7 Continuous-Time Fourier Series

Recommended Problems

P7.1

(a) Suppose that the signal $e^{j\omega t}$ is applied as the excitation to a linear, time-invariant system that has an impulse response $h(t)$. By using the convolution integral, show that the resulting output is $H(\omega)e^{j\omega t}$, where $H(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$.

(b) Assume that the system is characterized by a first-order differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t).$$

If $x(t) = e^{j\omega t}$ for all t, then $y(t) = H(\omega)e^{j\omega t}$ for all t. By substituting into the differential equation, determine $H(\omega)$.

P7.2

(a) Suppose that z^n, where z is a complex number, is the input to an LTI system that has an impulse response $h[n]$. Show that the resulting output is given by $H(z)z^n$, where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}.$$

(b) If the system is characterized by a first-order difference equation,

$$y[n] + ay[n-1] = x[n],$$

determine $H(z)$.

P7.3

Find the Fourier series coefficients for each of the following signals:

(a) $x(t) = \sin\left(10\pi t + \frac{\pi}{6}\right)$

(b) $x(t) = 1 + \cos(2\pi t)$

(c) $x(t) = [1 + \cos(2\pi t)]\left[\sin\left(10\pi t + \frac{\pi}{6}\right)\right]$

Hint: You may want to first multiply the terms and then use appropriate trigonometric identities.

P7.4

By evaluating the Fourier series analysis equation, determine the Fourier series for the following signals.

(a)

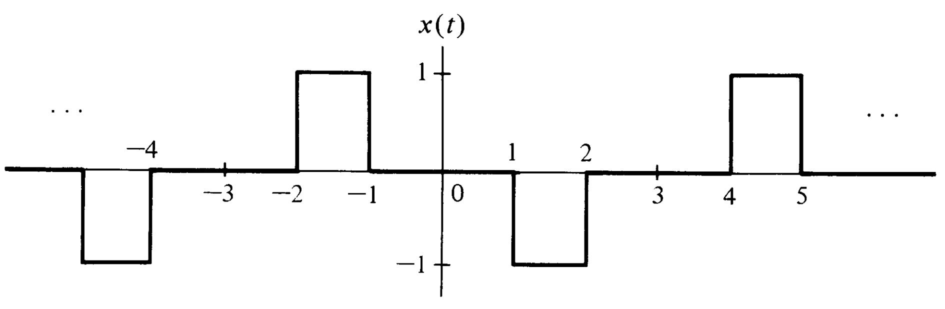

Figure P7.4-1

(b)

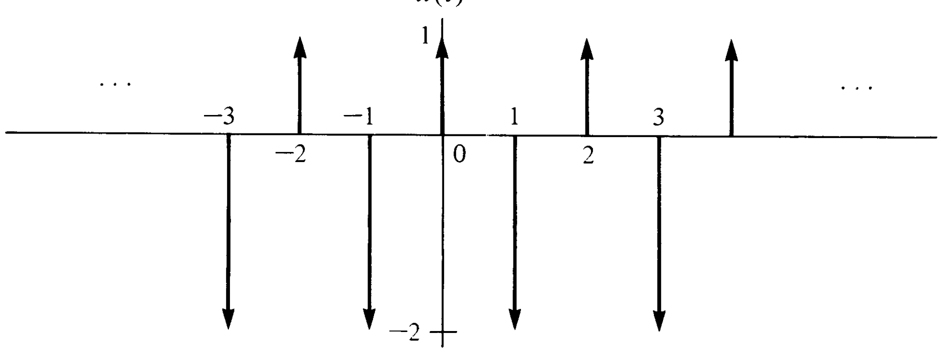

Figure P7.4-2

P7.5

Without explicitly evaluating the Fourier series coefficients, determine which of the periodic waveforms in Figures P7.5-1 to P7.5-3 have Fourier series coefficients with the following properties:

(i) Has only odd harmonics

(ii) Has only purely real coefficients

(iii) Has only purely imaginary coefficients

(a)

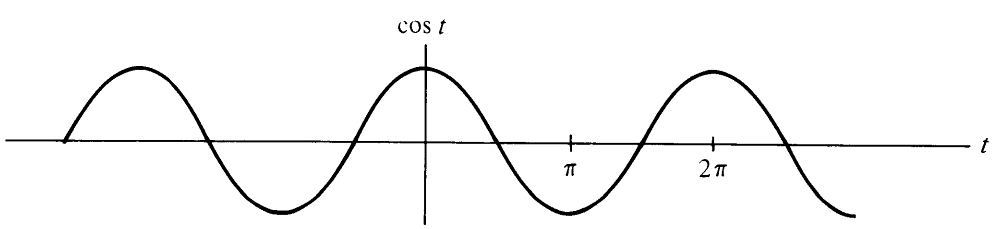

Figure P7.5-1

(b)

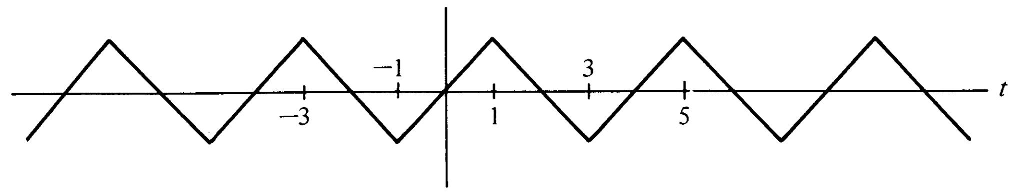

Figure P7.5-2

(c)

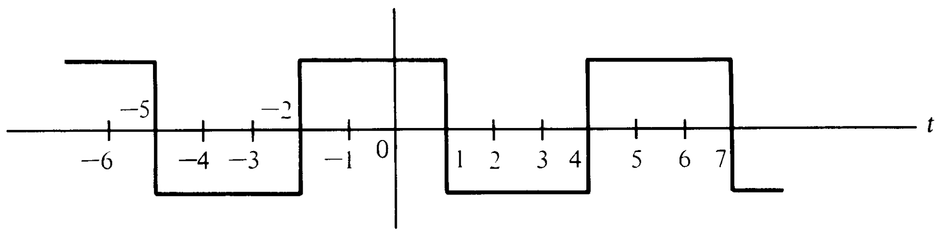

Figure P7.5-3

Optional Problems

P7.6

Suppose $x(t)$ is periodic with period T and is specified in the interval $0 < t < T/4$ as shown in Figure P7.6.

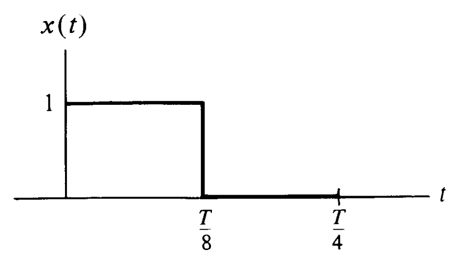

Figure P7.6

Sketch $x(t)$ in the interval $0 < t < T$ if

(a) the Fourier series has only odd harmonics and $x(t)$ is an even function;

(b) the Fourier series has only odd harmonics and $x(t)$ is an odd function.

P7.7

Let $x(t)$ be a periodic signal, with fundamental period T_0 and Fourier series coefficients a_k. Consider the following signals. The Fourier series coefficients for each can be expressed in terms of the a_k as in Table 4.2 (page 224) of the text. Show that the expression in Table 4.2 is correct for each signal.

(a) $x(t - t_0)$

(b) $x(-t)$

(c) $x^*(t)$

(d) $x(\alpha t)$, $\alpha > 0$ (Determine the period of the signal.)

P7.8

As we have seen in this lecture, the concept of an eigenfunction is an extremely important tool in the study of LTI systems. The same can also be said of linear but time-varying systems. Consider such a system with input $x(t)$ and output $y(t)$. We say that a signal $\phi(t)$ is an *eigenfunction* of the system if

$$\phi(t) \to \lambda\phi(t)$$

That is, if $x(t) = \phi(t)$, then $y(t) = \lambda\phi(t)$, where the complex constant λ is called the *eigenvalue associated with* $\phi(t)$.

(a) Suppose we can represent the input $x(t)$ to the system as a linear combination of eigenfunctions $\phi_k(t)$, each of which has a corresponding eigenvalue λ_k.

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \phi_k(t)$$

Express the output $y(t)$ of the system in terms of $\{c_k\}$, $\{\phi_k(t)\}$, and $\{\lambda_k\}$.

(b) Show that the functions $\phi_k(t) = t^k$ are eigenfunctions of the system characterized by the differential equation

$$y(t) = t^2 \frac{d^2x(t)}{dt^2} + t\frac{dx(t)}{dt}$$

For each $\phi_k(t)$, determine the corresponding eigenvalue λ_k.

P7.9

In the text and in Problem P4.10 in this manual, we defined the periodic convolution of two periodic signals $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ that have the same period T_0. Specifically, the periodic convolution of these signals is defined as

$$\tilde{y}(t) = \tilde{x}_1(t) \circledast \tilde{x}_2(t) = \int_{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t - \tau)\, d\tau \qquad \text{(P7.9-1)}$$

As shown in Problem P4.10, any interval of length T_0 can be used in the integral in eq. (P7.9-1), and $\tilde{y}(t)$ is also periodic with period T_0.

(a) If $\tilde{x}_1(t)$, $\tilde{x}_2(t)$, and $\tilde{y}(t)$ have Fourier series representations

$$\tilde{x}_1(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T_0)t}, \quad \tilde{x}_2(t) = \sum_{k=-\infty}^{+\infty} b_k e^{jk(2\pi/T_0)t}, \quad \tilde{y}(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk(2\pi/T_0)t},$$

show that $c_k = T_0 a_k b_k$.

(b) Consider the periodic signal $\tilde{x}(t)$ depicted in Figure P7.9-1. This signal is the result of the periodic convolution of another periodic signal, $\tilde{z}(t)$, with itself.

Find $\tilde{z}(t)$ and then use part (a) to determine the Fourier series representation for $\tilde{x}(t)$.

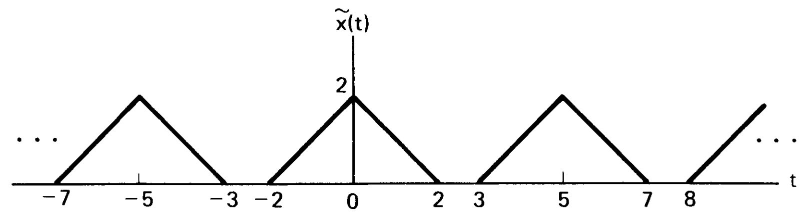

Figure P7.9-1

(c) Suppose now that $x_1(t)$ and $x_2(t)$ are the finite-duration signals illustrated in Figure P7.9-2(a) and (b). Consider forming the periodic signals $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$, which consist of periodically repeated versions of $x_1(t)$ and $x_2(t)$ as illustrated for $\tilde{x}_1(t)$ in Figure P7.9-2(c). Let $y(t)$ be the usual, aperiodic convolution of $x_1(t)$ and $x_2(t)$,

$$y(t) = x_1(t) * x_2(t),$$

and let $\tilde{y}(t)$ be the periodic convolution of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$,

$$\tilde{y}(t) = \tilde{x}_1(t) \circledast \tilde{x}_2(t)$$

Show that if T_0 is large enough, we can recover $y(t)$ completely from one period of $\tilde{y}(t)$, that is,

$$y(t) = \begin{cases} \tilde{y}(t), & |t| \leq T_0/2, \\ 0, & |t| > T_0/2 \end{cases}$$

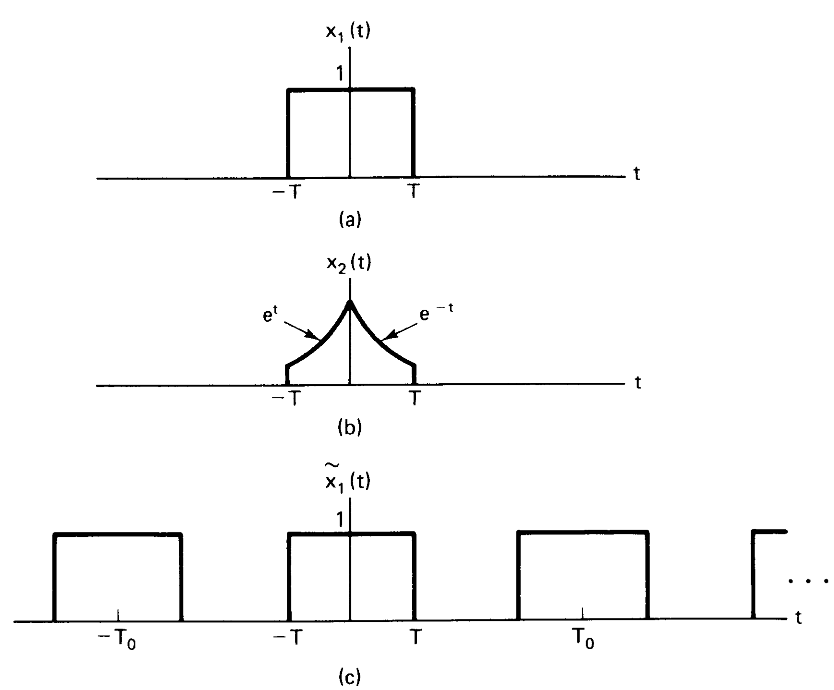

Figure P7.9-2

P7.10

The purpose of this problem is to show that the representation of an arbitrary periodic signal by a Fourier series, or more generally by a linear combination of any set of orthogonal functions, is computationally efficient and in fact is very useful for obtaining good approximations of signals. (See Problem 4.7 [page 254] of the text for the definitions of orthogonal and orthonormal functions.)

Specifically, let $\{\phi_i(t)\}$, $i = 0, \pm 1, \pm 2, \dots$, be a set of orthonormal functions on the interval $a \le t \le b$, and let $x(t)$ be a given signal. Consider the following approximation of $x(t)$ over the interval $a \le t \le b$:

$$\hat{x}_N(t) = \sum_{i=-N}^{+N} a_i \phi_i(t), \qquad \text{(P7.10-1)}$$

where the a_i are constants (in general, complex). To measure the deviation between $x(t)$ and the series approximation $\hat{x}_N(t)$, we consider the error $e_N(t)$ defined as

$$e_N(t) = x(t) - \hat{x}_N(t) \qquad \text{(P7.10-2)}$$

A reasonable and widely used criterion for measuring the quality of the approximation is the energy in the error signal over the interval of interest, that is, the integral of the squared-error magnitude over the interval $a \le t \le b$:

$$E = \int_a^b |e_N(t)|^2 \, dt \qquad \text{(P7.10-3)}$$

(a) Show that E is minimized by choosing

$$a_i = \int_a^b x(t)\phi_i^*(t) \, dt \qquad \text{(P7.10-4)}$$

Hint: Use eqs. (P7.10-1) to (P7.10-3) to express E in terms of a_i, $\phi_i(t)$, and $x(t)$. Then express a_i in rectangular coordinates as $a_i = b_i + jc_i$, and show that the equations

$$\frac{\partial E}{\partial b_i} = 0 \quad \text{and} \quad \frac{\partial E}{\partial c_i} = 0, \qquad i = 0, \pm 1, \pm 2, \dots, \pm N,$$

are satisfied by the a_i as given in eq. (P7.10-4).

(b) Determine how the result of part (a) changes if the $\{\phi_i(t)\}$ are orthogonal but not orthonormal, with

$$A_i = \int_a^b |\phi_i(t)|^2 \, dt$$

(c) Let $\phi_n(t) = e^{jn\omega_0 t}$ and choose any interval of length $T_0 = 2\pi/\omega_0$. Show that the a_i that minimize E are as given in eq. (4.45) of the text (page 180).

7 Continuous-Time Fourier Series

Solutions to Recommended Problems

S7.1

(a) For the LTI system indicated in Figure S7.1, the output $y(t)$ is expressed as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)\, d\tau,$$

where $h(t)$ is the impulse response and $x(t)$ is the input.

