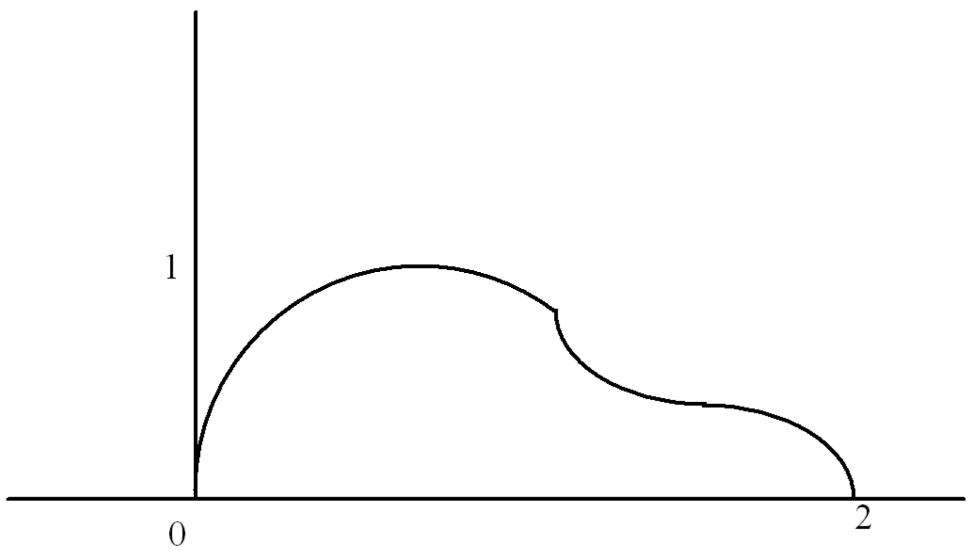
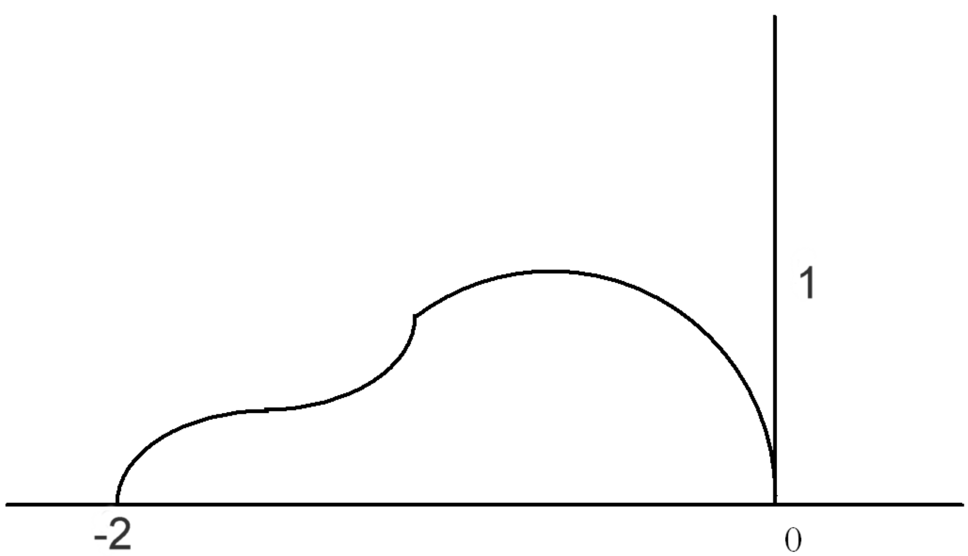
EE 261 The Fourier Transform and its Applications
Fall 2003, Handout 41
Midterm Solutions

