- Now, let us generalize the approximation & define the Trigonometric Fourier Series (TFS) given by:-

$$\boxed{\begin{aligned} x(t) &= a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \cdots \\ &\quad + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \cdots \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \end{aligned}}$$

- by minimizing the mean-square error (MSE) over all the parameters, we can show that:-

$$a_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t)\, dt \qquad \leftarrow \text{Average value (DC component)}$$

$$a_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \cos(n\omega_0 t)\, dt \qquad n = 1, 2, \ldots$$

$$b_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(n\omega_0 t)\, dt \qquad n = 1, 2, \ldots$$

a_0 : Called average value (DC value)

a_0, a_n, b_n : FS coeffecients

$\langle T_0 \rangle$ denotes integration over one period:

for example $[-\frac{T_0}{3}, \frac{2}{3}T_0]$, $[0, T_0]$, $[-\frac{T_0}{2}, \frac{T_0}{2}]$, ...

Choose whichever is more ~~convient~~ convenient to simplify the integration.

• <u>An alternative approach</u> for finding TFS coeffecients a_0, a_n, b_n (other than MSE approach) is just by integrating :-

$$\int_{\langle T_0 \rangle} x(t)\, dt \quad , \quad \int_{\langle T_0 \rangle} x(t) \cos(m\omega_0 t)\, dt, \quad \int_{\langle T_0 \rangle} x(t) \sin(m\omega_0 t)\, dt$$

Before evaluating the previous integrals, let us prove the following facts :-

① $\displaystyle\int_{\langle T_0 \rangle} \sin(m\omega_0 t)\, dt = 0$ for any integer m

~~proofe~~ The prove :-

$$\int_0^{T_0} \sin(m\omega_0 t)\, dt = \int_0^{T_0} \frac{-m\omega_0}{-m\omega_0} \sin(m\omega_0 t)\, dt$$

$$= -\frac{1}{m\omega_0} \Big[\cos(m\omega_0 t) \Big|_0^{T_0}$$

$$= -\frac{1}{m\omega_0} \Big[\cos(m\omega_0 T_0) - \cancel{\cos}(0) \Big]$$

$$= -\frac{1}{m\omega_0} \Big[\underbrace{\cos(\cancel{2\pi} m)}_{=1 \text{ for any } m=0,1,2,\ldots} - 1 \Big] \quad , \text{ where } \boxed{\omega_0 T_0 = 2\pi}$$

$$= 0$$

② $\displaystyle\int_{\langle T_0 \rangle} \cos(m\omega_0 t) = 0$ for non-zero integer m .

(3) $\displaystyle\int_{\langle T_0\rangle} \sin(m\omega_0 t)\cos(m\omega_0 t)\,dt = 0$ for any integers m, n

The prove :

$$\int_0^{T_0} \sin(m\omega_0 t)\cos(m\omega_0 t)\,dt = \frac{1}{2}\int_0^{T_0} \sin(m+n)\omega_0 t + \sin(m-n)\omega_0 t$$

$$= \frac{1}{2}\int_0^{T_0} \underbrace{\cancel{\sin \underbrace{(m+n)}_{\text{integer}}\omega_0 t}}_{\text{from (1)}}^{0} + \frac{1}{2}\int_0^{T_0} \underbrace{\cancel{\sin \underbrace{(m-n)}_{\text{integer}}\omega_0 t}}_{\text{from (1)}}^{0} = 0$$

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(4) $\displaystyle\int_{\langle T_0\rangle} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \begin{cases} 0 & \text{, when integers } m \neq n \text{ or } m \neq -n \\ \frac{T_0}{2} & \text{, when } m = n \neq 0 \end{cases}$

The prove :-

$$\int_{\langle T_0\rangle} \sin(m\omega_0 t)\sin(n\omega_0 t) = \frac{1}{2}\int_0^{T_0} \left(\cos\left[(m-n)\omega_0 t\right] - \cos\left[(m+n)\omega_0 t\right]\right)dt$$

$$= \frac{1}{2}\int_0^{T_0} \underbrace{\cancel{\cos\left[\underbrace{(m-n)}_{\substack{\text{non-zero}\\ \text{integer}}}\omega_0 t\right]}}_{\text{from (2)}}^{0} dt - \frac{1}{2}\int_0^{T_0} \underbrace{\cancel{\cos\left[\underbrace{(m+n)}_{\substack{\text{non-zero}\\ \text{integer}}}\omega_0 t\right]}}_{\text{from (2)}}^{0} dt = 0$$

