EE 261 The Fourier Transform and its Applications
Fall 2004
Final Exam, December 10, 2004

Notes:
There are 6 questions for a total of 120 points
Write all your answers in your exam booklets
Please be neat and indicate clearly the main parts of your solutions

1. (15 points) Suppose $f(t)$ and $g(t)$ are periodic functions, of period 1, with

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi int}, \quad g(t) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi int}.$$

The convolution of $f(t)$ and $g(t)$ is

$$(f * g)(t) = \int_0^1 f(t-\tau)g(\tau)\, d\tau = \int_0^1 f(\tau)g(t-\tau)\, d\tau.$$

(a) Working directly with the definition of convolution, show that

$$(f * g)(t) = \sum_{n=-\infty}^{\infty} a_n b_n e^{2\pi int}.$$

(b) Suppose we take a *windowed* version of $f(t)$, defined by

$$f_M(t) = \sum_{n=-M}^{M} w_n a_n e^{2\pi int},$$

where w_n is a finite sequence of numbers indexed from $-M$ to M. Write $f_M(t)$ as a convolution.

2. (20 points) Let X be a random variable whose probability density function is $f(x)$. Let $F(s)$ be the Fourier transform of $f(x)$.

   (a) On your formula sheet you will find expressions for the mean (expected value), $E[X; f(x)]$, and the second moment $E[X^2; f(x)]$:

$$E[X; f(x)] = \int_{-\infty}^{\infty} x f(x)\, dx = \frac{i}{2\pi} F'(0)$$

$$E[X^2; f(x)] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = (\frac{i}{2\pi})^2 F''(0)$$

   Derive these formulas.

   (b) If we consider a random variable X resulting from the sum of two independent random variables X_1 and X_2 ($X = X_1 + X_2$), where $f(x)$ is the pdf of X_1 and $g(x)$ is the pdf of X_2. Find the mean and the second moment of the random variable X in terms of the Fourier transforms of f and g.

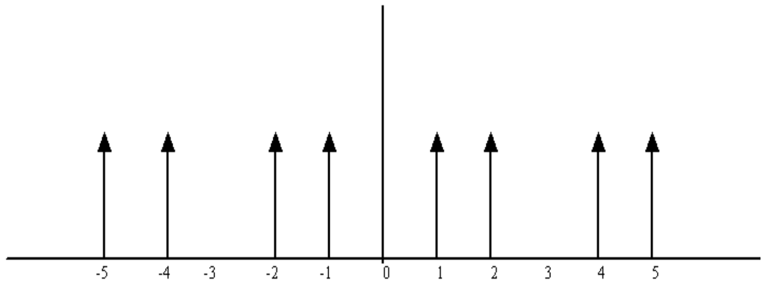
   (c) A shift of $f(x)$ to $f(x-a)$ is still a pdf, and

$$E[X; f(x-a)] = \int_{-\infty}^{\infty} x f(x-a)\, dx, \quad E[X^2; f(x-a)] = \int_{-\infty}^{\infty} x^2 f(x-a)\, dx$$

   Show that

$$\mathrm{Re}[E[X^2; f(x-a)] = E[X^2; f(x)] + a^2$$

3. (25 points) The signal $g(t) = \cos(\pi t/2)$ is sampled by multiplication with $\text{III}(t)$. However, due to an equipment malfunction every third sample is lost, including the sample at the origin, yielding effectively a sampling train $h(t)$ as illustrated below (but of infinite length).



(a) Sketch the *spectrum* of $h(t)$, labeling the location and strength of all the delta functions involved. **Hint**: Express $h(t)$ in terms of two III functions.