1. The signal $f(t)$ is shown below. Its Fourier transform is $F(s)$.

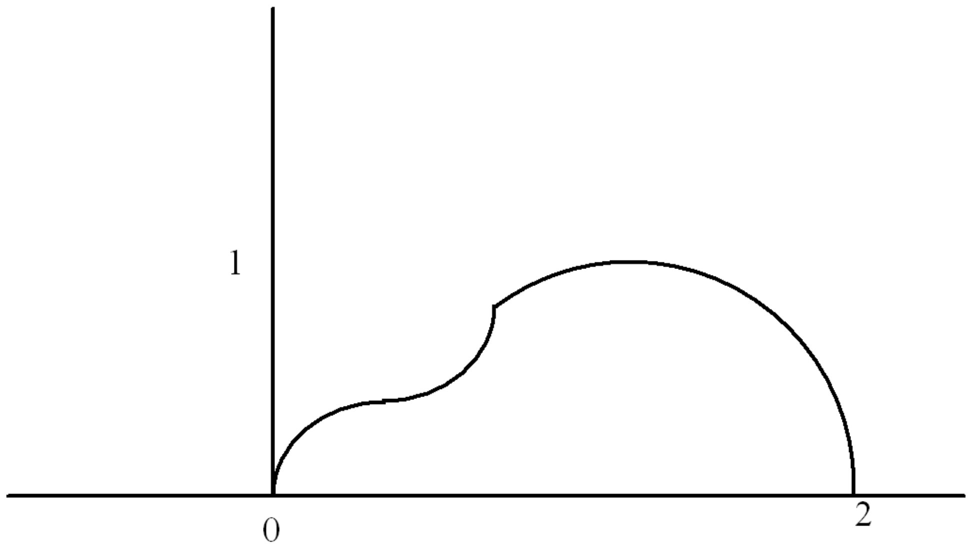


The first modified signal is



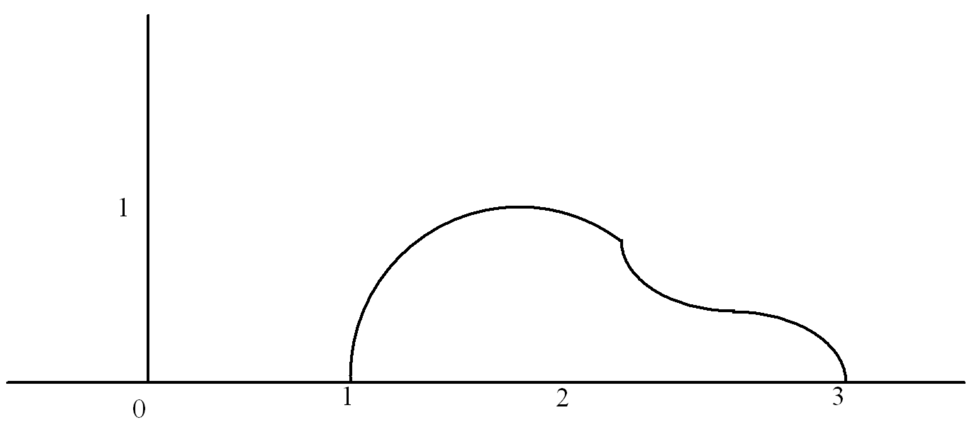
That's given by $f(-t)$, or, without writing the variable, f^-. Its Fourier transform is F^-, or, writing the variable, $F(-s)$.

