Q 3-3: first method:

a) $x_1(t) = \frac{1}{2} + \frac{1}{2}\cos 200\pi t$ by expansion of $\cos^2 100\pi t$

Second method:

If we apply Fourier-Series integrals to find $a_0, a_1, b_0, b_1 \dots$ we find that $a_0 = \frac{1}{2}$ $a_1 = \frac{1}{2}$

— $\cos^2 2\pi \cdot 100t$ is even signal $\Rightarrow$ only a's exist

$$a_0 = \frac{1}{T_0}\int_0^{T_0}\left(\frac{1}{2} + \underbrace{\cancel{\tfrac{1}{2}\cos 200\pi t}}_{0}\right)dt$$

$$= \frac{1}{T_0}\int_0^{T_0}\frac{1}{2}\,dt = \frac{1}{2}\frac{T_0}{T_0} = \frac{1}{2}$$

$$a_1 = \frac{2}{T_0}\int_0^{T_0}\left(\frac{1}{2} + \frac{1}{2}\cos 2\pi \cdot 100t\right)\cos 2\pi \cdot 100t$$

$$= \frac{2}{T_0}\int_0^{T_0}\left(\underbrace{\cancel{\tfrac{1}{2}\cos 2\pi \cdot 100t}}_{0} + \frac{1}{2}\cos^2 2\pi \cdot 100t\right)dt$$

$$= \frac{\cancel{2}}{T_0} \times \frac{1}{\cancel{2}}\int_0^{T_0}\left(\frac{1}{2} + \underbrace{\cancel{\tfrac{1}{2}\cos 2\pi \cdot 200t}}_{0}\right)dt$$

$$a_1 = \frac{1}{T_0}\int_0^{T_0}\frac{1}{2}\,dt = \frac{1}{2}\frac{\cancel{T_0}}{\cancel{T_0}} = \frac{1}{2}$$

$$a_2 = \frac{1}{T_0}\int_0^{T_0}\left(\frac{1}{2} + \frac{1}{2}\cos 2\pi \cdot 100t\right)\cos 2\pi \cdot 2 \cdot 100t\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0}\underbrace{\frac{1}{2}\cos 2\pi \cdot 2 \cdot 100t}_{\substack{=0 \\ \text{over the period}}} + \underbrace{\frac{1}{2}\cos 2\pi \cdot 100t \cdot \cos 2\pi \cdot 2 \cdot 100t}_{=0,\ \text{orthogonal}}$$

$$= 0$$

$$a_n = 0$$

$\Rightarrow$ $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{2}$ which is consistent with the first method.

Q 3-3 d)

$\cancel{\cos^3 20\pi t = \cos 20\pi t\left[\frac{1}{2} + \frac{1}{2}\cos 40\pi t\right]}$

$$\cos^3 20\pi t = \cos 20\pi t \cos^2 20\pi t$$

$$= \cos 20\pi t \left[\frac{1}{2} + \frac{1}{2}\cos 40\pi t\right]$$

$$= \frac{1}{2}\cos 20\pi t\left[1 + \cos 40\pi t\right]$$

$$= \frac{1}{2}\cos 20\pi t + \frac{1}{4}\cos 20\pi t + \frac{1}{4}\cos 60\pi t$$

$$= \frac{3}{4}\cos 20\pi t + \frac{1}{4}\cos 60\pi t$$

$$\cos^3 20\pi t\left[1 - \sin^2 10\pi t\right]$$

$$= \left[\frac{3}{4}\cos 20\pi t + \frac{1}{4}\cos 60\pi t\right]\left[1 - \left(\frac{1}{2} - \frac{1}{2}\cos 20\pi t\right)\right]$$

$$= \left(\frac{3}{4}\cos 20\pi t + \frac{1}{4}\cos 60\pi t\right)\left(\frac{1}{2} + \frac{1}{2}\cos 20\pi t\right)$$

$$= \frac{3}{8}\cos 20\pi t + \frac{3}{8}\underbrace{\cos 20\pi t \cos 20\pi t}_{\cos^2 20\pi t}$$