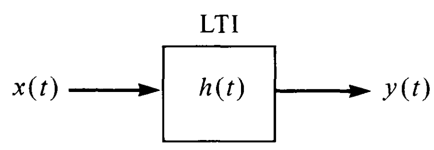

Figure S7.1

For $x(t) = e^{j\omega t}$,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}\, d\tau \\ &= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\, d\tau \\ &= e^{+j\omega t}H(\omega) \end{aligned}$$

(b) We are given that the first-order differential equation is of the form

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

From part (a), when $x(t) = e^{j\omega t}$, then $y(t) = e^{j\omega t}H(\omega)$. Also, by differentiating $y(t)$, we have

$$\frac{dy(t)}{dt} = j\omega e^{j\omega t}H(\omega)$$

Substituting, we get

$$j\omega e^{j\omega t}H(\omega) + ae^{j\omega t}H(\omega) = e^{j\omega t}$$

Hence,

$$j\omega H(\omega) + aH(\omega) = 1, \qquad \text{or}$$

$$H(\omega) = \frac{1}{a + j\omega}$$

S7.2

(a) The output of a discrete-time LTI system is given by the discrete-time convolution sum

$$y[n] = \sum_k h[k]x[n - k]$$

If $x[n] = z^n$, then

$$\begin{aligned} y[n] &= \sum_k h[k]z^{n-k} \\ &= z^n \sum_k h[k]z^{-k} \\ &= z^n H(z) \end{aligned}$$

(b) We are given that the first-order difference equation is of the form

$$y[n] + ay[n-1] = x[n]$$

From part (a), if $x[n] = z^n$, then $y[n] = z^n H(z)$. Hence,

$$y[n-1] = z^{n-1}H(z).$$

By substitution,

$$z^n H(z) + az^{n-1}H(z) = z^n,$$

which implies

$$\begin{aligned} (1 + az^{-1})H(z) &= 1, \\ H(z) &= \frac{1}{1 + az^{-1}} \end{aligned}$$

S7.3

(a) $x(t) = \sin\left(10\pi t + \frac{\pi}{6}\right)$

$$= \frac{e^{j\pi/6}}{2j} e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j} e^{-j2\pi t5}$$

We choose ω_0, the fundamental frequency, to be 2π.

$$x(t) = \sum_k a_k e^{jk\omega_0 t},$$

where

$$a_5 = \frac{e^{j\pi/6}}{2j}, \qquad a_{-5} = \frac{-e^{-j\pi/6}}{2j}$$

Otherwise $a_k = 0$.

(b) $x(t) = 1 + \cos(2\pi t)$

$$= 1 + \frac{e^{j2\pi t}}{2} + \frac{e^{-j2\pi t}}{2}$$

For $\omega_0 = 2\pi$, $a_{-1} = a_1 = \frac{1}{2}$, and $a_0 = 1$. All other a_k's $= 0$.

(c) $x(t) = [1 + \cos(2\pi t)]\left[\sin\left(10\pi t + \frac{\pi}{6}\right)\right]$

$$\begin{aligned} &= \sin\left(10\pi t + \frac{\pi}{6}\right) + \cos(2\pi t)\sin\left(10\pi t + \frac{\pi}{6}\right) \\ &= \left(\frac{e^{j\pi/6}}{2j} e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j} e^{-j2\pi t5}\right) + \left(\tfrac{1}{2}e^{j2\pi t} + \tfrac{1}{2}e^{-j2\pi t}\right)\left(\frac{e^{j\pi/6}}{2j} e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j} e^{-j2\pi t5}\right) \\ &= \frac{e^{j\pi/6}}{2j} e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j} e^{-j2\pi t5} + \frac{e^{j\pi/6}}{4j} e^{j2\pi t6} - \frac{e^{-j\pi/6}}{4j} e^{-j2\pi t4} \\ &\quad + \frac{e^{j\pi/6}}{4j} e^{j2\pi t4} - \frac{e^{-j\pi/6}}{4j} e^{-j2\pi t6} \end{aligned}$$

Therefore,

$$x(t) = \sum_k a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$.

$$a_4 = \frac{e^{j\pi/6}}{4j}, \qquad a_{-4} = \frac{-e^{-j\pi/6}}{4j},$$
$$a_5 = \frac{e^{j\pi/6}}{2j}, \qquad a_{-5} = \frac{-e^{-j\pi/6}}{2j},$$
$$a_6 = \frac{e^{j\pi/6}}{4j}, \qquad a_{-6} = \frac{-e^{-j\pi/6}}{4j}$$

All other a_k's = 0.

S7.4

(a)

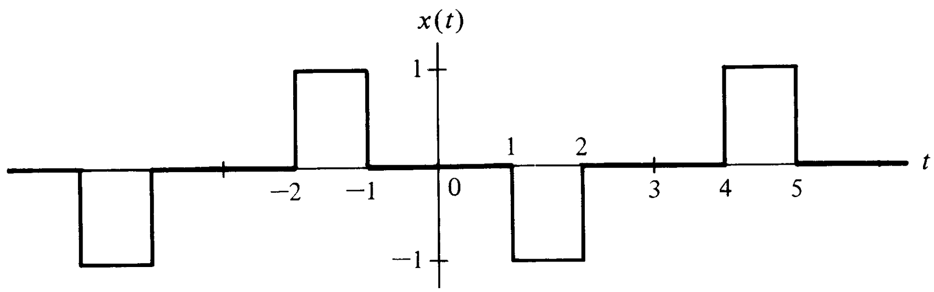

Figure S7.4-1

Note that the period is $T_0 = 6$. Fourier coefficients are given by

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

We take $\omega_0 = 2\pi/T_0 = \pi/3$. Choosing the period of integration as -3 to 3, we have

$$\begin{aligned}
a_k &= \frac{1}{6}\int_{-2}^{-1} e^{-jk(\pi/3)t}\,dt - \frac{1}{6}\int_{1}^{2} e^{-jk(\pi/3)t}\,dt \\
&= \frac{1}{6}\,\frac{1}{-jk(\pi/3)}\,e^{-jk(\pi/3)t}\Big|_{-2}^{-1} - \frac{1}{6}\,\frac{1}{-jk(\pi/3)}\,e^{-jk(\pi/3)t}\Big|_{1}^{2} \\
&= \frac{1}{-j2\pi k}\left[e^{+j(\pi/3)k} - e^{+j(2\pi/3)k} - e^{-j(2\pi/3)k} + e^{-j(\pi/3)k}\right] \\
&= \frac{\cos(2\pi/3)k}{j\pi k} - \frac{\cos(\pi/3)k}{j\pi k}
\end{aligned}$$

Therefore,

$$x(t) = \sum_k a_k e^{jk\omega_0 t}, \qquad \omega_0 = \frac{\pi}{3}$$

and

$$a_k = \frac{\cos(2\pi/3)k - \cos(\pi/3)k}{j\pi k}$$

Note that $a_0 = 0$, as can be determined either by applying L'Hôpital's rule or by noting that

$$a_0 = (1/T_0) \int_{T_0} x(t)dt.$$

(b)

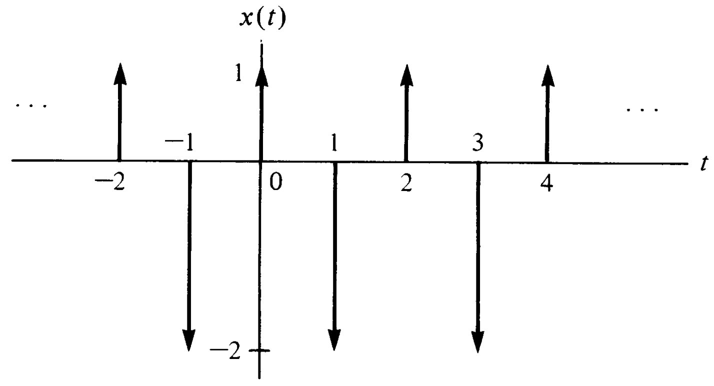

Figure S7.4-2

The period is $T_0 = 2$, with $\omega_0 = 2\pi/2 = \pi$. The Fourier coefficients are

$$a_k = \frac{1}{T_0} \int_{T_0} x(t)e^{-jk\omega_0 t}\, dt$$

Choosing the period of integration as $-\frac{1}{2}$ to $\frac{3}{2}$, we have

$$\begin{aligned} a_k &= \frac{1}{2} \int_{-1/2}^{3/2} x(t)e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{2} \int_{-1/2}^{3/2} [\delta(t) - 2\delta(t-1)]e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{2} - e^{-jk\omega_0} = \frac{1}{2} - (e^{-j\pi})^k \end{aligned}$$

Therefore,

$$a_0 = -\tfrac{1}{2}, \qquad a_k = \tfrac{1}{2} - (-1)^k$$

It is instructive to plot a_k, which we have done in Figure S7.4-3.

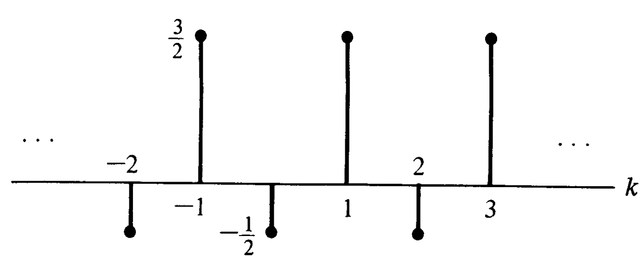

Figure S7.4-3

S7.5

(a) (i) and (ii)
From Problem 4.12 of the text (page 260), we have

$$x\left(t - \frac{T}{2}\right) = -x(t),$$

which means odd harmonics. Since $x(t)$ is real and even, the waveform has real coefficients.

(b) (i) and (iii)

$$-x(t) = x\left(t - \frac{T}{2}\right),$$

which means odd harmonics. Since $x(t)$ is real and odd, the waveform has imaginary coefficients.

(c) (i)

$$-x(t) = x\left(t - \frac{T}{2}\right),$$

which means odd harmonics. Also, $x(t)$ is neither even nor odd.

Solutions to Optional Problems

S7.6

$x(t)$ is specified in the interval $0 < t < T/4$, as shown in Figure S7.6-1.

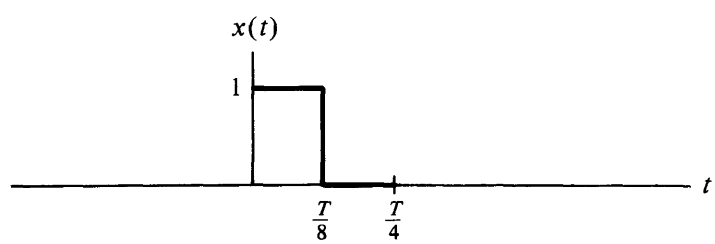

Figure S7.6-1

(a) Since $x(t)$ is even, we can extend Figure S7.6-1 as indicated in Figure S7.6-2.

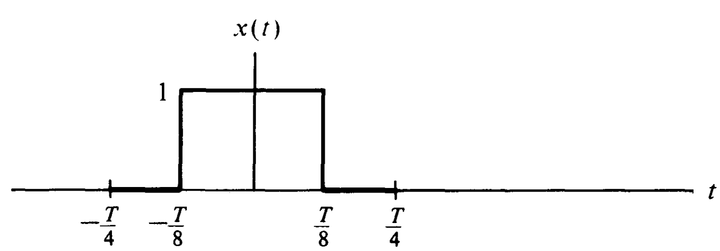

Figure S7.6-2

Since $x(t)$ has only odd harmonics, it must have the property that $x(t - T/2) = -x(t)$, as shown in Figure S7.6-3.

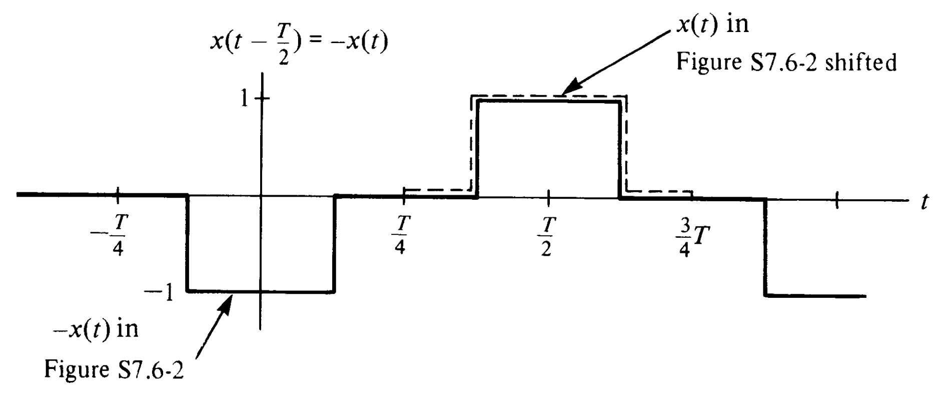

Figure S7.6-3

So we have $x(t)$ as in Figure S7.6-4.

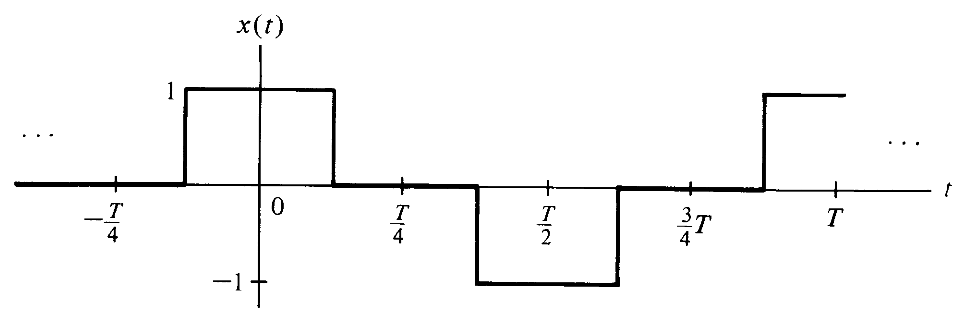

Figure S7.6-4

(b) In the interval from $t = 0$ to $t = T/4$, $x(t)$ is given as in Figure S7.6-5.

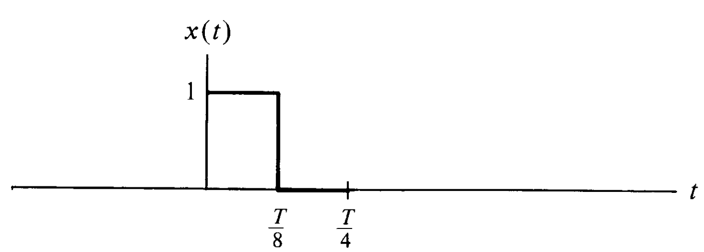

Figure S7.6-5

Since $x(t)$ is odd, for $-T/4 < t < T/4$ it must be as indicated in Figure S7.6-6.

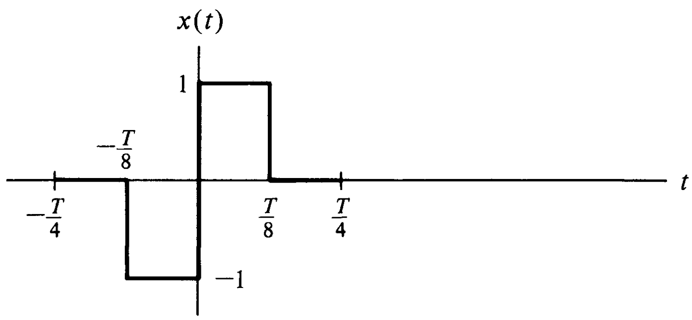

Figure S7.6-6

Since $x(t)$ has odd harmonics, $x[t - (T/2)] = -x(t)$. Consequently $x(t)$ is as shown in Figure S7.6-7.

