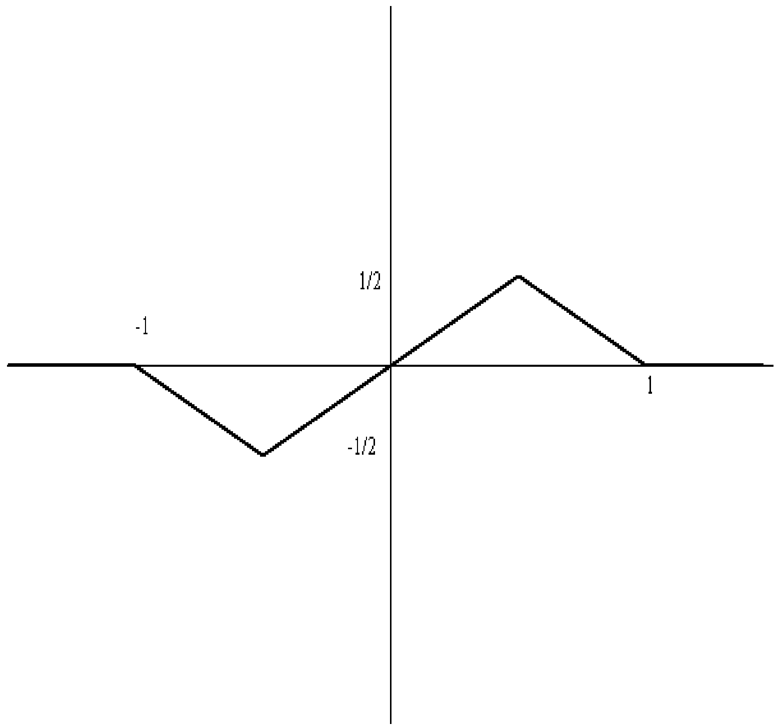
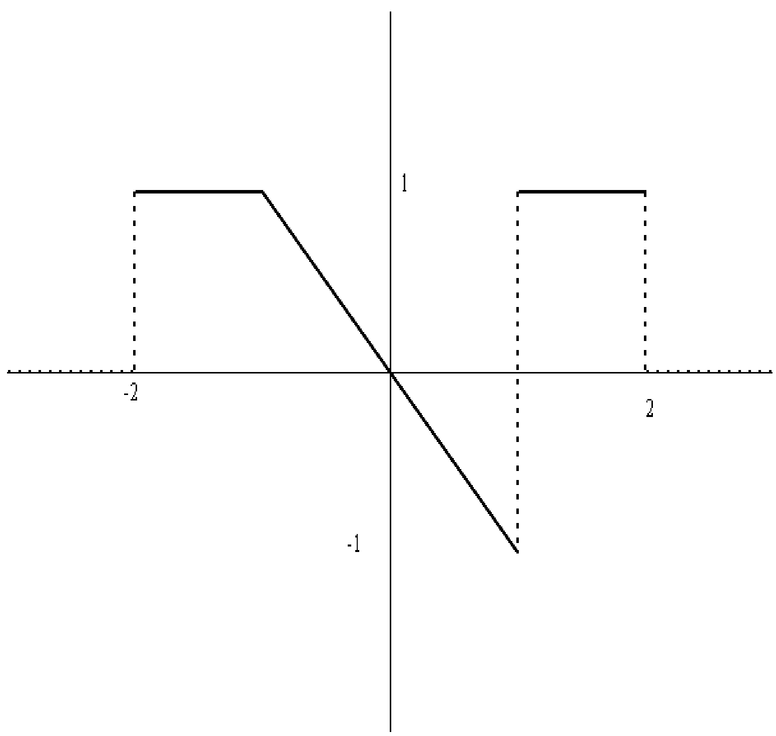
EE 261 The Fourier Transform and its Applications
Fall 2004
Midterm Solutions

1. (10 points) Two signals are plotted below. Without computing the Fourier transforms, determine if one can be the Fourier transform of the other. Explain your reasoning, and give at least two reasons for your conclusion.





Solution Call the top function $f(t)$ and the bottom function $g(t)$. Notice that $f(t)$ is odd, so it's Fourier transform is odd. The function $g(t)$ is not odd, so we can't have $\mathcal{F}f = g$. Another reason why we can't have $\mathcal{F}f = g$ is that g is not continuous, whereas the Fourier transform of an integrable function (and $f(t)$ is integrable) must be continuous. In the other direction, since $g(t)$ has no even-odd symmetries its Fourier transform will be complex, so it can't be $f(t)$.

2. (10 points each)

   (a) If $f(t) * g(t) = h(t)$ what is $f(t-1) * g(t+1)$?

