Home work: No. 2

Q 2-5,

The system is not Causal in fact if we Consider any $0<t<1$ we have. $y(t) = x(\sqrt{t})$ and we know that in this case $\sqrt{t} > t$ so the output depends on future values of the input $\rightarrow$ non Causal system.

Consider the two inputs $x_1(t) = r(t - 1/4)$

$x_2(t) = P(t - 1/4)$

for $t \leq 1/4$ $\quad P(t) = r(t) = 0$

but $y_1(t)\Big|_{t=\frac{1}{9}} = r(\sqrt{\tfrac{1}{9}} - 1/4) = \frac{1}{3} - \frac{1}{4} = 1/12$

$y_2(t)\Big|_{t=1/9} = P(\sqrt{1/9} - 1/4) = P(\frac{1}{3} - 1/4) = P(1/12) = \frac{1}{288}$

$\Longrightarrow y_1(t)\Big|_{t=1/9} \neq y_2(t)\Big|_{t=\frac{1}{9}}$

Q 2-6.

a) the system is not linear in fact

for $x_1(t)$ we have

$y_1(t) = 10\, x_1(t) + 5$

for $x_2(t)$ we have $y_2(t) = 10\, x_2(t) + 5$

for $x(t) = x_1(t) + x_2(t)$ we have.

$y(t) = 10(x_1(t) + x_2(t)) + 5 \neq y_1(t) + y_2(t) = 10 + 10(x_1(t) + x_2(t))$

b) the system is not Causal in fact the response $y(t)$ depends on future value of the input.

let $x_1(t) = u(t)$, $x_2(t) = r(t)$

$x_1(t) = x_2(t) = 0 \quad \forall\, t \leq 0$

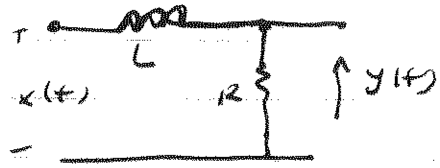
$y_1(t) = x_1(t+2) + 5 = u(t+2) + 5 = 6$

$y_2(t) = x_2(t+2) + 5 = r(t+2) + 5 = t + 2 + 5 = t + 7$

$y_1(t) \neq y_2(t) \rightarrow$ non Causal.

Q_{2-10}:

a) the system is oriented so that the output response is $y(t)$ the voltage on the resistor and the input is $x(t)$.



from electric network equations we have:

$$-x(t) + V_L + y(t) = 0$$

$$\Rightarrow \quad L\frac{di}{dt} + y(t) = x(t) \qquad \text{but } i(t) = \frac{y(t)}{R}$$

$$\Rightarrow \quad L\frac{d\frac{y(t)}{R}}{dt} + y(t) = x(t) \Rightarrow$$

$$\frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{R}{L}x(t)$$

b) ① multiplication property:

$$\alpha\frac{dy(t)}{dt} + \frac{R}{L}\alpha y(t) = \frac{R\alpha x(t)}{L}$$ multiplication of both sides by α

$$\frac{d\alpha y(t)}{dt} + \frac{R}{L}\alpha y(t) = \frac{R}{L}\alpha x(t)$$

$\Rightarrow$ $\alpha y(t)$ is the response for $\alpha x(t)$

② Addition property

$\forall$ $x_1(t)$, $x_2(t)$ we have.

$$\frac{dy_1(t)}{dt} + \frac{R}{L}y_1(t) = x_1(t)\frac{R}{L}$$

$$\frac{dy_2(t)}{dt} + \frac{R}{L}y_2(t) = x_2(t)R/L$$

$$\frac{dy_1(t)}{dt} + \frac{dy_2(t)}{dt} + \frac{R}{L}y_1(t) + \frac{R}{L}y_2(t) = \frac{R}{L}x_1(t) + \frac{R}{L}x_2(t)$$

$$\frac{d}{dt}(y_1+y_2) + \frac{R}{L}(y_1+y_2) = \frac{R}{L}(x_1+x_2)$$

$\Rightarrow$ the response for $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$

$\Rightarrow$ that the superposition is verified by (1 & 2) $\Rightarrow$ linear

(c) consider the differential equation:

$$\frac{dy(t)}{dt} + \frac{R}{L} y(t) = \frac{R}{L} x(t)$$

and the change of variable $t = \lambda - \tau \Rightarrow dt = d\lambda$
substituting in the equation we have.