$$+ \frac{1}{8}\cos 60\pi t + \frac{1}{8}\cos 20\pi t \cos 60\pi t$$

$$= \frac{3}{8}\cos 20\pi t + \frac{3}{16}\cos 40\pi t + \frac{3}{16}$$

$$+ \frac{1}{8}\cos 60\pi t + \frac{1}{16}\cos 40\pi t + \frac{1}{16}\cos 80\pi t$$

$$= \frac{3}{16} + \frac{3}{8}\cos 20\pi t + \frac{1}{4}\cos 40\pi t + \frac{1}{8}\cos 60\pi t + \frac{1}{16}\cos 80\pi t$$

$$a_0 = \frac{3}{16}, \quad a_1 = \frac{3}{8}, \quad a_2 = \frac{1}{4} \quad a_3 = \frac{1}{8} \quad a_4 = \frac{1}{16}$$

Q 3-11

$$\sum X_n e^{j\frac{3\pi}{2}nt} = \sum \frac{1}{1+jn\pi} e^{j\frac{3n\pi}{2}t}$$

$$X_n = \frac{1}{1+jn\pi}$$

a) $\frac{3\pi}{2} = \frac{6\pi}{4} = 2\pi \cdot \frac{3}{4} \Rightarrow f_0 = \frac{3}{4} \Rightarrow \frac{1}{f_0} = T_0 = \frac{4}{3}$

b) $X_0 = \frac{1}{1+j.0\pi} = 1$

$$1 = X_0 = \frac{1}{T_0}\int_{T_0} f(t)dt = \text{average of } f(t)$$

c) The third harmonic are the terms for $n = \pm 3$

$$X_3 = \frac{1}{1+3j\pi}, \quad X_{-3} = \frac{1}{1-3j\pi}$$

Third harmonic: $X_3 e^{j\frac{9\pi t}{2}} + X_{-3} e^{-j\frac{9\pi t}{2}}$

d) $X_3 = |X_3| e^{\tan^{-1}\phi}$

$$X_3 = \frac{(1-3j\pi)}{(1+3j\pi)(1-3j\pi)} = \frac{e^{-\tan^{-1}3\pi}}{1+9\pi^2}\sqrt{1+9\pi^2}$$

$$X_{-3} = \frac{1+3j\pi}{1+9\pi^2} e^{\tan^{-1}3\pi}\sqrt{1+9\pi^2}$$

$\therefore \phi = \pm 3\pi, \quad \tan^{-1}\pm(3\pi)$

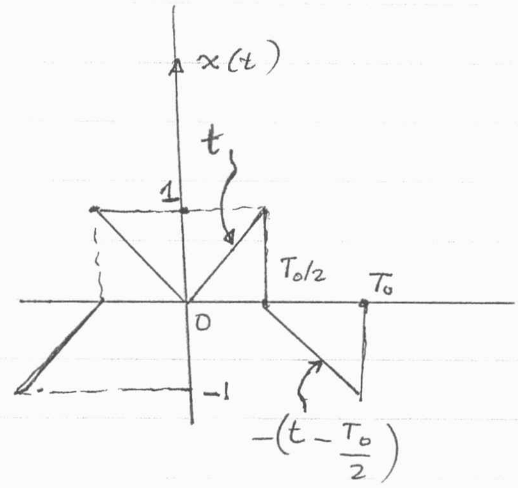
e) is given in c)

$$\boxed{X_3 e^{\frac{j9\pi t}{2}} + X_{-3} e^{-j9\pi t}}$$

Q 3-18 c)

