EE 261 The Fourier Transform and its Applications
Fall 2004
Solutions to Final Exam

1. (15 points) Suppose $f(t)$ and $g(t)$ are periodic functions, of period 1, with

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi int}, \quad g(t) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi int}.$$

The convolution of $f(t)$ and $g(t)$ is

$$(f * g)(t) = \int_0^1 f(t-\tau)g(\tau)\,d\tau = \int_0^1 f(\tau)g(t-\tau)\,d\tau.$$

(a) Working directly with the definition of convolution, show that

$$(f * g)(t) = \sum_{n=-\infty}^{\infty} a_n b_n e^{2\pi int}.$$

(b) Suppose we take a *windowed* version of $f(t)$, defined by

$$f_M(t) = \sum_{n=-M}^{M} w_n a_n e^{2\pi int},$$

where w_n is a finite sequence of numbers indexed from $-M$ to M. Write $f_M(t)$ as a convolution.

For part (a), plug the Fourier series for $f(t)$ and $g(t)$ into the formula for convolution:

$$\begin{aligned}
(f * g)(t) &= \int_0^1 f(t-\tau)g(\tau)\,d\tau \\
&= \int_0^1 \left(\sum_{n=-\infty}^{\infty} a_n e^{2\pi in(t-\tau)}\right)\left(\sum_{m=-\infty}^{\infty} b_m e^{2\pi im\tau}\right) d\tau \\
&\qquad \text{(Note that we use different indicies of summation for } f \text{ and } g) \\
&= \int_0^1 \sum_{n,m=-\infty}^{\infty} a_n b_m e^{2\pi in(t-\tau)} e^{2\pi im\tau}\,d\tau \\
&= \sum_{n,m=-\infty}^{\infty} a_n b_m e^{2\pi int} \int_0^1 e^{-2\pi in\tau} e^{2\pi im\tau}\,d\tau \\
&= \sum_{n,m=-\infty}^{\infty} a_n b_m e^{2\pi int} \int_0^1 e^{2\pi i(m-n)\tau}\,d\tau
\end{aligned}$$

We've seen that integral many times:

$$\int_0^1 e^{2\pi i(m-n)\tau}\,d\tau = \begin{cases} 0, & m \neq m \\ 1, & m = n \end{cases}$$

Thus

$$(f * g)(t) = \sum_{n=-\infty}^{\infty} a_n b_n e^{2\pi i n t}.$$

For part (b) we set

$$g(t) = \sum_{n=-M}^{M} w_n e^{2\pi i n t}.$$

Then since the Fourier coefficients of $g(t)$ are zero for $|n| > M$ we have from part (a),

$$(f * g)(t) = \sum_{n=-M}^{M} w_n a_n e^{2\pi i n t}.$$

2. (20 points) Let X be a random variable whose probability density function is $f(x)$. Let $F(s)$ be the Fourier transform of $f(x)$.

   (a) On your formula sheet you will find expressions for the mean (expected value), $E[X; f(x)]$, and the second moment $E[X^2; f(x)]$:

$$E[X; f(x)] = \int_{-\infty}^{\infty} x f(x)\, dx = \frac{i}{2\pi} F'(0)$$

$$E[X^2; f(x)] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = (\frac{i}{2\pi})^2 F''(0)$$

   Derive these formulas.

   (b) If we consider a random variable X resulting from the sum of two independent random variables X_1 and X_2 ($X = X_1 + X_2$), where $f(x)$ is the pdf of X_1 and $g(x)$ is the pdf of X_2. Find the mean and the second moment of the random variable X in terms of the Fourier transforms of f and g.

