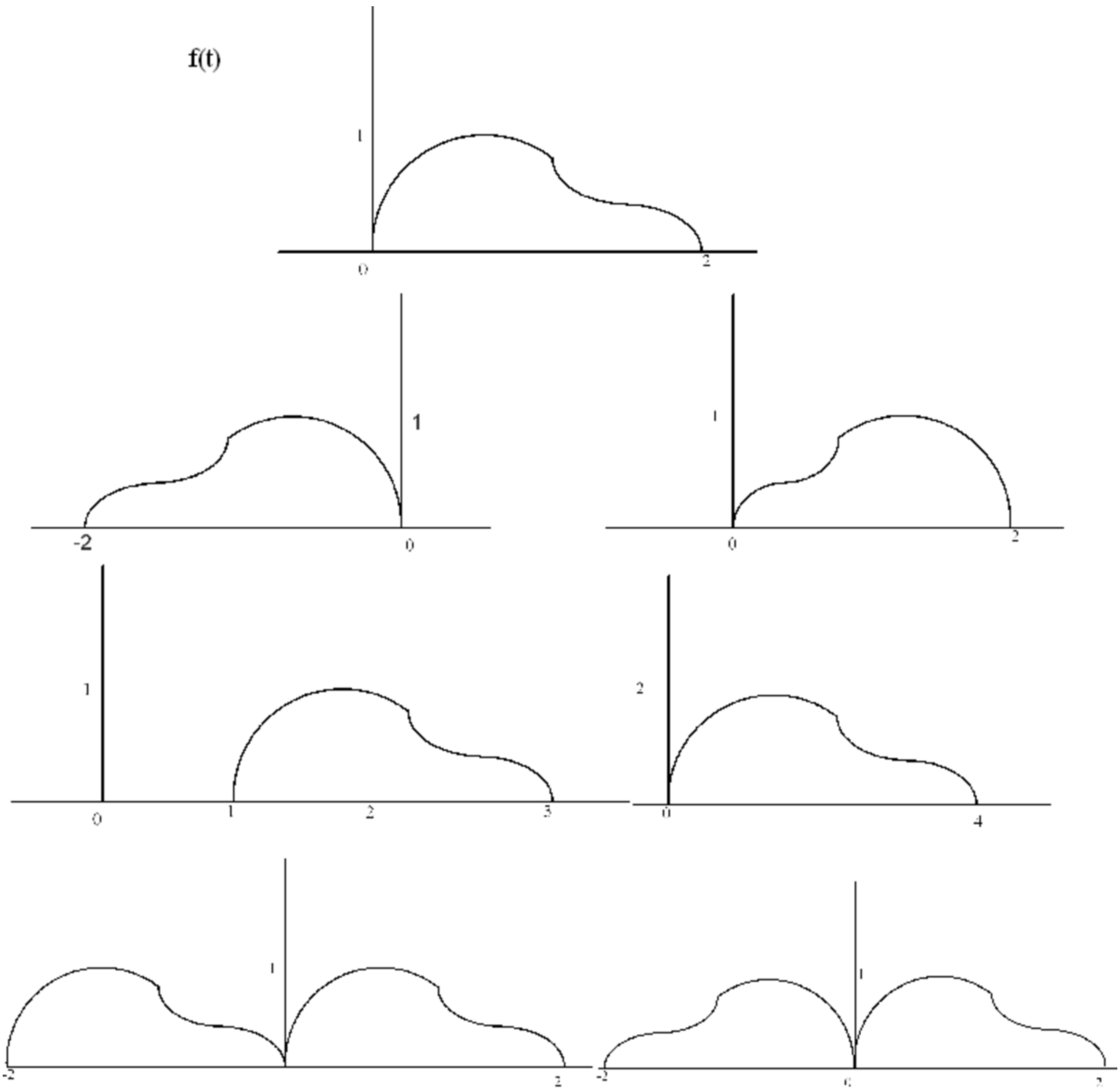
EE 261 The Fourier Transform and its Applications
Fall 2003, Handout 37
Midterm Exam

- There are 4 questions for a total of 100 points.
  - Problem 3 has three short, independent parts
- Please write your answers in the exam booklet provided, and make sure that your answers stand out.
- Don't forget to write your name on your exam book!

1. (30 points) The figures below show a signal $f(t)$ (the first one, on top) and six other signals derived from $f(t)$. Note the scales on the axes. Suppose $f(t)$ has Fourier transform $F(s)$. Express the Fourier transforms of the other six signals in terms of $F(s)$. When you write your answers in the exam booklet please sketch the curve you're working with in each case.