$$\frac{dy(\lambda-\tau)}{d\lambda} + \frac{R}{L} y(\lambda-\tau) = \frac{R}{L} x(\lambda-\tau)$$

so the form of the equation remains the same for the shift τ in the input $x(t)$ and the output $y(t)$ $\Longrightarrow$ The system is time invariant (fixed)

(D) To write $y(t)$ in the superposition integral form

$$y(t) = \int_{-\infty}^{+\infty} x(t-\tau)\, h(\tau)\, d\tau$$

we have to find $h(t)$ which is the impulse response.
first we note that this system is causal because
$\forall\, t \leq 0$ if $x(t) = 0$ then ~~y(t)~~ $h(t) = 0$
in fact $\frac{dh(t)}{dt} + \frac{R}{L} h(t) = 0 \rightarrow h(t) = A e^{-\frac{L}{R}t}$

$h(t)$ is the zero state response and so $h(0) = 0 = A e^{-\frac{R}{L}t}\Big|_{t=0} = A$
$\rightarrow h(t) = 0 \rightarrow$ Causal.

to determine $h(t)$ we have to consider $\delta(t)$ as input

$$* \quad \frac{dh(t)}{dt} + \frac{R}{L} h(t) = \frac{R}{L} \delta(t)$$

for $t > 0^+$ we have $x(t) = 0 \Rightarrow h(t) = \cancel{A e^{-\frac{R}{L}}}\ A e^{-\frac{R}{L}t} u(t)$
to find A we have to find $h(0^+)$.
consider the equation $*$, $h(t)$ can have no $\delta(t)$ or its derivatives otherwise the ~~left~~ left side will have $\dot{\delta}(t)$ while the right side doesn't have this order of singularity so $y(t)$ must verify the condition $\int_{0^-}^{0^+} y(t)\, dt = 0$ because at max it can

have a point of discontinuity in $t=0$ of which the integral is zero integrating the two sides of the differential equation we have

$$\int_{0^-}^{0^+} \frac{dy(t)}{dt}\,dt + \frac{R}{L}\int_{0^-}^{0^+} y(t)\,dt = \frac{R}{L}\int_{0^-}^{0^+} \delta(t)\,dt$$

$$y(0^+) - y(0^-) + \text{zero} = \frac{R}{L} * 1$$

but $y(0^-) = 0$ because $y(t) = h(t)$ in this case and we showed that it is causal.

$$y(0^+) = h(0^+) = \frac{R}{L}.$$

$$\begin{cases} h(t) = A e^{-\frac{R}{L}t} u(t) \\ h(0^+) = \frac{R}{L} \end{cases} \Rightarrow A = \frac{R}{L} \quad \text{so} \quad h(t) = \frac{R}{L} e^{-\frac{R}{L}t} u(t)$$

so in integral form we have

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)\,x(\tau)\,d\tau \xRightarrow{\text{causality}} \int_{-\infty}^{t} h(t-\tau)\,x(\tau)\,d\tau$$

because $h(t-\tau) = 0$ for $t-\tau < 0 \Rightarrow$ for $\tau > t$

and if $x(\tau)$ is zero $\forall \tau < 0$ then it becomes.

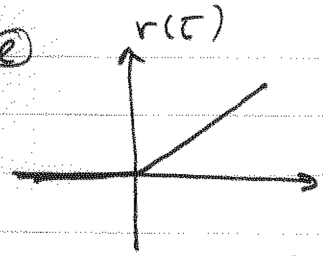
for causal system with $x(\tau) = 0 \quad \tau < 0$

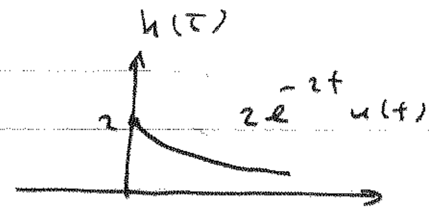
$$\int_{0}^{t} h(t-\tau)\,x(\tau)\,d\tau$$

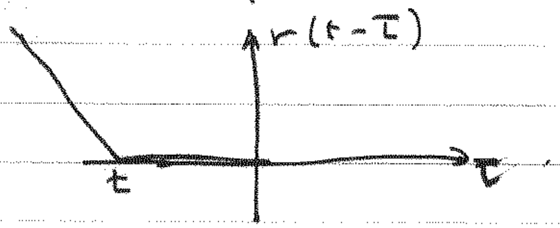
and in our case.

$$y(t) = \int_{0}^{t} \frac{R}{L} e^{-\frac{R}{L}(t-\tau)}\,x(\tau)\,d\tau$$

Q2-17 (e)