   (c) A shift of $f(x)$ to $f(x-a)$ is still a pdf, and

$$E[X; f(x-a)] = \int_{-\infty}^{\infty} x f(x-a)\, dx, \quad E[X^2; f(x-a)] = \int_{-\infty}^{\infty} x^2 f(x-a)\, dx$$

   Show that

$$\text{Re}[E[X^2; f(x-a)] = E[X^2; f(x)] + a^2$$

For part (a) we need to use the derivative property of the Fourier Transform. As written on the formula sheet:

$$\mathcal{F}(x^n f(x)) = \left(\frac{i}{2\pi}\right)^n F^{(n)}(s)$$

which when written out for $n = 1$ is

$$\int_{-\infty}^{\infty} e^{-2\pi i s x} x f(x)\, dx = \frac{i}{2\pi} F'(s).$$

Thus

$$E[X; f(x)] = \int_{-\infty}^{\infty} x f(x) dx = \frac{i}{2\pi} F'(0)$$

In a similar way, using the formula for the Fourier transform of the second derivative ($n = 2$),

$$E[X^2; f(x)] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = (\frac{i}{2\pi})^2 F''(0)$$

To do part (b) we need to remember that if a random variable X is the sum of two independent random variables, $X = X_1 + X_2$, then the pdf of X can is the convolution of the pdf of X_1 with the pdf of X_2. Thus X will have as pdf $f(x) * g(x)$, and its first and second moments will be:

$$E[X; f(x) * g(x)] = \int_{-\infty}^{\infty} x(f(x) * g(x)) dx = \frac{i}{2\pi}(FG)'(0) = \frac{i}{2\pi}(F'(0)G(0) + G(0)F'(0))$$

$$\begin{aligned} E[X^2; f(x) * g(x)] &= \int_{-\infty}^{\infty} x^2 (f(x) * g(x))\, dx \\ &= (\frac{i}{2\pi})^2 (FG)''(0) = (\frac{i}{2\pi})^2 (F''(0)G(0) + 2F'(0)G'(0) + G(0)F''(0)) \end{aligned}$$

Let's substitute the shifted signal in formula for the second moment and use with the previous exercises, including the shift property of the Fourier transform.

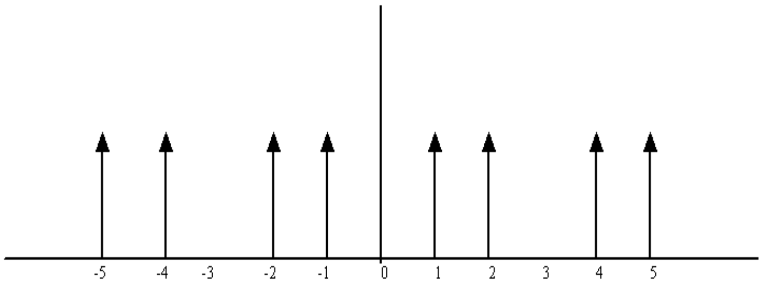
$$E[X^2; f(x-a)] = \int_{-\infty}^{\infty} x^2 f(x-a) dx = (\frac{i}{2\pi})^2 \left.\frac{d^2}{ds^2}\right|_{s=0} (F(s)e^{-2\pi i a s})$$

$$= (\frac{i}{2\pi})^2 [F''(s) + 2(-2\pi a i)e^{-i2\pi a s} F'(s) - 4\pi^2 a^2 e^{i2\pi a s}]|_{s=0}$$

Now it's just a matter of evaluating the real part of this result:

$$Re\{E[X^2; f(x-a)]\} = \frac{-1}{4\pi^2}[F''(s) - 4\pi^2 a^2 e^{i2\pi a s}]|_{s=0} = E[X^2; f(x)] + a^2$$

3. (25 points) The signal $g(t) = \cos(\pi t/2)$ is sampled by multiplication with $\text{III}(t)$. However, due to an equipment malfunction every third sample is lost, including the sample at the origin, yielding effectively a sampling train $h(t)$ as illustrated below (but of infinite length).



(a) Sketch the *spectrum* of $h(t)$, labeling the location and strength of all the delta functions involved. **Hint**: Express $h(t)$ in terms of two III functions.

