EE 261 The Fourier Transform and its Applications
Fall 2005
Midterm Solutions

1. (25 points)*Finding Fourier transforms:* The following three questions are independent.

   (a)(5) In communications theory the *analytic signal* $f_a(t)$ of a signal $f(t)$ is defined, via the Fourier transform, by

   $$\mathcal{F}f_a(s) = \begin{cases} \mathcal{F}f(s), & s \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

   For a real-valued signal $f(t)$, that is not identically zero, could the corresponding analytic signal $f_a(t)$ also be real? Why or why not?

   Solution No, the signal cannot be real, unless it is identically zero. If $f_a(t)$ were real its Fourier transform would have the property $\mathcal{F}f_a(-s) = \overline{\mathcal{F}f_a(s)}$, but we are told that $\mathcal{F}f_a(s) = 0$ for $s < 0$.

   (b)(10) Compute the Fourier transform of $f(x) = \cos(\pi x)\Pi(x)$, which is a half-cycle of a cosine.

   Solution We can do this directly from the definition:

   $$\begin{aligned} \mathcal{F}f(s) &= \int_{-1/2}^{1/2} \cos(\pi x) e^{-2\pi i s x}\, dx \\ &= \int_{-1/2}^{1/2} \frac{e^{\pi i x} + e^{-\pi i x}}{2} e^{-2\pi i s x}\, dx \\ &= \int_{1/2}^{1/2} \frac{1}{2} e^{\pi i x(1-2s)} + e^{-\pi i x(1+2s)}\, dx\,. \end{aligned}$$

   Integration then yields

   $$\mathcal{F}f(s) = \frac{2\cos(\pi s)}{\pi\,(1-4s^2)}\,.$$

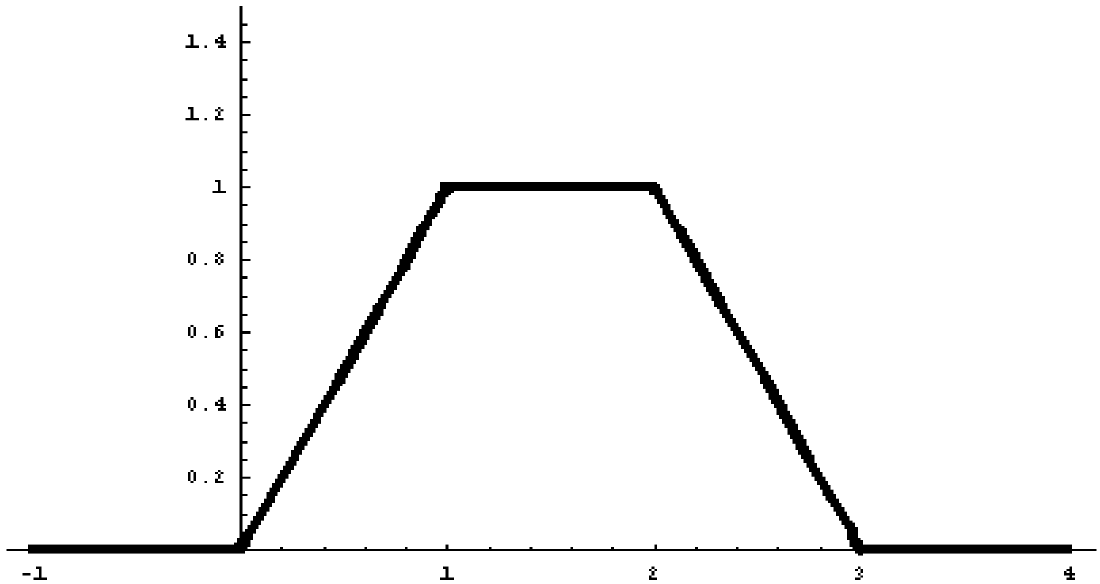
   Alternatively, using the convolution theorem

   $$\begin{aligned} \mathcal{F}(\cos \pi x\, \Pi(x)) &= \mathcal{F}(\cos \pi x) * \mathcal{F}\Pi(x) \\ &= \frac{1}{2}(\delta(s - \frac{1}{2}) + \delta(s + \frac{1}{2})) * \operatorname{sinc} s \\ &= \frac{1}{2}(\operatorname{sinc}(s - \frac{1}{2}) + \operatorname{sinc}(s + \frac{1}{2}))\,. \end{aligned}$$