I case: $t \leq 0 \Rightarrow r(t) * h(t) = 0$

II case: $t \geq 0 \Rightarrow r(t) * h(t) = 2\int_0^t (t-\tau)\, e^{-2\tau}\, d\tau$

$$= 2\left[t \int_0^t e^{-2\tau} d\tau - \int_0^t \tau e^{-2\tau} d\tau\right]$$

$$= 2\left[t \left.\frac{e^{-2\tau}}{-2}\right|_0^t - \left[\left.\tau \frac{e^{-2\tau}}{-2}\right|_0^t - \int_0^t \frac{e^{-2\tau}}{-2} d\tau\right]\right]$$

$$= 2\left\{t \frac{e^{-2t}}{-2} + \frac{t}{2} - \left[t \frac{e^{-2t}}{-2} + \frac{t}{2} + \left.\frac{e^{-2\tau}}{-4}\right|_0^t\right]\right\}$$

$$= 2\left\{\left.\frac{e^{-2\tau}}{+4}\right|_0^t\right\} = 2\left\{\frac{e^{-2t}}{+4} - \frac{1}{4}\right\} = \frac{-1}{2}\left(1 - e^{-2t}\right)$$

$$= \frac{1}{2}\left(e^{-2t} - 1\right)$$

Q2-20:

$$y(t) = \frac{1}{1+1+1} x(t) = \frac{1}{3} x(t)$$

for $x(t) = \delta(t)$ we have

$$h(t) = \frac{1}{3}\delta(t)$$

Q2-29:

for a relaxed linear time invariant system. if $y_{-k}(t)$ is the response of the system for the singular input function ~~the~~ $u_{-k}(t)$ then $\int_0^t y_{-k}(\tau)\,d\tau$ is the response of

the system for the Singularity input of order $u_{-k+1}(t)$

$$\text{so if } h(t) = \delta(t) - \frac{R}{L}\, e^{-\frac{R}{L}t}\, u(t)$$

a) the step response.

$$a(t) = \int_0^t h(\tau)\,d\tau = \int_{0^-}^{t} \delta(\tau)\,d\tau - \frac{R}{L}\int_0^t e^{-\frac{R}{L}\tau}\,d\tau$$

$$a(t) = u(t) - \frac{R}{L}\; \frac{e^{-\frac{R}{L}\tau}}{-\frac{R}{L}}\Bigg|_0^t$$

$$= u(t) + \left[e^{-\frac{R}{L}t} - 1\right]u(t)$$

$$= e^{-\frac{R}{L}t}\, u(t)$$

b) $$b(t) = \int_0^t e^{-\frac{R}{L}\tau}\,d\tau$$ response on $r(t)$

$$= \frac{e^{-\frac{R}{L}\tau}}{-\frac{R}{L}}\Bigg|_0^t = -\frac{L}{R}\left[e^{-\frac{R}{L}t} - 1\right]u(t)$$

$$= \frac{L}{R}\left[1 - e^{-\frac{R}{L}t}\right]u(t)$$

Q. 2-34

$$h(t) = \frac{1}{RC}\, e^{-\frac{t}{\tau}}\, u(t) \qquad \tau = \frac{R_1 R_2 C}{R_1 + R_2}$$

a) the frequency response is the Fourier transform of $h(t)$ so

$$H(f) = \int_0^{+\infty} \frac{1}{R_1 C}\, e^{-\frac{t}{\tau}}\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{R_1 C} \int_0^{+\infty} e^{-(j2\pi f + \frac{1}{\tau})t}\, dt$$

$$= \frac{1}{R_1 C} \left. \frac{e^{-(j2\pi f + \frac{1}{\tau})t}}{-(j2\pi f + \frac{1}{\tau})} \right|_0^{+\infty} = \frac{1}{R_1 C}\left[0 - \frac{1}{-(j2\pi f + \frac{1}{\tau})}\right]$$

$$= \frac{1}{R_1 C}\, \frac{1}{j2\pi f + \frac{1}{\tau}}$$

b) $$|H(f)| = \frac{1}{R_1 C}\, \frac{1}{\sqrt{(\frac{1}{\tau})^2 + (2\pi f)^2}}$$

$$\measuredangle H(f) = -\tan^{-1} \frac{2\pi f}{1/\tau}$$

c) Using superposition, the steady state response has the same frequency but modified amplitude & phase.

