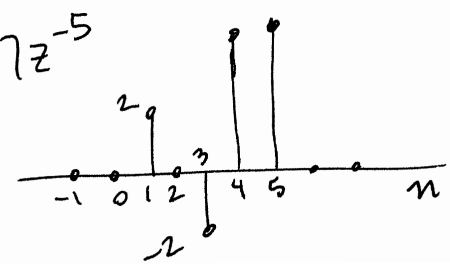
(6) $Y(z) = H(z)X(z) = (2 + 4z^{-1} + 2z^{-2})(z^{-1} - 2z^{-2} + 2z^{-3})$

$Y(z) = 2z^{-1} + 4z^{-2} - 4z^{-2} + 4z^{-3} - 8z^{-3} + 2z^{-3} + 8z^{-4} - 4z^{-4} + 4z^{-5}$

$= 2z^{-1} - 2z^{-3} + 4z^{-4} + 4z^{-5}$

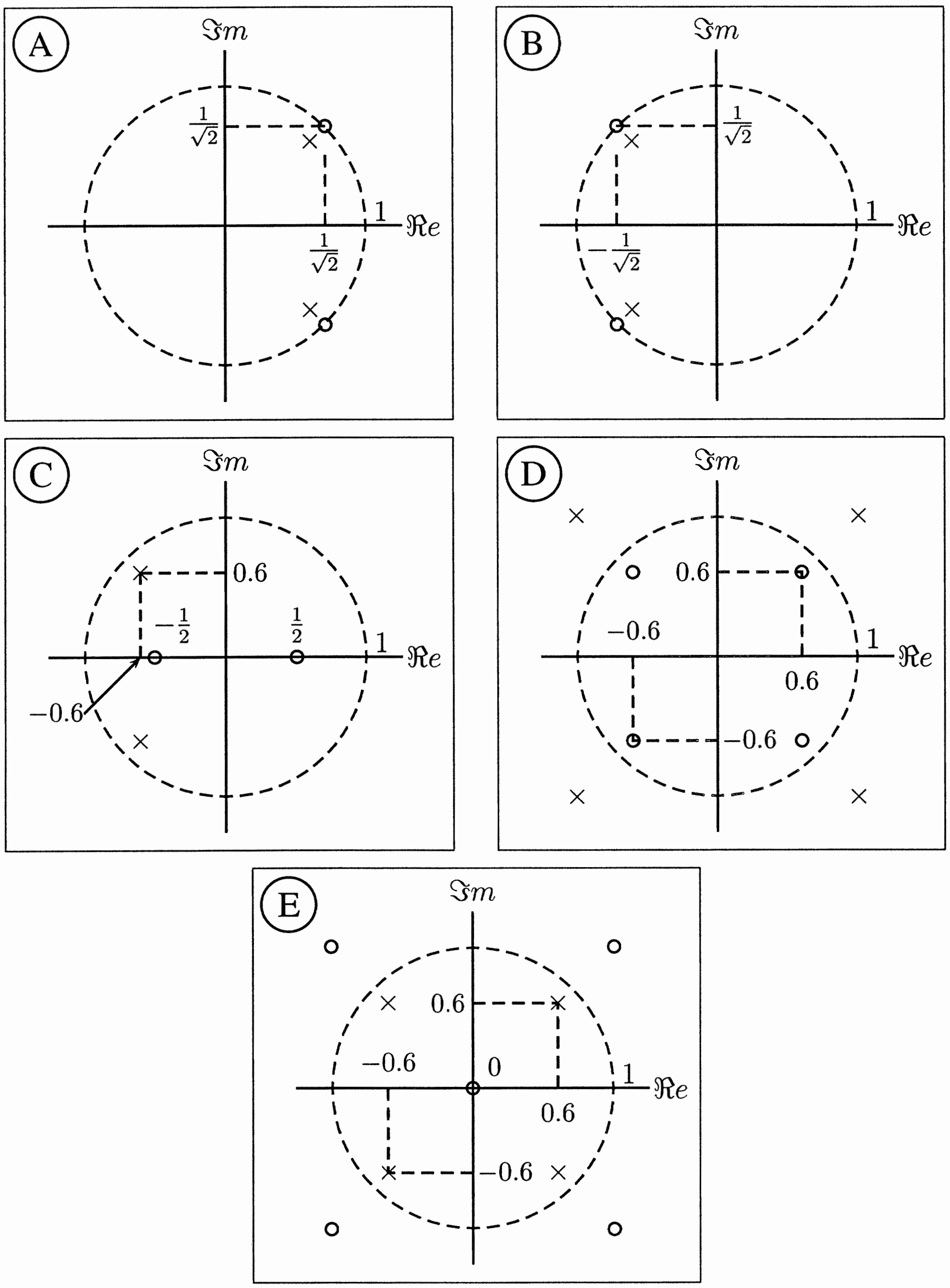
$= y[1]z^{-1} + y[3]z^{-3} + y[4]z^{-4} + y[5]z^{-5}$

$y[n] \rightarrow$



PROBLEM 7 (35 pts)

Consider the five pole-zero plots below. Each plot corresponds to a DT LTI system function whose unit sample response is real. Each plot is drawn to scale. *Note that you have all the information to solve the questions in this problem although some of the poles and zeros are not labeled.* **For your convenience, the identical pole-zero plots to the ones on this page are attached along with the transform tables.**



Work Page for Problem 7

Part a. Which plot(s) can have an ROC so that it corresponds to a causal and stable system ?