The signal has Half-wave symmetry $f(t) = -f(t+\frac{T}{2})$

Note: $(f_0 T_0 = 1 \quad f_0 = \frac{1}{T_0})$



$$a_0 = \frac{1}{T_0}\int_0^{T_0/2} t\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} -(t-\tfrac{T_0}{2})\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0/2} t\,dt - \frac{1}{T_0}\int_{T_0/2}^{T_0}(t-\tfrac{T_0}{2})\,dt$$

$$= \frac{1}{T_0}\frac{t^2}{2}\Big|_0^{T_0/2} - \frac{1}{T_0}\left(\frac{t^2}{2}-\frac{T_0 t}{2}\right)\Big|_{T_0/2}^{T_0}$$

$$= \frac{T}{8} - \frac{T}{8} = 0$$

$n>0$

$$a_n = \underbrace{\frac{2}{T_0}\int_0^{T_0/2} t\cos n\omega_0 t\,dt}_{1} - \underbrace{\frac{2}{T_0}\int_{T_0/2}^{T_0}(t-\tfrac{T_0}{2})\cos n\omega_0 t\,dt}_{2}$$

by parts

(1): $$\frac{2}{T_0}\int_0^{\frac{T_0}{2}} t\cos n\omega_0 t\,dt = \frac{2t\sin(n\omega_0 t)}{T_0 n\omega_0}\Big|_0^{T_0/2} - \frac{1}{T_0}\frac{1}{n\omega_0}\int_0^{T_0}\sin n\omega_0 t\,dt$$

$$= \frac{2T_0}{2n\omega_0}\underbrace{\cancel{\sin n\omega_0 \tfrac{T_0}{2}}}_{0} - 0 - \frac{1}{n\omega_0}\frac{2}{T_0}\int_0^{T_0/2}\sin n\omega_0 t\,dt$$

$$= -\frac{2}{T_0}\frac{1}{(n\omega_0)^2}(-\cos n\omega_0 t)\Big|_0^{T_0/2}$$

$$-\frac{2}{T_0}\frac{1}{(2\pi n f_0)^2}\left(-\cos \cancel{2}\pi n \underbrace{f_0 T_0}_{=1}/\cancel{2} + 1\right)$$

n even $\Longrightarrow$ $(-\cos n\pi + 1) = 0$

n odd $(-\cos n\pi + 1) = 2$

$$-\frac{2}{T_0}\cdot\frac{1}{4\pi^2 n^2 f_0^2}(+2) = -\frac{T_0}{\pi^2 n^2} = -\frac{T_0}{\pi^2 n^2}$$

(2): $$-\frac{2}{T_0}\int_{T_0/2}^{T_0} t\cos n\omega_0 t\,dt + \underbrace{\cancel{\frac{T_0}{2}\frac{2}{T_0}\int_{T_0/2}^{T_0}\cos n\omega_0 t\,dt}}_{=0}$$

$$= -\frac{2}{T_0}\cancel{\frac{t}{n\omega_0 t}\sin n\omega_0 t} + \frac{1}{n\omega_0}\frac{2}{T_0}\int_{T_0/2}^{T_0}\sin n\omega_0 t\,dt = -\frac{2}{T_0}\left(\frac{1}{n\omega_0}\right)^2\cos n\omega_0 t\Big|_{T_0/2}^{T_0}$$

$$= -\frac{2}{T_0}\frac{1}{(2\pi n f_0)^2}\underbrace{(\underbrace{\cos 2\pi n f_0 T_0}_{=1} - \cos \cancel{2}\pi n f_0 \tfrac{T_0}{\cancel{2}})}_{=2 \text{ for odd}} = -\frac{4T_0}{4\pi^2 n^2} \quad n \text{ odd}$$

(2) $= -\frac{T_0}{\pi^2 n^2}$, n odd

n even

Q 3-18 c (contd')

