EE 261 The Fourier Transform and its Applications
Fall 2005
Final Exam Solutions

1. (15 points) (a) Let $f(t)$ be periodic of period 1 and $g(t)$ be periodic of period 2. Find the Fourier series of $h(t) = f(t) + g(t)$, that is, find the Fourier coefficients of $h(t)$ in terms of the Fourier coefficients of $f(t)$ and $g(t)$.

   (b) Suppose $f(t)$ is periodic of period T. Find a delay τ_n so that the n'th Fourier coefficient of $f(t - \tau_n)$ is real and positive; real and negative; purely imaginary. The answers should be expressed in terms of the n'th Fourier coefficient of $f(t)$.

   Solution for (a) $h(t) = f(t) + g(t)$ is periodic of period 2 so has a Fourier series of the form

$$h(t) = \sum_{n=-\infty}^{\infty} c_n e^{\pi int} .$$

   Write the Fourier series for $f(t)$ and $g(t)$, respectively, as

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi int}, \quad g(t) = \sum_{n=-\infty}^{\infty} b_n e^{\pi int},$$

   so that

$$\sum_{n=-\infty}^{\infty} c_n e^{\pi int} = \sum_{n=-\infty}^{\infty} a_n e^{2\pi int} + \sum_{n=-\infty}^{\infty} b_n e^{\pi int} .$$

   Compare like terms on both sides. For the constant term we have

$$c_0 = a_0 + b_0 .$$

   The left-hand-side (the series we want) has all multiples of π in the exponential, as does the series for $g(t)$ (the b_n's), while the series for $f(t)$ (the a_n's) contributes only to the even multiples of π in the exponentials. That is,

$$\begin{aligned} n \text{ odd} \quad &\Longrightarrow c_n = b_n ; \\ n \text{ even} \quad &\Longrightarrow c_n = a_{n/2} + b_n . \end{aligned}$$

   Solution for (b) Write

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt/T} .$$

   Then $f(t - \tau_n)$ has Fourier series

$$f(t - \tau_n) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ik(t-\tau_n)/T} = \sum_{k=-\infty}^{\infty} c_k e^{-2\pi ik\tau_n/T} e^{2\pi ikt/T},$$

   and the n'th Fourier coefficient is

$$e^{-2\pi in\tau_n/T} c_n .$$

   Now write

$$c_n = |c_n| e^{i \arg c_n},$$

   so that

$$e^{-2\pi in\tau_n/T} c_n = |c_n| e^{i(\arg c_n - 2\pi n\tau_n/T)} .$$

To make this real and positive we want τ_n so that $\arg c_n - 2\pi n\tau_n/T = 0$, in which case the Fourier coefficient of the delayed signal is $|c_n|$. We get this by taking

$$\tau_n = \frac{T \arg c_n}{2\pi n}\,.$$

We can make the new coefficient real and negative, equal to $-|c_n|$, by making $\arg c_n - 2\pi n\tau_n/T = \pi$, and so

$$\tau_n = \frac{T(\arg c_n - \pi)}{2\pi n}\,.$$

We can make the new coefficient purely imaginary by making $\arg c_n - 2\pi n\tau_n/T = \pm\pi/2$, giving

$$\tau_n = \frac{T(\arg c_n \mp \pi/2)}{2\pi n}\,.$$

2. (15 points) “Hey”, said a student excited by the sampling theorem, “I’m not so sure you need infinitely many sample points. Suppose a signal $f(t)$ is bandlimited, like always, with $\mathcal{F}f(s) \equiv 0$ for $|s| \geq p/2$, like always. Now use a *finite* version of the Ш function, say

$$Ш_p^N(x) = \sum_{k=-N}^{N} \delta(x - kp)\,,$$

and we still have

$$\mathcal{F}f = \Pi_p(\mathcal{F}f * Ш_p^N)$$

just like in the derivation of the usual sampling theorem. Now if we take the inverse Fourier transform don’t we get $f(t)$ back using just finitely many samples?”

Do you? What formula do you get?