The next one is



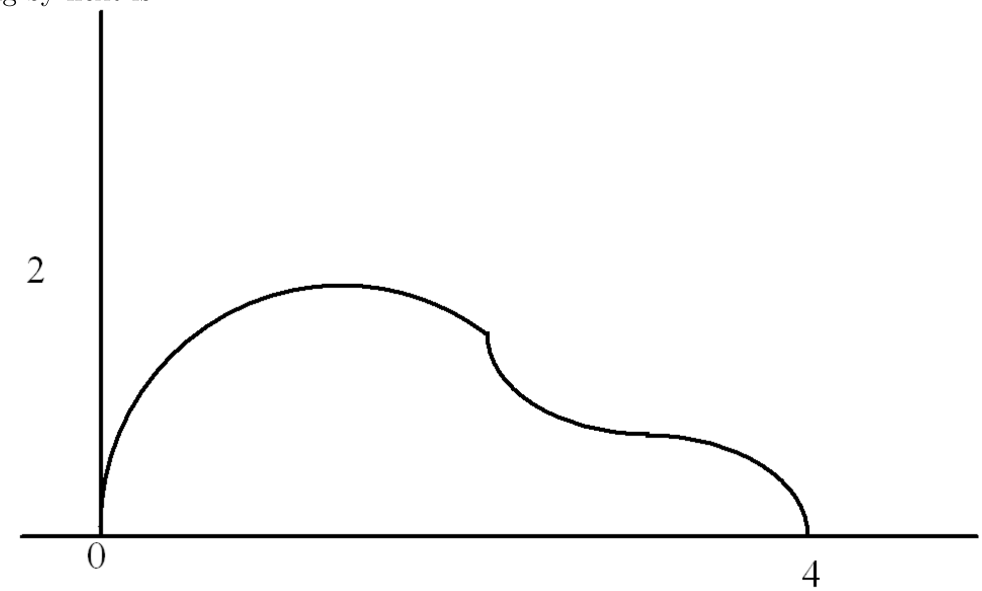
To express this in terms of f we first reverse f to f^- (which gives the preceding picture) and then shift to the right by 2, giving $f^-(t-2)$. The Fourier transform of this is $e^{-2\pi i 2s}\mathcal{F}(f^-)(s) = e^{-4\pi is}(\mathcal{F}f)^-(s) = e^{-4\pi is}F(-s)$.

And next we have



That's just $f(t-1)$, and the Fourier transform is $e^{-2\pi is}F(s)$.