$x_1(t) = \cos(2\pi t) \longrightarrow f = 1\ \text{Hz} \qquad \Theta_1 = \text{zero}.$

$y_1(t) = |x_1(t)|\ |H(f)|_{f=1}$

$$\boxed{\begin{array}{l} (R_1 = R_2 = 1\,\Omega) \\ \quad C = 1F \\ \text{but } \tau = \frac{1 \cdot 1 \cdot 1}{1+1} = \frac{1}{2} \end{array}}$$

$$= 1 * \frac{1}{1*1}\, \frac{1}{\sqrt{(\frac{1}{\tau})^2 + (2\pi)^2}}$$

$$= \frac{1}{\sqrt{\frac{1}{4} + 4\pi^2}}$$

$$\theta_{x_1} = \Theta_1 + \measuredangle H(f) = -\tan^{-1} 2\pi * \tfrac{1}{2} = -\tan^{-1}\pi$$

$x_2(t) = \sin 5\pi t \longrightarrow f = \frac{5}{2} \qquad \Theta_2 = -\frac{\pi}{2}$

$|y_2(t)| =$

$$\measuredangle y_2(t) = \theta_2 + \measuredangle H(f)\Big|_{f=\frac{5}{2}\,\text{Hz}}$$

$$= -\frac{\pi}{2} - \tan^{-1} 2\pi * \frac{5}{2} * \frac{1}{2}$$

$$= -\frac{\pi}{2} - \tan^{-1}\left(\frac{5\pi}{2}\right)$$

Q2-35:

from Q2-10 we have $h(t) = \frac{R}{L} e^{-\frac{R}{L}t} u(t)$

it is necessary & sufficient for the system to be BIBO stable that $\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$

in our case we have

$$\int_{0}^{+\infty} |h(t)|\, dt = \int_{0}^{+\infty} \left|\frac{R}{L} e^{-\frac{R}{L}t}\right| dt = \int_{0}^{+\infty} \frac{R}{L} e^{-\frac{R}{L}t}\, dt$$

$$= \frac{R}{L} \int_{0}^{+\infty} e^{-\frac{R}{L}t}\, dt = \frac{R}{L} \left.\frac{e^{-\frac{R}{L}t}}{-\frac{R}{L}}\right|_{0}^{+\infty}$$

$$= -\left[0 - 1\right] = 1 < \infty \Longrightarrow$$ the system is BIBO stable.

Q. 2-2

Homework No. 2

c) $4y(t) + 10 = \frac{dx}{dt} + 5x(t)$

$$a(4y_1(t)+10) + b(4y_2(t)+10) = a\left(\frac{dx_1}{dt}+5x_1(t)\right) + b\left(\frac{dx_2}{dt}+5x_2(t)\right)$$

$(4ay_1 + 4by_2) + (10a + 10b) = ..$ has different form as the original ~~(a+b)10~~

$\cancel{(a+b)4y(t) + 10(a+b) = (a+b)\left(\frac{dx}{dt}+5x(t)\right)}$

~~changes the form of the system~~

$10(a+b)$ changes the form of the eq., thus it is nonlinear → nonlinear

e) $\frac{d^2y(t)}{dt^2} + \frac{1}{2}\left[\frac{dy}{dt}\right]^2 + y(t) = 5x(t)$

$$\frac{d\,y^2}{dt} = 2y\frac{dy}{dt}$$ is a nonlinear term.

d) is linear

$$\frac{dy}{dt} + t^2y(t) = \int_{-\infty}^{t} x(\lambda)d\lambda$$

$$a\frac{dy_1}{dt} + at^2y_1(t) + b\frac{dy_2}{dt} + bt^2y_2(t)$$

$$= \cancel{(a+b)\frac{dy}{dt}} = \frac{d}{dt}(ay_1+by_2) + t^2(ay_1+by_2) = a\int x_1(\lambda) + b\int x_2(\lambda)$$

linear.

a, b, c, e are time invariant

e.g., a: $2\frac{dy(t-\tau)}{dt} + 3y(t-\tau) = \frac{d^2x(t-\tau)}{dt^2} + x(t-\tau)$

d) is time-variant:

$$\frac{dy(t-\tau)}{dt} + (t-\tau)^2y(t-\tau) = \int_{-\infty}^{t-\tau} x(\lambda)d\lambda$$

$$= \frac{dy(t-\tau)}{dt} + [t^2-2t\tau+\tau^2]y(t-\tau) = \int_{-\infty}^{t-\tau} x(\lambda)d\lambda$$

