Text book

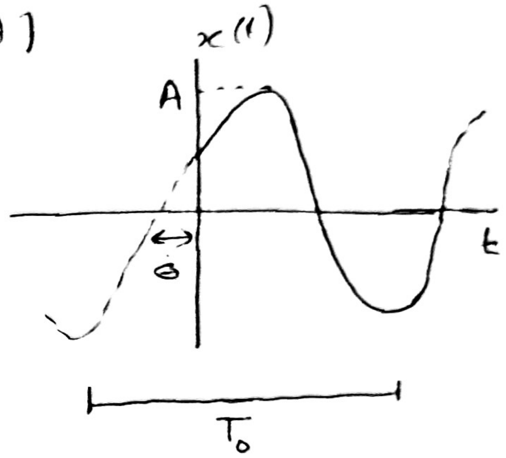
1.3 Phasor Representation of a Sinusoids

- The basic Representation of a real sinusoids is written :

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

$$= A\cos\left(\frac{2\pi}{T_0} t + \theta\right)$$

A : Amplitude
T_0 : Fundamental period
θ : phase shift (angle)



- It is often mathematically convenient to represent real signals interms of Complex quantities

- From euler identity $e^{jz} = \cos z + j\sin z$

complex exponential signal (rotating phasor)

- $x(t)$ can be represented by phasors (or Complex exp.)

$$x(t) = \text{Real}\left\{A e^{j(2\pi f_0 t + \theta)}\right\} = Re[x_p(t)]$$

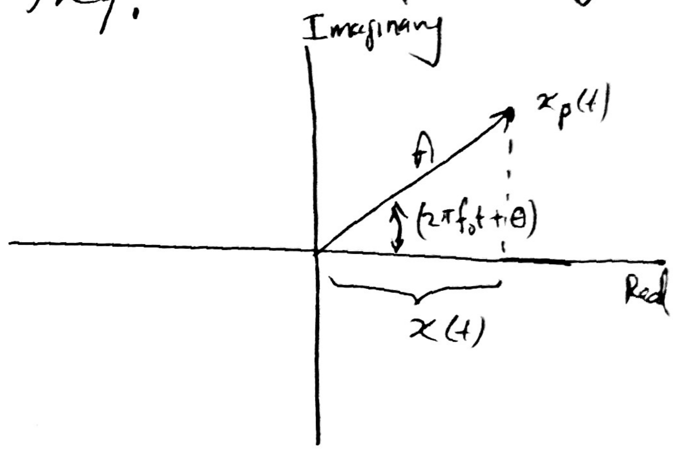
$x_p(t)$ is a complex exponential ~~pha~~ signal and it is periodic

$$x_p(t) = \underbrace{A e^{j\theta}}_{\text{Fixed part}} \underbrace{e^{j2\pi f_0 t}}_{\text{rotating part}}$$

Hence, $x_p(t) = A e^{j\theta} e^{j2\pi f_0 t} = X e^{j2\pi f_0 t}$

$X = A e^{j\theta}$ is called phasor represents and specify $x(t)$ excepts for the frequency information

- phasor representation can be used in combining signals with same freq.



- $X = A e^{j\theta} = A \angle \theta$

- example, let us add 2 currents in parallel

$$i_1 = 6.5 \cos(2\pi \cdot 60 \cdot t - \pi/6)$$
$$i_2 = 2 \cos(2\pi \cdot 60 \cdot t + \pi/40)$$

the phasor representation $I_1 = 6.5 \angle -\pi/6 = 6.5\, e^{-j\frac{\pi}{6}}$

$I_2 = 2 \angle \pi/40 = 2\, e^{j\pi/40}$

also, in rectangular format $I_1 = 5.63 - j3.25$

$I_2 = 1.99 + j0.16$

$\Rightarrow I_{total} = I_1 + I_2 = 7.62 - j3.1 = 8.23 \angle -0.38$

$\Rightarrow i(t) = 8.23 \cos(2\pi \cdot 60 \cdot t - 0.38)$ #

- An alternative phasor representation can be obtained by

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

$$= A\left[\frac{e^{j(2\pi f_0 t + \theta)} + e^{-j(2\pi f_0 t - \theta)}}{2}\right]$$