Driving by next is

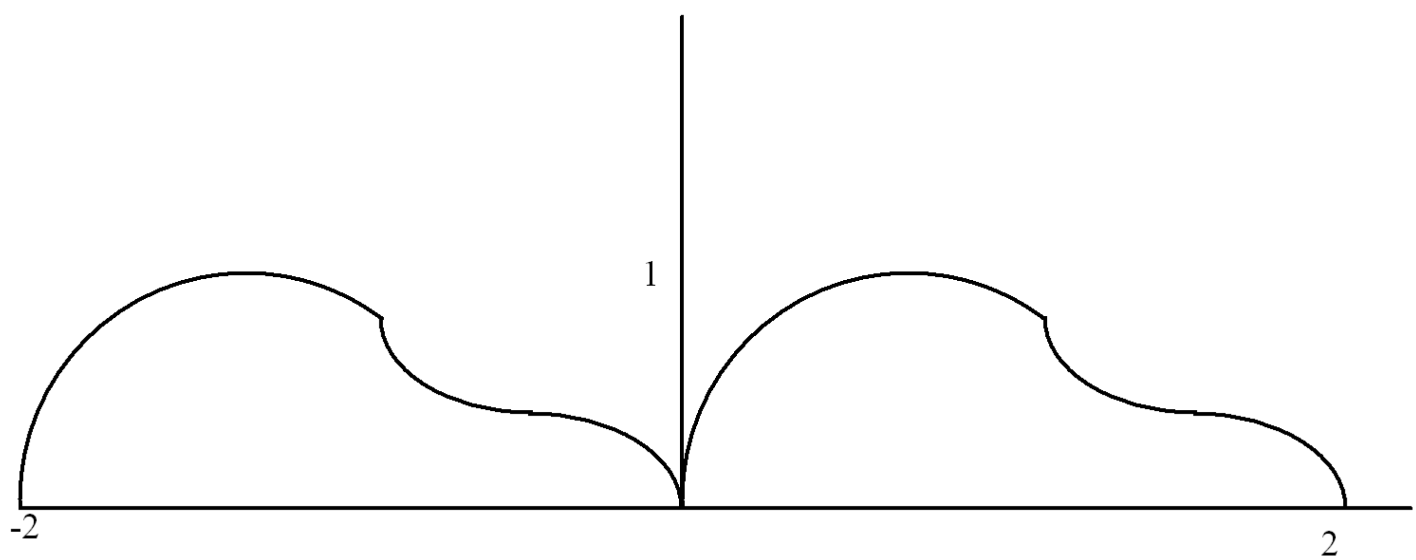


This is a little trickier. The figure is twice as large in the horizontal direction, which is accomplished by forming $f(t/2)$ and also scaled up by 2 in the vertical direction, so that makes $2f(x/2)$. The combination of the two scalings gives a curve of the same shape, just twice as large. Leaving either of the two scalings out would distort the shape.

The Fourier transform is

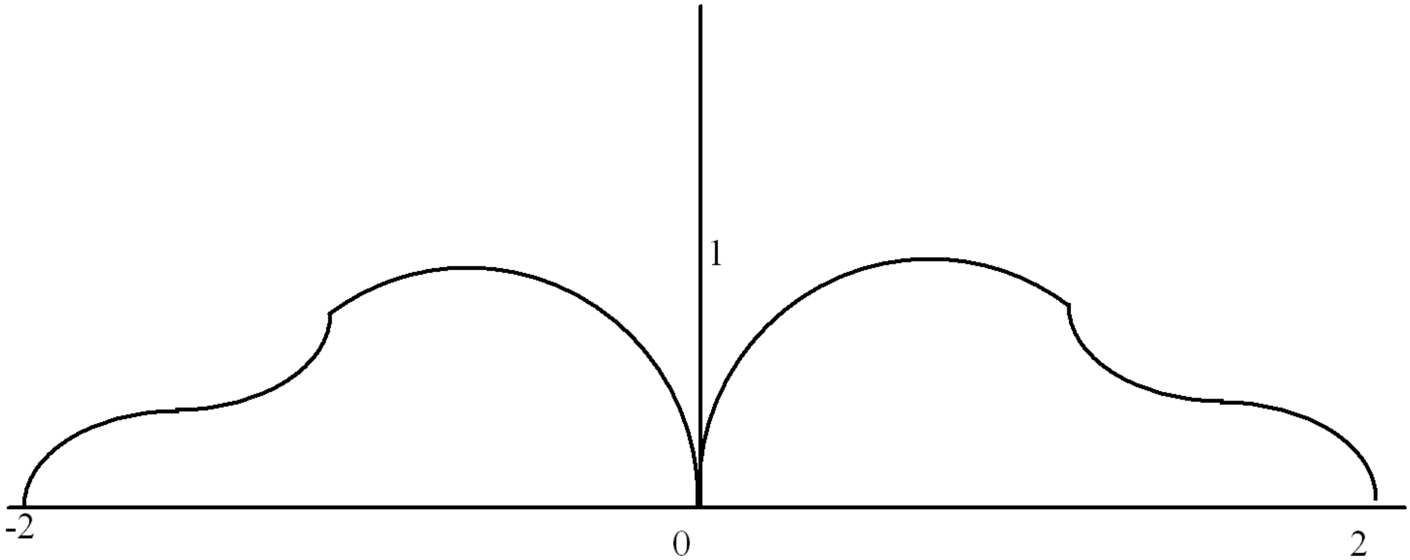
$$2 \times \frac{1}{\frac{1}{2}} F(2s) = 4F(2s).$$

The traffic picks up with:



This is given by $f(t) + f(t+2)$ so its Fourier transform is $F(s) + e^{4\pi is}F(s) = (1 + e^{4\pi is})F(s)$.