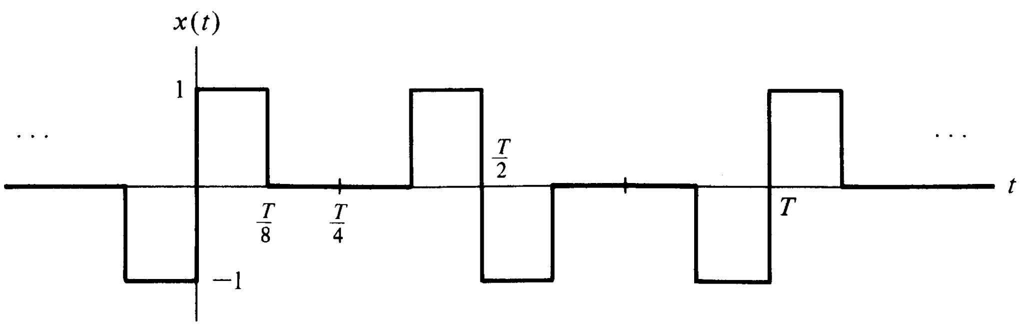

Figure S7.6-7

S7.7

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}\, dt$$

(a) $\hat{a}_k = \dfrac{1}{T_0} \displaystyle\int_{T_0} x(t - t_0) e^{-jk\omega_0 t}\, dt$

Substituting $\tau = t - t_0$, we obtain

$$\begin{aligned} \hat{a}_k &= \frac{1}{T_0} \int_{T_0} x(\tau) e^{jk\omega_0 \tau}\, d\tau \cdot e^{-jk\omega_0 t_0} \\ &= a_k e^{-jk\omega_0 t_0} \end{aligned}$$

(b) $\hat{a}_k = \dfrac{1}{T_0} \displaystyle\int_{T_0} x(-t) e^{-jk\omega_0 t}\, dt$

Substituting $\tau = -t$, we have

$$\hat{a}_k = \frac{1}{T_0} \int_{T_0} x(\tau) e^{jk\omega_0 \tau}\, d\tau = a_{-k}$$

(c) $\hat{a}_k = \dfrac{1}{T_0} \displaystyle\int_{T_0} x^*(t)e^{-jk\omega_0 t}\, dt$

$$\hat{a}_k^* = \frac{1}{T_0} \int_{T_0} x(t)e^{jk\omega_0 t}\, dt = a_{-k},$$

$$\hat{a}_k = a_{-k}^*$$

(d) $\hat{a}_k = \dfrac{\alpha}{T_0} \displaystyle\int_{T_0/\alpha} x(\alpha t)e^{-jk(2\pi\alpha/T_0)t}\, dt$

Let $\tau = \alpha t$. Then

$$\hat{a}_k = \frac{1}{T_0} \int_{T_0} x(\tau)e^{-jk(2\pi/T_0)\tau}\, d\tau = a_k$$

Therefore,

$$\hat{T}_0 = \frac{T_0}{\alpha}$$

S7.8

(a) Since $\phi_k(t)$ are eigenfunctions and the system is linear, the output is

$$y(t) = \sum_{k=-\infty}^{\infty} \lambda_k c_k \phi_k(t).$$

(b) $y(t) = t^2 \dfrac{d^2x(t)}{dt^2} + t\, \dfrac{dx(t)}{dt},$

$$\phi_k(t) = t^k,$$

$$\frac{d\phi_k(t)}{dt} = kt^{k-1},$$

$$\frac{d^2\phi_k(t)}{dt^2} = k(k-1)t^{k-2}$$

So if $\phi_k(t) = x(t)$, then

$$\begin{aligned} y(t) &= t^2k(k-1)t^{k-2} + tkt^{k-1} \\ &= k(k-1)t^k + kt^k \\ &= k^2t^k = k^2\phi_k(t) \end{aligned}$$

The eigenfunction $\phi_k(t)$ has eigenvalue $\lambda_k = k^2$.

S7.9

(a) $\tilde{y}(t) = \tilde{x}_1(t) \circledast \tilde{x}_2(t)$

$$= \int_{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\, d\tau$$

The Fourier coefficients for $\tilde{y}(t)$ are given by

$$\begin{aligned} c_k &= \frac{1}{T_0} \int_{T_0} \int_{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\, d\tau e^{-jk(2\pi/T_0)t}\, dt \\ &= \frac{1}{T_0} \int_{T_0} \tilde{x}_1(\tau)e^{-jk(2\pi/T_0)\tau}\, d\tau \int_{T_0} \tilde{x}_2(t-\tau)e^{-jk(2\pi/T_0)(t-\tau)}\, dt \\ &= T_0 a_k b_k \end{aligned}$$

(b) Since $z(t) * z(t) = x(t)$, as shown in Figure S7.9-1, then $\tilde{z}(t)$ is shown in Figure S7.9-2.

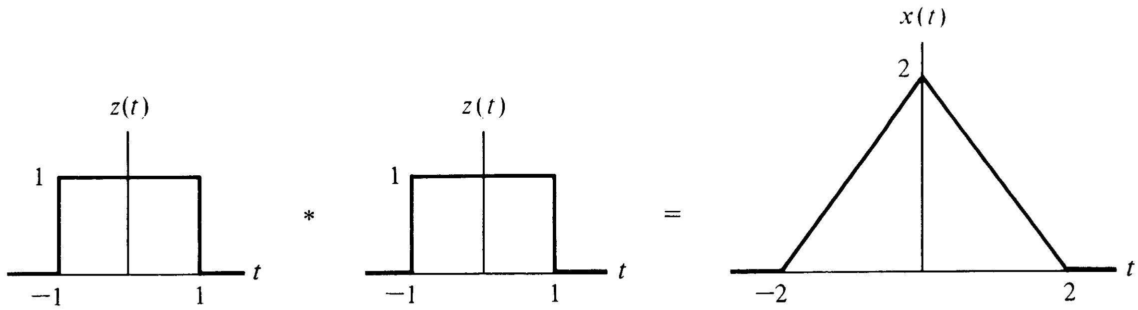

Figure S7.9-1

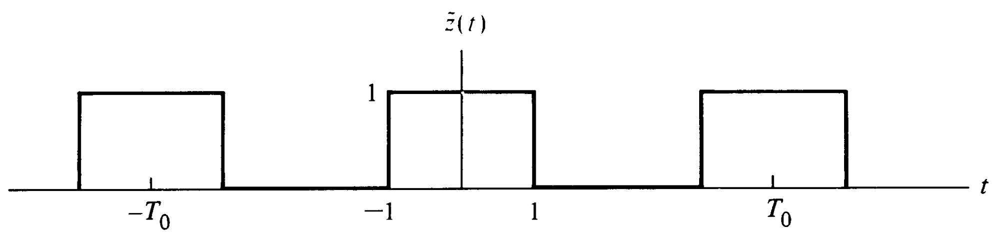

Figure S7.9-2

In Figure S7.9-2, $T_0 = 5$. Hence,

$$\tilde{x}(t) \longleftrightarrow T_0 z_k^2 = \frac{4}{5}\left[\text{sinc}\left(\frac{2\pi k}{5}\right)\right]^2$$

(c) Without explicitly carrying out the convolutions, we can argue that the aperiodic convolution of $x_1(t)$ and $x_2(t)$ will be symmetric about the origin and is nonzero from $t = -2T$ to $t = 2T$. Now, if $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are periodic with period T_0, then the periodic convolution, $\tilde{y}(t)$, will be periodic with period T_0. If T_0 is large enough, then $\tilde{y}(t)$ is the periodic version of $y(t)$ with period T_0. Hence, to recover $y(t)$ from $\tilde{y}(t)$ we should extract only one period of $\tilde{y}(t)$ from $t = -T_0/2$ to $t = T_0/2$ and set $y(t) = 0$ for $|t| > T_0/2$, where $T_0/2 \geq 2T$, or $T_0 \geq 4T$.

S7.10

(a) The approximation is

$$\hat{x}_N(t) = \sum_{k=-N}^{N} a_k \phi_k(t)$$

with the corresponding error signal

$$\begin{aligned} e_N(t) &= x(t) - \hat{x}_N(t) \\ &= x(t) - \sum_{k=-N}^{N} a_k \phi_k(t) \end{aligned}$$

Hence,

$$|e_N(t)|^2 = \left[x(t) - \sum_k a_k\phi_k(t)\right]\left[x^*(t) - \sum_k a_k^*\phi_k^*(t)\right]$$
$$= |x(t)|^2 - \sum_k a_k^* x(t)\phi_k^*(t) - \sum_k a_k x^*(t)\phi_k(t) + \sum_k\sum_l a_k a_l^* \phi_k(t)\phi_l^*(t)$$

If we integrate, $\int_a^b |e_N(t)|^2\,dt$, and use the property that

$$\int_a^b \phi_k(t)\phi_l^*(t)\,dt = \begin{cases} 1, & k = l, \\ 0, & \text{otherwise,} \end{cases}$$

we get

$$E = \int_a^b |x(t)|^2\,dt - \sum_k a_k^* \int_a^b x(t)\phi_k^*(t)\,dt$$
$$- \sum_k a_k \int_a^b x^*(t)\phi_k(t)\,dt + \sum_k |a_k|^2$$

Since $a_i = b_i + jc_i$,

$$\frac{\partial E}{\partial b_i} = -\int_a^b x(t)\phi_i^*(t)\,dt - \int_a^b x^*(t)\phi_i(t)\,dt + 2b_i$$

and

$$\frac{\partial E}{\partial c_i} = j\int_a^b x(t)\phi_i^*(t)\,dt - j\int_a^b x^*(t)\phi_i(t)\,dt + 2c_i$$

Setting

$$\frac{\partial E}{\partial b_i} = 0 \quad \text{and} \quad \frac{\partial E}{\partial c_i} = 0,$$

we can multiply the second equation by j and add the two equations to get

$$\frac{\partial E}{\partial b_i} + j\frac{\partial E}{\partial c_i} = 0$$

By substitution, we get

$$b_i + jc_i = \int_a^b x(t)\phi_i^*(t)\,dt$$
$$= a_i$$

(b) If $\{\phi_i(t)\}$ are orthogonal but not orthonormal, then the only thing that changes from the result of part (a) is

$$\int_a^b \sum_k \sum_l a_k a_l^* \phi_k(t)\phi_l^*(t)\,dt = \sum_k |a_k|^2 A_k$$

It is easy to see that we will now get

$$a_i = \frac{1}{A_i}\int_a^b x(t)\phi_i^*(t)\,dt$$

(c) Since

$$\int_{a}^{T_0+a} e^{jn\omega_0 t} e^{-jn\omega_0 t}\, dt = T_0$$

for all values of a, using parts (a) and (b) we can write

$$a_i = \frac{1}{T_0} \int_{a}^{T_0+a} x(t) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t}\, dt$$

8 Continuous-Time Fourier Transform

Recommended Problems

P8.1

Consider the signal $x(t)$, which consists of a single rectangular pulse of unit height, is symmetric about the origin, and has a total width T_1.

(a) Sketch $x(t)$.

(b) Sketch $\tilde{x}(t)$, which is a periodic repetition of $x(t)$ with period $T_0 = 3T_1/2$.

(c) Compute $X(\omega)$, the Fourier transform of $x(t)$. Sketch $|X(\omega)|$ for $|\omega| \leq 6\pi/T_1$.

(d) Compute a_k, the Fourier series coefficients of $\tilde{x}(t)$. Sketch a_k for $k = 0, \pm 1, \pm 2, \pm 3$.

(e) Using your answers to (c) and (d), verify that, for this example,

$$a_k = \frac{1}{T_0} X(\omega) \Big|_{\omega = (2\pi k)/T_0}$$

(f) Write a statement that indicates how the Fourier series for a periodic function can be obtained if the Fourier transform of one period of this periodic function is given.

P8.2

Find the Fourier transform of each of the following signals and sketch the magnitude and phase as a function of frequency, including both positive and negative frequencies.

(a) $\delta(t - 5)$

(b) $e^{-at}u(t)$, $\quad a$ real, positive

(c) $e^{(-1+j2)t}u(t)$

P8.3

In this problem we explore the definition of the Fourier transform of a periodic signal.

(a) Show that if $x_3(t) = ax_1(t) + bx_2(t)$, then $X_3(\omega) = aX_1(\omega) + bX_2(\omega)$.

(b) Verify that

$$e^{j\omega_0 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\, d\omega$$

From this observation, argue that the Fourier transform of $e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$.

(c) Recall the synthesis equation for the Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

By taking the Fourier transform of both sides and using the results to parts (a) and (b), show that

$$\tilde{X}(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

(d) Sketch $\tilde{X}(\omega)$ for your answer to Problem P8.1(d) for $|\omega| \leq 4\pi/T_0$.

P8.4

(a) Consider the often-used alternative definition of the Fourier transform, which we will call $X_a(f)$. The forward transform is written as

$$X_a(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt,$$

where f is the frequency variable in hertz. Derive the inverse transform formula for this definition. Sketch $X_a(f)$ for the signal discussed in Problem P8.1.

(b) A second, alternative definition is

$$X_b(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-jvt}\, dt$$

Find the inverse transform relation.

P8.5

Consider the periodic signal $\tilde{x}(t)$ in Figure P8.5-1, which is composed solely of impulses.

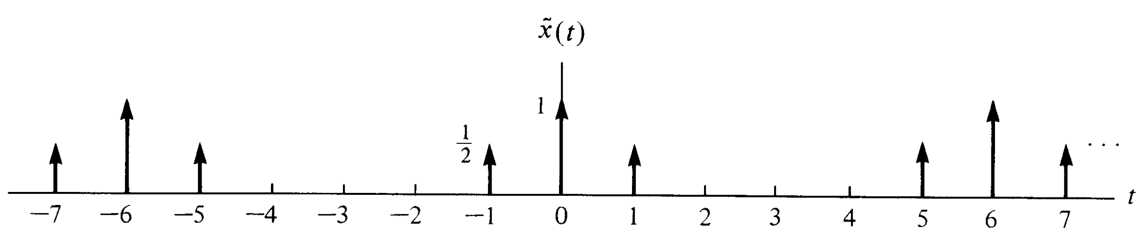

Figure P8.5-1

(a) What is the fundamental period T_0?

(b) Find the Fourier series of $\tilde{x}(t)$.

(c) Find the Fourier transform of the signals in Figures P8.5-2 and P8.5-3.

(i)

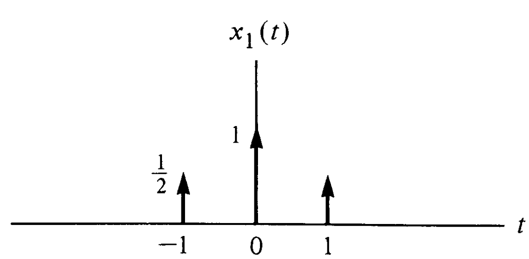

Figure P8.5-2

(ii)

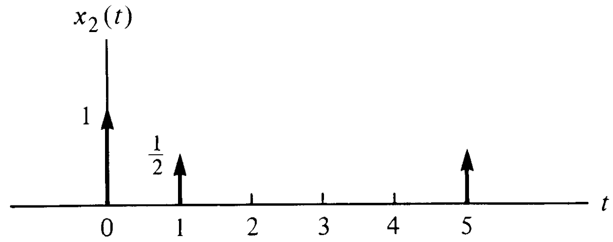

Figure P8.5-3

(d) $\tilde{x}(t)$ can be expressed as either $x_1(t)$ periodically repeated or $x_2(t)$ periodically repeated, i.e.,

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x_1(t - kT_1), \qquad \text{or} \tag{P8.5-1}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x_2(t - kT_2) \tag{P8.5-2}$$

Determine T_1 and T_2 and demonstrate graphically that eqs. (P8.5-1) and (P8.5-2) are valid.

(e) Verify that the Fourier series of $\tilde{x}(t)$ is composed of scaled samples of either $X_1(\omega)$ or $X_2(\omega)$.

P8.6

Find the signal corresponding to the following Fourier transforms.

(a) $X_a(\omega) = \dfrac{1}{7 + j\omega}$

(b)

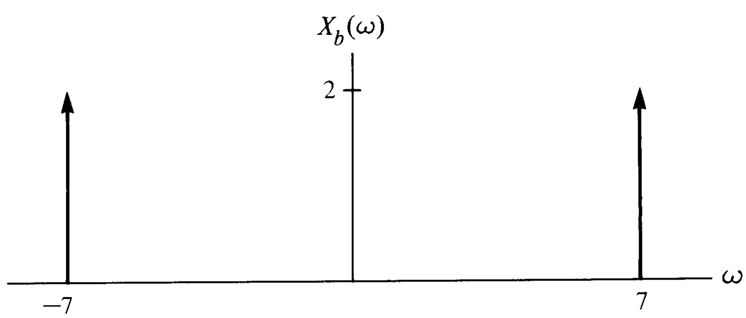

Figure P8.6-1

(c) $X_c(\omega) = \dfrac{1}{9 + \omega^2}$

See Example 4.8 in the text (page 191).

(d) $X_d(\omega) = X_a(\omega)X_b(\omega)$, where $X_a(\omega)$ and $X_b(\omega)$ are given in parts (a) and (b), respectively. Try to simplify as much as possible.

(e)

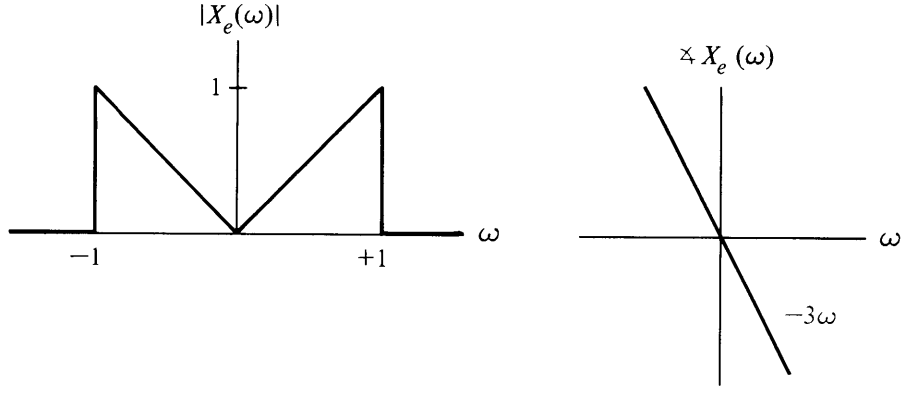

Figure P8.6-2

Optional Problems

P8.7

In earlier lectures, the response of an LTI system to an input $x(t)$ was shown to be $y(t) = x(t) * h(t)$, where $h(t)$ is the system impulse response.