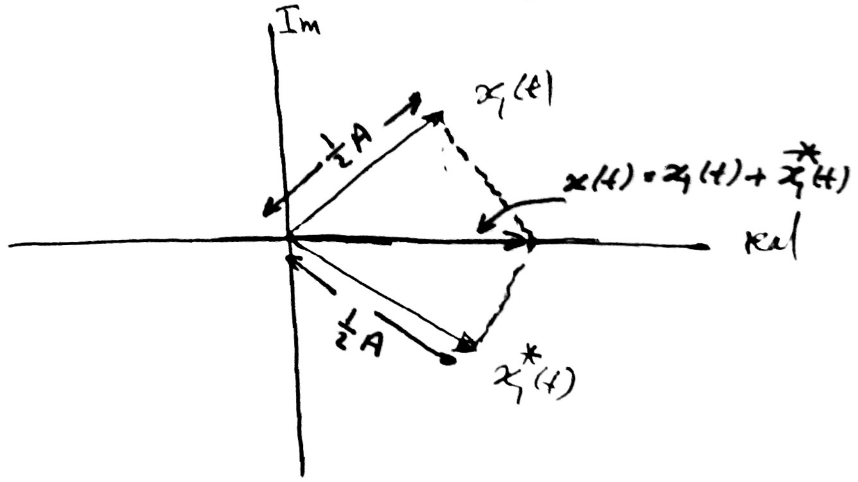
we used eular again

$$\cos z = \frac{e^{jz} + e^{-jz}}{2}$$

$$\Rightarrow \quad x(t) = \frac{A e^{j\theta}}{2} e^{j2\pi f_0 t} + \frac{A e^{-j\theta}}{2} e^{-j2\pi f_0 t}$$

$$= \frac{X}{2} e^{j2\pi f_0 t} + \frac{X^*}{2} e^{-j2\pi f_0 t} = x_1(t) + x_1^*(t)$$

at $+f$ at $-f$



Summary: two ways to express $x(t)$: interms of phasor and complex exponential signal. (will be useful in freq. domain)

Spectra

- Alternative representation for $x(t)$ which is provided in the freq. domain, whereby, the amplitude and the phase of the signal is ploted with respect to the freq.

- $x(t) = A\cos(2\pi f_0 t + \theta) = Re\{A e^{j(2\pi f_0 t + \theta)}\}$

  $x(t)$ is completely specified by A and θ for a given f_0

  also $x_p(t) = A e^{j(2\pi f_0 t + \theta)}$ is completely specified by A and θ for a given f_0.

  However, dealing with complex quantities ~~some~~ most of the time is preferable.

- Frequency domain Representation can take the form of two plots :-

  ① A as a function of f_0.

  ② θ as a function of f_0.

- Based on the two ways discussed earlier to express $x(t) = A\cos(2\pi f_0 t + \theta)$ in terms of complex exponential and phasor, ~~we show~~ such that

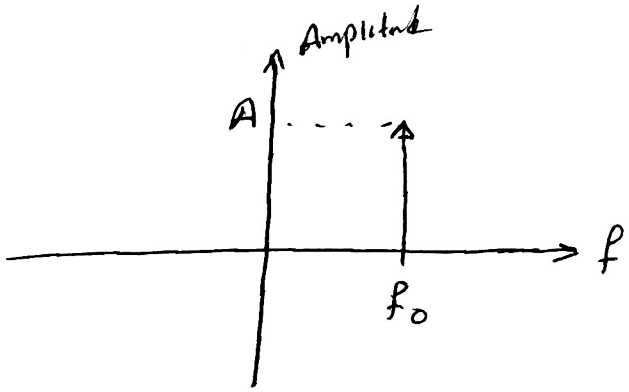
$$x(t) = \mathrm{Re}\{A e^{j\theta} e^{j2\pi f_0 t}\}$$