Solution To obtain the Fourier transform of $Ш_p^N$ we use $\mathcal{F}(\delta(x-a)) = e^{-2\pi i s a}$. This gives

$$\mathcal{F}Ш_p^N(t) = \sum_{n=-N}^{N} \mathcal{F}(\delta(t - kp)) = \sum_{n=-N}^{N} e^{-2\pi i n p s}\,.$$

The Poisson summation formula miraculously converts the Fourier transform of the *full* Ш to a sum of impulses, but for the finite Ш the best we can do is to use the formula for a geometric series to find a simpler expression. From the formula sheet this is

$$\sum_{n=-N}^{N} e^{-2\pi i n p s} = \frac{\sin(\pi(2N+1)s)}{\sin \pi s}\,.$$

Since $Ш_p^N$ is even we also have

$$\mathcal{F}^{-1}Ш_p^N(t) = \frac{\sin(\pi(2N+1)s)}{\sin \pi s}\,.$$

To see what happens with the derivation of the sampling formula, we still have

$$\mathcal{F}f(s) = \Pi_p(s)(\mathcal{F}f(s) * Ш_p^N(s)),$$

from which

$$\begin{aligned} f(t) &= (p \operatorname{sinc} pt) * (f \cdot \mathcal{F}^{-1}Ш_p^N)(t) \\ &= (p \operatorname{sinc} pt) * \left(f(t) \frac{\sin(\pi(2N+1)t)}{\sin \pi t} \right) \\ &= \int_{-\infty}^{\infty} p \operatorname{sinc} p(t-\tau) \left(f(\tau) \frac{\sin(\pi(2N+1)\tau)}{\sin \pi \tau} \right) d\tau\,. \end{aligned}$$

This is no longer a ‘sampling formula’ because $\mathcal{F}^{-1}Ш_p^N$ is not a Ш on the right hand side of the expression $f(t) = (p \operatorname{sinc} pt) * (f \cdot \mathcal{F}^{-1}Ш_p^N)(t)$, and hence no samples of f are taken.

3. (20 points) *Fourier transforms*

(a) Find the Fourier transform of

$$f(t) = \frac{t}{1+2t^2+t^4} = \frac{t}{(1+t^2)^2}\,.$$

Use

$$\mathcal{F}(e^{-|t|}) = \frac{2}{1+4\pi^2 s^2}\,.$$

(b) Find the Fourier transform of the polynomial $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$. (This was one of Professor Osgood's quals questions a few years ago.)

(c) Given that

$$\mathcal{F}(|t|) = -\frac{1}{2\pi^2 s^2}\,,$$

find the Fourier transform of the unit ramp

$$r(t) = \begin{cases} t\,, & t \geq 0, \\ 0\,, & t \leq 0\,. \end{cases}$$

(d) Find $\underline{1} * \underline{f}$, where $\underline{1} = (1, 1, \cdots, 1)$ and $\underline{f}$ is a discrete signal.

Solutions

(a) From the formula sheet:

$$\mathcal{F}\{e^{-|t|}\} = \frac{2}{1+4\pi^2 s^2}$$

Then, by duality,

$$\mathcal{F}\{\frac{1}{1+4\pi^2 t^2}\} = \frac{1}{2}e^{-|-s|} = \frac{1}{2}e^{-|s|}$$

Let $g(t) = \frac{1}{1+t^2}$. By the stretch theorem we get:

$$\mathcal{F}g(s) = \pi e^{-2\pi|s|}$$

Let us compute the derivative of g:

$$g'(t) = \frac{-2t}{(1+t^2)^2}$$

$$\mathcal{F}\{g'\}(s) = 2\pi i s \mathcal{F}g(s) = 2\pi^2 i s e^{-2\pi|s|}$$

Notice that $f(t) = -\frac{1}{2}g'(t)$ then:

$$\mathcal{F}f(s) = -\pi^2 i s e^{-2\pi|s|}$$

This makes sense: f is real and odd and its Fourier transform is pure imaginary and odd.