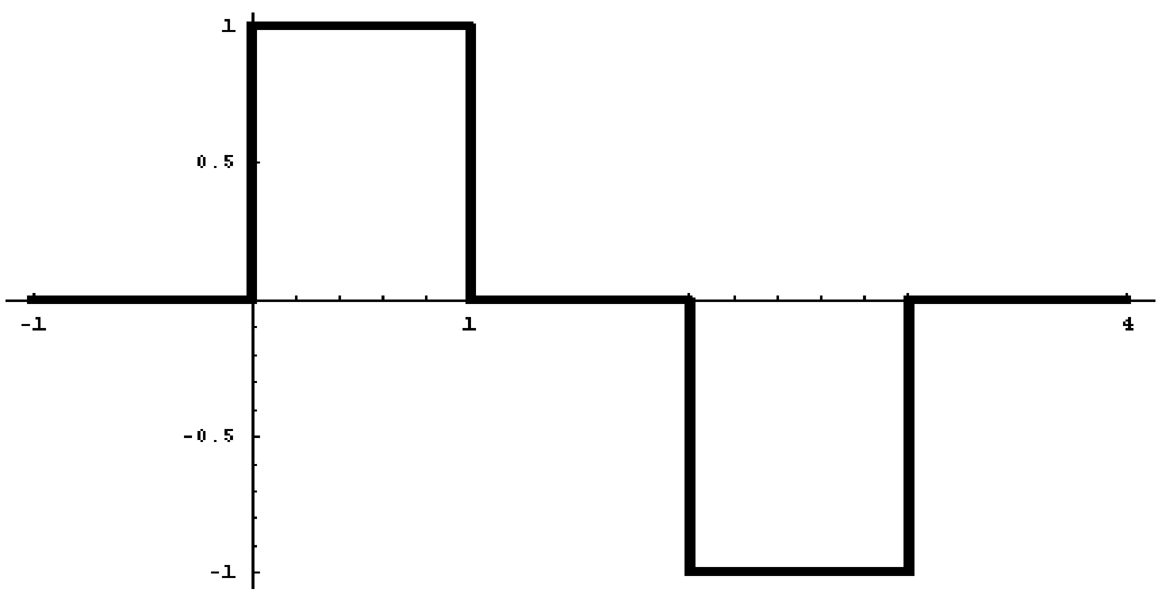
Let's check that these are the same.

$$
\begin{aligned}
\frac{1}{2}(\operatorname{sinc}(s-\frac{1}{2})+\operatorname{sinc}(s+\frac{1}{2})) &= \frac{1}{2}\left(\frac{\sin\pi(s-\frac{1}{2})}{\pi(s-\frac{1}{2})}+\frac{\sin\pi(s+\frac{1}{2})}{\pi(s+\frac{1}{2})}\right) \\
&= \frac{1}{2}\left(\frac{-\cos\pi s}{\pi(s-\frac{1}{2})}+\frac{\cos\pi s}{\pi(s+\frac{1}{2})}\right) \\
&= \frac{\cos\pi s}{2\pi}\left(\frac{1}{s+\frac{1}{2}}-\frac{1}{s-\frac{1}{2}}\right) \\
&= \frac{\cos\pi s}{2\pi}\left(\frac{-1}{s^2-\frac{1}{4}}\right) \\
&= \frac{2\cos\pi s}{\pi}\left(\frac{1}{1-4s^2}\right).
\end{aligned}
$$

(c)(10) Use the derivative theorem for Fourier transforms to find the Fourier transform of the function sketched below.