or

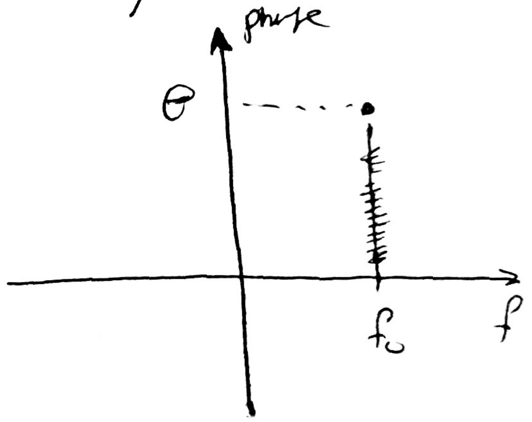
$$x(t) = \frac{A}{2} e^{j\theta} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j\theta} e^{-j2\pi f_0 t}$$

we classify the frequency representation into two forms

(A) "Single-sided" spectra
which considers only +ve freq.

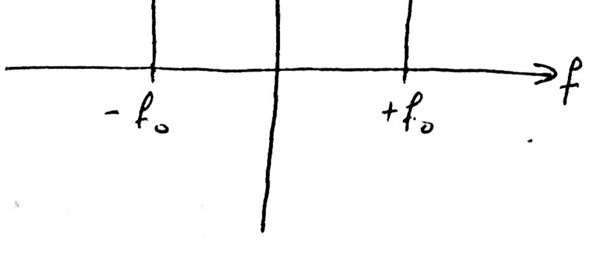


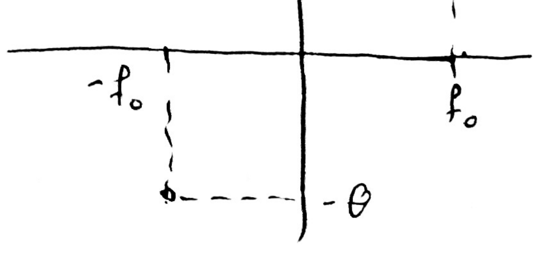
single-sided Amplitude spectra



single-sided phase spectra

(b) "Double-sided" spectra.
which considers both $+f_0$ and $-f_0$





Example: Given the signal

$$x(t) = 6\cos\left(20\pi t - \frac{\pi}{3}\right) + 4\sin^2\left(30\pi t - \frac{\pi}{6}\right)$$

① sketch its single-sided amplitude and phase spectra
② " " Double-sided " " " "

Sol.

$$x(t) = 6\cos\left(20\pi t - \frac{\pi}{3}\right) + 4\left[\frac{1}{2} - \frac{1}{2}\cos 2\left(30\pi t - \frac{\pi}{6}\right)\right]$$

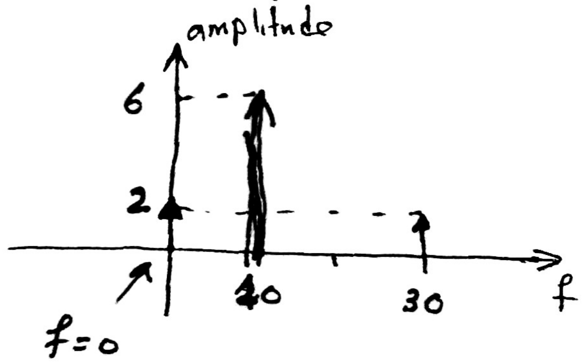
$$= 6\cos\left(20\pi t - \frac{\pi}{3}\right) + 2 - 2\cos\left(60\pi t - \frac{\pi}{3}\right)$$