(b) We use the derivative formula

$$\mathcal{F}(t^n G(t)) = \left(\frac{i}{2\pi}\right)^n G^{(n)}(s)\,,$$

where $G = \mathcal{F}g$. Apply this with $g(t) = 1$, for which $G = \delta$ and we have

$$\mathcal{F}(t^n) = \left(\frac{i}{2\pi}\right)^n \delta^{(n)}(s)\,.$$

Thus the Fourier transform of the polynomial is

$$\mathcal{F}(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = a_0\delta + a_1 \frac{i}{2\pi}\delta' + a_2 \left(\frac{i}{2\pi}\right)^2 \delta'' + \cdots + a_n \left(\frac{i}{2\pi}\right)^n \delta^{(n)}\,.$$

(c) We can write

$$r(t) = \frac{1}{2}(t + |t|)$$

From Part (b) (or from the formula sheet), the Fourier transform of the function $f(t) = t$ is

$$\mathcal{F}(t) = \frac{i}{2\pi}\delta'$$

and from the formula given for $\mathcal{F}(|t|)$ we have

$$\mathcal{F}r(s) = \frac{1}{2}\left(\frac{i}{2\pi}\delta' - \frac{1}{2\pi^2 s^2}\right)\,.$$

(d) In convolving with the constant vector $\underline{1}$ the m'th component of the convolution is given by

$$(\underline{1} * \underline{f})[m] = \sum_{n=0}^{N-1} \underline{f}[n]\,,$$

i.e., it is a constant, independent of m and, in fact, is simply $\underline{\mathcal{F}}\underline{f}[0]$. Thus

$$\underline{1} * \underline{f} = (\underline{\mathcal{F}}\,\underline{f}[0], \underline{\mathcal{F}}\,\underline{f}[0], \ldots, \underline{\mathcal{F}}\,\underline{f}[0]) = \underline{\mathcal{F}}\,\underline{f}[0]\underline{1}\,.$$

This result also follows easily by taking the discrete Fourier transform. For

$$\underline{\mathcal{F}}\,(\underline{1} * \underline{f}) = \underline{\mathcal{F}}\,\underline{1}\,\underline{\mathcal{F}}\,\underline{f} = \underline{\delta}_0\,\underline{\mathcal{F}}\,\underline{f} = \underline{\mathcal{F}}\,\underline{f}[0]\underline{\delta}_0\,,$$

and then taking the inverse discrete Fourier transform gives the earlier result:

$$\underline{1} * \underline{f} = \underline{\mathcal{F}}^{-1}(\underline{\mathcal{F}}\,\underline{f}[0]\underline{\delta}_0) = \underline{\mathcal{F}}\,\underline{f}[0]\underline{\mathcal{F}}^{-1}\underline{\delta}_0 = \underline{\mathcal{F}}\,\underline{f}[0]\underline{1}\,.$$

Note that in the continuous case convolving a signal $f(t)$ with the constant function 1 gives the constant value

$$(1 * f)(t) = \int_{-\infty}^{\infty} f(t)\,dt = \mathcal{F}f(0)\,.$$

The integral of $f(t)$, as compared to the sum of the values $\underline{f}[n]$, and its value $\mathcal{F}f(0)$ may be considered as the continuous analog of the result of this problem.

4. (20 points) Let

$$\underline{f} = (\underline{f}[0], \underline{f}[1], \underline{f}[2], \underline{f}[3], \underline{f}[4])$$

be a real-valued five-point signal. Append 2 zeros to $\underline{f}$, making the seven-point signal

$$(\underline{f}[0], \underline{f}[1], \underline{f}[2], \underline{f}[3], \underline{f}[4], 0, 0)\,,$$

and let $\underline{F}$ denote the DFT of this seven-point signal. Let

$$\underline{G}[m] = \mathrm{Re}\,\underline{F}[m] \quad \text{(real part)},$$

and let $\underline{g}$ be the seven-point signal whose DFT is $\underline{G}$.

(a) Show that $\underline{g}[0] = \underline{f}[0]$.

(b) Show that $\underline{g}[1] = \frac{1}{2}\underline{f}[1]$.