(a) Using the fact that

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau,$$

show that

$$Y(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h(t - \tau)e^{-j\omega t}\, d\tau\, dt$$

(b) By appropriate change of variables, show that

$$Y(\omega) = X(\omega)H(\omega),$$

where $X(\omega)$ is the Fourier transform of $x(t)$, and $H(\omega)$ is the Fourier transform of $h(t)$.

P8.8

Consider the impulse train

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

shown in Figure P8.8-1.

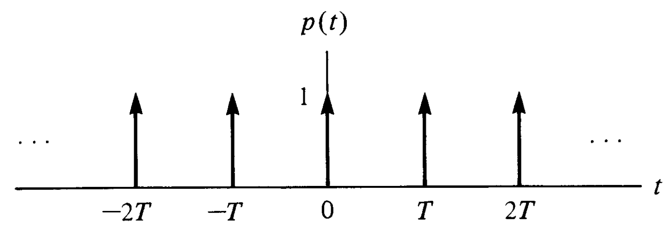

Figure P8.8-1

(a) Find the Fourier series of $p(t)$.

(b) Find the Fourier transform of $p(t)$.

(c) Consider the signal $x(t)$ shown in Figure P8.8-2, where $T_1 < T$.

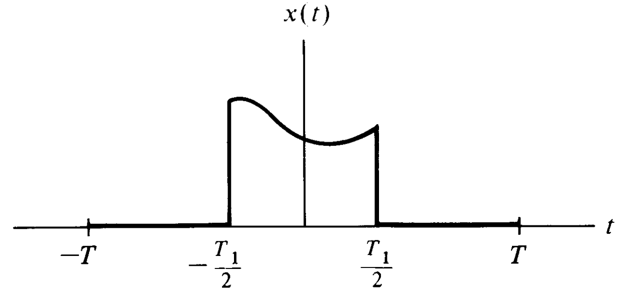

Figure P8.8-2

Show that the periodic signal $\tilde{x}(t)$, formed by periodically repeating $x(t)$, satisfies

$$\tilde{x}(t) = x(t) * p(t)$$

(d) Using the result to Problem P8.7 and parts (b) and (c) of this problem, find the Fourier transform of $\tilde{x}(t)$ in terms of the Fourier transform of $x(t)$.

8 Continuous-Time Fourier Transform

Solutions to Recommended Problems

S8.1

(a)

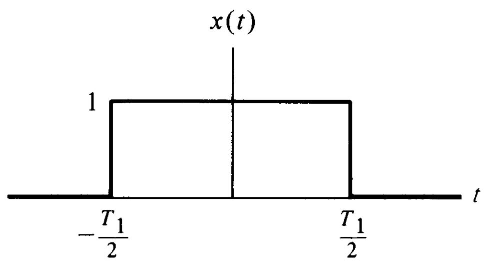

Figure S8.1-1

Note that the *total* width is T_1.

(b)

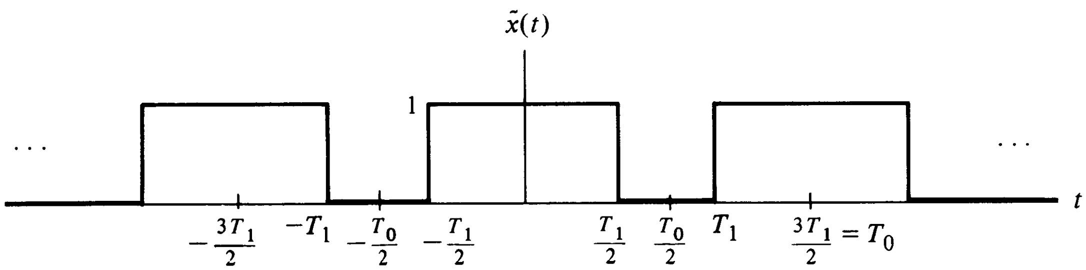

Figure S8.1-2

(c) Using the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-T_1/2}^{T_1/2} 1e^{-j\omega t}\,dt \quad \text{since } x(t) = 0 \quad \text{for} \quad |t| > \frac{T_1}{2}$$

$$= \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-T_1/2}^{T_1/2} = \frac{-1}{j\omega}\left(e^{-j\omega T_1/2} - e^{j\omega T_1/2}\right) = \frac{2\sin\frac{\omega T_1}{2}}{\omega}$$

See Figure S8.1-3.

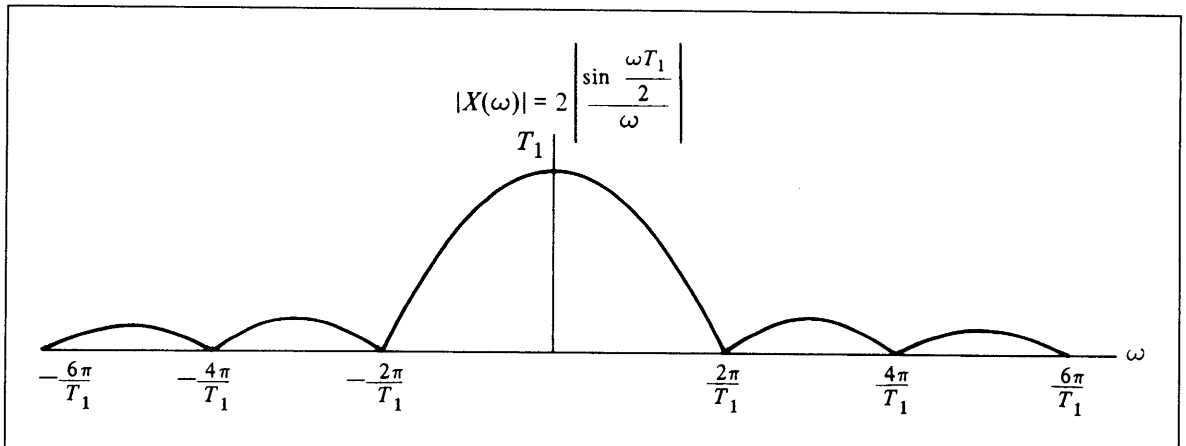

Figure S8.1-3

(d) Using the analysis formula, we have

$$a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)e^{-jk\omega_0 t}\,dt,$$

where we integrate over *any* period.

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)e^{-jk(2\pi/T_0)t}\,dt = \frac{1}{T_0}\int_{-T_1/2}^{T_1/2} e^{-jk(2\pi/T_0)t}\,dt,$$

$$a_k = \frac{1}{T_0}\left(\frac{1}{-jk\dfrac{2\pi}{T_0}}\right)(e^{-jk\pi T_1/T_0} - e^{jk\pi T_1/T_0}) = \frac{\sin k\pi(T_1/T_0)}{\pi k} = \frac{\sin \pi(2k/3)}{\pi k}$$

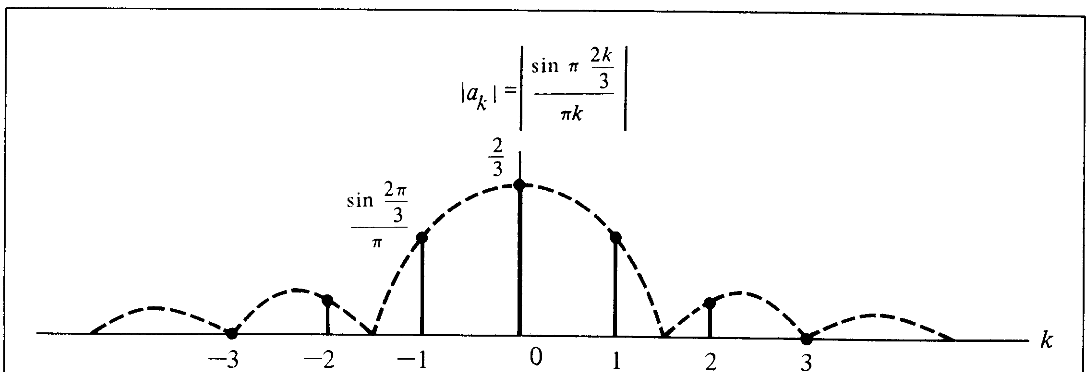

Figure S8.1-4

Note that $a_k = 0$ whenever $(2\pi k)/3 = \pi m$ for m a nonzero integer.

(e) Substituting $(2\pi k)/T_0$ for ω, we obtain

$$\frac{1}{T_0}X(\omega)\bigg|_{\omega=(2\pi k)/T_0} = \frac{1}{T_0}\,\frac{2\sin(\pi k T_1/T_0)}{2\pi k/T_0} = \frac{\sin \pi k(T_1/T_0)}{\pi k} = a_k$$

(f) From the result of part (e), we sample the Fourier transform of $x(t)$, $X(\omega)$, at $\omega = 2\pi k/T_0$ and then scale by $1/T_0$ to get a_k.

S8.2

(a) $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \delta(t-5)e^{-j\omega t}\,dt = e^{-j5\omega} = \cos 5\omega - j\sin 5\omega,$

by the sifting property of the unit impulse.

$$|X(\omega)| = |e^{j5\omega}| = 1 \qquad \text{for all } \omega,$$

$$\sphericalangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = \tan^{-1}\left(\frac{-\sin 5\omega}{\cos 5\omega}\right) = -5\omega$$

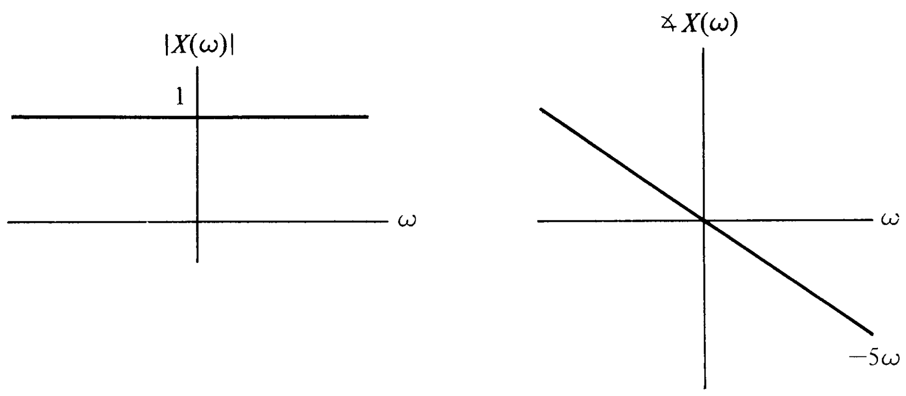

Figure S8.2-1

(b) $X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$

$$= \int_0^{\infty} e^{-(a+j\omega)t}\,dt = \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}$$

Since $Re\{a\} > 0$, e^{-at} goes to zero as t goes to infinity. Therefore,

$$X(\omega) = \frac{-1}{a+j\omega}(0-1) = \frac{1}{a+j\omega},$$

$$|X(\omega)| = [X(\omega)X^*(\omega)]^{1/2} = \left[\frac{1}{a+j\omega}\left(\frac{1}{a-j\omega}\right)\right]^{1/2} \frac{1}{\sqrt{a^2+\omega^2}},$$

$$Re\{X(\omega)\} = \frac{X(\omega)+X^*(\omega)}{2} = \frac{a}{a^2+\omega^2},$$

$$Im\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2} = \frac{-\omega}{a^2+\omega^2},$$

$$\sphericalangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = -\tan^{-1}\frac{\omega}{a}$$

The magnitude and angle of $X(\omega)$ are shown in Figure S8.2-2.

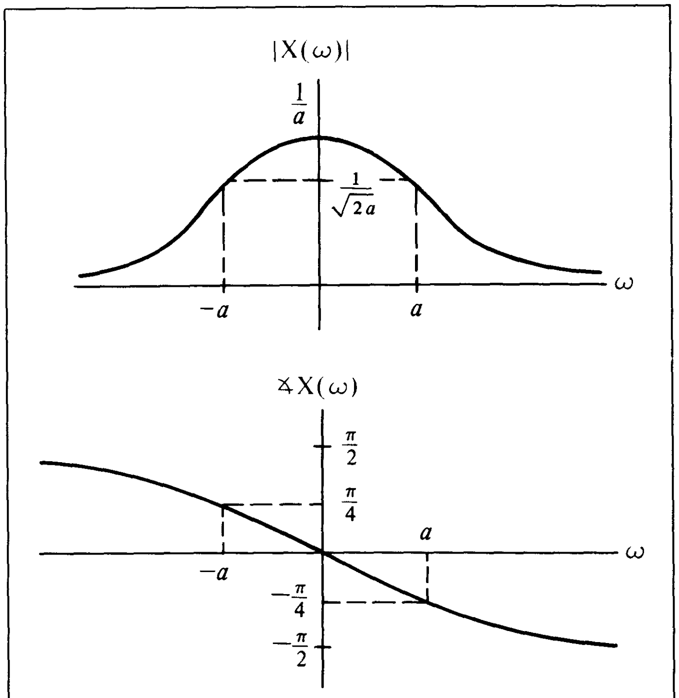

Figure S8.2-2

(c) $X(\omega) = \int_{-\infty}^{\infty} e^{(-1+j2)t}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{(-1+j2)t}e^{-j\omega t}\,dt$

$$= \frac{1}{-1 + j(2-\omega)} e^{[-1+j(2-\omega)]t} \Big|_0^{\infty}$$

Since $Re\{-1 + j(2-\omega)\} < 0$, $\lim_{t\to\infty} e^{[-1+j(2-\omega)]t} = 0$. Therefore,

$$X(\omega) = \frac{1}{1 + j(\omega - 2)}$$

$$|X(\omega)| = [X(\omega)X^*(\omega)]^{1/2} = \frac{1}{\sqrt{1 + (\omega-2)^2}}$$

$$Re\{X(\omega)\} = \frac{X(\omega) + X^*(\omega)}{2} = \frac{1}{1 + (\omega-2)^2}$$

$$Im\{X(\omega)\} = \frac{X(\omega) - X^*(\omega)}{2} \quad \frac{-(\omega-2)}{1 + (\omega-2)^2}$$

$$\sphericalangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = -\tan^{-1}(\omega - 2)$$

The magnitude and angle of $X(\omega)$ are shown in Figure S8.2-3.

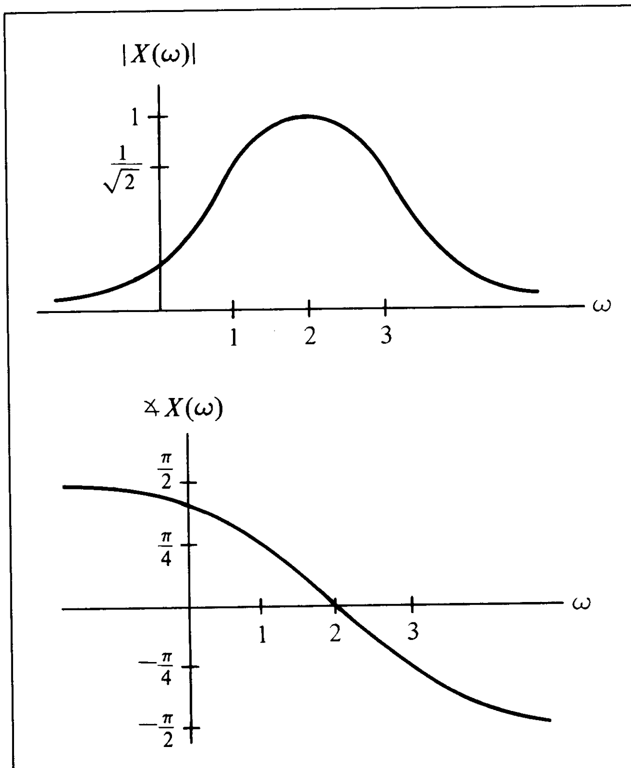

Figure S8.2-3

Note that there is no symmetry about $\omega = 0$ since $x(t)$ is not real.