$\boxed{\text{if } m = n} \Rightarrow \displaystyle\int_0^{T_0} \sin^2(m\omega_0 t)\,dt =$

$$= \frac{1}{2}\int_0^{T_0} \cancel{\cos\left[\underbrace{(m-n)}_{=0}\omega_0 t\right]}^{1} dt - \frac{1}{2}\int_0^{T_0} \cancel{\cos\left[\underbrace{(m+n)}_{\substack{\text{non-zero}\\ \text{integer}}}\omega_0 t\right]}^{0} dt$$

$$= \frac{1}{2}\int_0^{T_0} 1\,dt = \frac{T_0}{2} \quad \#$$

(5) product of cosines :

$$\int_{\langle T_0 \rangle} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0 & \text{, when integers } m \neq n \text{ or } m \neq -n \\ \frac{T_0}{2} & \text{, when } m = n\text{, non-zero integer } (m = n \neq 0) \end{cases}$$

Back into <u>First term</u> of Fourier series (a_0) :-

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Now evaluate $\int_{\langle T_0 \rangle} x(t)\,dt = ??$

$$x(t) = a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + \dots + a_n\cos(n\omega_0 t) + b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + \dots + b_n\sin(n\omega_0 t)$$

$$\int_{\langle T_0 \rangle} x(t)\,dt = \int_{\langle T_0 \rangle} a_0\,dt + \cancelto{0}{\int_{\langle T_0 \rangle} a_1\cos(\omega_0 t)\,dt} + \cancelto{0}{\int a_2\cos(2\omega_0 t)\,dt} + \dots + \cancelto{0}{\int_{\langle T_0 \rangle} a_n\cos(n\omega_0 t)} + \cancelto{0}{b_1\int \sin(\omega_0 t)\,dt} + \cancelto{0}{b_2\int \sin(2\omega_0 t)\,dt} + \dots + \cancelto{0}{\int_{\langle T_0 \rangle} b_n\sin(n\omega_0 t)}$$

$$\Rightarrow \int_0^{T_0} x(t)\,dt = \int_0^{T_0} a_0\,dt = a_0(T_0 - 0)$$

$$\Rightarrow \boxed{a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt}$$

Back into the second set of terms (a_n's):

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum b_n \sin(n\omega_0 t)$$

Evaluate $\displaystyle\int_{\langle T_0 \rangle} x(t) \cos(m\omega_0 t)\, dt = ??$

$$\Rightarrow \int_0^{T_0} x(t) \cos(m\omega_0 t)\, dt = \overset{0}{\cancel{\int_0^{T_0} a_0 \cos(m\omega_0 t)\, dt}}$$

$$+ \sum_{n=1}^{\infty} a_n \int_0^{T_0} \cos(n\omega_0 t) \cos(m\omega_0 t)\, dt \quad = \begin{cases} 0 & m \neq n \\ \frac{T_0}{2} & m = n \neq 0 \end{cases}$$

$$+ \sum b_n \overset{0}{\cancel{\int_0^{T_0} \sin(n\omega_0 t) \cos(m\omega_0 t)\, dt}} \quad = 0 \quad \forall\, m, n$$

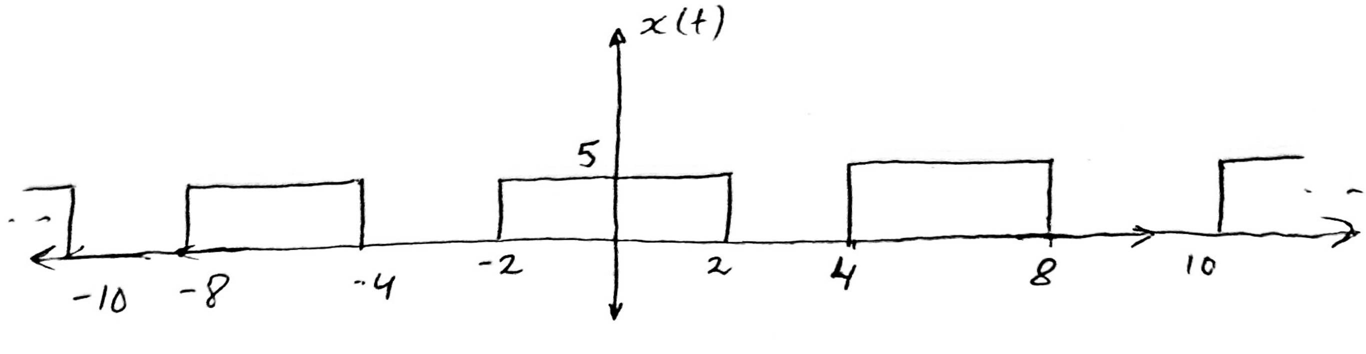
$$\int_0^{T_0} x(t) \cos(m\omega_0 t)\, dt = a_m \cdot \frac{T_0}{2}$$

$$\Rightarrow \boxed{a_m = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \cos(m\omega_0 t)\, dt}, \quad m = 1, 2, \ldots$$