$a_n = 1 + 2$

$$\Rightarrow \boxed{a_n = -\frac{2T_0}{\pi^2 n^2}} \quad , \quad n = 1, 3, 5, \ldots$$

$$b_n = \underbrace{\frac{2}{T_0}\int_0^{T_0/2} t \sin n\omega_0 t \, dt}_{1} - \underbrace{\frac{2}{T_0}\int_{T_0/2}^{T_0} \left(t - \frac{T_0}{2}\right) \sin n\omega_0 t \, dt}_{2}$$

$$1 : -\frac{2t}{T_0 n\omega_0} \cos n\omega_0 t \Big|_0^{T_0/2} + \frac{2}{T_0}\int_0^{T_0/2} \frac{1}{n\omega_0} \cos n\omega_0 t \, dt$$

$$= -\left(\frac{\cancel{2}\frac{T_0}{\cancel{2}}}{T_0 n\omega_0} \cos n\omega_0 \frac{T_0}{2} - 0\right) + \frac{2}{T_0 (n\omega_0)^2} \sin n\omega_0 t \Big|_0^{T_0/2}$$

$$= -\frac{1}{2\pi n f_0}\left(\cos \cancel{2}\pi n f_0 \overbrace{\frac{T_0}{\cancel{2}}}^{=1} - 0\right)$$

$$= -\frac{1}{2\pi n f_0}(-1 - 0) = \frac{T_0}{2\pi n}, \quad n = \text{odd}$$

$$= -\frac{1}{2\pi n f_0}(1 - 0 = -\frac{T_0}{2\pi n} \quad n \text{ even}$$

$$2 : -\frac{2}{T_0}\int_{T_0/2}^{T_0} \left(t - \frac{T_0}{2}\right) \sin n\omega_0 t \, dt = -\underbrace{\frac{2}{T_0}\int_{T_0/2}^{T_0} t \sin n\omega_0 t \, dt}_{a} + \underbrace{\frac{2}{T_0}\int_{T_0/2}^{T_0} \frac{T_0}{2} \sin n\omega_0}_{b}$$

$$a) -\left(-\frac{2}{T_0 n\omega_0} t \cos n\omega_0 t\right)_{T_0/2}^{T_0} - \cancel{\int_{T_0/2}^{T_0} -(-\cos n\omega_0 t \, dt)} \; a$$

$$= \frac{2\cancel{T_0}}{n\omega_0 \cancel{T_0}} \cos 2\pi n f_0 T_0 - \frac{\cancel{2}}{\cancel{T_0}} \cdot \frac{T_0}{\cancel{2}} \frac{1}{n\omega_0} \cos 2\pi n f_0 \overbrace{\frac{T_0}{2}}^{=1} = \frac{2T_0}{\pi n}$$

$$= \frac{2}{2\pi n f_0} \cos 2\pi n - \frac{1}{2\pi n f_0} \cos \pi n = \frac{3T_0}{2\pi n} \quad n \text{ odd}$$

$$\frac{1}{\pi n f_0}(1) + \frac{1}{2\pi n f_0} \qquad = 0 \quad n \text{ even.}$$

$$\frac{3T_0}{\pi n f_0}$$

Q. 3-18c (contd')

6

b) $\frac{2}{T_0}\int_{T_0/2}^{T_0} \frac{T_0}{2}\sin n\omega_0 t\,dt$

$$= \int_{T_0/2}^{T_0} \sin n\omega_0 t\,dt = -\frac{1}{n\omega_0}\cos n\omega_0 t\Big|_{T_0/2}^{T_0}$$

$$= -\frac{1}{2\pi n f_0}\Big[\cos 2\pi n \underbrace{f_0 T_0}_{=1} - \cos \cancel{2}\pi n \overbrace{f_0 \frac{T_0}{\cancel{2}}}^{=1}\Big)$$

$$= -\frac{T_0}{2\pi n}\left[1-1\right] = 0 \qquad n \text{ even}$$

$$-\frac{T_0}{2\pi n}\left[1+1\right] = -\frac{2T_0}{2\pi n} \qquad n \text{ odd}$$

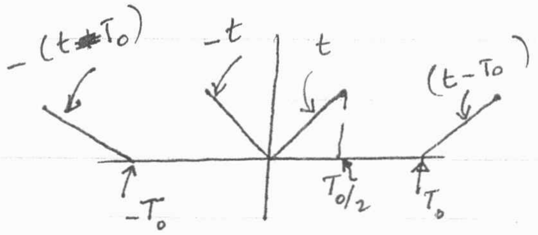
$\Rightarrow$ $b_n =$ result of 1 + result of 2 (2 = a+b)

$$= \frac{T_0}{2\pi n} + \frac{3T_0}{2\pi n} + \left(-\frac{\cancel{2}T_0}{\cancel{2}\pi n}\right) = \frac{\cancel{2}T_0}{\cancel{2}\pi n}, \; n \text{ odd}$$