S8.3

(a) $X_3(\omega) = \int_{-\infty}^{\infty} x_3(t)e^{-j\omega t}\, dt$

Substituting for $x_3(t)$, we obtain

$$\begin{aligned} X_3(\omega) &= \int_{-\infty}^{\infty} [ax_1(t) + bx_2(t)]e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} ax_1(t)e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} bx_2(t)e^{-j\omega t}\, dt \\ &= a\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}\, dt + b\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\, dt = aX_1(\omega) + bX_2(\omega) \end{aligned}$$

(b) Recall the sifting property of the unit impulse function:

$$\int_{-\infty}^{\infty} h(t)\delta(t - t_0)\, dt = h(t_0)$$

Therefore,

$$\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\, d\omega = 2\pi e^{j\omega_0 t}$$

Thus,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

Note that the integral relating $2\pi\delta(\omega - \omega_0)$ and $e^{j\omega_0 t}$ is exactly of the form

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega,$$

where $x(t) = e^{j\omega_0 t}$ and $X(\omega) = 2\pi\delta(\omega - \omega_0)$. Thus, we can think of $e^{j\omega_0 t}$ as the inverse Fourier transform of $2\pi\delta(\omega - \omega_0)$. Therefore, $2\pi\delta(\omega - \omega_0)$ is the Fourier transform of $e^{j\omega_0 t}$.

(c) Using the result of part (a), we have

$$X(\omega) = \mathcal{F}\{\tilde{x}(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}\right\} = \sum_{k=-\infty}^{\infty} a_k\,\mathcal{F}\left\{e^{jk(2\pi/T)t}\right\}$$

From part (b),

$$\mathcal{F}\{e^{jk(2\pi/T)t}\} = 2\pi\delta\left(\omega - \frac{2\pi k}{T}\right)$$

Therefore,

$$\tilde{X}(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\delta\left(\omega - \frac{2\pi k}{T}\right)$$

(d)

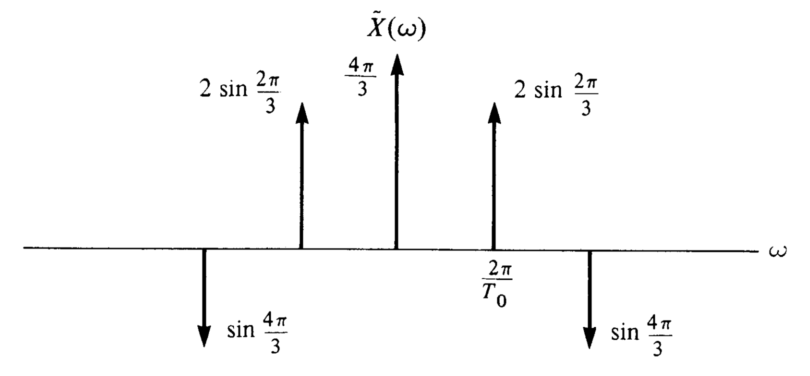

Figure S8.3

S8.4

(a) We see that the new transform is

$$X_a(f) = X(\omega)\Big|_{\omega=2\pi f}$$

We know that

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Let $\omega = 2\pi f$. Then $d\omega = 2\pi df$, and

$$x(t) = \frac{1}{2\pi}\int_{f=-\infty}^{\infty} X(2\pi f)e^{j2\pi ft}2\pi\, df = \int_{f=-\infty}^{\infty} X_a(f)e^{j2\pi ft}\, df$$

Thus, there is no factor of 2π in the inverse relation.

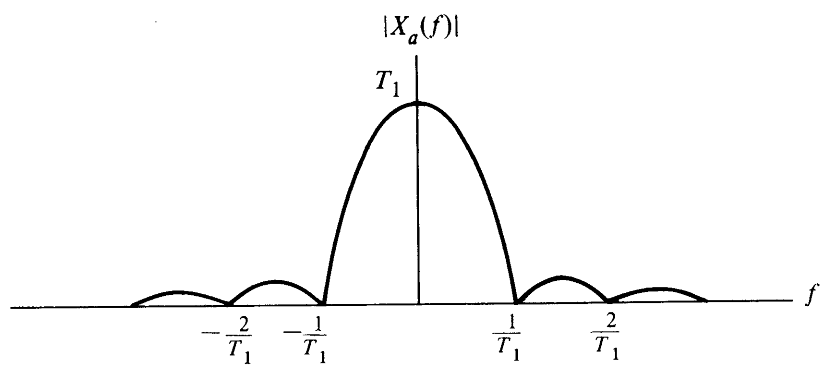

Figure S8.4

(b) Comparing

$$X_b(v) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)e^{-jvt}\, dt \qquad \text{and} \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt,$$

we see that

$$X_b(v) = \frac{1}{\sqrt{2\pi}}X(\omega)\bigg|_{\omega=v} \qquad \text{or} \qquad X(\omega) = \sqrt{2\pi}\, X_b(\omega)$$

The inverse transform relation for $X(\omega)$ is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{2\pi}\, X_b(\omega)e^{j\omega t}\, d\omega$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X_b(v)e^{jvt}\, dv,$$

where we have substituted v for ω. Thus, the factor of $1/2\pi$ has been distributed among the forward and inverse transforms.

S8.5

(a) By inspection, $T_0 = 6$.

(b) $a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)\, e^{-jk(2\pi/T_0)t}\, dt$

We integrate from -3 to 3:

$$a_k = \frac{1}{6}\int_{-3}^{3}\left[\frac{1}{2}\delta(t+1) + \delta(t) + \frac{1}{2}\delta(t-1)\right]e^{-jk(2\pi/6)t}dt$$
$$= \frac{1}{6}\left(\frac{1}{2}e^{j2\pi k/6} + 1 + \frac{1}{2}e^{-j2\pi k/6}\right) = \frac{1}{6}\left(1 + \cos\frac{2\pi k}{6}\right)$$

so

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{6}\left(1 + \cos\frac{2\pi k}{6}\right) e^{jk(2\pi/6)t}$$

(c) (i) $X_1(\omega) = \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} [\frac{1}{2}\delta(t+1) + \delta(t) + \frac{1}{2}\delta(t-1)]e^{-j\omega t}\,dt$

$= \frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega} = 1 + \cos\omega$

(ii) $X_2(\omega) = \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} [\delta(t) + \frac{1}{2}\delta(t-1) + \frac{1}{2}\delta(t-5)]e^{-j\omega t}\,dt$

$= 1 + \frac{1}{2}e^{-j\omega} + \frac{1}{2}e^{-j5\omega}$

(d) We see that by periodically repeating $x_1(t)$ with period $T_1 = 6$, we get $\tilde{x}(t)$, as shown in Figure S8.5-1.

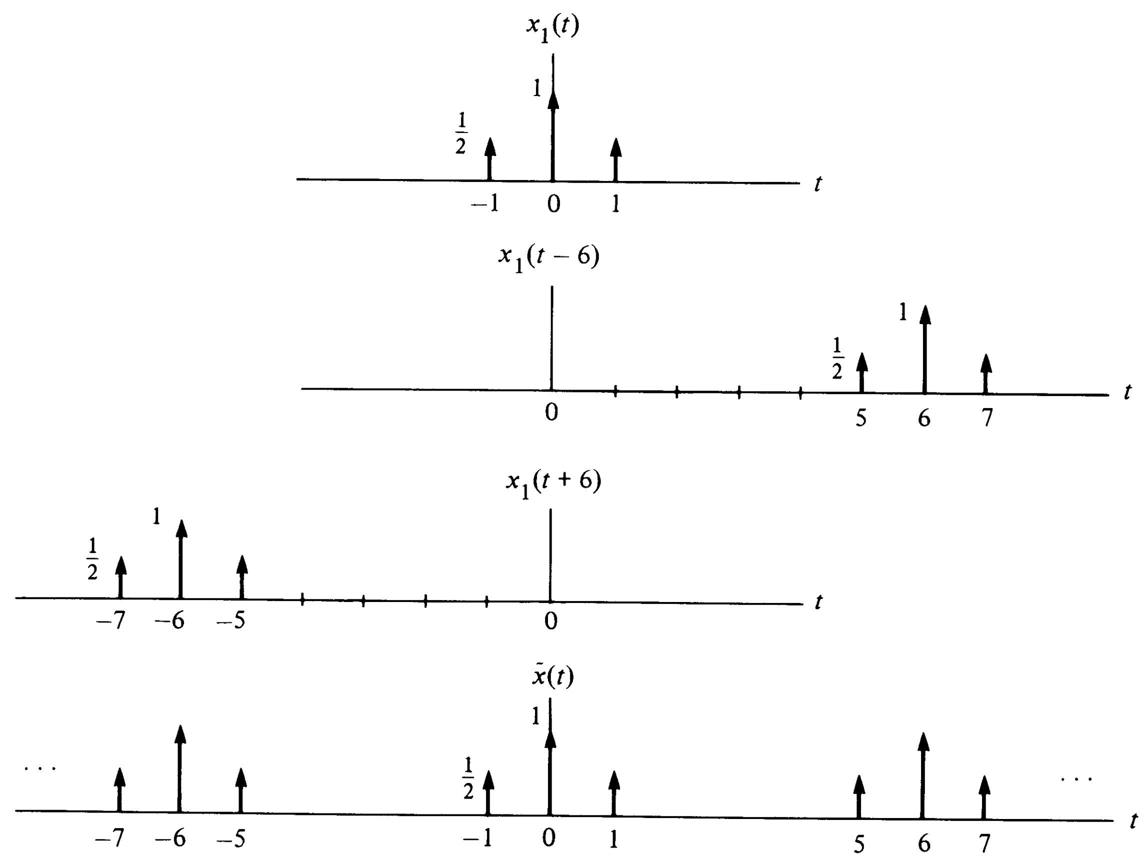

Figure S8.5-1

Similarly, we can periodically repeat $x_2(t)$ to get $\tilde{x}(t)$. Thus $T_2 = 6$. See Figure S8.5-2.

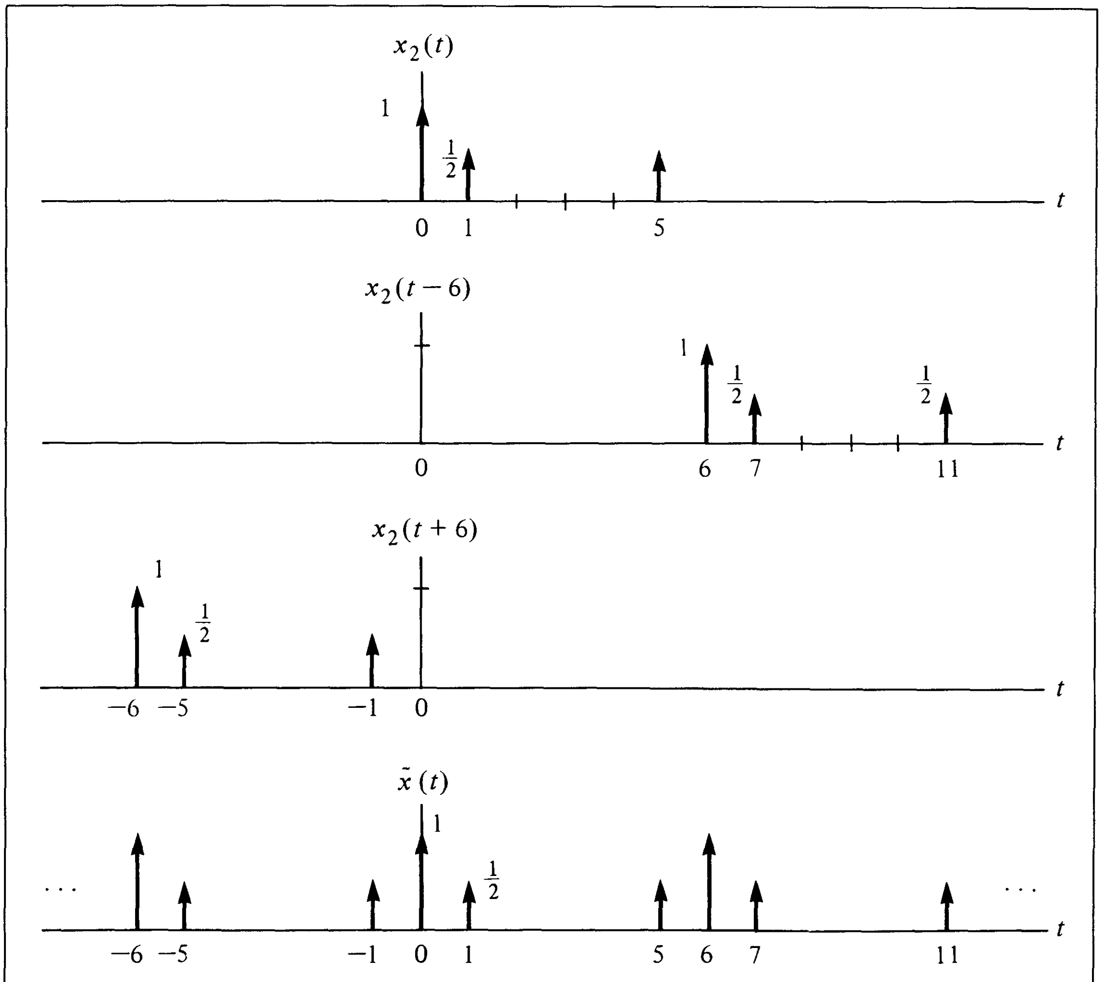

Figure S8.5-2

(e) Since $\tilde{x}(t)$ is a periodic repetition of $x_1(t)$ or $x_2(t)$, the Fourier series coefficients of $\tilde{x}(t)$ should be expressible as scaled samples of $X_1(\omega)$. Evaluate $X_1(\omega)$ at $\omega = 2\pi k/6$. Then

$$X_1(\omega)\Big|_{\omega=2\pi k/6} = 1 + \cos\frac{2\pi k}{6} = 6a_k \Longrightarrow a_k = \frac{1}{6}X_1\left(\frac{2\pi k}{6}\right)$$

Similarly, we can get a_k as a scaled sample of $X_2(\omega)$. Consider $X_2(2\pi k/6)$:

$$X_2\left(\frac{2\pi k}{6}\right) = 1 + \frac{1}{2}e^{-j2\pi k/6} + \frac{1}{2}e^{-j10\pi k/6}$$

But $e^{-j10\pi k/6} = e^{-j(10\pi k/6 - 2\pi k)} = e^{j2\pi k/6}$. Thus,

$$X_2\left(\frac{2\pi k}{6}\right) = 1 + \cos\frac{2\pi k}{6} = 6a_k.$$

Although $X_1(\omega) \neq X_2(\omega)$, they are equal for $\omega = 2\pi k/6$.

S8.6

(a) By inspection,

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$$

Thus,

$$e^{-7t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{7+j\omega}$$

Direct inversion using the inverse Fourier transform formula is very difficult.