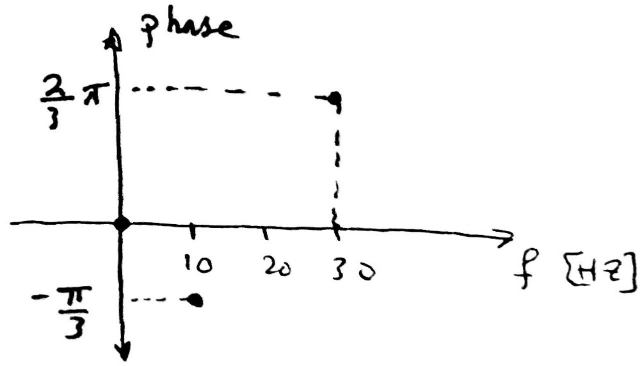
$$= Re\left\{6e^{-j\frac{\pi}{3}} e^{j20\pi t}\right\} + 2 + Re\left\{-2e^{-j\frac{\pi}{3}} e^{j60\pi t}\right\}$$

$$= Re\left\{6e^{-j\frac{\pi}{3}} e^{j20\pi t}\right\} + 2 + Re\left\{2e^{-j\frac{\pi}{3}+\pi} e^{j60\pi t}\right\}$$

note that $-1 = e^{j\pi} = \cos\pi + j\sin\pi$

$$= Re\left\{6e^{-j\frac{\pi}{3}} e^{j20\pi t}\right\} + 2 + Re\left\{2e^{j\frac{2}{3}\pi} e^{j60\pi t}\right\}$$





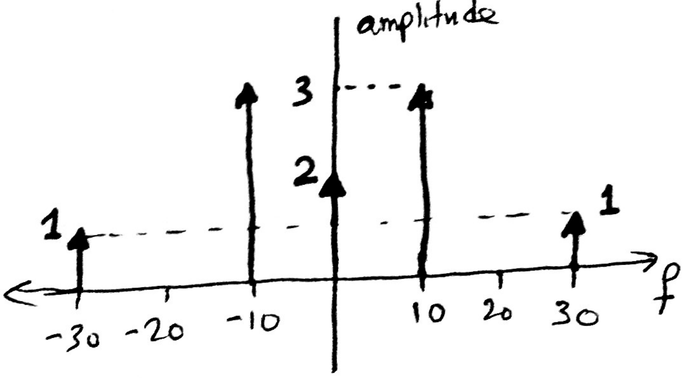
at	Amplitud	phase
$f = 0$	2	0.
$f = 10$	6	$-\frac{\pi}{3}$
$f = 30$	2	$\frac{2}{3}\pi$

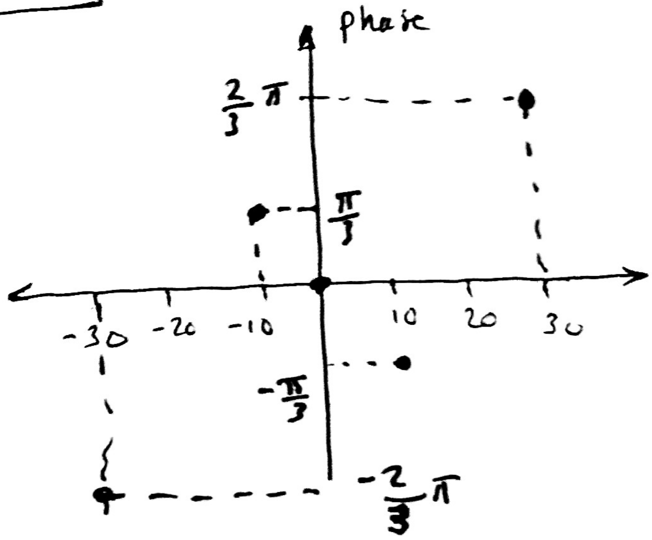
② to sketch the double sided :

$$x(t) = 6\left[\frac{e^{-j\frac{\pi}{3}} e^{j20\pi t} + e^{j\frac{\pi}{3}} e^{-j20\pi t}}{2}\right] + 2$$

$$\boxed{-1 = e^{j\pi}} \qquad + 2e^{j\pi}\left[\frac{e^{-j\frac{\pi}{3}} e^{j60\pi t} + e^{j\frac{\pi}{3}} e^{-j60\pi t}}{2}\right]$$