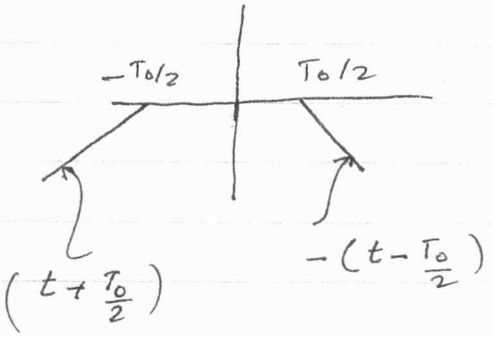
$$= \frac{T_0}{\pi n}$$

another method:

Splitt the signal into two even signals and find the Coefficients then add the two signals together.



periodi



periodic

In general:

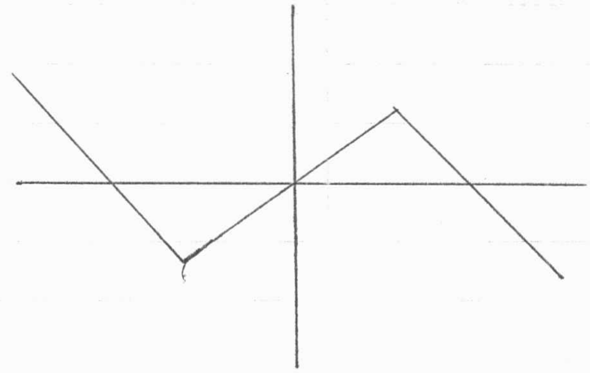
$a_n = \frac{4}{T_0}\int^{T_0/2} f(t)\cos n\omega_0 t$ n odd

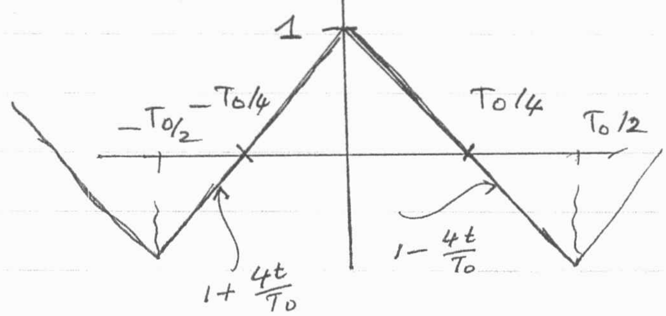
$= 0$ n even

$b_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin n\omega_0 t\,dt$ n odd, n even

Q. 3-18 f)

We first solve the even or odd symmetry similar function. We mean

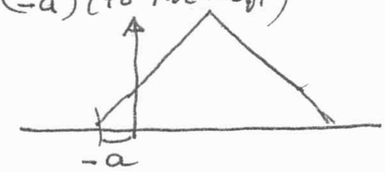




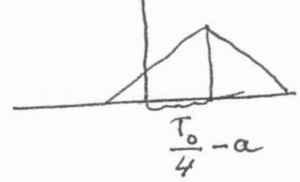
a) odd representation

b) even representation

$x(t)$ in figure f) represents the odd a) shifted back by $(-a)$ (to the left)



$x(t)$ in Fig f) represents the even b) shifted by $\frac{T_0}{4}-a$ to the right