(b) Sketch two or more periods of the spectrum of the sampled signal $h(t)g(t)$, again labeling all delta functions. (You may use the identity $\delta(x-a) * \delta(x-b) = \delta(x-a-b)$.)

(c) Suppose we attempt to reconstruct the sampled signal by passing it through a filter whose transfer function is $\Pi(s)$ (following the derivation of the sampling formula). What is the resulting signal?

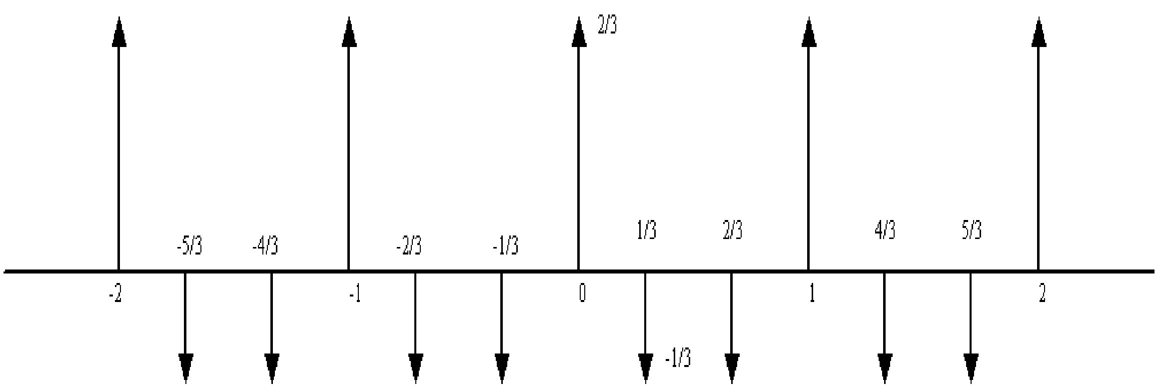
For part (a), the gap-toothed shah function arises by deleting the impulses at $0, \pm 3, \pm 6, \ldots$. This can be written as the difference of two shahs, one the shah with impulses at the integers and the other the shah with impulses at the multiples of 3:

$$h(t) = \text{III}(t) - \text{III}_3(t).$$

The Fourier transform of this is

$$\mathcal{F}h(s) = \text{III}(s) - \frac{1}{3}\text{III}_{1/3}(s).$$

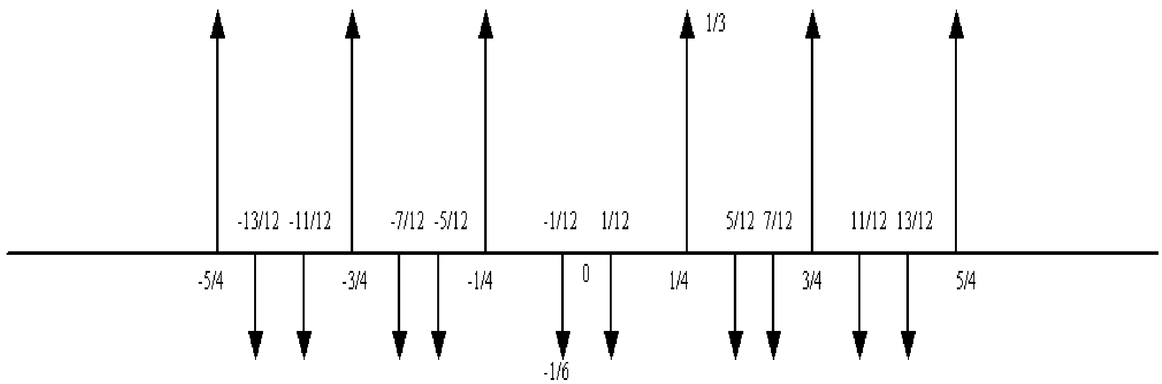
Here's a plot.