Solutions: Let $\tilde{\underline{f}}$ be the seven-point sequence defined by:

$$\tilde{\underline{f}}[n] = \begin{cases} \underline{f}[n], & n = 0, 1, \ldots 4 \\ 0, & n = 5, 6 \end{cases}$$

The easiest approach to the problem is to reindex, using both negative and positive indices. Using that $\tilde{\underline{f}}$ is real and hence that $\underline{F}$ is Hermitian, we find

$$\begin{aligned} \underline{G}[m] &= \mathrm{Re}(\underline{F}[m]) \\ &= \frac{\underline{F}[m] + \overline{\underline{F}[m]}}{2} \\ &= \frac{\underline{F}[m] + \underline{F}[-m]}{2} \quad \text{(reindexing)}\,. \end{aligned}$$

Taking the inverse DFT,

$$\underline{g}[n] = \frac{\tilde{\underline{f}}[n] + \tilde{\underline{f}}[-n]}{2}$$

Hence

$$\begin{aligned} \underline{g}[0] &= \frac{\tilde{\underline{f}}[0] + \tilde{\underline{f}}[0]}{2} \\ &= \tilde{\underline{f}}[0] \\ &= \underline{f}[0] \end{aligned}$$

the last line holding since $\underline{f}[0] = \tilde{\underline{f}}[0]$. This is part (a). For part (b),

$$\begin{aligned} \underline{g}[1] &= \frac{\tilde{\underline{f}}[1] + \tilde{\underline{f}}[-1]}{2} \\ &= \frac{\tilde{\underline{f}}[1] + \tilde{\underline{f}}[6]}{2} \\ &= \frac{\tilde{\underline{f}}[1] + 0}{2} \\ &= \frac{\underline{f}\tilde{[1]}}{2} \end{aligned}$$

If you didn't reindex, the solutions go as follows: The Hermitian property of $\underline{\mathrm{F}}$ implies that $\underline{\mathrm{F}}[7-m] = \overline{\underline{\mathrm{F}}[m]}$ which is equivalent to:

$$\begin{aligned} \mathrm{Re}(\underline{\mathrm{F}}[m]) &= \mathrm{Re}(\underline{\mathrm{F}}[7-m]) \\ \mathrm{Im}(\underline{\mathrm{F}}[m]) &= -\mathrm{Im}(\underline{\mathrm{F}}[7-m]) \end{aligned}$$

Note also that $\underline{\mathrm{F}}[0] = \mathrm{Re}(\underline{\mathrm{F}}[0])$ since $\underline{\mathrm{F}}[0] = \sum_{n=0}^{6} \tilde{\underline{\mathrm{f}}}[n]$.
Using again $\underline{\mathrm{f}}[0] = \tilde{\underline{\mathrm{f}}}[0]$ we obtain

$$\begin{aligned} \underline{\mathrm{f}}[0] = \tilde{\underline{\mathrm{f}}}[0] &= \frac{1}{7}\sum_{m=0}^{6} \underline{\mathrm{F}}[m] \\ &= \frac{1}{7}\left\{\underline{\mathrm{F}}[0] + \sum_{m=1}^{3}(\underline{\mathrm{F}}[m] + \underline{\mathrm{F}}[7-m])\right\} \\ &= \frac{1}{7}\left\{\underline{\mathrm{F}}[0] + \sum_{m=1}^{3}(\underline{\mathrm{F}}[m] + \overline{\underline{\mathrm{F}}[m]})\right\} \\ &= \frac{1}{7}\left\{\mathrm{Re}(\underline{\mathrm{F}}[0]) + 2\sum_{m=1}^{3}\mathrm{Re}(\underline{\mathrm{F}}[m])\right\} \\ &= \frac{1}{7}\sum_{m=0}^{6}\mathrm{Re}(\underline{\mathrm{F}}[m]) \\ &= \underline{\mathrm{g}}[0] \end{aligned}$$

