CH.4 The Fourier Transforms & its Applications

Introduction

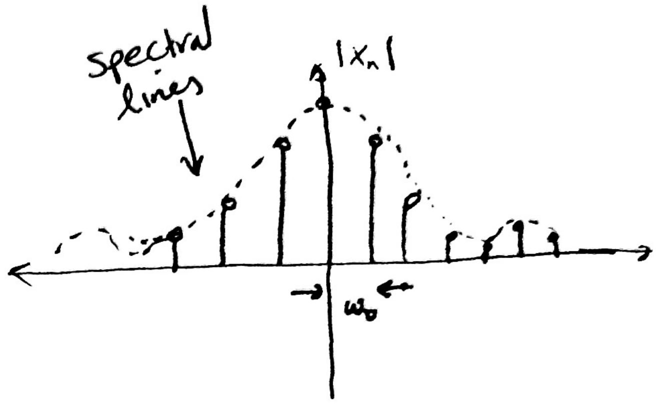
- The basic idea of FT is to expand the concept of FS to general non periodic signals.
- it is also middle step between FS & the more general Laplace Transform.
- The FT is useful to identify which freq. components are present in a certain signal.
- FT is useful in cct. Analysis
- The most Fundamental idea behained FT is to start with FS representation, then let period, T_o, goes to infinity.
- Recall, for a periodic signal with period T_o

$$\begin{cases} x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_o t} \\ X_o = \frac{1}{T_o}\int_{\langle T_o \rangle} x(t)\, dt \\ X_n = \frac{1}{T_o}\int_{\langle T_o \rangle} x(t) e^{-jn\omega_o t}\, dt \qquad n = \pm 1, \pm 2, \ldots \end{cases}$$

* Let us Relate Fourier Transform to Fourier series

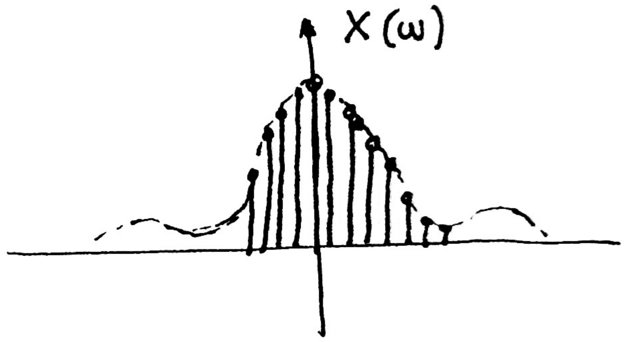
- Complex FS

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$



we note the the corresponding spectrum of ~~series~~ periodic signal is Discrete

- As $T_0 \longrightarrow \infty$, $\omega_0 \longrightarrow 0$
the spectrum becomes continuous
while the signal $x(t)$ becomes non periodic (aperiodic)