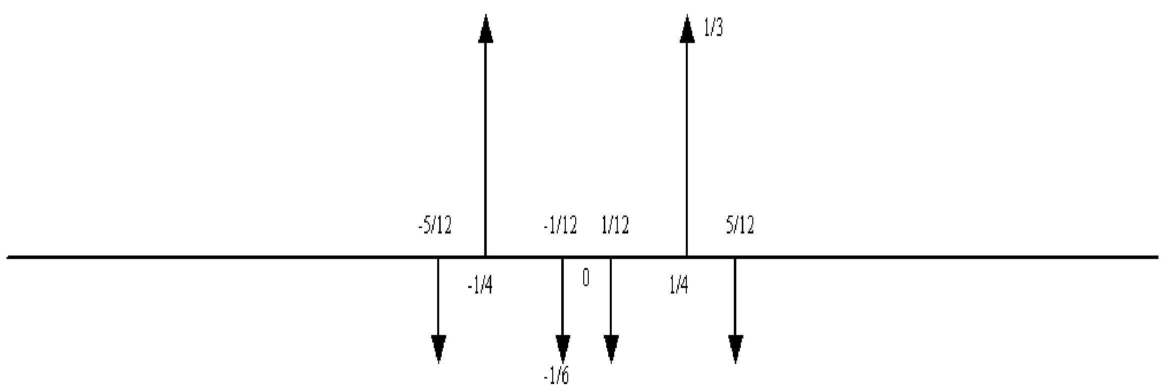
For part (b), the sampled signal is $(\cos(\pi t/2))h(t)$ and its Fourier transform is

$$\begin{aligned}\mathcal{F}(\cos(\pi t/2)) * \mathcal{F}h &= \frac{1}{2}((\delta(s-\frac{1}{4}) + \delta(s+\frac{1}{4})) * (\text{III}(s) - \frac{1}{3}\text{III}_{1/3}(s))\\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}(\delta(s-n-\frac{1}{4}) - \delta(s-n+\frac{1}{4})) - \frac{1}{6}\sum_{n=-\infty}^{\infty}(\delta(s-\frac{n}{3}-\frac{1}{4}) + \delta(s-\frac{n}{3}+\frac{1}{4}))\end{aligned}$$

This looks like



In part (c) we cut this off by $\Pi(s)$, leaving



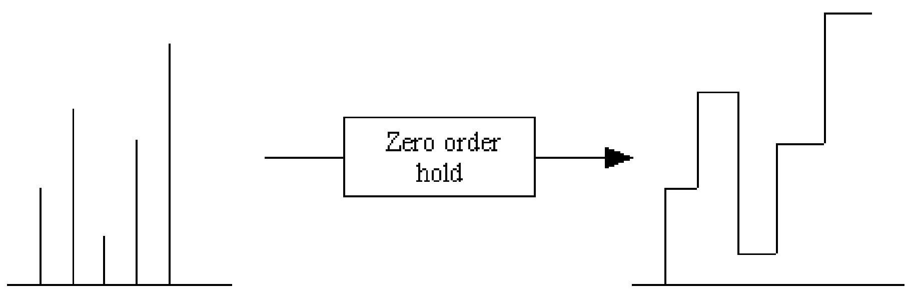
If we now take the inverse Fourier transform we obtain the signal

$$\frac{2}{3}\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{\pi t}{6} - \frac{1}{3}\cos\frac{5\pi t}{6}$$

4. (20 points) **Zero Order Hold**: The music on your CD has been sampled at the rate 44.1 kHz. This sampling rate comes from the Nyquist theorem together with experimental observations that your ear cannot respond to sounds with frequencies above about 20 kHz. (The precise value 44.1 kHz comes from the technical specs of the earlier audio tape machines that were used when CDs were first getting started.)

A problem with reconstructing the original music from samples is that interpolation based on the sinc function is not physically realizable – for one thing, the sinc function is not time-limited. Cheap CD players use what is known as 'zero-order hold'. This means that the value of a given sample is *held* until the next sample is read, at which point that sample value is held, and so on.