For Part (b):

$$\begin{aligned} \underline{\mathrm{g}}[1] &= \frac{1}{7}\sum_{m=0}^{6}\mathrm{Re}(\underline{\mathrm{F}}[m])e^{2\pi i m/7} \\ &= \frac{1}{7}\left\{\mathrm{Re}(\underline{\mathrm{F}}[0]) + \sum_{m=1}^{3}(\mathrm{Re}(\underline{\mathrm{F}}[m])e^{2\pi i m/7} + \mathrm{Re}(\underline{\mathrm{F}}[7-m])e^{-2\pi i m/7})\right\} \\ &= \frac{1}{7}\left\{\mathrm{Re}(\underline{\mathrm{F}}[0]) + \sum_{m=1}^{3}(\mathrm{Re}(\underline{\mathrm{F}}[m])e^{2\pi i m/7} + \mathrm{Re}(\underline{\mathrm{F}}[m])e^{-2\pi i m/7})\right\} \\ &= \frac{1}{7}\left\{\mathrm{Re}(\underline{\mathrm{F}}[0]) + 2\sum_{m=1}^{3}\mathrm{Re}(\underline{\mathrm{F}}[m])\cos(\frac{2\pi m}{7})\right\} \end{aligned}$$

Next,

$$\tilde{\underline{\mathrm{f}}}[1] = \frac{1}{7}\sum_{m=0}^{6}\underline{\mathrm{F}}[m]e^{2\pi i m/7}$$

and

$$\tilde{\underline{\mathrm{f}}}[6] = \frac{1}{7}\sum_{m=0}^{6}\underline{\mathrm{F}}[m]e^{2\pi i(6m)/7} = \frac{1}{7}\sum_{m=0}^{6}\underline{\mathrm{F}}[m]e^{-2\pi i m/7}$$

Let us check what $\underline{\tilde{f}}[1] + \underline{\tilde{f}}[6]$ is:

$$\begin{aligned}
\underline{\tilde{f}}[1] + \underline{\tilde{f}}[6] &= \frac{1}{7}\sum_{m=0}^{6} \underline{F}[m](e^{2\pi im/7} + e^{-2\pi im/7}) \\
&= \frac{2}{7}\sum_{m=0}^{6} \underline{F}[m]\cos(\frac{2\pi m}{7}) \\
&= \frac{2}{7}\left\{\underline{F}[0] + \sum_{m=1}^{3}(\underline{F}[m] + \underline{F}[7-m])\cos(\frac{2\pi m}{7})\right\} \\
&= \frac{2}{7}\left\{\underline{F}[0] + 2\sum_{m=1}^{3}\mathrm{Re}(\underline{F}[m])\cos(\frac{2\pi m}{7})\right\} \\
&= 2\underline{g}[1]
\end{aligned}$$

Now recall that $\underline{\tilde{f}}[1] = \underline{f}[1]$ and $\underline{\tilde{f}}[6] = 0$, so

$$\underline{g}[1] = \frac{1}{2}f[1]$$

5. (20 points) Consider an LTI system $y(t) = Lx(t)$. When the input is a pulse $x(t) = u(t) - u(t-1)$, where $u(t)$ is the unit step, the output is $y(t) = e^{-t}u(t) - e^{-(t-1)}u(t-1)$.

   (a) Find the impulse response of the system and the transfer function. Is the system (essentially) a low pass or a high pass filter? That is, if $H(s)$ is the transfer function, examine the behavior of $|H(s)|$ as $s \to 0$ and as $s \to \infty$.

   (c) Suppose the input is given by $x(t) = \sin(2\pi\nu t)$. Find the output $y(t)$ in terms of *real* functions.

Solution (a) Since the system is LTI, it's not difficult to see by inspection that the unit step response is given by $v(t) = e^{-t}u(t)$. If this wouldn't be the case, then assuming the system is LTI would lead us to a contradiction. Another way of finding $v(t)$ is by construction of a telescopic sum (middle terms would cancel and the last one would tend to zero). Now:

$$h(t) = v'(t) = \frac{d}{dt}e^{-t}u(t) = -e^{-t}u(t) + e^{-t}\delta(t) = -e^{-t}u(t) + \delta(t)$$

