EE 261 Helpful Formulas

Fourier series. If $f(t)$ is periodic with period T then its Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

where

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi i n t/T} f(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt$$

Rayleigh (Parseval). If $f(t)$ is periodic of period T then

$$\frac{1}{T}\int_0^T |f(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

The Fourier transform.

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x}\,dx$$

The inverse Fourier transform.

$$\mathcal{F}^{-1}f(x) = \int_{-\infty}^{\infty} f(s) e^{2\pi i s x}\,ds$$

Fourier transform theorems.

Linearity.

$$\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha F(s) + \beta G(s)$$

Stretch.

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|}F(\frac{s}{a})$$

Shift.

$$\mathcal{F}\{f(x-a)\} = e^{-i2\pi a s}F(s)$$

Shift and stretch.

$$\mathcal{F}\{f(ax-b)\} = \frac{1}{|a|}e^{-i2\pi s b/a}F\left(\frac{s}{a}\right)$$

Rayleigh (Parseval).

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} F(s)\overline{G(s)}\,ds$$

Modulation.

$$\mathcal{F}\{f(x)\cos(2\pi s_0 x)\} = \tfrac{1}{2}\big(F(s-s_0) + F(s+s_0)\big)$$

Derivative.

$$\mathcal{F}\{f'(x)\} = 2\pi i s F(s)$$

$$\mathcal{F}\{f^{(n)}(x)\} = (2\pi i s)^n F(s)$$

$$\mathcal{F}\{x^n f(x)\} = (\tfrac{i}{2\pi})^n F^{(n)}(s)$$

Moments.

$$\int_{-\infty}^{\infty} f(x)dx = F(0)$$

$$\int_{-\infty}^{\infty} x f(x)dx = \frac{i}{2\pi}F'(0)$$

$$\int_{-\infty}^{\infty} x^n f(x)dx = \left(\frac{i}{2\pi}\right)^n F^{(n)}(0)$$

Autocorrelation.

$$\mathcal{F}\{\overline{g} \star g\} = |G(s)|^2$$

Crosscorrelation.

$$\mathcal{F}\{\overline{g} \star f\} = \overline{G(s)}F(s)$$

Miscellaneous:

$$\mathcal{F}\left(\int_{-\infty}^{x} g(\xi)d\xi\right) = \tfrac{1}{2}G(0)\delta(s) + \frac{G(s)}{i2\pi s}$$

Symmetry and duality properties.

The *reversal* of $f(x)$ is $f^-(x) = f(-x)$.

$\mathcal{F}^{-1}f = \mathcal{F}f^-$

$\mathcal{F}f^- = (\mathcal{F}f)^-$

If f is even (odd) then $\mathcal{F}f$ is even (odd)

If f is real valued, then $\overline{\mathcal{F}f} = (\mathcal{F}f)^-$

Convolution.

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

$f * g = g * f$

$(f * g) * h = (f * g) * h$

$f * (g + h) = f * g + f * h$

Smoothing: If f is k-times continuously differentiable, then so is $f * g$ and

$$\frac{d^k}{dx^k}(f * g) = \left(\frac{d^k}{dx^k}f\right) * g$$

Convolution theorem.

$$\mathcal{F}(f*g) = (\mathcal{F}f)(\mathcal{F}g)$$
$$\mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g$$

Autocorrelation. If $f(x)$ has finite energy, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty\,,$$

then

$$(\overline{f} \star f)(x) = \int_{-\infty}^{\infty} f(y)\overline{f(y-x)}\, dy = f(x) * \overline{f(-x)}$$

Cross correlation. Let $g(x)$ and $h(x)$ be functions with finite energy. Then

$$\begin{aligned}(\overline{g} \star h)(x) &= \int_{-\infty}^{\infty} \overline{g(y)}h(y+x)\, dy \\ &= \int_{-\infty}^{\infty} \overline{g(y-x)}h(y)\, dy \\ &= \overline{(\overline{h} \star g)(-x)}\end{aligned}$$

Rectangle and triangle functions

$$\Pi(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & |x| \geq \frac{1}{2}\end{cases} \qquad \Lambda(x) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & |x| > 1\end{cases}$$

$$\mathcal{F}\Pi(s) = \text{sinc} s = \frac{\sin \pi s}{\pi s} \qquad \mathcal{F}\Lambda(s) = \text{sinc}^2 s$$

Scaled rect function

$$\Pi_T(x) = \Pi\Big(\frac{x}{T}\Big) = \begin{cases} 1 & |x| < \frac{T}{2} \\ 0 & |x| \geq \frac{T}{2}\end{cases} \qquad \mathcal{F}\Pi_T(s) = T\,\text{sinc} Ts$$

The delta function $\boldsymbol{\delta(x)}$

Scaling.

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

Sifting.

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$$
$$\int_{-\infty}^{\infty} \delta(x)f(x+a)dx = f(a)$$

Convolution.

$$\delta(x) * f(x) = f(x)$$
$$\delta(x-a) * f(x) = f(x-a)$$
$$\delta(x-a) * \delta(x-b) = \delta(x-(a+b))$$

Product.

$$f(x)\delta(x) = f(0)\delta(x)$$

Fourier transforms.

$$\mathcal{F}\delta = 1$$
$$\mathcal{F}(\delta(x-a)) = e^{-2\pi i s a}$$

Derivatives.

$$\int_{-\infty}^{\infty} \delta^{(n)}(x)f(x)dx = (-1)^n f^{(n)}(0)$$
$$\delta'(x) * f(x) = f'(x)$$
$$x\delta(x) = 0$$
$$x\delta'(x) = -\delta(x)$$

Fourier transform of cosine and sine

$$\mathcal{F}\cos 2\pi a t = \frac{1}{2}(\delta(s-a) + \delta(s+a))$$
$$\mathcal{F}\sin 2\pi a t = \frac{1}{2i}(\delta(s-a) - \delta(s+a))$$

Unit step and sgn

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0\end{cases} \qquad \mathcal{F}H(s) = \tfrac{1}{2}\left(\delta(s) + \frac{1}{\pi i s}\right)$$

$$\text{sgn}\, t = \begin{cases} -1 & t < 0 \\ +1 & t > 0\end{cases} \qquad \mathcal{F}\text{sgn}\,(s) = \frac{1}{\pi i s}$$