Suppose the input is represented by a train of δ-functions, spaced $T = 1/44.1$ msec apart with strengths determined by the sampled values of the music, and the output looks like a staircase function. The system for carrying out zero-order hold then looks like the diagram, below. (The scales on the axes are the same for both the input and the output.)



(a) Is this a linear system? Is it time invariant for shifts of integer multiples of the sampling period?

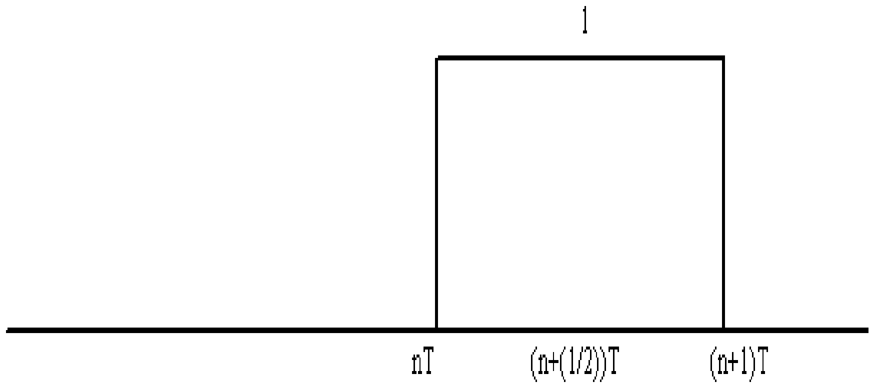
(b) Find the impulse response for this system.

(c) Find the transfer function.

Say the music is represented by a signal $u(t)$. We sample u at a rate T and so the input can be written

$$v(t) = \sum_{n=-\infty}^{\infty} u(nT)\delta(t - nT).$$

The output, $w(t)$, takes the value $u(nT)$ and holds it for T seconds, then takes the value $u((n+1)T))$ and holds it for T seconds, and so on. We can express this as follows. Start with the rect function of width T, that's $\Pi_T(t)$, and center it on the interval from nT to $(n+1)T$, that's $\Pi_T(t - (n + \frac{1}{2})T)$; the shifted rect looks like



Then the output is

$$w(t) = Lu(t) = \sum_{n=-\infty}^{\infty} u(nT)\Pi_T(t - (n + \frac{1}{2})T).$$

From this expression we can easily see that the system is linear, for

$$\begin{aligned} L(u_1(t) + u_2(t)) &= \sum_{n=-\infty}^{\infty} (u_1(nT) + u_2(nT))\Pi_T(t - (n + \frac{1}{2})T) \\ &= \sum_{n=-\infty}^{\infty} (u_1(nT)\Pi_T(t - (n + \frac{1}{2})T) + \sum_{n=-\infty}^{\infty} u_2(nT)\Pi_T(t - (n + \frac{1}{2})T) \\ &= Lu_1(t) + Lu_2(t) \end{aligned}$$

Similarly

$$L(\alpha u(t)) = \alpha Lu(t).$$

Next, suppose we shift the time by an integer multiple of T, say mT. Then the input is

$$\begin{aligned} u(t - mT) &= \sum_{n=-\infty}^{\infty} u(nT)\delta(t - mT - nT) \\ &= \sum_{n=-\infty}^{\infty} u(nT)\delta(t - (n + m)T) \quad \text{(now let } k = m + n \text{ to write the sum as)} \\ &= \sum_{k=-\infty}^{\infty} u((k - m)T)\delta(t - kT) \end{aligned}$$

If $w(t) = Lv(t)$ then the output resulting from the shift is

$$\begin{aligned}
L(u(t-mT)) &= \sum_{k=-\infty}^{\infty} u((k-m)T)\Pi_T(t-(k+\frac{1}{2})T) \\
&\quad \text{(now, undoing what we did before, let } n = k-m \text{ to write the sum as)} \\
&= \sum_{n=-\infty}^{\infty} u(nT)\Pi_T(t-(n+m+\frac{1}{2})T) \\
&= \sum_{n=-\infty}^{\infty} u(nT)\Pi_T(t-(n+\frac{1}{2})T-mT) \\
&= w(t-mT)
\end{aligned}$$

Thus the system is time invariant for shifts by integer multiples of T.