In a similar way, we can show that

$$\boxed{b_m = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(m\omega_0 t)\, dt} \quad m = 1, 2, \ldots$$

Example: Find the TFS coeffecients for the following signal $x(t)$ [Periodic Pulses]



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* it is periodic pulse signal
* it is an Energy signal (check that)!
* Fundamental period $T_0 = 6$ seconds
* = radian Frequency $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{3}$ rad/sec.

$$a_0 = \frac{1}{T_0}\int_{\langle T_0\rangle} x(t)\,dt = \frac{1}{6}\int_{-2}^{4} x(t)\,dt = \frac{1}{6}\left[\int_{-2}^{2} 5\,dt + \int_{2}^{4} \cancel{(0)}\,dt\right]$$

$$= \frac{1}{6}\cdot 5t\Big|_{-2}^{2} = \frac{1}{6}\cdot 5\,[2 - -2] = \frac{20}{6} = \frac{10}{3}$$

or a_0 = DC value (Average value) $= \frac{4 \times 5}{6} = \frac{10}{3}$

$$a_n = \frac{2}{T_0}\int_{\langle T_0\rangle} x(t)\cos(n\omega_0 t)\,dt = \frac{2}{T_0}\int_{-2}^{2} 5\cdot\cos(n\omega_0 t)\,dt$$

$$= \frac{10}{T_0}\int_{-2}^{2} \frac{n\omega_0}{n\omega_0}\cos(n\omega_0 t)\,dt = \frac{10}{T_0}\cdot\frac{1}{n\omega_0}\cdot\sin(n\omega_0 t)\Big|_{-2}^{2}$$

$$= \frac{10}{n T_0 \omega_0}\left[\sin(2n\omega_0) - \sin(-2n\omega_0)\right]$$

$$= \frac{10}{2\pi n}\left[\sin(2n\omega_0) + \sin(2n\omega_0)\right] = \frac{5}{n\pi}\cdot 2\cdot\sin\left(\frac{2}{3}\pi n\right)$$

$$b_n = \frac{2}{T_0}\int_{\langle T_0\rangle} x(t)\sin(n\omega_0 t)\,dt = \frac{2}{T_0}\int_{-2}^{2} 5\sin(n\omega_0 t)\,dt$$

$$= \frac{10}{T_0}\int_{-2}^{2} \frac{-n\omega_0}{-n\omega_0}\sin(n\omega_0 t)\,dt = \frac{-10}{\underbrace{n\omega_0 T_0}_{2\pi}}\cos(n\omega_0 t)\Big|_{-2}^{2}$$

$$= \frac{-10}{2\pi n}\left[\cos(2n\omega_0) - \cos(-2n\omega_0)\right]$$

$$= -\frac{5}{n\pi}\left[\cos(2n\omega_0) - \cos(2n\omega_0)\right] = \underline{\underline{\text{zero}}}$$

$\Rightarrow$ All b_n's are zeros (Why)

$$\Rightarrow \begin{cases} a_0 = \frac{10}{3} \\ a_n = \frac{10}{n\pi}\sin\left(\frac{2}{3}\pi n\right) \Rightarrow \begin{array}{l} a_1 = \frac{10}{\pi}\sin\left(\frac{2}{3}\pi\right) = 2.757 \\ a_2 = \frac{10}{2\pi}\sin\left(\frac{4}{3}\pi\right) = -1.379 \\ a_3 = \\ \vdots \end{array} \\ b_n = 0, \ \forall n \end{cases}$$

$$\Rightarrow x(t) = \frac{10}{3} + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2}{3}\pi n\right)\cos\left(\frac{\pi}{3}nt\right)$$

$$= \frac{10}{3} + 2.757\cos\left(\frac{\pi}{3}t\right) + -1.379\cos\left(2.\frac{\pi}{3}t\right) + \ldots$$