2. (15 points) Recall that the inner product of two functions in $L^2([0,1])$ is defined by

$$(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt.$$

Let $\{\varphi_n(t)\}$ be an orthonormal basis for $L^2([0,1])$. Remember that orthonormality means

$$(\varphi_n, \varphi_m) = \begin{cases} 1, & , \quad n = m \\ 0, & \quad n \neq m \end{cases}$$

To say that the $\{\varphi_n\}$ form a basis means we can write any function $f(t)$ in $L^2([0,1])$ as

$$f(t) = \sum_{n=1}^{\infty} (f, \varphi_n)\varphi_n(t).$$

Define a function of two variables

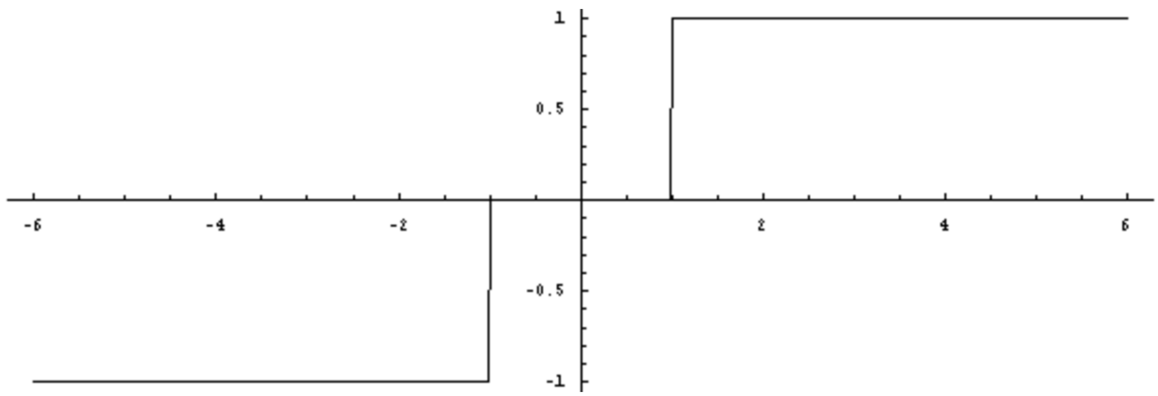
$$K(t,\tau) = \sum_{n=1}^{\infty} \varphi_n(t)\overline{\varphi_n(\tau)}.$$

Show that

$$(f(t), K(t,\tau)) = f(\tau).$$

(Ignore all questions of convergence of series, *etc.* Because of this property, $K(t,\tau)$ is called a *reproducing kernel*; taking the inner product of $f(t)$ with $K(t,\tau)$ 'reproduces' the value of f at τ. In Electrical Engineering literature sampling theorems are often expressed in terms of reproducing kernels.)

3(a) (10 points) Find the Fourier transform of the signal shown below.



3(b) (10 points) Recall the triangle function

$$\Lambda(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

and consider the signal

$$f(t) = 1 + \Lambda(3t) * \text{III}_{1/3}(t).$$

Sketch the graph of $f(t)$, find the Fourier transform $\mathcal{F}f(t)$, and sketch its graph. Comment on what you see.

3(c) (10 points) As one of our applications of Fourier transforms and convolutions to differential equations (Lecture 8) we showed that the ordinary differential equation

$$u'' - u = f$$

has solution

$$u(t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|t-\tau|} f(\tau)\, d\tau.$$

We know from elementary courses that the general solution of the equation should include a solution of the homogeneous equation $u'' - u = 0$, and so be of the form

$$u(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-|t-\tau|} f(\tau)\, d\tau.$$

A student sent an email pointing this out and wondering why methods based on the Fourier transform will not produce such a solution. Why do you think that is? (You *need not* look back at Lecture 8.)

4. (25 points) Let $f(t)$ be a band-limited signal whose Fourier transform $\mathcal{F}f(s)$ is shown below. The spectrum is nonzero only for $2 < |s| < 3$.

According to the sampling theorem, we can reconstruct this signal by a sinc interpolation with sample points taken at a rate of 6 Hz. Since the spectrum is concentrated in two islands, effectively with what one might say is a bandwidth of 1 Hz each, it would seem we should be able to do better. Arguing as in the derivation of the sampling theorem, find an interpolation formula that uses a sampling rate of 2 Hz. Can you do better still?