Finally, heading off in opposite directions:



That's just $f(t) + f(-t)$ so the Fourier transform is $F(s) + F(-s)$

2. (15 points) Recall that the inner product of two functions in $L^2([0,1])$ is defined by

$$(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt.$$

Let $\{\varphi_n(t)\}$ be an orthonormal basis for $L^2([0,1])$. Remember that orthonormality means

$$(\varphi_n, \varphi_m) = \begin{cases} 1, & , \quad n = m \\ 0, & \quad n \neq m \end{cases}$$

To say that the $\{\varphi_n\}$ form a basis means we can write any function $f(t)$ in $L^2([0,1])$ as

$$f(t) = \sum_{n=1}^{\infty} (f, \varphi_n)\varphi_n(t).$$

Define a function of two variables

$$K(t,\tau) = \sum_{n=1}^{\infty} \varphi_n(t)\overline{\varphi_n(\tau)}.$$

Show that

$$(f(t), K(t,\tau)) = f(\tau).$$

(Ignore all questions of convergence of series, *etc.* Because of this property, $K(t,\tau)$ is called a *reproducing kernel*; taking the inner product of $f(t)$ with $K(t,\tau)$ 'reproduces' the value of f at τ. In Electrical Engineering literature sampling theorems are often expressed in terms of reproducing kernels.)

We compute the inner product of $f(t)$ and $K(t,\tau)$. You can write this out in terms of integrals or you can use simply the properties of (complex) inner products. I'll show you both approaches.

$$\begin{aligned}
(f(t), K(t,\tau)) &= \left(f(t), \sum_{n=1}^{\infty} \varphi_n(t)\overline{\varphi_n(\tau)}\right) \\
&= \sum_{n=1}^{\infty} (f(t), \varphi_n(t)\overline{\varphi_n(\tau)}) \\
&= \sum_{n=1}^{\infty} \varphi_n(\tau)(f(t), \varphi_n(t))
\end{aligned}$$

(The inner product is 'with respect to t', and so the $\overline{\varphi_n(\tau)}$ are constants in each inner product. Being in the second slot, they come out with a conjugate, *i.e.* as $\varphi_n(\tau)$)

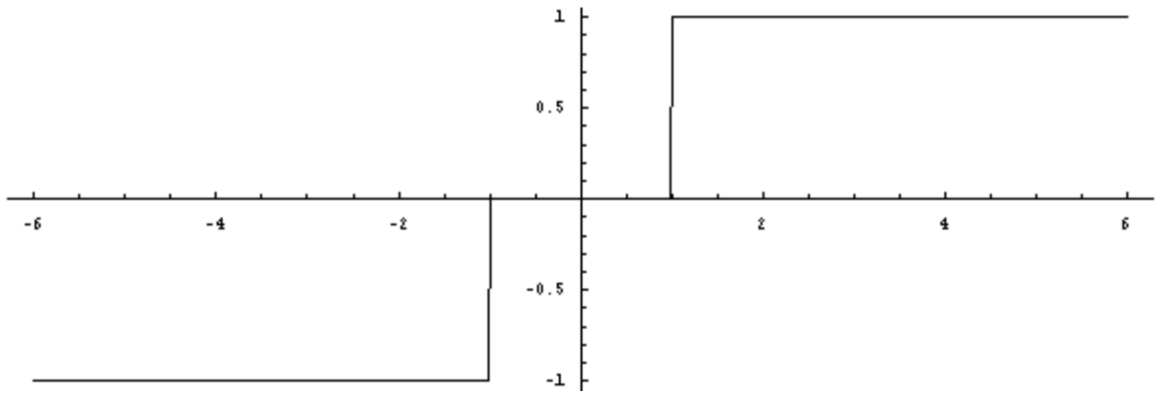
But this last expression is exactly how to write $f(\tau)$ in terms of the orthonormal basis,

$$\sum_{n=1}^{\infty} (f, \varphi_n)\varphi_n(\tau) = f(\tau).$$

If you had trouble keeping your t's and τ's straight in what we just did, here's how the calculation looks with integrals.

$$\begin{aligned}
(f(t), K(t,\tau)) &= \int_0^1 f(t)\overline{K(t,\tau)}\,dt \\
&= \int_0^1 f(t)\overline{\left(\sum_{k=1}^{\infty} \varphi_n(t)\overline{\varphi_n(\tau)}\right)}\,dt \\
&= \int_0^1 f(t)\left(\sum_{k=1}^{\infty} \overline{\varphi_n(t)}\varphi_n(\tau)\right)\,dt \\
&= \sum_{k=1}^{\infty}\left(\int_0^1 f(t)\overline{\varphi_n(t)}\,dt\right)\varphi_n(\tau) \\
&= \sum_{k=1}^{\infty}(f,\varphi_n)\varphi(\tau) \\
&= f(\tau)
\end{aligned}$$

3(a) (10 points) Find the Fourier transform of the signal shown below.