Solution Call the function $f(x)$. The horizontal segments have slope 0 and the oblique segments have slopes 1 and -1, respectively. Here's a plot of the derivative:



We see that the derivative consists of two shifted rect functions:

$$f'(x) = \Pi(x - \frac{1}{2}) - \Pi(x - \frac{5}{2}) \, .$$

Now the derivative formula for the Fourier transform states

$$\mathcal{F}f'(x) = 2\pi i s \mathcal{F}f(s)$$

and so in the present case

$$\begin{aligned}\mathcal{F}f(s) &= \frac{1}{2\pi i s}\mathcal{F}(\Pi(x-\frac{1}{2}) - \Pi(x-\frac{5}{2})) \\ &= \frac{1}{2\pi i s}(e^{-\pi i s}\operatorname{sinc} s - e^{-5\pi i s}\operatorname{sinc} s)\end{aligned}$$

Another approach is to differentiate $f(x)$ twice, and from $f'(x)$ above we see that this yields

$$f''(x) = \delta(x) - \delta(x-1) - \delta(x-2) + \delta(x-3)\,.$$

For the Fourier transform:

$$\mathcal{F}f'' = -(1 - e^{-2\pi i(1)s} - e^{-2\pi i(2)s} + e^{-2\pi i(3)s})\,.$$

Now use

$$\mathcal{F}f'' = (2\pi i s)^2\mathcal{F}f = -4\pi s^2\mathcal{F}f$$

to obtain

$$\mathcal{F}f(s) = \frac{-1}{4\pi^2 s^2}\left(1 - e^{-2\pi i s} - e^{-4\pi i s} + e^{-6\pi i s}\right)\,.$$

Combining the complex exponentials into sines will produce the earlier answer.

2. (30 points) *Finding Fourier series: Sometimes one is enough.*

(a)(5) Let $f(t)$ be periodic of period 1 with Fourier coefficients $\hat{f}(n)$, $n = 0, \pm 1, \pm 2, \ldots$. Let $g(t) = f(t-a)$. Express $\hat{g}(n)$ in terms of $\hat{f}(n)$.

Solution The n'th Fourier coefficient is given by

$$\hat{f}(n) = \int_0^1 f(t) e^{-2\pi int} dt \, .$$

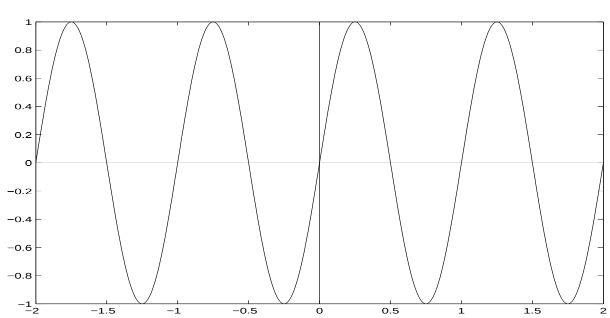
Therefore

$$\begin{aligned}
\hat{g}(n) &= \int_0^1 g(t) e^{-2\pi int} dt \\
&= \int_0^1 f(t-a) e^{-2\pi int} dt \\
&= \int_{-a}^{1-a} f(u) e^{-2\pi in(u+a)} du \\
&= e^{-2\pi ina} \int_{-a}^{1-a} f(u) e^{-2\pi inu} du \\
&= e^{-2\pi ina} \int_0^1 f(u) e^{-2\pi inu} du \\
&= e^{-2\pi ina} \hat{f}(n)
\end{aligned}$$

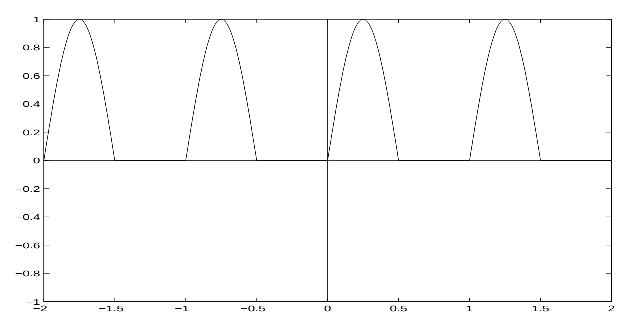
(b)(25) Signals $f_1(t), \ldots, f_6(t)$, together with their graphs, are as below. Your task is to find the Fourier series for each of these six signals, *but* you are allowed to compute *only one* Fourier series and you must deduce the others. For this, you choose one signal to work with and you express the remaining five signals in terms of the one you chose. Carry this out, explaining how the Fourier coefficients of each of the five remaining signals are related to the Fourier coefficients of the one you chose. (You will need part (a).)

Note! You *do not* have to compute the Fourier coefficients of the signal you choose, you only have to give expressions *relating* the Fourier coefficients of the other signals to those of the one you chose.