(b) $X_b(\omega) = 2\delta(\omega + 7) + 2\delta(\omega - 7)$,

$$x_b(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_b(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2[\delta(\omega+7) + \delta(\omega-7)]e^{j\omega t}\,d\omega$$
$$= \frac{1}{\pi}e^{-j7t} + \frac{1}{\pi}e^{j7t} = \frac{2}{\pi}\cos 7t$$

(c) From Example 4.8 of the text (page 191), we see that

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

However, note that

$$\alpha x(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha X(\omega)$$

since

$$\int_{-\infty}^{\infty} \alpha x(t)e^{-j\omega t}\,dt = \alpha\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \alpha X(\omega)$$

Thus,

$$\frac{1}{2a}e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a^2+\omega^2} \qquad \text{or} \qquad \frac{1}{9+\omega^2} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{6}e^{-3|t|}$$

(d) $X_a(\omega)X_b(\omega) = X_a(\omega)[2\delta(\omega+7) + 2\delta(\omega-7)]$

$$= 2X_a(-7)\delta(\omega+7) + 2X_a(7)\delta(\omega-7)$$

$$X_d(\omega) = \frac{2}{7-j7}\delta(\omega+7) + \frac{2}{7+j7}\delta(\omega-7)$$

$$x_d(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2}{7-j7}\delta(\omega+7) + \frac{2}{7+j7}\delta(\omega-7)\right]e^{j\omega t}\,d\omega$$

$$x_d(t) = \frac{1}{\pi}\frac{1}{7-j7}e^{-j7t} + \frac{1}{\pi}\frac{1}{7+j7}e^{j7t}$$

Note that

$$\frac{1}{7+j7} = \frac{1}{7}\left(\frac{\sqrt{2}}{2}\right)e^{-j\pi/4}, \qquad \frac{1}{7-j7} = \frac{1}{7}\left(\frac{\sqrt{2}}{2}\right)e^{+j\pi/4}$$

Thus

$$x_d(t) = \frac{1}{\pi}\left(\frac{1}{7}\right)\frac{\sqrt{2}}{2}[e^{-j(7t-\pi/4)} + e^{j(7t-\pi/4)}] = \frac{\sqrt{2}}{7\pi}\cos\left(7t - \frac{\pi}{4}\right)$$

(e) $$X_e(\omega) = \begin{cases} \omega e^{-j3\omega}, & 0 \le \omega \le 1, \\ -\omega e^{-j3\omega}, & -1 \le \omega \le 0, \\ 0, & \text{elsewhere,} \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi} \left[\int_0^1 \omega e^{-j3\omega} e^{j\omega t}\, d\omega - \int_{-1}^{0} \omega e^{-j3\omega} e^{j\omega t}\, d\omega \right]$$

Note that

$$\int x e^{\alpha x}\, dx = \frac{e^{\alpha x}}{\alpha^2}(\alpha x - 1)$$

Substituting $\alpha = j(t-3)$ into the integrals, we obtain

$$x(t) = \frac{1}{2\pi} \left[\frac{e^{j(t-3)\omega}}{(j(t-3))^2}(j(t-3)\omega - 1) \Big|_0^1 - \frac{e^{j(t-3)\omega}}{(j(t-3))^2}(j(t-3)\omega - 1) \Big|_{-1}^{0} \right],$$

which can be simplified to yield

$$x(t) = \frac{1}{\pi} \left[\frac{\cos(t-3) - 1}{(t-3)^2} + \frac{\sin(t-3)}{(t-3)} \right]$$

Solutions to Optional Problems

S8.7

(a) $$Y(\omega) = \int_{t=-\infty}^{\infty} y(t) e^{-j\omega t}\, dt = \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau\, e^{-j\omega t}\, dt$$
$$= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau) e^{-j\omega t}\, d\tau\, dt$$

(b) Let $r = t - \tau$ and integrate for all τ and r. Then

$$Y(\omega) = \int_{\tau=-\infty}^{\infty} \int_{r=-\infty}^{\infty} x(\tau) h(r) e^{-j\omega(r+\tau)}\, dr\, d\tau$$
$$= \int_{\tau=-\infty}^{\infty} x(\tau) e^{-j\omega\tau}\, d\tau \int_{r=-\infty}^{\infty} h(r) e^{-j\omega r}\, dr$$
$$= X(\omega) H(\omega)$$

S8.8

(a) Using the analysis equation, we obtain

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk(2\pi/T)t}\, dt = \frac{1}{T}$$

Thus all the Fourier series coefficients are equal to $1/T$.

(b) For periodic signals, the Fourier transform can be calculated from a_k as

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

In this case,

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

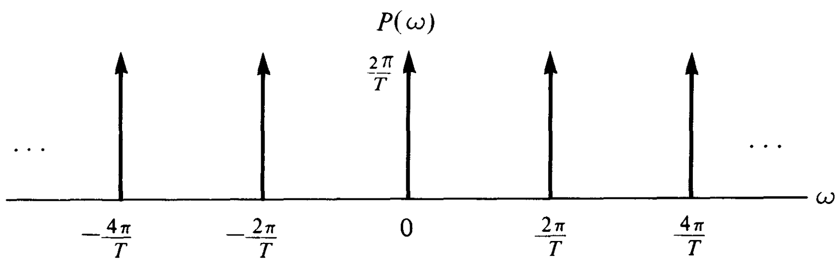

Figure S8.8

(c) We are required to show that

$$\tilde{x}(t) = x(t) * p(t)$$

Substituting for $p(t)$, we have

$$x(t) * p(t) = x(t) * \left[\sum_{k=-\infty}^{\infty} \delta(t - kT)\right]$$

Using the associative property of convolution, we obtain

$$x(t) * p(t) = \sum_{k=-\infty}^{\infty} [x(t) * \delta(t - kT)]$$

From the sifting property of $\delta(t)$, it follows that

$$x(t) * p(t) = \sum_{k=-\infty}^{\infty} x(t - kT) = \tilde{x}(t)$$

Thus, $x(t) * p(t)$ is a periodic repetition of $x(t)$ with period T.

(d) From Problem P8.7, we have

$$\begin{aligned}\tilde{X}(\omega) &= X(\omega)P(\omega)\\ &= X(\omega) \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta\left(\omega - \frac{2\pi k}{T}\right)\\ &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} X(\omega)\delta\left(\omega - \frac{2\pi k}{T}\right)\end{aligned}$$

Since each summation term is nonzero *only* at $\omega = 2\pi k/T$,

$$\tilde{X}(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} X\left(\frac{2\pi k}{T}\right) \delta\left(\omega - \frac{2\pi k}{T}\right)$$

From this expression we see that the Fourier series coefficients of $\tilde{x}(t)$ are

$$a_k = \frac{1}{T} X\left(\frac{2\pi k}{T}\right),$$

which is consistent with our previous discussions.

9 Fourier Transform Properties

Recommended Problems

P9.1

Determine the Fourier transform of $x(t) = e^{-t/2}u(t)$ and sketch

(a) $|X(\omega)|$

(b) $\sphericalangle X(\omega)$

(c) $Re\{X(\omega)\}$

(d) $Im\{X(\omega)\}$

P9.2

Figure P9.2 shows real and imaginary parts of the Fourier transform of a signal $x(t)$.

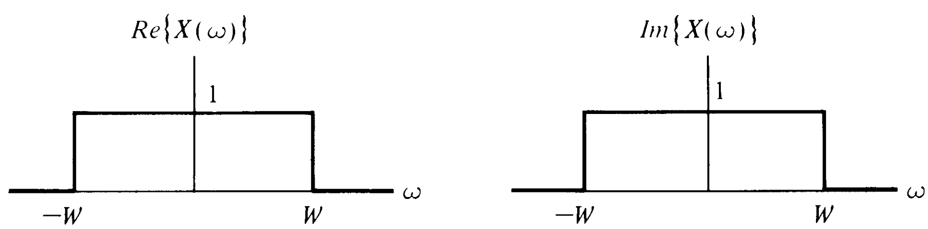

Figure P9.2

(a) Sketch the magnitude and phase of the Fourier transform $X(\omega)$.

(b) In general, if a signal $x(t)$ is real, then $X(-\omega) = X^*(\omega)$. Determine whether $x(t)$ is real for the Fourier transform sketched in Figure P9.2.

P9.3

Determine which of the Fourier transforms in Figures P9.3-1 and P9.3-2 correspond to real-valued time functions.

(a)

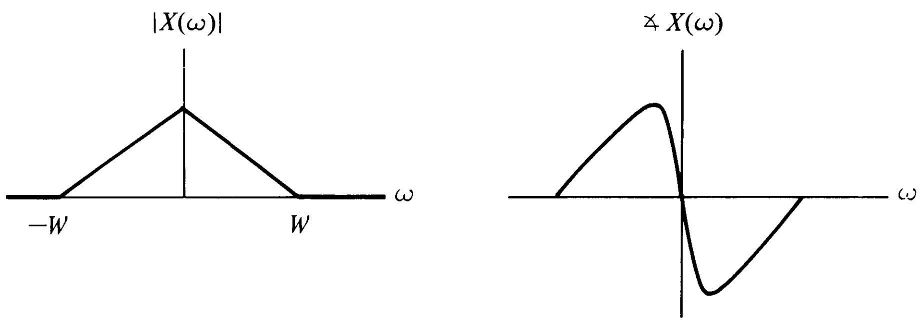

Figure P9.3-1

(b)

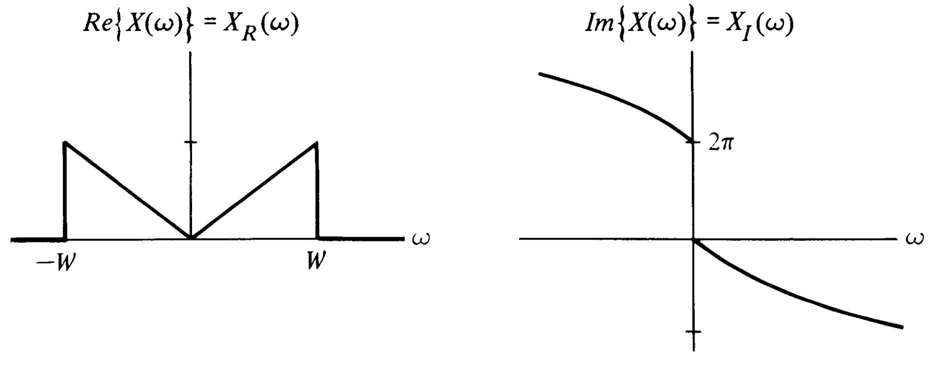

Figure P9.3-2

P9.4

(a) By considering the Fourier analysis equation or synthesis equation, show the validity in general of each of the following statements:

(i) If $x(t)$ is real-valued, then $X(\omega) = X^*(-\omega)$.

(ii) If $x(t) = x^*(-t)$, then $X(\omega)$ is real-valued.

(b) Using the statements in part (a), show the validity of each of the following statements:

(i) If $x(t)$ is real and even, then $X(\omega)$ is real and even.

(ii) If $x(t)$ is real and odd, then $X(\omega)$ is imaginary and odd.

P9.5

(a) In the lecture, we derived the transform of $x(t) = e^{-at}u(t)$. Using the linearity and scaling properties, derive the Fourier transform of $e^{-\alpha|t|} = x(t) + x(-t)$.

(b) Using part (a) and the duality property, determine the Fourier transform of $1/(1 + t^2)$.

(c) If

$$r(t) = \frac{1}{1 + (3t)^2},$$

find $R(\omega)$.

(d) $x(t)$ is sketched in Figure P9.5. If $y(t) = x(t/2)$, sketch $y(t)$, $Y(\omega)$, and $X(\omega)$.

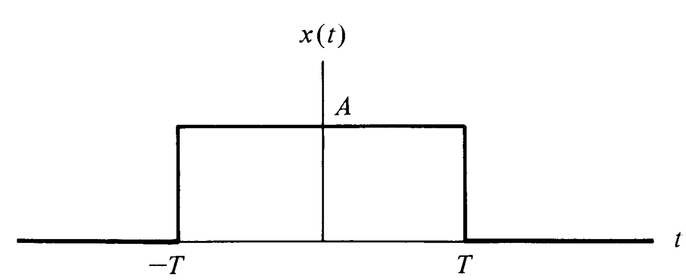

Figure P9.5

P9.6

Show the validity of the following statements:

(a) $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$

(b) $X(0) = \int_{-\infty}^{\infty} x(t)\,dt$

P9.7

The output of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

(a) Determine the frequency response $H(\omega) = Y(\omega)/X(\omega)$ and sketch the phase and magnitude of $H(\omega)$.

(b) If $x(t) = e^{-t}u(t)$, determine $Y(\omega)$, the Fourier transform of the output.

(c) Find $y(t)$ for the input given in part (b).

P9.8

By first expressing the triangular signal $x(t)$ in Figure P9.8 as the convolution of a rectangular pulse with itself, determine the Fourier transform of $x(t)$.

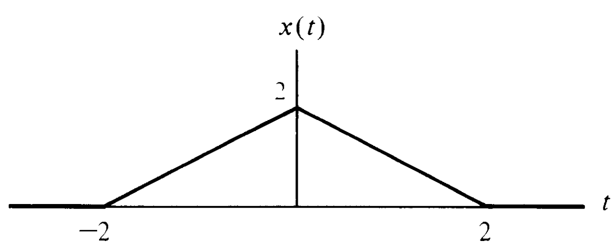

Figure P9.8

Optional Problems

P9.9

Using Figure P9.9-1, determine $y(t)$ and sketch $Y(\omega)$ if $X(\omega)$ is given by Figure P9.9-2. Assume $\omega_c > \omega_0$.

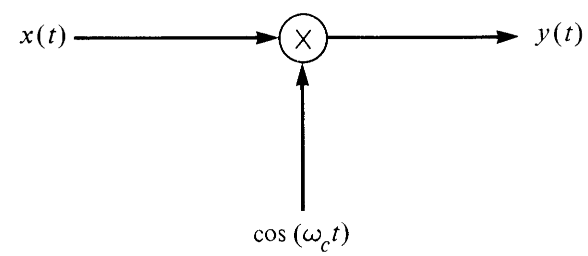

Figure P9.9-1

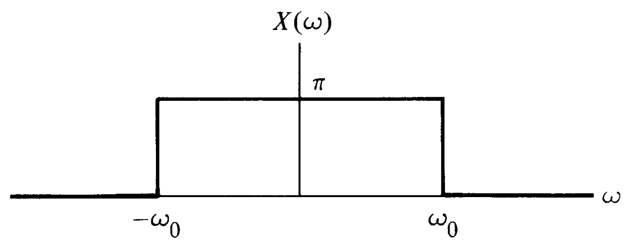

Figure P9.9-2

P9.10

Compute the Fourier transform of each of the following signals:

(a) $[e^{-\alpha t} \cos \omega_0 t]u(t), \qquad \alpha > 0$

(b) $e^{-3|t|} \sin 2t$

(c) $\left(\dfrac{\sin \pi t}{\pi t}\right)\left(\dfrac{\sin 2\pi t}{\pi t}\right)$

P9.11

Consider the following linear constant-coefficient differential equation (LCCDE):

$$\frac{dy(t)}{dt} + 2y(t) = A \cos \omega_0 t$$

Find the value of ω_0 such that $y(t)$ will have a maximum amplitude of $A/3$. Assume that the resulting system is linear and time-invariant.

P9.12

Suppose an LTI system is described by the following LCCDE:

$$\frac{d^2y(t)}{dt^2} + \frac{2dy(t)}{dt} + 3y(t) = \frac{4dx(t)}{dt} - x(t)$$

(a) Show that the left-hand side of the equation has a Fourier transform that can be expressed as

$$A(\omega)Y(\omega), \qquad \text{where } Y(\omega) = \mathcal{F}\{y(t)\}$$

Find $A(\omega)$.

(b) Similarly, show that the right-hand side of the equation has a Fourier transform that can be expressed as

$$B(\omega)X(\omega), \qquad \text{where } X(\omega) = \mathcal{F}\{x(t)\}$$

(c) Show that $Y(\omega)$ can be expressed as $Y(\omega) = H(\omega)X(\omega)$ and find $H(\omega)$.

P9.13

From Figure P9.13, find $y(t)$ where

$$x(t) = \frac{\sin(\omega_0 t)}{t} \qquad \text{and} \qquad h(t) = \frac{\sin(2\omega_0 t)}{t}$$

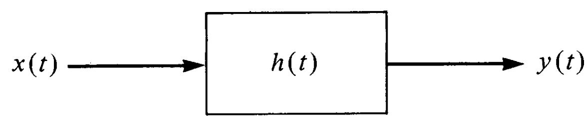

Figure P9.13

P9.14

(a) Determine the energy in the signal $x(t)$ for which the Fourier transform $X(\omega)$ is given by Figure P9.14.

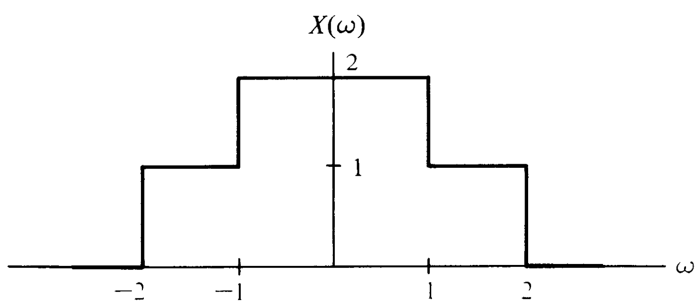

Figure P9.14

(b) Find the inverse Fourier transform of $X(\omega)$ of part (a).

P9.15

Suppose that the system F takes the Fourier transform of the input, as shown in Figure P9.15-1.

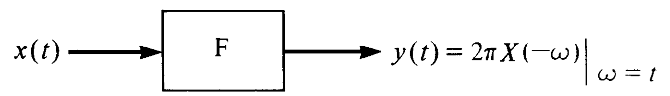

Figure P9.15-1

What is $w(t)$ calculated as in Figure P9.15-2?