2-7

$$y(t) = x(t^2)$$

a) linear, ~~ox~~ because:

$$a\, x_1(t^2) + b\, x_2(t^2) = a\, y_1(t) + b\, y_2(t)$$

the same form

b) causal ? (non-causal)

$$y(t) = x(t^2)$$

Since t at the output is the root of ~~the~~ t at the input, that is the output depends on future values of ~~output~~ input

$$y(2) = x(2^2) = x(4)$$

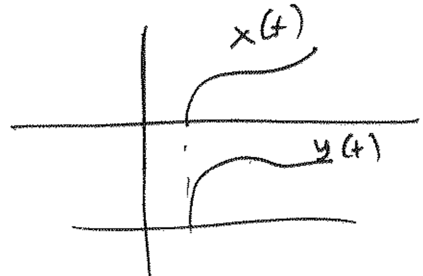
in other words the future input does not affect the output or

$y(t_0)$ cannot be affected by $x(t_0 + \tau)$

$y(0)$ cannot affected by $x(+\tau)$

that is $y(0)$ cannot be $\neq 0$ if x is applied after 0 or at $\tau > 0$

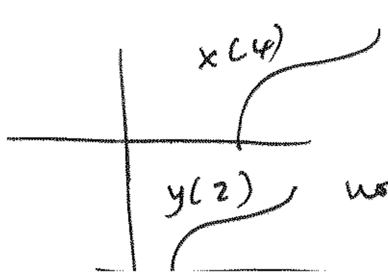
The output does not depend on the future values of the inpu



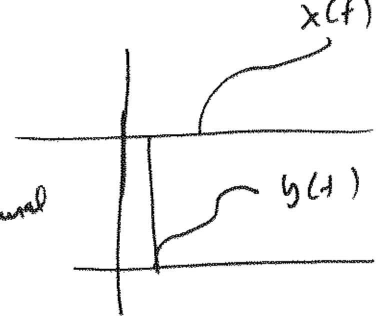
Causal ✓

$y(2) = x(4)$

is not causal



non-causal



X non-causal

let $x(t-\tau)$ be delayed input and $\tilde{y}(t)$ is the output

$$\frac{d\tilde{y}(t)}{dt} + t^2 \frac{\cancel{d}\tilde{y}(t)}{\cancel{dt}} = \int_{-\infty}^{t-\tau} x(\lambda)\,d\lambda \qquad (1)$$

The delayed output is

$$\cancel{d\tilde{y}} \quad \frac{dy(t-\tau)}{dt} + (t-\tau)^2 y(t) = \int_{-\infty}^{t-\tau} x(\lambda)\,d\lambda \qquad (2)$$

(1) and (2) are different.

<u>2-5</u>

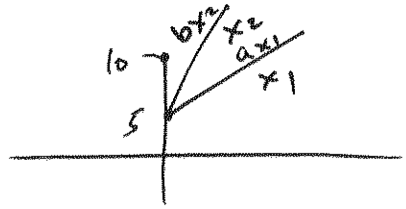
$y(t) = x(t+10)$

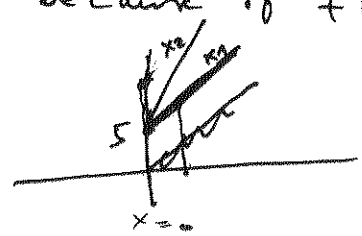
$y(t) > 0 \qquad t < 0 \qquad$ non-causal.

<u>2-6</u>

$$a\,y_1(t) + b\,y_2(t) = a(10x_1(t)) + b(10x_2(t)) + 5 + 5$$
$$= 10(ax_1 + bx_2) + 10$$

The system is <u>non-linear</u> because of $+10$





c) time varying

$\tilde{y}(t) = x(t^2 - \tau)$ the response to delayed input.

The delayed output to undelayed input (time shifted input)

$y(t-\tau) = x((t-\tau)^2)$

$y(t-\tau) \neq \tilde{y}(t)$ it is not fixed

note: delayed does not mean time-shifted.

d) not zero memory

$x((0.1)^2) = y(0.01)$

or $y(0.1) = x(0.01)$

that is the system depends on future and past, that is not zero memory.

Q. 2-8 b) $y(t) = x(t) + \alpha y(t-\tau_0)$ (recursive)

$y(0) = x(0) + \alpha y(-\tau_0)$

not zero memory since it depend on the input at $t = -\tau_0$

Zero memory only if $\tau_0 = 0$

c) $y(t) = x(t) + \alpha y(t-\tau_0)$

Causal only if $\tau_0 \geq 0$

because the output does not respond before the input is applied.