$f_1(t) = \sin(2\pi t)$

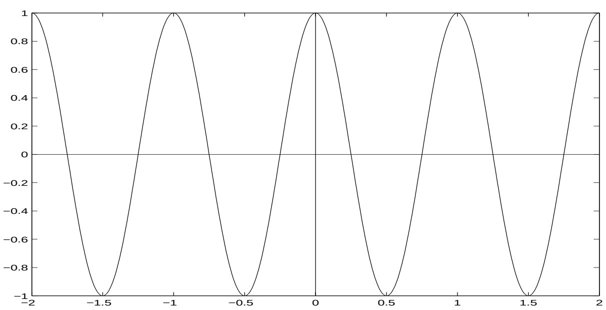


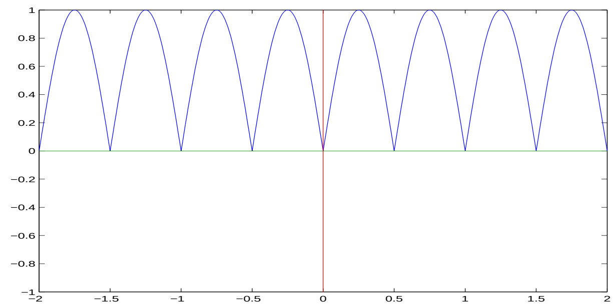
$f_2(t) = \max\{\sin(2\pi t), 0\}$



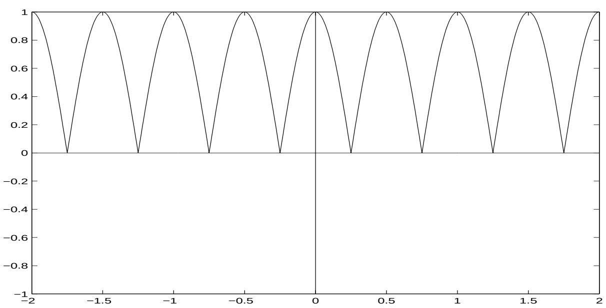
$f_3(t) = \cos(2\pi t)$

$f_4(t) = |\sin(2\pi t)|$

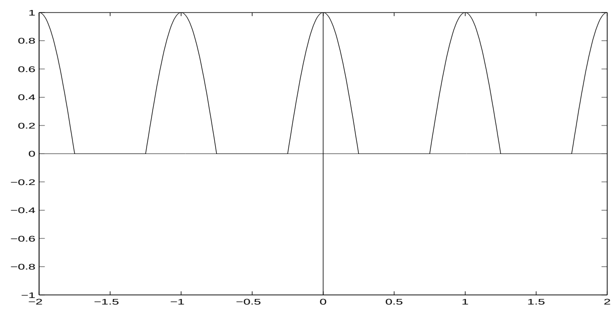




$f_5(t) = |\cos(2\pi t)|$



$f_6(t) = \max\{\cos(2\pi t), 0\}$



Solution The trick in this problem is to see that we can express these functions in terms of each others.
Let's express all of the functions in terms of f_2:

$$\begin{aligned}
f_1(t) &= f_2(t) - f_2(t - \frac{1}{2}) \\
f_3(t) &= f_1(t + \frac{1}{4}) = f_2(t + \frac{1}{4}) - f_2(t - \frac{1}{4}) \\
f_4(t) &= f_2(t) + f_2(t - \frac{1}{2}) \\
f_5(t) &= f_4(t + \frac{1}{4}) = f_2(t + \frac{1}{4}) + f_2(t - \frac{1}{4}) \\
f_6(t) &= f_2(t + \frac{1}{4})
\end{aligned}$$

So now assuming that we find the Fourier series of f_2, i.e. that we find the coefficients $\hat{f}_2$, we can express $\hat{f}_1$, $\hat{f}_3$, $\hat{f}_4$, $\hat{f}_5$ and $\hat{f}_6$ in terms of $\hat{f}_2$.