   Solution Take the Fourier transform. Convolution becomes multiplication and the result is:That gives

$$e^{-2\pi is}\mathcal{F}f(s)\, e^{2\pi is}\mathcal{F}g(s) = \mathcal{F}f(s)\mathcal{F}g(s) = \mathcal{F}(f * g)(s) = \mathcal{F}h(s)$$

   Thus we get back what we started with:

$$f(t-1) * g(t+1) = h(t).$$

   The next three parts are related.

   (b) Show that the following relation holds for any two functions u and v:

$$\int_{-\infty}^{\infty} u(t)v(-t)dt \quad = \quad \int_{-\infty}^{\infty} \mathcal{F}u(s)\mathcal{F}v(s)ds$$

   Solution Let $w(t) = (u * v)(t)$ then

$$w(t) = \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau$$

   We also know that $w = \mathcal{F}^{-1}(\mathcal{F}u \cdot \mathcal{F}v)$ by the convolution theorem. This means that

$$w(t) = \int_{-\infty}^{\infty} \mathcal{F}u(s)\mathcal{F}v(s)e^{2\pi ist}ds$$

   Hence

$$\int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau = \int_{-\infty}^{\infty} \hat{u}(s)\hat{v}(s)e^{2\pi ist}ds$$

   Evaluating this equality at $t = 0$, we obtain the desired relation

$$\int_{-\infty}^{\infty} u(\tau)v(-\tau)d\tau = \int_{-\infty}^{\infty} \mathcal{F}u(s)\mathcal{F}v(s)ds$$

   (Replace the variable τ with the variable t.)

   (c) Using the result derived in the previous part (even if you couldn't derive it), show that the following holds for any two functions f and g:

$$\int_{-\infty}^{\infty} f(t)\mathcal{F}g(t)dt \quad = \quad \int_{-\infty}^{\infty} \mathcal{F}f(s)g(s)ds$$

   Solution Notice that $v^{-} = \mathcal{F}\mathcal{F}v$. Let $g = \mathcal{F}v$ and $f = u$ then $\mathcal{F}g = v^{-}$ and $\mathcal{F}f = \hat{u}$. Therefore the relation derived in the previous part becomes

$$\int_{-\infty}^{\infty} f(t)\hat{g}(t)dt = \int_{-\infty}^{\infty} \mathcal{F}f(s)g(s)ds$$

(d) Calculate the following integral:

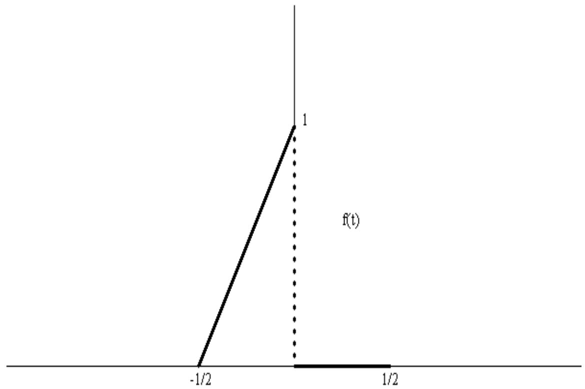
$$\int_{-\infty}^{\infty} \frac{e^{\pi i t}\,\mathrm{sinc}(t)}{1+4\pi^2 t^2} dt$$

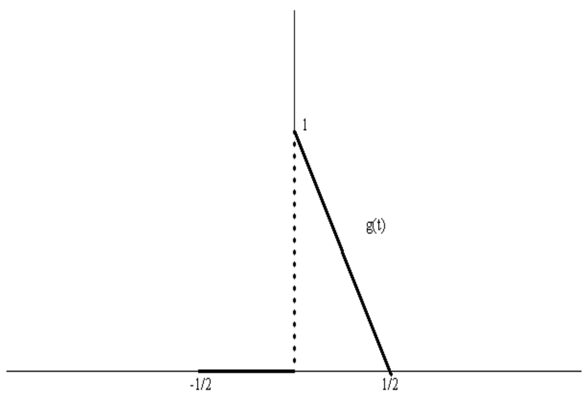
Solution Take $f(t) = e^{j\pi t}\,\mathrm{sinc}(t)$ and $\mathcal{F}g(t) = \frac{1}{1+4\pi^2 t^2}$. Then $\mathcal{F}f(s) = \Pi(s-\frac{1}{2})$ and $g(s) = \frac{1}{2}e^{-|s|}$.