$f = 0$	2	0
$f = 10$	$6/2$	$-\pi/3$
$f = -10$	$6/2$	$\pi/3$
$f = 30$	$2/2$	$\frac{2}{3}\pi$
$f = -30$	$2/2$	$-\frac{2}{3}\pi$





Energy and Power Spectral Densities :

- it is useful for some applications to define Functions of Frequency that when integrated over all frequencies give

  total Energy if it is energy signal
  Average power " " " power signal

- For Energy signal, a fun. of freq. when integrated that gives total energy is referred to as an " Energy spectral Density (ESD)"

denoting the ESD of $x(t)$ by $G(f)$

$$\Rightarrow \quad E = \int_{-\infty}^{\infty} G(f)\,df$$

note: How to obtain ESD will be discussed in CH.4

- For power signal, denoting the "Power spectral density (PSD)" of a power signal $x(t)$ by $S(f)$

$$\Rightarrow \quad P = \int_{-\infty}^{\infty} S(f)\,df$$

average power

Generalizing :-

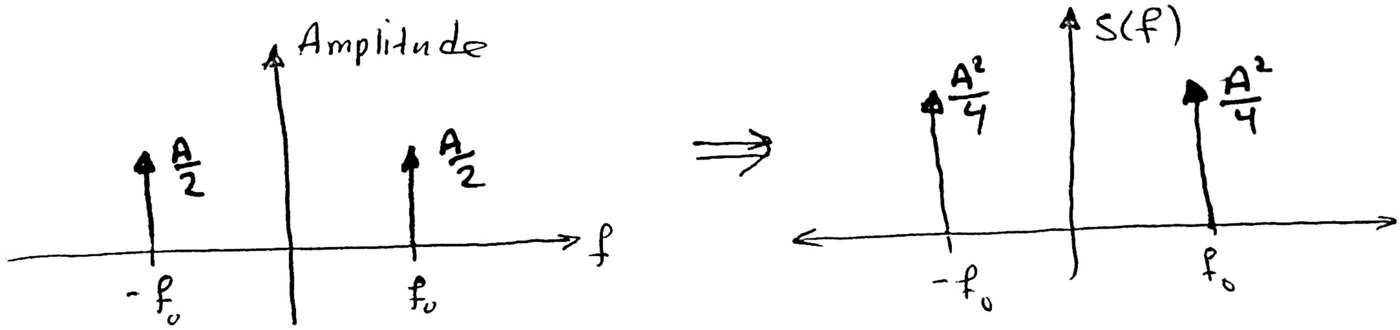
to obtain the PSD : for any signal possessing a two sided line (amplitude) spectrum, we obtain the corresponding PSD by :

1. taking each line of the amplitude spectrum
2. squaring the line value
3. Multiply the squared value by unit impulse located at that particular frequency.

Example : The amplitude (Double-sided spectrum) for periodic signal

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

$$= \frac{A}{2} e^{j\theta} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j\theta} e^{-j2\pi f_0 t}$$



$$\boxed{S(f) = \frac{A^2}{4}\delta(f+f_0) + \frac{A^2}{2}\delta(f-f_0)}$$

One can show that

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt = \frac{A^2}{2}$$

Also $\quad P = \int_{-\infty}^{\infty} S(f)\, df = \int \frac{A^2}{4}\delta(f+f_0) + \int \frac{A^2}{4}\delta(f+f_0)\, df$

$$= \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

Note that :-

① PSD of any signal is an <u>even function</u> of frequency

② PSD possesses no phase information about the signal.

Excercise : ① Find and plot the PSD for

$$x(t) = 6\cos\left(20\pi t - \frac{\pi}{3}\right) + 4\sin^2\left(30\pi t - \frac{\pi}{6}\right)$$

② Find the average power.