$$
\begin{aligned}
\hat{f}_1(n) &= \hat{f}_2(n) - e^{-2\pi i n \frac{1}{2}} \hat{f}_2(n) \\
&= \hat{f}_2(n)(1 - e^{-\pi i n}) \\
&= \begin{cases} 2\hat{f}_2(n) & \text{if } n \text{is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\hat{f}_3(n) &= e^{2\pi i n \frac{1}{4}} \hat{f}_2(n) - e^{-2\pi i n \frac{1}{4}} \hat{f}_2(n) \\
&= (e^{\frac{\pi}{2} i n} - e^{-\frac{\pi}{2} i n}) \hat{f}_2(n) \\
&= 2i \sin(\frac{\pi}{2} n) \hat{f}_2(n)
\end{aligned}
$$

$$
\begin{aligned}
\hat{f}_4(n) &= \hat{f}_2(n) + e^{-2\pi i n \frac{1}{2}} \hat{f}_2(n) \\
&= \hat{f}_2(n)(1 + e^{-\pi i n}) \\
&= \begin{cases} 2\hat{f}_2(n) & \text{if } n \text{is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\hat{f}_5(n) &= e^{2\pi i n \frac{1}{4}} \hat{f}_2(n) + e^{-2\pi i n \frac{1}{4}} \hat{f}_2(n) \\
&= (e^{\frac{\pi}{2} i n} + e^{-\frac{\pi}{2} i n}) \hat{f}_2(n) \\
&= 2 \cos(\frac{\pi}{2} n) \hat{f}_2(n)
\end{aligned}
$$

$$
\begin{aligned}
\hat{f}_5(n) &= e^{2\pi i n \frac{1}{4}} \hat{f}_2(n) \\
&= e^{\frac{\pi}{2} i n} \hat{f}_2(n)
\end{aligned}
$$

3. (15 points) *Could be convolution:* The following two questions are independent.

(a)(5) Find all functions $f(t)$ satisfying $(f * f)(t) = e^{-\pi t^2}$.

Solution Taking the Fourier transform of both sides of the equation,

$$\begin{aligned}
\mathcal{F}f(s).\mathcal{F}f(s) &= \mathcal{F}\{e^{-\pi t^2}\} \\
\Rightarrow \mathcal{F}f(s)^2 &= e^{-\pi s^2} \\
\Rightarrow \mathcal{F}f(s) &= \pm e^{-\pi s^2/2} \\
\Rightarrow \mathcal{F}f(s) &= \pm e^{-\pi(s/\sqrt{2})^2} \\
\Rightarrow f(t) &= \mathcal{F}^{-1}\{\pm e^{-\pi(s/\sqrt{2})^2}\} \\
\Rightarrow f(t) &= \pm\sqrt{2}e^{-\pi(t\sqrt{2})^2} \\
\Rightarrow f(t) &= \pm\sqrt{2}e^{-2\pi t^2}
\end{aligned}$$

(b)(10) Solve the equation

$$x(t) + 4\int_{-\infty}^{\infty} e^{-|\tau|}x(t-\tau)\,d\tau = e^{-|t|}\,.$$

Solution The equation can be rewritten in the following form:

$$x(t) + 4e^{-|t|} * x(t) = e^{-|t|}$$

Taking the Fourier transform, we get:

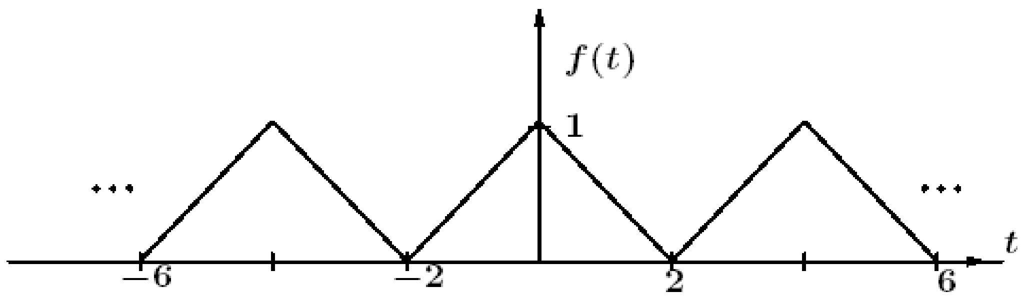
$$\begin{aligned}
\mathcal{F}x(s) + 4\frac{2}{1+4\pi^2 s^2}\mathcal{F}x(s) &= \frac{2}{1+4\pi^2 s^2} \\
\Rightarrow \mathcal{F}x(s)(1+4\pi^2 s^2 + 8) &= 2 \\
\Rightarrow \mathcal{F}x(s) &= \frac{2}{9+4\pi^2 s^2} \\
\Rightarrow \mathcal{F}x(s) &= \frac{1}{9}.\frac{2}{1+4\pi^2(\frac{s}{3})^2}
\end{aligned}$$