Plugging into the expression derived in the previous part, we obtain the following result

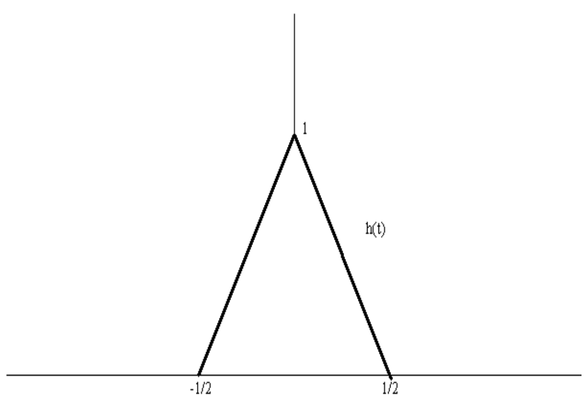
$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{j\pi t}\,\mathrm{sinc}(t)}{1+4\pi^2 t^2} dt &= \frac{1}{2}\int_{-\infty}^{\infty} \Pi(s-\frac{1}{2})e^{-|s|} ds \\
&= \frac{1}{2}\int_{0}^{1} e^{-s} ds \\
&= -\frac{1}{2}[e^{-s}]_0^1 \\
&= -\frac{1}{2}(e^{-1}-1) \\
&= \frac{1}{2}(1-\frac{1}{e})
\end{aligned}$$

3. (15 points) (a) Find the Fourier coefficients of the signals $f(t)$ and $g(t)$ shown below; assume period 1 for each. (Hint: it should be sufficient to do the calculation for just one of the signals.)





(b) Find the Fourier series for the signal $h(t)$ shown below. Again assume period 1.



Solution Formulas for the two functions are:

$$f(t) = \begin{cases} 2t+1, & -\frac{1}{2} \le t \le 0, \\ 0, & else. \end{cases}$$

$$g(t) = \begin{cases} -2t+1, & 0 \le t \le \frac{1}{2}, \\ 0, & else. \end{cases}$$

From looking at the graphs, or from the formulas, we can see that $g(t) = f(-t)$. From this, and from

$$\widehat{f}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)e^{-i2\pi nt}dt$$

we find that

$$\widehat{g}(n) = \widehat{f}(-n).$$

Thus if we calculate $\hat{f}(n)$ we can get $\hat{g}(n)$ with no extra work. We proceed:

$$\widehat{f}(n) = \int_{-\frac{1}{2}}^{0} (2t+1)e^{-i2\pi nt}dt = \int_{-\frac{1}{2}}^{0} 2t\ e^{-i2\pi nt}dt + \int_{-\frac{1}{2}}^{0} e^{-i2\pi nt}dt =$$

Integrating by parts, leads us to:

$$= 2\left\{\left[\frac{t}{(-i2\pi n)}e^{-i2\pi nt}\right]_{-\frac{1}{2}}^{0} - \int_{-\frac{1}{2}}^{0} \frac{1}{(-i2\pi n)}e^{-i2\pi nt}dt\right\} + \left[\frac{1}{(-i2\pi n)}e^{-i2\pi nt}\right]_{-\frac{1}{2}}^{0}$$

To simplify the computations, let's study each part independently so we do not get confused with all the different signs.

$$\left[\frac{t}{(-i2\pi n)}e^{-i2\pi nt}\right]_{-\frac{1}{2}}^{0} = -0 + \frac{-1/2}{i2\pi n}e^{i2\pi n1/2} = \frac{1}{-i4\pi n}e^{i\pi n}$$

(Note that we have written $e^{i\pi n}$ but if in your solution you have specified $e^{i\pi n} = (-1)^n$ the answer is also valid.)
For the second term:

$$\int_{-\frac{1}{2}}^{0} \frac{1}{(-i2\pi n)}e^{-i2\pi nt}dt = \left[\frac{1}{(-i2\pi n)^2}e^{-i2\pi nt}\right]_{-\frac{1}{2}}^{0} = \frac{-1}{(2\pi n)^2} + \frac{e^{i\pi n}}{(2\pi n)^2}$$

Combining the terms leads to

$$2[\frac{1}{-i4\pi n}e^{i\pi n} - [\frac{-1}{(2\pi n)^2} + \frac{e^{i\pi n}}{(2\pi n)^2}]] - \frac{1}{i2\pi n} + \frac{e^{i\pi n}}{i2\pi n} = \frac{1}{2\pi^2 n^2}[1 - e^{i\pi n}] - \frac{1}{i2\pi n},$$

that is,

$$\widehat{f}(n) = \frac{1}{2\pi^2 n^2}[1 - e^{i\pi n}] - \frac{1}{i2\pi n}$$

Then,

$$\widehat{g}(n) = \widehat{f}(-n) = \frac{1}{2\pi^2 n^2}[1 - e^{-i\pi n}] + \frac{1}{i2\pi n}$$

