1. (a) Using KVL, we have

$$V_c(s) + sI_1(s) = V_1(s) \tag{1}$$

$$sI_1(s) + I_2(s) = sV_c(s) - I_1(s) \;\Rightarrow\; I_1(s) = \frac{s}{s+1}V_c(s) \tag{2}$$

Substituting (2)→(1), we have

$$V_c(s) + \frac{s^2}{s+1}V_c(s) = V_1(s) \quad \text{or} \quad V_c(s) = \frac{s+1}{s^2+s+1}V_1(s)$$

$$\therefore\; V_2(s) = I_2(s) = sI_1(s) = \frac{s^2}{s+1}V_c(s) = \frac{s^2}{\cancel{s+1}}\cdot\frac{\cancel{s+1}}{s^2+s+1}V_1(s)$$

Since $V_1(s) = \frac{1}{s}$, we have

$$V_2(s) = \frac{s^2}{s^2+s+1}V_1(s) = \frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \frac{1}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$\therefore\; v_2(t) = \left(e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t\right)u(t)$$

(b)

$$H(j\omega) = \frac{V_2(j\omega)}{V_1(j\omega)} = \frac{(j\omega)^2}{(j\omega)^2+(j\omega)+1} = \frac{-\omega^2}{(1-\omega^2)+j\omega}$$

$$|H(j\omega)| = \frac{\omega^2}{\sqrt{(1-\omega^2)^2+\omega^2}} = \frac{\omega^2}{\sqrt{\omega^4-\omega^2+1}}$$

$$\therefore\; |H(j0)| = 0, \quad |H(j1)| = 1, \quad |H(j\infty)| = 1$$

$$\frac{d}{d\omega}|H(j\omega)| = \frac{\omega(2-\omega^2)}{(\omega^4-\omega^2+1)^{\frac{3}{2}}}$$

$$\therefore\; |H(j\omega)|_{\max} = \frac{2}{\sqrt{3}} \quad \text{at } \omega = \sqrt{2}$$

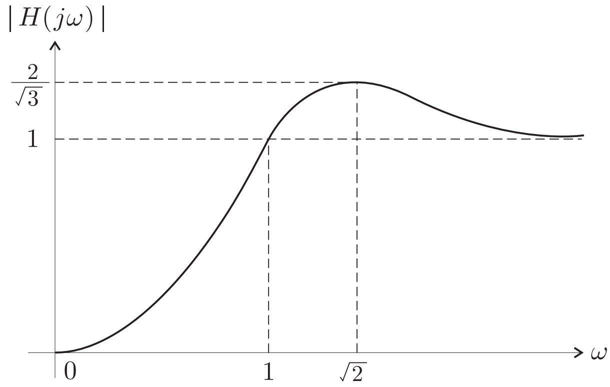

Figure 1: Plot for $|H(j\omega)|$ in original ω-scale.

2. $y(t) = \sum_{k=-\infty}^{\infty} x(kT)h(t-kT)$

$$\therefore\ y(t) = x(nT-T)\cdot\frac{1}{2}\left\{1+\cos\left(\frac{\pi t}{T} - n\pi + \pi\right)\right\} + x(nT)\cdot\frac{1}{2}\left\{1+\cos\left(\frac{\pi t}{T} - n\pi\right)\right\}$$

3. (a)
$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} x_1(k)x_2(n-k)\right)\cdot z^{-n}$$
$$= \sum_{k=0}^{\infty} x_1(k)z^{-k}\cdot\sum_{n=0}^{\infty} x_2(n-k)z^{-n+k} = X_1(z)\cdot\sum_{n=0}^{\infty} x_2(n-k)z^{-n+k}$$

Let $n-k=m$ and noting that $x_2(m)=0$ when $m<0$, we have

$$X(z) = X_1(z)\cdot\sum_{m=0}^{\infty} x_2(m)z^{-m} = X_1(z)X_2(z)$$

(b) $\mathscr{Z}[\sin bnT] = \dfrac{(\sin bT)z^{-1}}{1-2(\cos bT)z^{-1}+z^{-2}}$

If $bT = \frac{\pi}{2}$, then $\mathscr{Z}\left[\sin\left(\frac{\pi}{2}n\right)\right] = \dfrac{z^{-1}}{1+z^{-2}}$

$$\therefore\ \frac{2z^{-2}}{1+z^{-2}} = 2z^{-1}\cdot\frac{z^{-1}}{1+z^{-2}}$$

$$\therefore\ \mathscr{Z}^{-1}\left[\frac{2z^{-2}}{1+z^{-2}}\right] = \sin\frac{(n-1)\pi}{2} = x(n)$$