Now taking the inverse Fourier transform gives the result:

$$x(t) = \frac{1}{9}.3e^{-3|t|} = \frac{1}{3}e^{3|t|}$$

4. (20 points) *Periodizing and filtering: Smoothing a triangle train.*

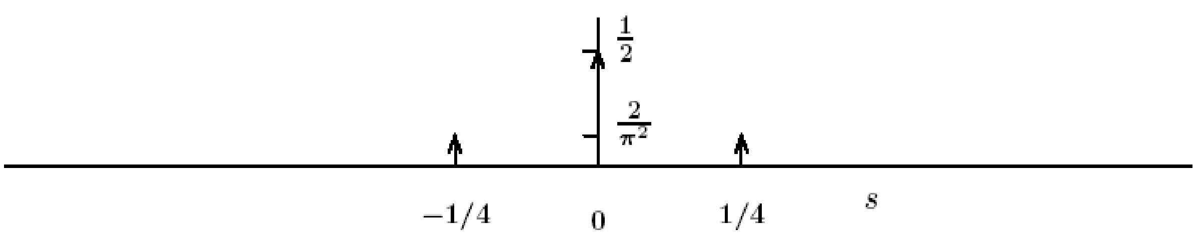
(a)(10) Find the Fourier transform of the periodic function $f(t)$ sketched below; the period is 4 and three complete cycles are shown. Sketch $\mathcal{F}f(s)$ for $-0.5 \le s \le 0.5$.



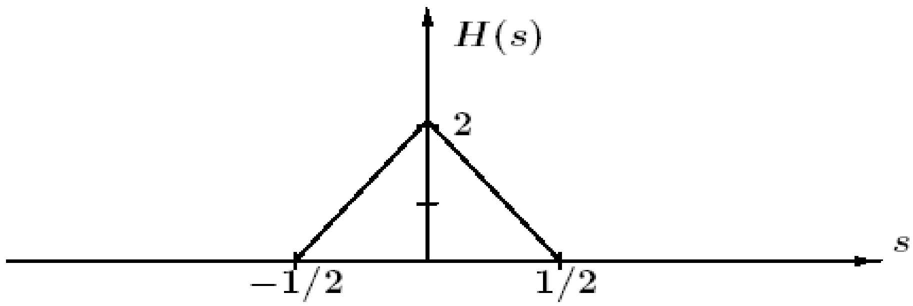
Solution The function $f(t)$ can be written as $f(t) = \Lambda(t/2) * \text{III}_4(t)$. Applying Fourier Transform we obtain:

$$\begin{aligned}
\mathcal{F}f(s) &= \mathcal{F}\{\Lambda(t/2)\} \cdot \mathcal{F}\{\text{III}_4(t)\} \\
&= 2\text{sinc}^2(2s) \cdot \frac{1}{4}\text{III}_{\frac{1}{4}}(s) \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} \text{sinc}^2(2s)\delta(s - \frac{k}{4}) \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} \text{sinc}^2(\frac{k}{2})\delta(s - \frac{k}{4})
\end{aligned}$$

A sketch of $\mathcal{F}f(s)$ for $-0.5 \le s \le 0.5$ is:



(b)(10) We filter $f(t)$ by multiplying its Fourier transform $\mathcal{F}f(s)$ by the function $H(s)$ sketched below. Find the filtered version of $f(t)$; that is, find $g(t) = (h * f)(t)$ where $h = \mathcal{F}^{-1}H$.



Solution Note that $H(-\frac{1}{4}) = H(\frac{1}{4}) = 1$. Also note that $H(0) = 2$. Thus

$$G(s) = F(s)H(s) = \delta(s) + \frac{2}{\pi^2}(\delta(s - \frac{k}{4}) + \delta(s + \frac{k}{4})$$

Taking the inverse Fourier transform gives,

$$g(t) = \mathcal{F}^{-1}G = 1 + \frac{4}{\pi^2}\cos(\frac{\pi t}{2})$$