(b) Once we have solved the first part, the second one is not that complicated, since it is clear that $\widehat{h}(n) = \widehat{f}(n) + \widehat{g}(n)$. Thus,

$$\widehat{h}(n) = \frac{1}{2\pi^2 n^2}[2 - (e^{i\pi n} + e^{i\pi n})] = \frac{1}{\pi^2 n^2}[1 - cos(\pi n)] = \frac{1}{\pi^2 n^2}[1 - cos(2\pi(\frac{n}{2}))]$$

$$\widehat{h}(n) = \frac{1}{\pi^2 n^2} 2\sin^2(\frac{n\pi}{2})$$

The Fourier Series for h is then

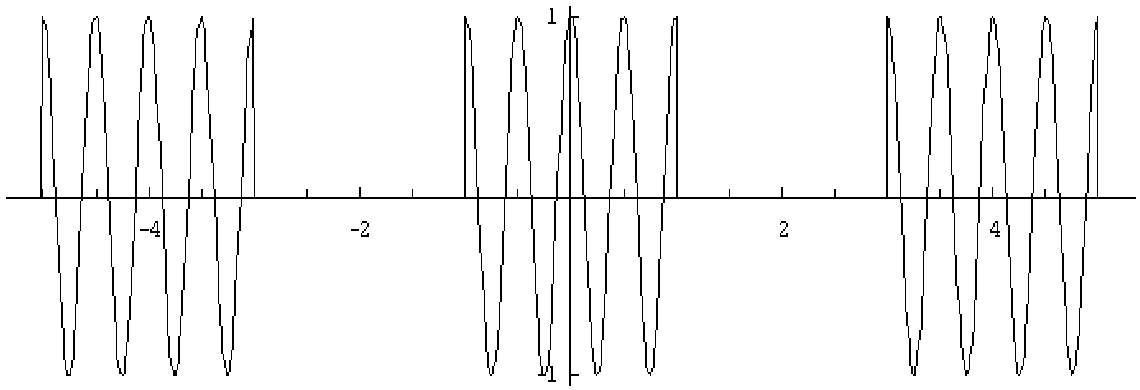
$$h(t) = \sum_{n=-\infty}^{\infty} \frac{1}{\pi^2 n^2} 2\sin^2(\frac{n\pi}{2}) e^{i2\pi nt}$$

which can be written

$$h(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2}\text{sinc}^2(\frac{n}{2}) e^{i2\pi nt}$$

For those who separated terms when n is odd, even, or 0, the results are: $\widehat{h}(n) = \frac{2}{\pi^2 n^2}$ if n is odd, $\widehat{h}(n) = 0$ for n even and $\widehat{h}(0) = \frac{1}{2}$

4. (15 points) Sketch the Fourier transform of the periodic cosine pulse shown below. Explain your work.



Solution The cosine has frequency 2, so it's given by $\cos 4\pi t$. To cut it off at ± 1 (width 2) we multiply by $\Pi_2(t)$, producing the single cosine pulse $\Pi_2(t)\cos 4\pi t$. Finally, we periodize by convolving with $Ш_4$ (spacing 4) to produce the repeating cosine pulse given by

$$f(t) = Ш_4(t) * (\Pi_2(t)\cos 4\pi t).$$

Using the convolution theorem, twice, the Fourier transform is then

$$\mathcal{F}f = \mathcal{F}Ш_4 \cdot (\mathcal{F}\Pi_2 * \mathcal{F}\cos 4\pi t).$$

Look at the convolution first:

$$\begin{aligned}
\mathcal{F}\Pi_2 * \mathcal{F}\cos 4\pi t &= 2\,\mathrm{sinc}\,2s * \frac{1}{2}(\delta(s-2) + \delta(s+2)) \\
&= \mathrm{sinc}\,2s * \delta(s-2) + \mathrm{sinc}\,2s * \delta(s+2) \\
&= \mathrm{sinc}(2(s-2)) + \mathrm{sinc}(2(s+2)) \\
&= \mathrm{sinc}(2s-4) + \mathrm{sinc}(2s+4)
\end{aligned}$$

Next, with

$$\mathcal{F}Ш_4 = \frac{1}{4}Ш_{1/4}$$

we have

$$\begin{aligned}
\mathcal{F}f(s) &= \frac{1}{4}Ш_{1/4}(s)(\mathrm{sinc}(2s-4) + \mathrm{sinc}(2s+4)) \\
&= \frac{1}{4}\sum_{k=-\infty}^{\infty}(\mathrm{sinc}(\frac{k}{2}-4) + \mathrm{sinc}(\frac{k}{2}+4))\delta(s-\frac{k}{4})
\end{aligned}$$

There are many values of k for which both sincs are zero. If k is even and $\neq \pm 8$ then

$$\frac{k}{2} - 4 \quad \text{and} \quad \frac{k}{2} + 4$$

are nonzero integers and

$$\mathrm{sinc}(\frac{k}{2} - 4) = 0 \quad \text{and} \quad \mathrm{sinc}(\frac{k}{2} + 4) = 0$$

Here's a plot of the Fourier transform. Your answer should have this basic shape, but need not be so carefully rendered, of course.