Which plot(s) ? A, B, C

Brief explanation:

For causal & stable systems, poles need to be inside the unit circle $\Rightarrow$ A, B, C, E are candidates

E has a pole at infinity as well since there is one more finite zero than finite poles. So E cannot be causal

Part b. Consider the following block diagram

$$x[n] \longrightarrow \boxed{H(z)} \xrightarrow{y[n]} \boxed{G(z)} \longrightarrow w[n] = x[n]$$

$H(z)$ is described by one or more of the pole-zero plots A-E. $G(z)$, which does not correspond to any of the pole-zero plots A-E, is a system such that $w[n] = x[n]$. Which plot(s) corresponds to $H(z)$ such that both $H(z)$ and $G(z)$ are causal and stable ?

Which plot(s) ? C

Brief explanation:

$G(z) = \dfrac{1}{H(z)}$ $\therefore$ Poles of $G(z)$ are at the locations of the zeros of $H(z)$.

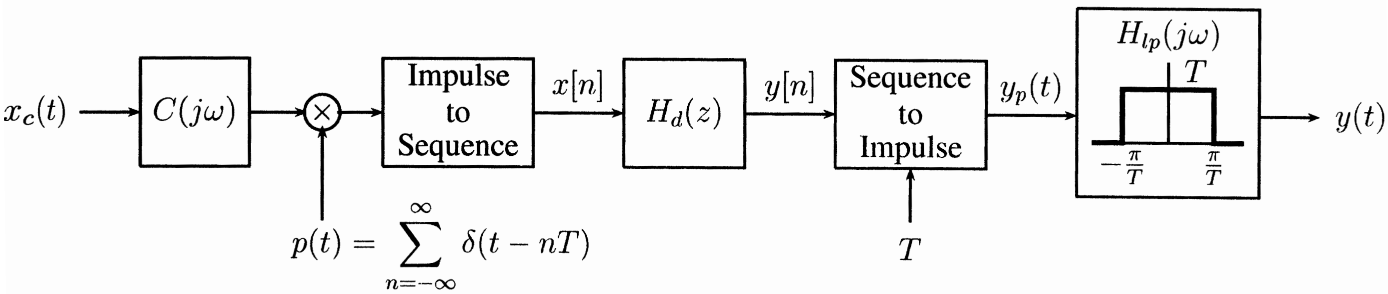
Thus, for poles of $G(z)$ to be within the unit circle the zeros of $H(z)$ have to be within the unit circle. So, for causal & stable $H(z)$ and $G(z)$; all the poles and zeros of $H(z)$ have to be inside the unit circle $\Rightarrow$ C

 Problem 7 continues on the following page.

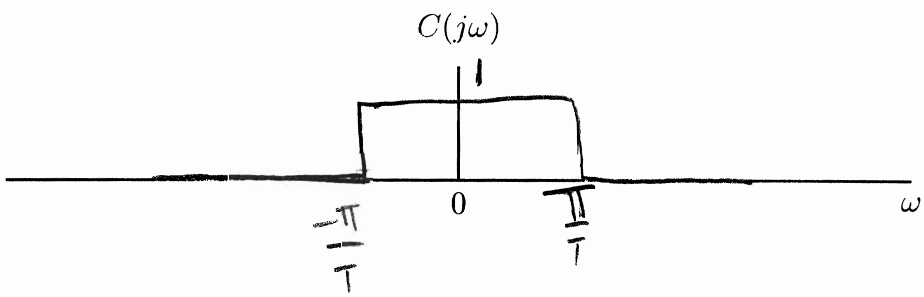
Work Space for Problem 7

 Problem 7 continues on the following page.

Part c. Consider the following system with $T = \frac{1}{480}$ sec.



(i) Plot the frequency response $C(j\omega)$ such that the entire system is LTI with the largest possible bandwidth.



(ii) Assume that $C(j\omega)$ is 1 for all ω and $x_c(t)$ is sufficiently band-limited so that the Nyquist criteria is met. $x_c(t)$ consists of the superposition of $s(t)$ which is the signal you are interested in and a $60Hz$ sinusoidal interference, i.e.,

$$x_c(t) = s(t) + \cos(2\pi \cdot 60t).$$

Which pole-zero plot corresponds to the best choice for $H_d(z)$ such that $|Y(j\omega)|$, the magnitude of the Fourier transform of the overall output $y(t)$ is approximately equal to $|S(j\omega)|$, the magnitude of the Fourier transform of $s(t)$?

Which plot ? $\boxed{\text{A}}$

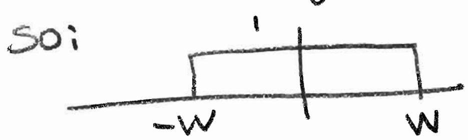
Brief explanation:

Work Page for Problem 7

(c)

(i) We are not told that the input signal is band-limited. So, we use $C(j\omega)$ to force the input to the impulse train multiplier to meet the Nyquist criteria

So:



$$W = \frac{1}{2}\omega_s = \frac{1}{2}\cdot\frac{2\pi}{T} = \frac{\pi}{T}$$

Note that any height of the box works. Any shape that forces Nyquist works too.

(ii) We need a notch filter to remove the 60 Hz interference while not attenuating the rest of the signal by much. A & B are p-z plots of notch filters.

The 60Hz interference in the DT frequency domain is a pair of impulses at $\pm 2\pi \cdot 60 \cdot T$ rad/s $= \pm 2\pi \cdot 60 \cdot \frac{1}{480} = \pm\frac{\pi}{4}$

$\therefore$ We need the notches at $\pm\frac{\pi}{4}$

Since $\tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = \frac{\pi}{4}$, we pick A

Additional Work Page

There are no additional problems from this page on. Pages 34 to 37 are provided solely as additional work pages.