For part (b) we input the delta function, $\delta(t)$, and have to determine the output. The input is concentrated at $t = 0$ with strength 1 and is thereafter 0. So, starting at $t = 0$, the output holds the value 1 for T seconds and is then 0 for $t \geq T$. The output is also 0 for $t < 0$. In other words, the impulse response is

$$h(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases}$$

From the impulse response in part (b) we can now calculate the transfer function asked for in part (c). This is

$$\begin{aligned}
H(s) &= \mathcal{F}h(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} h(t)\, dt \\
&= \int_0^T e^{-2\pi ist}\, dt \\
&= \left[\frac{e^{-2\pi ist}}{-2\pi is}\right]_0^T \\
&= \frac{1-e^{-2\pi usT}}{2\pi is} \\
&= Te^{-\pi isT}\frac{\sin \pi sT}{\pi sT} \\
&= Te^{\pi isT}\operatorname{sinc} sT
\end{aligned}$$

5. (20 points) Let $\underline{X}$ be a 14-point DFT of a length-14 real sequence $\underline{\mathrm{x}}$. The first 8 values of $\underline{X}[k]$ are given by:

$$\underline{X}[0] = 12, \quad \underline{X}[1] = -1+3i, \quad \underline{X}[2] = 3+4i, \quad \underline{X}[3] = 1-5i,$$
$$\underline{X}[4] = -2+2i, \quad \underline{X}[5] = 6+3i, \quad \underline{X}[6] = -2-3i, \quad \underline{X}[7] = 10.$$

(a) Determine the remaining values of $\underline{X}[k]$.

(b) Evaluate the following without computing the inverse DFT of $\underline{X}$. Justify your answers.

i. $\underline{\mathrm{x}}[0]$

ii. $\underline{\mathrm{x}}[7]$

iii. $\sum_{n=0}^{13} \underline{\mathrm{x}}[n]$

iv. $\sum_{n=0}^{13} \underline{\mathrm{x}}[n] e^{\frac{4\pi i n}{7}}$

v. $\sum_{n=0}^{13} |\underline{\mathrm{x}}[n]|^2$

For part (a), since $x[n]$ is real then $\underline{\mathrm{X}}$[k] is Hermitian. Hence $\underline{X}[-k] = \overline{\underline{X}[k]}$. Since the DFT is periodic, this also means that $\underline{X}[N-k] = \overline{\underline{X}[k]}$ where $N = 14$ in our case. This gives us that:

$$\begin{aligned}
\underline{X}[8] &= \overline{\underline{X}[6]} = -2+3i \\
\underline{X}[9] &= \overline{\underline{X}[5]} = 6-3i \\
\underline{X}[10] &= \overline{\underline{X}[4]} = -2-2i \\
\underline{X}[11] &= \overline{\underline{X}[3]} = 1+5i \\
\underline{X}[12] &= \overline{\underline{X}[2]} = 3-4i \\
\underline{X}[13] &= \overline{\underline{X}[0]} = -1-3i.
\end{aligned}$$

Now for part (b),

i. $\underline{\mathrm{x}}[0]=\frac{1}{14}\sum_{n=0}^{13} \underline{X}[n] = \frac{16}{7}$

ii. $\underline{\mathrm{x}}[7] = \frac{1}{14}\sum_{n=0}^{13} \underline{X}[n](-1)^n = -\frac{6}{7}$

iii. $\sum_{n=0}^{13} \underline{\mathrm{x}}[n] = \underline{X}[0] = 12$

iv. Since the DFT of $y[n] = \underline{\mathrm{x}}[n]e^{\frac{4\pi i n}{7}}$ is $Y[k] = \underline{X}[k-4]$ then $\sum_{n=0}^{13} \underline{\mathrm{x}}[n]e^{\frac{4\pi i n}{7}} = Y[0] = \underline{X}[-4] = -2-2i$

v. $\sum_{n=0}^{13} |\underline{\mathrm{x}}[n]|^2 = \frac{1}{14}\sum_{k=0}^{13} |\underline{X}[k]|^2 = \frac{249}{7}$

