EE 261 The Fourier Transform and its Applications
Fall 2005
Final Exam, December 12, 2005

Notes:
There are 6 questions for a total of 100 points
Write all your answers in your exam booklets
Please be neat and indicate clearly the main parts of your solutions

1. (15 points) (a) Let $f(t)$ be periodic of period 1 and $g(t)$ be periodic of period 2. Find the Fourier series of $h(t) = f(t) + g(t)$, that is, find the Fourier coefficients of $h(t)$ in terms of the Fourier coefficients of $f(t)$ and $g(t)$.

   (b) Suppose $f(t)$ is periodic of period T. Find a delay τ_n so that the n'th Fourier coefficient of $f(t - \tau_n)$ is real and positive; real and negative; purely imaginary. The answers should be expressed in terms of the n'th Fourier coefficient of $f(t)$.

2. (15 points) "Hey", said a student excited by the sampling theorem, "I'm not so sure you need infinitely many sample points. Suppose a signal $f(t)$ is bandlimited, like always, with $\mathcal{F}f(s) \equiv 0$ for $|s| \geq p/2$, like always. Now use a *finite* version of the Ш function, say

$$Ш_p^N(x) = \sum_{k=-N}^{N} \delta(x - kp)\,,$$

and we still have

$$\mathcal{F}f = \Pi_p(\mathcal{F}f * Ш_p^N)$$

just like in the derivation of the usual sampling theorem. Now if we take the inverse Fourier transform don't we get $f(t)$ back using just finitely many samples?"

Do you? What formula do you get?

3. (20 points) *Fourier transforms*

(a) Find the Fourier transform of

$$f(t) = \frac{t}{1 + 2t^2 + t^4} = \frac{t}{(1 + t^2)^2} .$$

Use

$$\mathcal{F}(e^{-|t|}) = \frac{2}{1 + 4\pi^2 s^2} .$$

(b) Find the Fourier transform of the polynomial $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$. (This was one of Professor Osgood's quals questions a few years ago.)

(c) Given that

$$\mathcal{F}(|t|) = -\frac{1}{2\pi^2 s^2} ,$$

find the Fourier transform of the unit ramp

$$r(t) = \begin{cases} t\,, & t \geq 0, \\ 0\,, & t \leq 0\,. \end{cases}$$

(d) Find $\underline{1} * \underline{\mathrm{f}}$, where $\underline{1} = (1, 1, \cdots, 1)$ and $\underline{\mathrm{f}}$ is a discrete signal.

4. (20 points) Let

$$\underline{\mathrm{f}} = (\underline{\mathrm{f}}[0], \underline{\mathrm{f}}[1], \underline{\mathrm{f}}[2], \underline{\mathrm{f}}[3], \underline{\mathrm{f}}[4])$$

be a real-valued five-point signal. Append 2 zeros to $\underline{\mathrm{f}}$, making the seven-point signal

$$(\underline{\mathrm{f}}[0], \underline{\mathrm{f}}[1], \underline{\mathrm{f}}[2], \underline{\mathrm{f}}[3], \underline{\mathrm{f}}[4], 0, 0)\,,$$

and let $\underline{\mathrm{F}}$ denote the DFT of this seven-point signal. Let

$$\underline{\mathrm{G}}[m] = \operatorname{Re} F[m] \quad \text{(real part)},$$

and let $\underline{\mathrm{g}}$ be the seven-point signal whose DFT is $\underline{\mathrm{G}}$.

(a) Show that $\underline{\mathrm{g}}[0] = \underline{\mathrm{f}}[0]$.

(b) Show that $\underline{\mathrm{g}}[1] = \frac{1}{2}\underline{\mathrm{f}}[1]$.

5. (20 points) Consider an LTI system $y(t) = Lx(t)$. When the input is a pulse $x(t) = u(t) - u(t-1)$, where $u(t)$ is the unit step, the output is $y(t) = e^{-t}u(t) - e^{-(t-1)}u(t-1)$.

   (a) Find the impulse response of the system and the transfer function. Is the system (essentially) a low pass or a high pass filter? That is, if $H(s)$ is the transfer function, examine the behavior of $|H(s)|$ as $s \to 0$ and as $s \to \infty$.

   (c) Suppose the input is given by $x(t) = \sin(2\pi\nu t)$. Find the output $y(t)$ in terms of *real* functions.

6. (10 points) *Projections and 3D Fourier transforms*

Let $f(x_1, x_2, x_3)$ be a 3-dimensional function whose Fourier transform is $\mathcal{F}f(\xi_1, \xi_2, \xi_3)$

(a) Let

$$g(x_1, x_2) = \int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, dx_3\,.$$

One says that g is the projection of f *along* the x_3 direction. Find $\mathcal{F}g(\xi_1, \xi_2)$ in terms of $\mathcal{F}f$

(b) Let

$$h(x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, dx_1 dx_2\,.$$

One says that h is the projection of f *onto* the x_3 direction. Find $\mathcal{F}h(\xi_3)$ in terms of $\mathcal{F}f$