The transfer function is then given by:

$$H(s) = -\frac{1}{2\pi i s + 1} + 1 = \frac{2\pi i s}{2\pi i s + 1}$$

A different approach would be to first compute $H(s)$ and then take the inverse FT to compute $h(t)$, obtaining the same result as before:

$$\begin{aligned}
Y(s) &= \frac{1}{2\pi i s + 1} - \frac{e^{-2\pi i s}}{2\pi i s + 1} \\
&= \frac{1}{2\pi i s + 1}(1 - e^{-2\pi i s}) \\
X(s) &= (\frac{\delta(s)}{2} + \frac{1}{2\pi i s})(1 - e^{-2\pi i s}) \\
&= \frac{1}{2\pi i s}(1 - e^{-2\pi i s}) \\
\Rightarrow H(s) &= \frac{Y(s)}{X(x)} \\
&= \frac{2\pi i s}{1 + 2\pi i s} = 1 - \frac{1}{1 + 2\pi i s}
\end{aligned}$$

This is a high-pass filter since $|H(s)| \to 0$ as $s \to 0$ and $|H(s)| \to 1$ as $s \to \infty$

(b) Note that

$$\begin{aligned}
x(t) &= \sin(2\pi\nu t) \\
&= \frac{1}{2i}(e^{2\pi i\nu t} - e^{-2\pi i\nu t})
\end{aligned}$$

Now,

$$
\begin{aligned}
Y(s) &= H(s)X(s) \\
&= \frac{1}{2i}H(s)\left(\delta(s-\nu)-\delta(s+\nu)\right) \\
&= \frac{1}{2i}\left(H(\nu)\delta(s-\nu)-H(-\nu)\delta(s+\nu)\right) \\
&= \frac{1}{2i}\left(\frac{2\pi i\nu}{1+2\pi i\nu}\delta(s-\nu)+\frac{2\pi i\nu}{1-2\pi i\nu}\delta(s+\nu)\right) \\
&= \frac{1}{2i(1+4\pi^2\nu^2)}\big((2\pi i\nu+4\pi^2\nu^2)\delta(s-\nu)+(2\pi i\nu-4\pi^2\nu^2)\delta(s+\nu)\big)
\end{aligned}
$$

Hence,

$$
y(t) = \frac{1}{(1+4\pi^2\nu^2)}\big(2\pi\nu\cos(2\pi\nu t)+4\pi^2\nu^2\sin(2\pi\nu t)\big)
$$

6. (10 points) *Projections and 3D Fourier transforms*

Let $f(x_1, x_2, x_3)$ be a 3-dimensional function whose Fourier transform is $\mathcal{F}f(\xi_1, \xi_2, \xi_3)$

(a) Let

$$g(x_1, x_2) = \int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, dx_3 \,.$$

One says that g is the projection of f *along* the x_3 direction. Find $\mathcal{F}g(\xi_1, \xi_2)$ in terms of $\mathcal{F}f$

(b) Let

$$h(x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, dx_1 dx_2 \,.$$

One says that h is the projection of f *onto* the x_3 direction. Find $\mathcal{F}h(\xi_3)$ in terms of $\mathcal{F}f$

Solution:

(a)

$$\begin{aligned}
\mathcal{F}g(\xi_1, \xi_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2) e^{-2\pi i (x_1\xi_1 + x_2\xi_2)} dx_1 dx_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x_1, x_2, x_3) dx_3 \right) e^{-2\pi i (x_1\xi_1 + x_2\xi_2)} dx_1 dx_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_3) e^{-2\pi i (x_1\xi_1 + x_2\xi_2)} dx_1 dx_2 dx_3 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_3) e^{-2\pi i (x_1\xi_1 + x_2\xi_2 + x_3 \cdot 0)} dx_1 dx_2 dx_3 \\
&= \mathcal{F}f(\xi_1, \xi_2, 0)
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{F}h(\xi_3) &= \int_{-\infty}^{\infty} h(x_3) e^{-2\pi i x_3 \xi_3} dx_3 \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_2) dx_1 dx_2 \right) e^{-2\pi i x_3 \xi_3} dx_3 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_2) e^{-2\pi i x_3 \xi_3} dx_1 dx_2 dx_3 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_2) e^{-2\pi i (x_1 \cdot 0 + x_2 \cdot 0 + x_3 \xi_3)} dx_1 dx_2 dx_3 \\
&= \mathcal{F}f(0, 0, \xi_3)
\end{aligned}$$