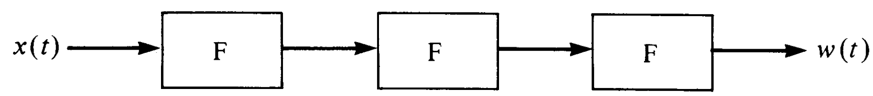

Figure P9.15-2

P9.16

Use properties of the Fourier transform to show by induction that the Fourier transform of

$$x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at} u(t), \qquad a > 0$$

is

$$X(\omega) = \frac{1}{(a + j\omega)^n}$$

9 Fourier Transform Properties

Solutions to Recommended Problems

S9.1

The Fourier transform of $x(t)$ is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-t/2}u(t)e^{-j\omega t}\,dt \tag{S9.1-1}$$

Since $u(t) = 0$ for $t < 0$, eq. (S9.1-1) can be rewritten as

$$X(\omega) = \int_{0}^{\infty} e^{-(1/2+j\omega)t}\,dt$$
$$= \frac{+2}{1 + j2\omega}$$

It is convenient to write $X(\omega)$ in terms of its real and imaginary parts:

$$X(\omega) = \frac{2}{1 + j2\omega}\left(\frac{1 - j2\omega}{1 - j2\omega}\right) = \frac{2 - j4\omega}{1 + 4\omega^2}$$
$$= \frac{2}{1 + 4\omega^2} - j\frac{4\omega}{1 + 4\omega^2}$$

$$\text{Magnitude of } X(\omega) = \frac{2}{\sqrt{1 + 4\omega^2}}$$
$$\measuredangle X(\omega) = \tan^{-1}(-2\omega) = -\tan^{-1}(2\omega)$$
$$Re\{X(\omega)\} = \frac{+2}{1 + 4\omega^2}, \qquad Im\{X(\omega)\} = \frac{-4\omega}{1 + 4\omega^2}$$

(a)

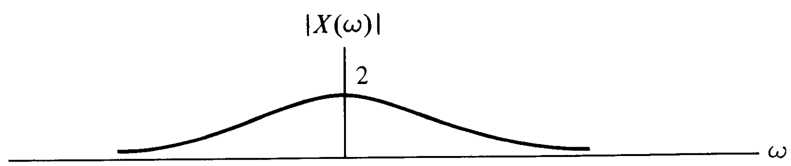

Figure S9.1-1

(b)

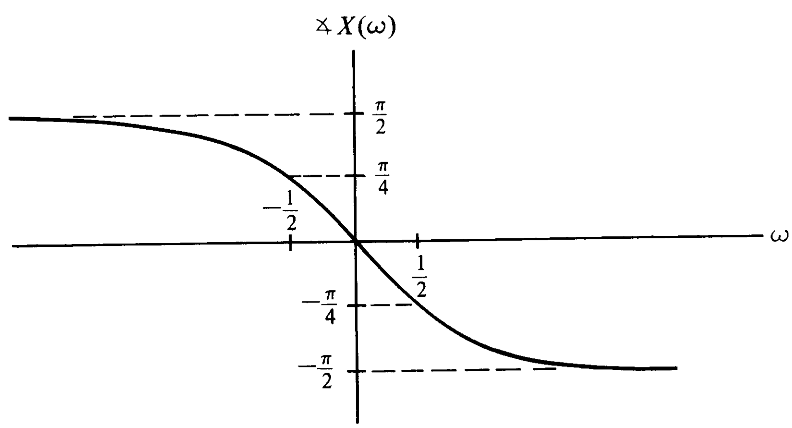

Figure S9.1-2

(c)

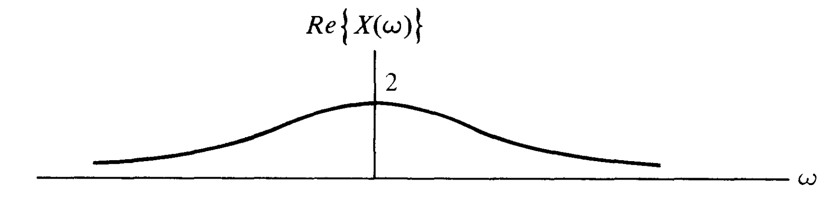

Figure S9.1-3

(d)

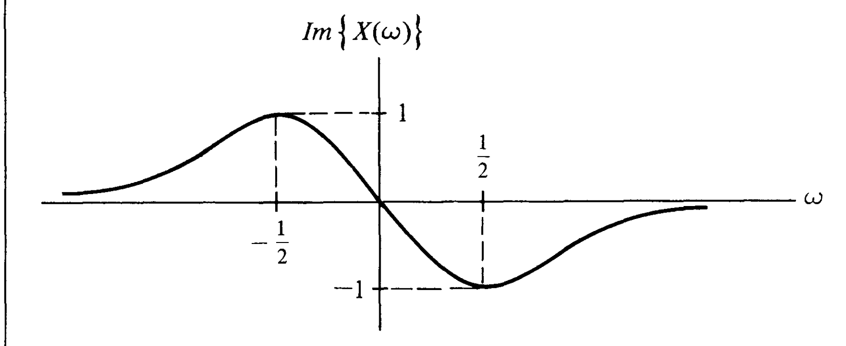

Figure S9.1-4

S9.2

(a) The magnitude of $X(\omega)$ is given by

$$|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)},$$

where $X_R(\omega)$ is the real part of $X(\omega)$ and $X_I(\omega)$ is the imaginary part of $X(\omega)$. It follows that

$$|X(\omega)| = \begin{cases} \sqrt{2}, & |\omega| < W, \\ 0, & |\omega| > W \end{cases}$$

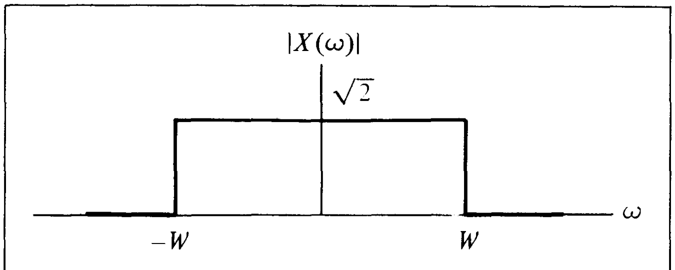

Figure S9.2-1

The phase of $X(\omega)$ is given by

$$\sphericalangle X(\omega) = \tan^{-1}\left(\frac{X_I(\omega)}{X_R(\omega)}\right) = \tan^{-1}(1), \qquad |\omega| < W$$

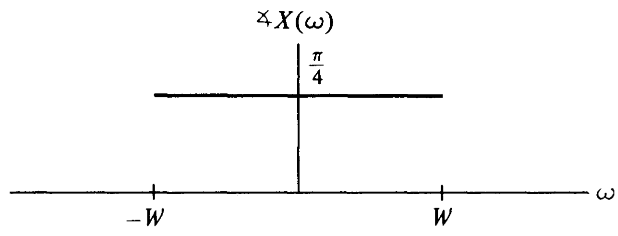

Figure S9.2-2

(b) $$X(\omega) = \begin{cases} 1+j, & |\omega| < W \\ 0, & \text{otherwise} \end{cases}$$

$$X(-\omega) = \begin{cases} 1+j, & |\omega| < W \\ 0, & \text{otherwise} \end{cases}$$

$$X^*(\omega) = \begin{cases} 1-j, & |\omega| < W \\ 0, & \text{otherwise} \end{cases}$$

Hence, the signal is not real.

S9.3

For $x(t)$ to be real-valued, $X(\omega)$ is conjugate symmetric:

$$X(-\omega) = X^*(\omega)$$

(a) $$\begin{aligned} X(\omega) &= |X(\omega)|e^{j\measuredangle X(\omega)} \\ &= |X(\omega)|\cos(\measuredangle X(\omega)) + j|X(\omega)|\sin(\measuredangle X(\omega)) \end{aligned}$$

Therefore,

$$\begin{aligned} X(-\omega) &= |X(-\omega)|\cos(\measuredangle X(-\omega)) + j|X(-\omega)|\sin(\measuredangle X(-\omega)) \\ &= |X(\omega)|\cos(\measuredangle X(\omega)) - j|X(\omega)|\sin(\measuredangle X(\omega)) \\ &= X^*(\omega) \end{aligned}$$

Hence, $x(t)$ is real-valued.

(b) $$\begin{aligned} X(\omega) &= X_R(\omega) + jX_I(\omega) \\ X(-\omega) &= X_R(-\omega) + jX_I(-\omega) \\ &= X_R(\omega) + j[-X_I(\omega) + 2\pi] \qquad \text{for } \omega > 0 \\ X^*(\omega) &= X_R(\omega) - jX_I(\omega) \end{aligned}$$

Therefore,

$$X^*(\omega) \neq X(-\omega)$$

Hence, $x(t)$ is not real-valued.

S9.4

(a) (i) $$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

We take the complex conjugate of both sides to get

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}\,dt$$

Since $x(t)$ is real-valued,

$$X^*(\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt$$

Therefore,

$$X^*(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$
$$= X(\omega)$$

(ii) $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$

Taking the complex conjugate of both sides, we have

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}\,d\omega$$

Therefore,

$$x^*(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{j\omega t}\,d\omega$$

Since $x(t) = x^*(-t)$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{j\omega t}\,d\omega$$

This shows that $X(\omega)$ must be real-valued.

(b) (i) Since $x(t)$ is real, $X(\omega) = X^*(-\omega)$. Since $x(t)$ is real and even, it satisfies $x(t) = x^*(-t)$ and, therefore, $X(\omega)$ is real. Hence, $X(\omega) = X^*(-\omega) = X(-\omega)$. It follows that $X(\omega)$ is real and even.

(ii) If $x(t)$ is real, $X(\omega) = X^*(-\omega)$. Since $x(t)$ is real and odd, $x(t) = -x^*(-t)$; an analysis similar to part (a)(ii) proves that $X(\omega)$ must be imaginary. Hence, $X(\omega) = X^*(-\omega) = -X(-\omega)$. It follows that $X(\omega)$ is also odd.

S9.5

(a) $\mathcal{F}\{e^{-\alpha|t|}\} = \mathcal{F}\{e^{-\alpha t}u(t) + e^{\alpha t}u(-t)\}$

$$= \frac{1}{\alpha + j\omega} + \frac{1}{\alpha - j\omega}$$
$$= \frac{2\alpha}{\alpha^2 + \omega^2}$$

(b) Duality states that

$$g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(\omega)$$
$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi g(-\omega)$$

Since

$$e^{-\alpha|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\alpha}{\alpha^2 + \omega^2},$$

we have

$$\frac{1}{1+t^2} \stackrel{\mathcal{F}}{\longleftrightarrow} \pi e^{-|\omega|}$$

(c) $\dfrac{1}{1+(3t)^2} \stackrel{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{3}\pi e^{-|\omega/3|}$ since $x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

(d) We are given Figure S9.5-1.

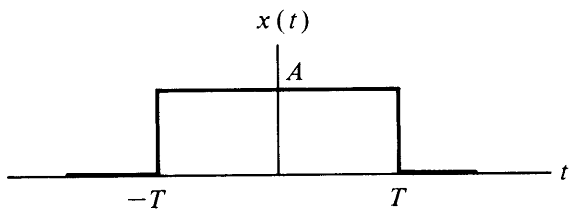

Figure S9.5-1

$$\begin{aligned}X(\omega) = A\int_{-T}^{T} e^{-j\omega t}\,dt &= \frac{A}{-j\omega}(e^{-j\omega T} - e^{j\omega T})\\ &= A\,\frac{-2j\sin\omega T}{-j\omega}\\ &= 2TA\,\frac{\sin(\omega T)}{\omega T}\end{aligned}$$

Sketches of $y(t)$, $Y(\omega)$, and $X(\omega)$ are given in Figure S9.5-2.

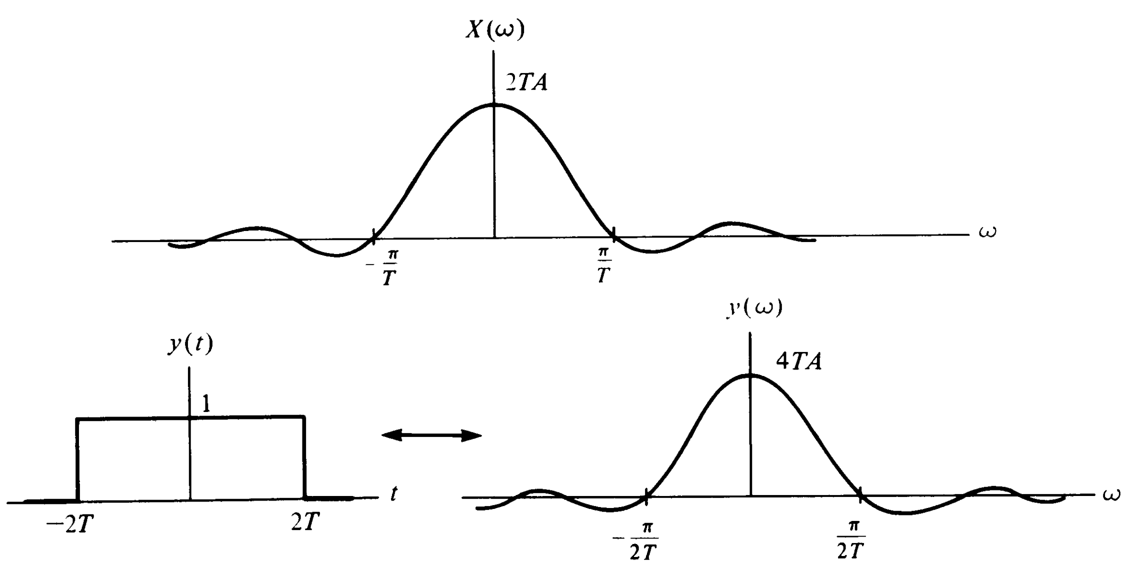

Figure S9.5-2

Substituting $2T$ for T in $X(\omega)$, we have

$$Y(\omega) = 2(2T)\frac{\sin(\omega 2T)}{\omega 2T}$$

The zero crossings are at

$$\omega_z 2T = n\pi, \qquad \text{or} \qquad \omega_z = n\frac{\pi}{2T}$$

S9.6

(a) $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$

Substituting $t = 0$ in the preceding equation, we get

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$$

(b) $X(\omega) = \displaystyle\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

Substituting $\omega = 0$ in the preceding equation, we get

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$$

S9.7

(a) We are given the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \tag{S9.7-1}$$

Taking the Fourier transform of eq. (S9.7-1), we have

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

Hence,

$$Y(\omega)[2 + j\omega] = X(\omega)$$

and

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2 + j\omega},$$

$$H(\omega) = \frac{1}{2 + j\omega} = \frac{1}{2 + j\omega}\left(\frac{2 - j\omega}{2 - j\omega}\right) = \frac{2 - j\omega}{4 + \omega^2}$$

$$= \frac{2}{4 + \omega^2} - j\frac{\omega}{4 + \omega^2},$$

$$|H(\omega)|^2 = \frac{4}{(4+\omega^2)^2} + \frac{\omega^2}{(4+\omega^2)^2} = \frac{4+\omega^2}{(4+\omega^2)^2},$$
$$|H(\omega)| = \frac{1}{\sqrt{4+\omega^2}}$$

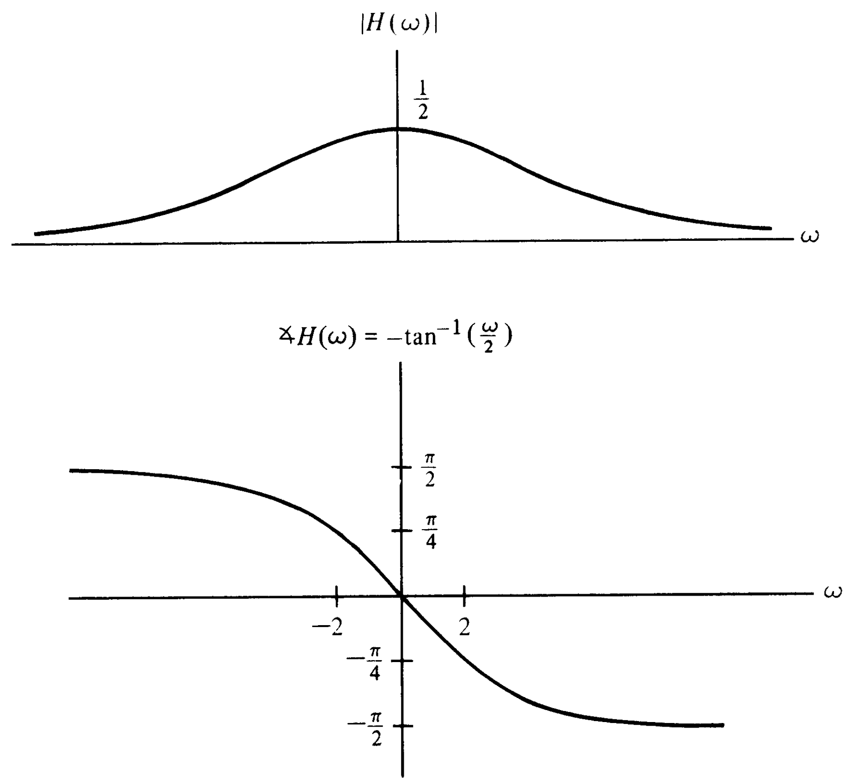

Figure S9.7

(b) We are given $x(t) = e^{-t}u(t)$. Taking the Fourier transform, we obtain

$$X(\omega) = \frac{1}{1+j\omega}, \qquad H(\omega) = \frac{1}{2+j\omega}$$

Hence,

$$Y(\omega) = \frac{1}{(1+j\omega)(2+j\omega)} = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

(c) Taking the inverse transform of $Y(\omega)$, we get

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

S9.8

A triangular signal can be represented as the convolution of two rectangular pulses, as indicated in Figure S9.8.