4. (a) $\dfrac{v_1(t)-v_2(t)}{R} = C\dfrac{d}{dt}v_2(t) \implies \dfrac{V_1(s)-V_2(s)}{R} = CsV_2(s)$

Since $R=1,\ C=1$, we have $V_1(s)-V_2(s) = sV_2(s)$.

$$\therefore\ H_a(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{s+1}$$

(b)
$$H(z) = (1-z^{-1})\mathscr{Z}\left[\mathscr{L}^{-1}\left\{\frac{1}{s}H_a(s)\right\}\Bigg|_{t=nT}\right]$$

$$\frac{1}{s}H_a(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s}H_a(s)\right\}\Bigg|_{t=nT} = [u(t)-e^{-t}u(t)]\Big|_{t=nT} = (1-e^{nT})u(n)$$

$$\begin{aligned}\therefore\ \mathscr{Z}\left[\mathscr{L}^{-1}\left\{\frac{1}{s}H_a(s)\right\}\Bigg|_{t=nT}\right] &= \sum_{n=0}^{\infty}(1-e^{-nT})z^{-n} \\ &= \sum_{n=0}^{\infty} z^{-n} - \sum_{n=0}^{\infty}(e^{-T}\cdot z^{-1})^n \\ &= \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}}\end{aligned}$$

$$\begin{aligned}\therefore\ H(z) &= (1-z^{-1})\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}}\right) \\ &= 1 - \frac{1-z^{-1}}{1-e^{-T}z^{-1}} = \frac{z^{-1}-e^{-T}z^{-1}}{1-e^{-T}z^{-1}}\end{aligned}$$

(c) 3-dB bandwidth of $|H_a(j\omega)| = \left|\dfrac{1}{1+j\omega}\right| = \dfrac{1}{\sqrt{\omega^2+1}}$ is $\omega_c = 1$.

$$\therefore\ C = \omega_c\cdot\cot\frac{\omega_c T}{2} = 1\cdot\cot\frac{1\cdot(0.1)}{2} = \cot\left(\frac{1}{20}\right) \cong 19.98 \cong 20$$

$$\therefore\ H(z) = \frac{1}{\left(20\cdot\dfrac{1-z^{-1}}{1+z^{-1}}\right)+1} = \frac{1+z^{-1}}{21-19z^{-1}}$$

(d) $|H_a(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}}$, $\quad |H(e^{j\omega T})| = \left|\dfrac{1+e^{-j(0.1)\omega}}{21-19e^{-j(0.1)\omega}}\right|$

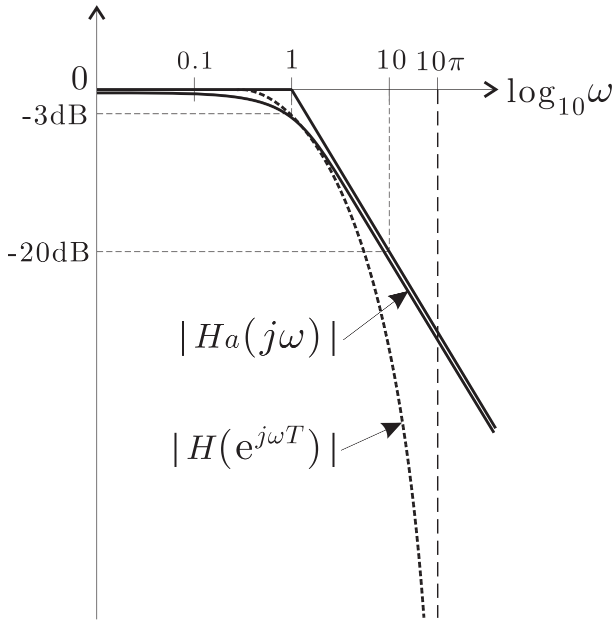

Figure 2: Plot for $|H_a(j\omega)|$ and $|H(e^{j\omega T})|$.