6. (20 points) Let $\underline{f}[n]$ be a sequence of length N, where N is even. Let $\underline{F}[m]$ be its DFT. Let $\underline{g}[n]$ be the $2N$-element sequence obtained by adding N trailing zeros to $\underline{f}[n]$. Let $\underline{G}[m]$ be its DFT.

   (a) Show that $\underline{G}[2m] = \underline{F}[m]$ where $m = 0, 1, ..., N-1$

   (b) Show that the odd-indexed elements of $\underline{G}$ can be obtained as follows:

$$\underline{G}[v] = \sum_{k=-\infty}^{\infty} \underline{F}[\frac{v}{2} - k - \frac{1}{2}] \operatorname{sinc}(k + \frac{1}{2})$$

   where $v = 1, 3, ..., 2N-1$

For part (a)

$$\begin{aligned} G[m] &= \sum_{k=0}^{k=2N-1} g[n] e^{-2\pi i \frac{nm}{2N}} \\ &= \sum_{k=0}^{k=N-1} g[n] e^{-\pi i \frac{nm}{N}} \end{aligned}$$

Hence,

$$\begin{aligned} G[2m] &= \sum_{k=0}^{k=N-1} g[n] e^{-2\pi i \frac{nm}{N}} \\ &= \sum_{k=0}^{k=N-1} f[n] e^{-2\pi i \frac{nm}{N}} \\ &= F[m] \end{aligned}$$

For part (b), let's find the formula for midpoint interpolation. We know that the general formula for interpolating a function from its samples is given by:

$$x(t) = \sum_{m=-\infty}^{\infty} x(\frac{m}{p}) \operatorname{sinc} p(t - \frac{m}{p})$$

The midpoints in this case will be at $\frac{n}{2p}$ and the formula becomes:

$$x(\frac{n}{2p}) = \sum_{m=-\infty}^{\infty} x(\frac{m}{p}) \operatorname{sinc} p(\frac{n}{2p} - \frac{m}{p})$$

Let $u = \frac{n}{2} - m$. Then

$$x(\frac{n}{2p}) = \sum_{u=-\infty}^{\infty} x(\frac{n}{2p} - \frac{u}{p}) \operatorname{sinc}(u)$$

Now if $p = 1$ then

$$x(\frac{n}{2}) = \sum_{u=-\infty}^{\infty} x(\frac{n}{2} - u)\,\mathrm{sinc}(u)$$

Note here that u is not an integer: $u = \frac{2k+1}{2}$. Thus we have

$$x(\frac{n}{2}) = \sum_{k=-\infty}^{\infty} x(\frac{n}{2} - k - \frac{1}{2})\,\mathrm{sinc}(k + \frac{1}{2})$$

Now the trick is to see that when we zero pad to 2N points, we are effectively sampling the Fourier transform of the $f[n]$ at double sampling rate in the frequency domain to get the DFT of G. Hence we can reconstitute the odd-indexed points of G by using the previous formula, considering that the given points are $G[0], G[2], ...G[2N-2]$ (*i.e.*, assuming that the sampling period is 1). Equivalently the odd-indexed points of G will be obtained by mid-point interpolation between the values $F[0], F[1], F[2], \ldots$ Hence we get the following:

$$G[v] = \sum_{k=-\infty}^{\infty} F[\frac{n}{2} - k - \frac{1}{2}]\,\mathrm{sinc}(k + \frac{1}{2})$$

which is the result we want.