We consider the even representation b), then apply the rule of time-shift

Since it is even, then $b_n \equiv 0 \implies a_n \neq 0$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\cos n\omega_0 t\, dt$$

$$f(t) = \begin{cases} 1+\frac{4}{T_0} & -\frac{T_0}{2} \leq t \leq 0 \\ 1-\frac{4t}{T_0} & 0 \leq t \leq \frac{T_0}{2} \end{cases}$$

$$\frac{2}{T_0}\int_{-T_0/2}^{0}\left(1+\frac{4t}{T_0}\right)\cos n\omega_0 t + \int_{0}^{T_0/2}\left(1-\frac{4t}{T_0}\right)\cos n\omega_0 t\, dt$$

$$= \frac{2}{T_0}\int_{-T_0/2}^{0}\cos n\omega_0 t + \frac{2}{T_0}\int_{0}^{T_0/2}\cos n\omega_0 t\, dt = 0 \qquad = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2}\cos n\omega_0 t\, dt = 0$$

X

$$\frac{2}{T_0}\int_{-T_0/2}^{0}\frac{4t}{T_0}\cos n\omega_0 t\,dt - \frac{2}{T_0}\int_{0}^{T_0/2}\frac{4t}{T_0}\cos n\omega_0 t\,dt$$

$$= \frac{8}{T_0^2}\int_{-T_0/2}^{0} t\cos n\omega_0 t\,dt - \frac{8}{T_0^2}\int_{0}^{T_0/2} t\cos n\omega_0 t\,dt$$

1 integration by parts

2 integration by parts

(see (3-18c)

$$\Rightarrow \quad 1 = -\frac{8}{T_0^2}\int_{-T_0/2}^{0} t\cos n\omega_0 t\,dt$$

Substitute in 1 $\quad t = -\tau \Rightarrow dt = -d\tau$

$t = -T_0/2 \Rightarrow \tau = T_0/2$

$t = 0 \quad , \quad \tau = 0$

$$\Rightarrow \quad \frac{8}{T_0^2}\int_{-T_0/2}^{0} t\cos n\omega_0 t\,dt = \frac{8}{T_0^2}\int_{T_0/2}^{0} -\tau\cos n\omega_0(-\tau)(-d\tau)$$

$$= \frac{8}{T_0^2}\int_{T_0/2}^{0} \tau\cos n\omega_0\tau\,d\tau$$

$$= -\frac{8}{T_0^2}\int_{0}^{T_0/2} \tau\cos n\omega_0\tau\,d\tau$$

The integrals 1 + 2 become

$$-\frac{8}{T_0^2}\int_{0}^{T_0/2} \tau\cos n\omega_0\tau\,d\tau - \frac{8}{T_0^2}\int_{0}^{T_0/2} t\cos n\omega_0 t\,dt$$

$$= -\frac{16}{T_0^2}\int_{0}^{T_0/2} t\cos n\omega_0 t\,dt$$

$$= -\frac{16}{T_0^2}\cdot\cancel{\frac{t}{n\omega_0}\sin n\omega_0 t\Big|_0^{T_0/2}} - \frac{16}{T_0^2}\frac{1}{n\omega_0}\int_{0}^{T_0/2}\sin n\omega_0 t\,dt$$

$$= -\frac{16}{T_0}\cdot\left(\frac{1}{n\omega_0}\right)^2 \cos n\omega_0 t\Big|_0^{T_0/2} =$$

Q 3-18 f) Contd' (9)

$$-\frac{16}{T_0^2}\cdot\frac{1}{(2\pi n f_0)^2}\left[\cos 2\pi n f_0 \frac{T_0}{2} - 1\right]$$

$$-\frac{16}{T_0^2}\cdot\frac{1}{4\pi^2 n^2 f_0^2}\left[\cos\pi n - 1\right) = -\frac{16}{4\pi^2 n^2}\left(\cos\pi n - 1\right)$$

$= 1 = T_0^2 f_0^2$

$$a_n = \frac{4}{\pi^2 n^2}\left(1 - \cos\pi n\right)$$

$$a_n = \begin{cases} 0 & n \text{ even} \\ \frac{8}{n^2\pi^2} & n \text{ odd} \end{cases}$$

$$f(t) = \sum_{n=1,3,\dots} a_n \cos n\omega_0 t$$

The shifted version f) is now shifted by a time

then $f(t-t_0) = \sum a_n \cos n\omega_0 (t-t_0)$