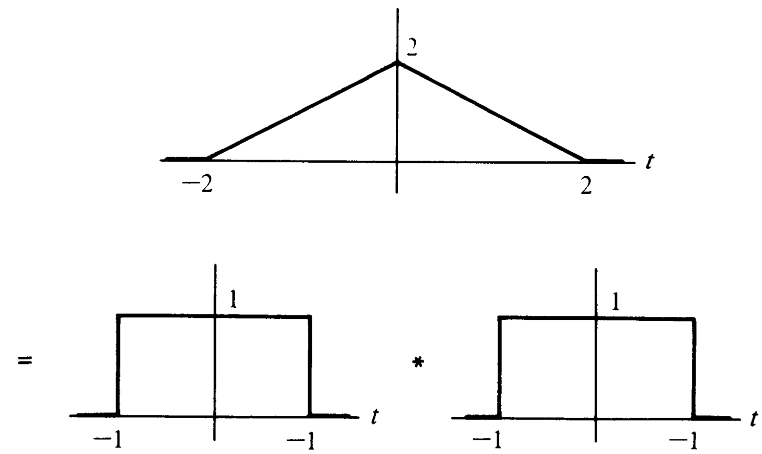

Figure S9.8

Since each of the rectangular pulses on the right has a Fourier transform given by $(2 \sin \omega)/\omega$, the convolution property tells us that the triangular function will have a Fourier transform given by the square of $(2 \sin \omega)/\omega$:

$$X(\omega) = \frac{4 \sin^2\omega}{\omega^2}$$

Solutions to Optional Problems

S9.9

We can compute the function $x(t)$ by taking the inverse Fourier transform of $X(\omega)$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} \pi e^{j\omega t}\, d\omega \\ &= \frac{1}{2}\left(\frac{1}{jt}\right)(e^{j\omega_0 t} - e^{-j\omega_0 t}) \\ &= \frac{\sin \omega_0 t}{t} \end{aligned}$$

Therefore,

$$y(t) = \cos(\omega_c t)\left[\frac{\sin(\omega_0 t)}{t}\right]$$

From the multiplicative property, we have

$$Y(\omega) = X(\omega) * [\pi\delta(\omega - \omega_c) - \pi\delta(\omega + \omega_c)]$$

$Y(\omega)$ is sketched in Figure S9.9.

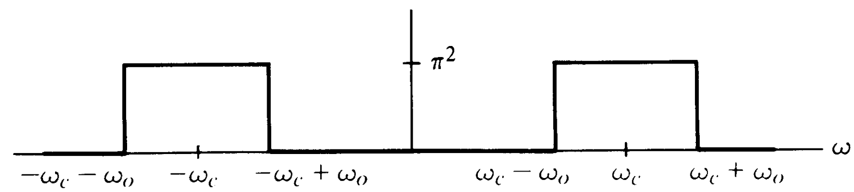

Figure S9.9

S9.10

(a) $x(t) = e^{-\alpha t} \cos \omega_0 t u(t), \qquad \alpha > 0$
$\qquad = e^{-\alpha t}u(t)\cos(\omega_0 t)$

Therefore,

$$X(\omega) = \frac{1}{2\pi}\frac{1}{\alpha + j\omega} * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$
$$= \frac{1/2}{\alpha + j(\omega - \omega_0)} + \frac{1/2}{\alpha + j(\omega + \omega_0)}$$

(b) $x(t) = e^{-3|t|} \sin 2t$

$$e^{-3|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{6}{9 + \omega^2}$$

$$\sin 2t \overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)], \qquad \omega_0 = 2$$

Therefore,

$$X(\omega) = \frac{1}{2\pi}\left(\frac{6}{9 + \omega^2}\right) * \left\{\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]\right\}$$
$$= \frac{j3}{9 + (\omega + 2)^2} - \frac{j3}{9 + (\omega - 2)^2}$$

(c) $x(t) = \dfrac{\sin \pi t}{\pi t}\left(\dfrac{\sin 2\pi t}{\pi t}\right),$

$$X(\omega) = \frac{1}{2\pi}X_1(\omega) * X_2(\omega),$$

where

$$X_1(\omega) = \begin{cases} 1, & |\omega| < \pi, \\ 0, & \text{otherwise} \end{cases}$$

$$X_2(\omega) = \begin{cases} 1, & |\omega| < 2\pi, \\ 0, & \text{otherwise} \end{cases}$$

Hence, $X(\omega)$ is given by the convolution shown in Figure S9.10.

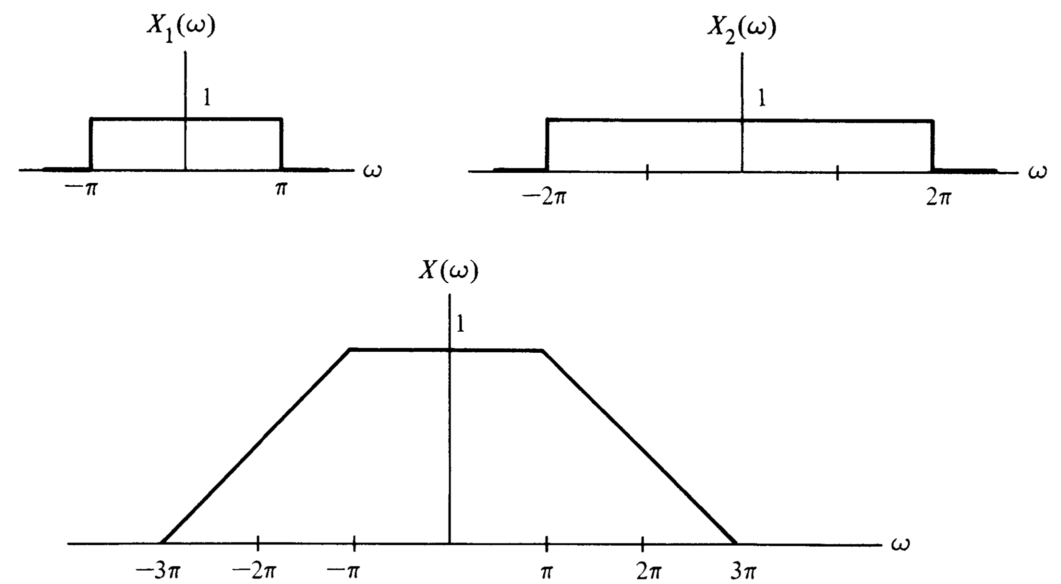

Figure S9.10

S9.11

We are given the LCCDE

$$\frac{dy(t)}{dt} + 2y(t) = A \cos \omega_0 t$$

We can view the LCCDE as

$$\frac{dy(t)}{dt} + 2y(t) = x(t),$$

the transfer function of which is given by

$$H(\omega) = \frac{1}{2 + j\omega} \qquad \text{and} \qquad x(t) = A \cos \omega_0 t$$

We have already seen that for LTI systems,

$$\begin{aligned} y(t) &= |H(\omega_0)|\, A \cos(\omega_0 t + \phi), \qquad \text{where } \phi = \sphericalangle H(\omega_0) \\ &= \frac{1}{\sqrt{4 + \omega_0^2}} A \cos(\omega_0 t + \phi) \end{aligned}$$

For the maximum value of $y(t)$ to be $A/3$, we require

$$\frac{1}{4 + \omega_0^2} = \frac{1}{9}$$

Therefore, $\omega_0 = \pm\sqrt{5}$.

S9.12

(a) $$\mathcal{F}\left\{\frac{d^2y(t)}{dt^2} + \frac{2dy(t)}{dt} + 3y(t)\right\} = -\omega^2 Y(\omega) + 2j\omega Y(\omega) + 3Y(\omega)$$

$$= (-\omega^2 + j2\omega + 3)Y(\omega),$$

$$A(\omega) = -\omega^2 + j2\omega + 3$$

(b) $$\mathcal{F}\left\{\frac{4dx(t)}{dt} - x(t)\right\} = 4j\omega X(\omega) - X(\omega)$$

$$= (j4\omega - 1)X(\omega),$$

$$B(\omega) = j4\omega - 1,$$

$$A(\omega)Y(\omega) = B(\omega)X(\omega),$$

$$Y(\omega) = \frac{B(\omega)}{A(\omega)}X(\omega)$$

$$= H(\omega)X(\omega)$$

Therefore,

$$H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{-1 + j4\omega}{-\omega^2 + 3 + j2\omega}$$

$$= \frac{1 - j4\omega}{\omega^2 - 3 - j2\omega}$$

S9.13

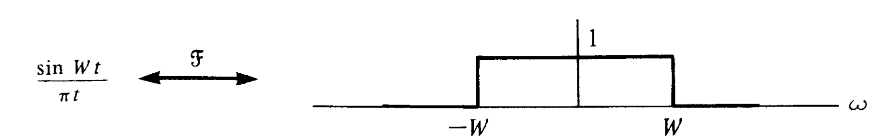

Figure S9.13-1

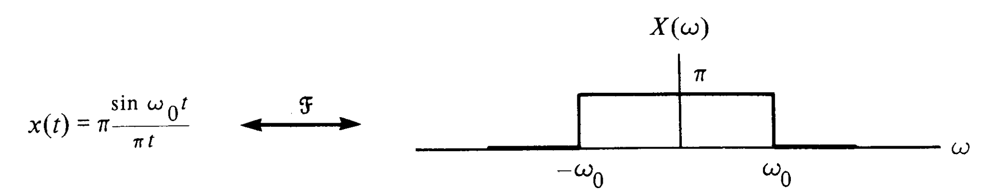

Figure S9.13-2

$$h(t) = \pi\frac{\sin(2\omega_0 t)}{\pi t} \overset{\mathfrak{F}}{\longleftrightarrow}$$

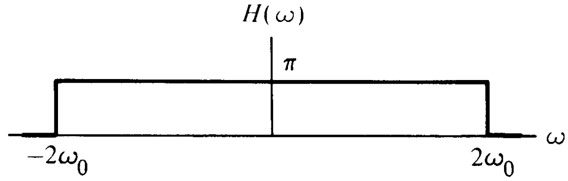

Figure S9.13-3

$$Y(\omega) = X(\omega)H(\omega) =$$

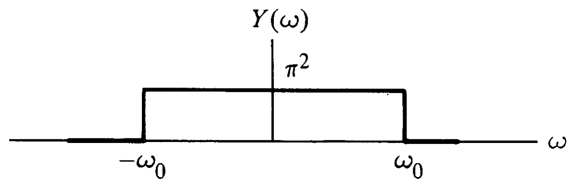

Figure S9.13-4

Therefore, $y(t) = \pi\dfrac{\sin(\omega_0 t)}{t}$.

S9.14

(a) Energy $= \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$

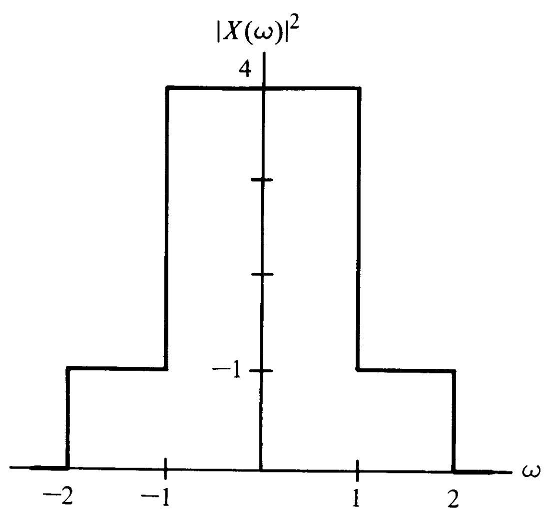

Figure S9.14-1

$$\begin{aligned} \text{Area} &= (4)(2) + (2)(1)(1) \\ &= 10 \\ \text{Energy} &= \frac{5}{\pi} \end{aligned}$$

(b)

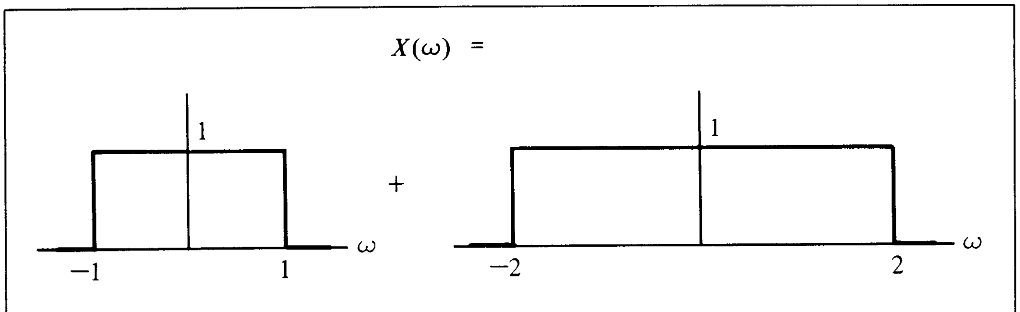

Figure S9.14-2

$$x(t) = \frac{\sin t}{\pi t} + \frac{\sin 2t}{\pi t}$$

S9.15

Given that

$$y_1(t) = 2\pi X(-\omega)|_{\omega=t}$$

we have

$$y_1(t) = 2\pi \int_{u=-\infty}^{\infty} x(u)e^{jtu}\, du$$

Similarly, let $y_2(t)$ be the output due to passing $x(t)$ through F twice.

$$\begin{aligned}
y_2(t) &= 2\pi \int_{v=-\infty}^{\infty} 2\pi \int_{u=-\infty}^{\infty} x(u)e^{jvu}\, du\, e^{jtv}\, dv \\
&= (2\pi)^2 \int_{u=-\infty}^{\infty} x(u) \int_{v=-\infty}^{\infty} e^{j(t+u)v}\, dv\, du \\
&= (2\pi)^2 \int_{u=-\infty}^{\infty} x(u)(2\pi)\delta(t+u)\, du \\
&= (2\pi)^3\, x(-t)
\end{aligned}$$

Finally, let $y_3(t)$ be the output due to passing $x(t)$ through F three times.

$$\begin{aligned}
y_3(t) = w(t) &= 2\pi \int_{u=-\infty}^{\infty} (2\pi)^3 x(-u)e^{jtu}\, du \\
&= (2\pi)^4 \int_{-\infty}^{\infty} e^{-jtu}\, x(u)\, du \\
&= (2\pi)^4 X(t)
\end{aligned}$$

S9.16

We are given

$$x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at}u(t), \qquad a > 0$$

Let $n = 1$:

$$x(t) = e^{-at}u(t), \qquad a > 0,$$
$$X(\omega) = \frac{1}{a + j\omega}$$

Let $n = 2$:

$$x(t) = te^{-at}u(t),$$
$$X(\omega) = j\frac{d}{d\omega}\left(\frac{1}{a + j\omega}\right) \qquad \text{since} \quad tx(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega)$$
$$= \frac{1}{(a + j\omega)^2}$$

Assume it is true for n:

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t),$$
$$X(\omega) = \frac{1}{(a + j\omega)^n}$$

We consider the case for $n + 1$:

$$x(t) = \frac{t^n}{n!}e^{-at}u(t),$$
$$X(\omega) = \frac{j}{n}\frac{d}{d\omega}\left[\frac{1}{(a + j\omega)^n}\right]$$
$$= \frac{j}{n}\frac{d}{d\omega}[(a + j\omega)^{-n}]$$
$$= \frac{j}{n}(-n)(a + j\omega)^{-n-1}j$$
$$= \frac{1}{(a + j\omega)^{n+1}}$$

Therefore, it is true for all n.