The signal is made out of the unit step function $H(t)$, so for sure we'll need to use

$$\mathcal{F}H(s) = \frac{1}{2}(\delta + \frac{1}{\pi i s}).$$

The part of the signal on the right is just a shift of $H(t)$ to the right, and that's $H(t-1)$. To get the part on the left we first reverse to H^-, then shift to the left, making $H^-(t+1)$ (which you could also write as $H(-(t+1)) = H(-t-1)$, but I won't) and then flip it upside down, to $-H^-(t+1)$. The whole signal is then

$$f(t) = H(t-1) - H^-(t+1)$$

The Fourier transform of this is

$$\begin{aligned}
\mathcal{F}f(s) &= e^{-2\pi i s}\mathcal{F}H(s) - e^{2\pi i s}(\mathcal{F}H^-)(s) \\
&= e^{-2\pi i s}\mathcal{F}H(s) - e^{2\pi i s}(\mathcal{F}H)^-(s) \\
&= e^{-2\pi i s}\mathcal{F}H(s) - e^{2\pi i s}\mathcal{F}H(-s) \\
&= e^{-2\pi i s}(\frac{1}{2}\delta(s) + \frac{1}{2\pi i s}) - e^{2\pi i s}(\frac{1}{2}\delta(-s) + \frac{1}{2\pi i(-s)}) \\
&= \frac{1}{2}(e^{-2\pi i s} - e^{2\pi i s})\delta(s) + \frac{1}{2\pi i s}(e^{-2\pi i s} + e^{2\pi i s}) \\
&= -i(\sin 2\pi s)\delta(s) + \frac{1}{\pi i s}\cos 2\pi s \\
&= \frac{1}{\pi i s}\cos 2\pi s
\end{aligned}$$

As a check, notice that the Fourier transform is purely imaginary and odd, which it should be.

3(b) (10 points) Recall the triangle function

$$\Lambda(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

and consider the signal

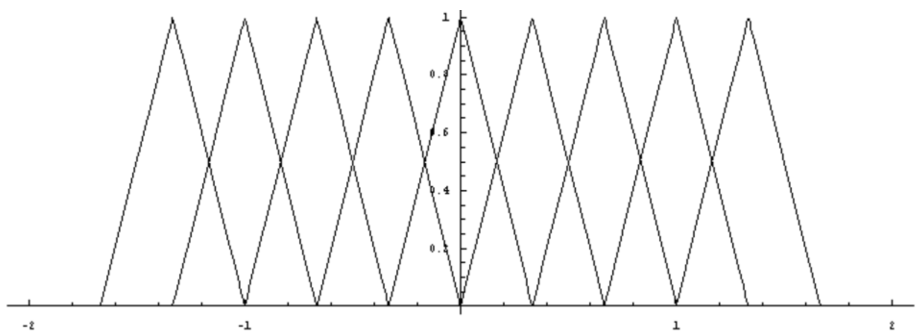
$$f(t) = 1 + \Lambda(3t) * \text{III}_{1/3}(t).$$

Sketch the graph of $f(t)$, find the Fourier transform $\mathcal{F}f(t)$, and sketch its graph. Comment on what you see.