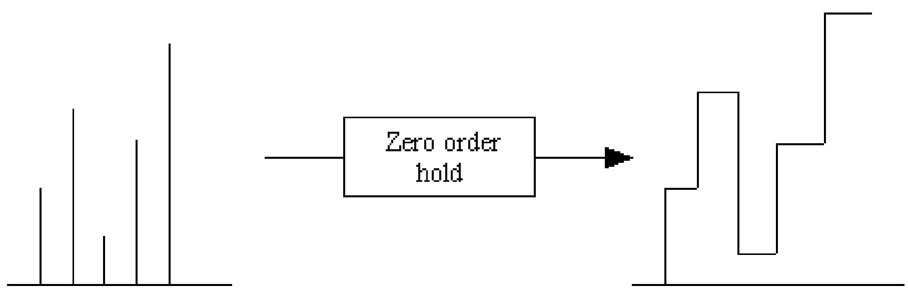
(b) Sketch two or more periods of the spectrum of the sampled signal $h(t)g(t)$, again labeling all delta functions. (You may use the identity $\delta(x-a) * \delta(x-b) = \delta(x-a-b)$.)

(c) Suppose we attempt to reconstruct the sampled signal by passing it through a filter whose transfer function is $\Pi(s)$ (following the derivation of the sampling formula). What is the resulting signal?

4. (20 points) **Zero Order Hold**: The music on your CD has been sampled at the rate 44.1 kHz. This sampling rate comes from the Nyquist theorem together with experimental observations that your ear cannot respond to sounds with frequencies above about 20 kHz. (The precise value 44.1 kHz comes from the technical specs of the earlier audio tape machines that were used when CDs were first geting started.)

   A problem with reconstructing the original music from samples is that interpolation based on the sinc function is not physically realizable – for one thing, the sinc function is not time-limited. Cheap CD players use what is known as 'zero-order hold'. This means that the value of a given sample is *held* until the next sample is read, at which point that sample value is held, and so on.

   Suppose the input is represented by a train of δ-functions, spaced $T = 1/44.1$ msec apart with strengths determined by the sampled values of the music, and the output looks like a staircase function. The system for carrying out zero-order hold then looks like the diagram, below. (The scales on the axes are the same for both the input and the output.)



   (a) Is this a linear system? Is it time invariant for shifts of integer multiples of the sampling period?

   (b) Find the impulse response for this system.

   (c) Find the transfer function.

5. (20 points) Let $\underline{X}$ be a 14-point DFT of a length-14 real sequence $\underline{\mathrm{x}}$. The first 8 values of $\underline{X}[k]$ are given by:

$$\underline{X}[0] = 12, \quad \underline{X}[1] = -1 + 3i, \quad \underline{X}[2] = 3 + 4i, \quad \underline{X}[3] = 1 - 5i,$$
$$\underline{X}[4] = -2 + 2i, \quad \underline{X}[5] = 6 + 3i, \quad \underline{X}[6] = -2 - 3i, \quad \underline{X}[7] = 10.$$

(a) Determine the remaining values of $\underline{X}[k]$.

(b) Evaluate the following without computing the inverse DFT of $\underline{X}$. Justify your answers.

i. $\underline{\mathrm{x}}[0]$

ii. $\underline{\mathrm{x}}[7]$

iii. $\sum_{n=0}^{13} \underline{\mathrm{x}}[n]$

iv. $\sum_{n=0}^{13} \underline{\mathrm{x}}[n] e^{\frac{4\pi i n}{7}}$

v. $\sum_{n=0}^{13} |\underline{\mathrm{x}}[n]|^2$

6. (20 points) Let $\underline{\text{f}}[n]$ be a sequence of length N, where N is even. Let $\underline{\text{F}}[m]$ be its DFT. Let $\underline{\text{g}}[n]$ be the $2N$-element sequence obtained by adding N trailing zeros to $\underline{\text{f}}[n]$. Let $\underline{\text{G}}[m]$ be its DFT.

   (a) Show that $\underline{\text{G}}[2m] = \underline{\text{F}}[m]$ where $m = 0, 1, ..., N-1$

   (b) Show that the odd-indexed elements of $\underline{\text{G}}$ can be obtained as follows:

$$\underline{\text{G}}[v] = \sum_{k=-\infty}^{\infty} \underline{\text{F}}[\frac{v}{2} - k - \frac{1}{2}] \operatorname{sinc}(k + \frac{1}{2})$$

   where $v = 1, 3, ..., 2N-1$