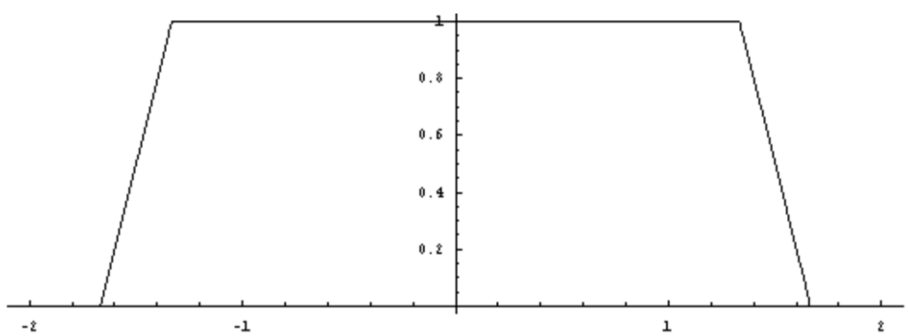
Taking the convolution $\Lambda(3t) * \text{III}_{1/3}(t)$ produces the sum

$$\Lambda(3t) * \text{III}_{1/3}(t) = \sum_{k=-\infty}^{\infty} \Lambda(3(t - \frac{k}{3})) = \sum_{k=-\infty}^{\infty} \Lambda(3t - k)$$

A plot of the shifted triangles from $k = -4$ to 4 looks like this:



If you draw your graph carefully enough (or actually do the algebra) you'll see that the overlaps are such that when you *add* the shifted triangles you get a plot that looks like:



And you would have concluded that $\Lambda(3t) * \text{III}_{1/3}(t) = 1$, and that $f(t) = 2$, and, so, for that matter that $\mathcal{F}f(s) = 2\delta$

However, if you didn't draw the graph carefully enough you had a second chance to come to the same conclusion by taking the Fourier transform:

$$\begin{aligned} \mathcal{F}f &= \delta + (\frac{1}{3}\text{sinc}^2(\frac{1}{3}s))(3\text{III}_3(s)) \\ &= \delta + \sum_{k=-\infty}^{\infty} \text{sinc}^2(\frac{k}{3})\delta(s - 3k) \end{aligned}$$

Now the thing you have to realize is that $\mathrm{sinc}^2(k/3)$ is 1 at $k = 0$ and is zero at the points where $k/3$ is an integer, *i.e.* where k is a multiple of 3. The zeros of sinc^2 line up with the impulses $\delta(s - 3k)$, $k \neq 0$ and hence kill them off. What remains is

$$\mathcal{F}f = \delta + \delta = 2\delta.$$

I won't plot this. But having seen it, if you didn't get the graph of $f(t)$ right you should now see that it's just $2\mathcal{F}^{-1}\delta = 2$.

3(c) (10 points) As one of our applications of Fourier transforms and convolutions to differential equations (Lecture 8) we showed that the ordinary differential equation

$$u'' - u = f$$

has solution

$$u(t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|t-\tau|} f(\tau)\, d\tau.$$

We know from elementary courses that the general solution of the equation should include a solution of the homogeneous equation $u'' - u = 0$, and so be of the form

$$u(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-|t-\tau|} f(\tau)\, d\tau.$$

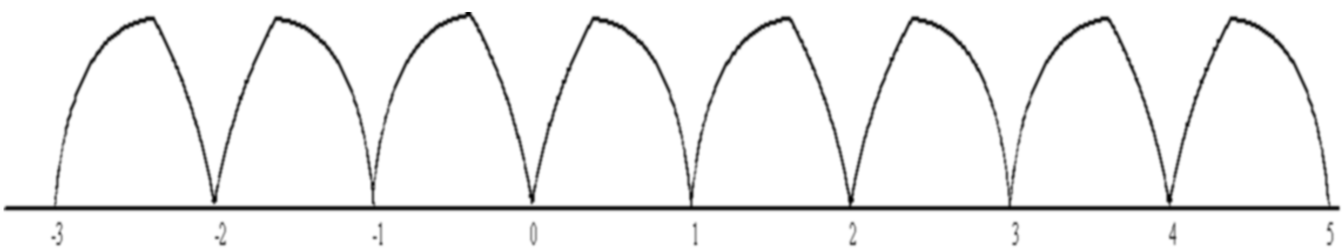
A student sent an email pointing this out and wondering why methods based on the Fourier transform will not produce such a solution. Why do you think that is? (You *need not* look back at Lecture 8.)

The reason is that neither e^t nor e^{-t} have Fourier transforms; e^t grows too fast at $+\infty$ and e^{-t} grows too fast at $-\infty$. No argument based on using the Fourier transform can handle these functions as solutions to the differential equation.

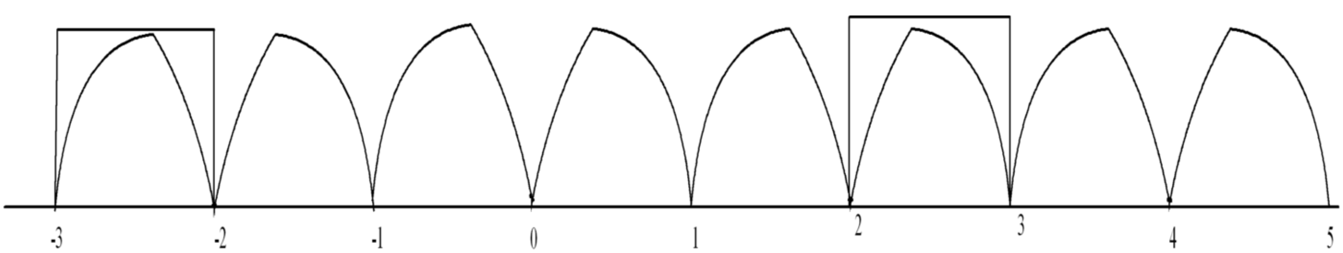
4. (25 points) Let $f(t)$ be a band-limited signal whose Fourier transform $\mathcal{F}f(s)$ is shown below. The spectrum is nonzero only for $2 < |s| < 3$.

According to the sampling theorem, we can reconstruct this signal by a sinc interpolation with sample points taken at a rate of 6 Hz. Since the spectrum is concentrated in two islands, effectively with what one might say is a bandwidth of 1 Hz each, it would seem we should be able to do better. Arguing as in the derivation of the sampling theorem, find an interpolation formula that uses a sampling rate of 2 Hz. Can you do better still?

If we convolve $\mathcal{F}f$ with $Ш_2$ each 'spectral island' is shifted enough so that there is no overlap and the copies of the islands are still distinct when added up. The (partial) picture (squeezed slightly) is:



We still have to recover the original Fourier transform (and then take the inverse). To do that we multiply by the sum of *two* Π's of width 1, one centered at 2.5 and one centered at -2.5. The picture is:



We thus have

$$\mathcal{F}f = (\Pi(s - \frac{5}{2}) + \Pi(s + \frac{5}{2}))(Ш_2 * \mathcal{F}f).$$

If we now take the inverse Fourier transform we get

$$\begin{aligned} f(t) &= (2\cos 5\pi t \ \mathrm{sinc}\, t) * (\frac{1}{2}\text{Ш}_{1/2}(t)\, f(t)) \\ &= (\cos 5\pi t \ \mathrm{sinc}\, t) * \left(\sum_{k=-\infty}^{\infty} f(k/2)\delta(t-k/2) \right) \\ &= \sum_{k=-\infty}^{\infty} f(k/2)(\cos 5\pi t \ \mathrm{sinc}\, t) * \delta(t-k/2) \\ &= \sum_{k=-\infty}^{\infty} f(k/2)\cos(5\pi(t-k/2))\,\mathrm{sinc}(t-k/2) \end{aligned}$$

We also see that if we tried to periodize $\mathcal{F}f$ with Ш_p for any $p < 2$ then the islands would overlap. We could not recover $\mathcal{F}f$ by cutting off, nor could we then recover f.