$\Rightarrow$ Aperiodic signal Representation by Fourier transform

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

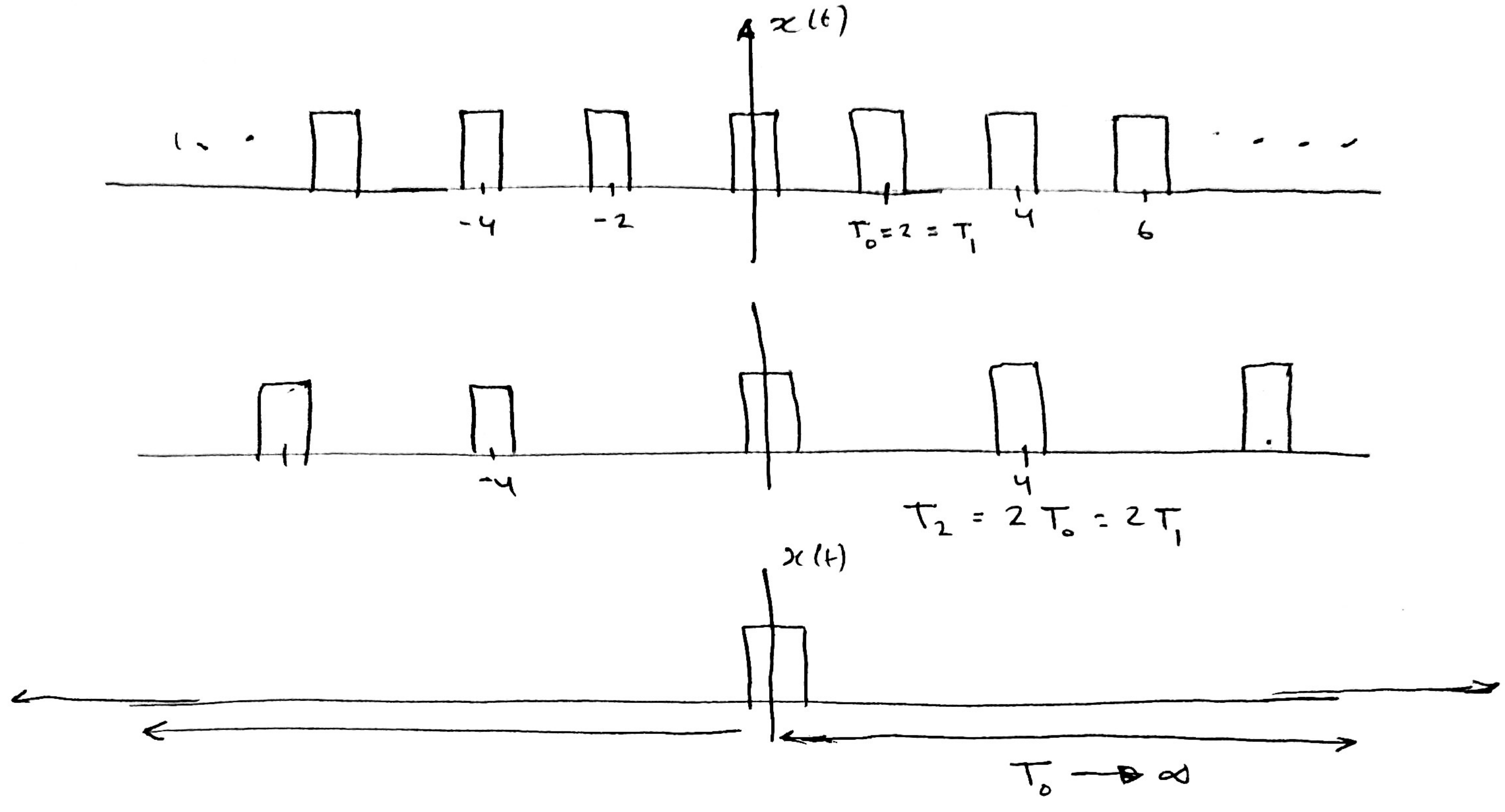
$$= \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

while the Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

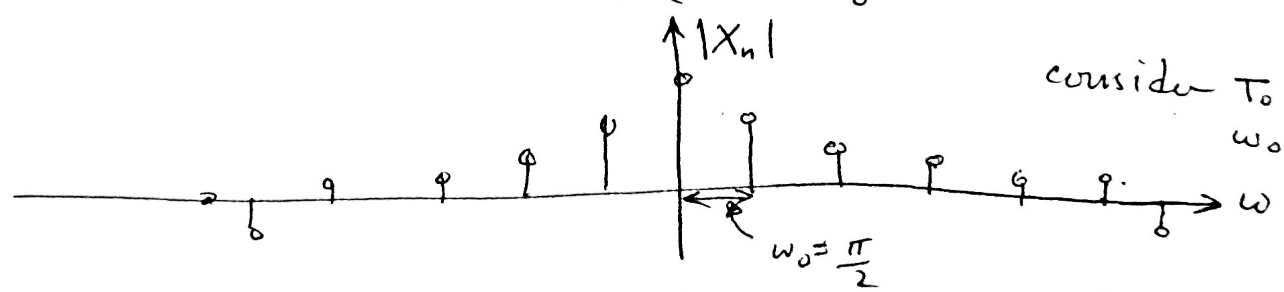
• consider just a sequence of pulse.



so, if we start with a periodic signal $x(t)$ with period T_0, then we let T_0 increase and go to infinity, we end up with only one pulse, [it repeats after ∞]

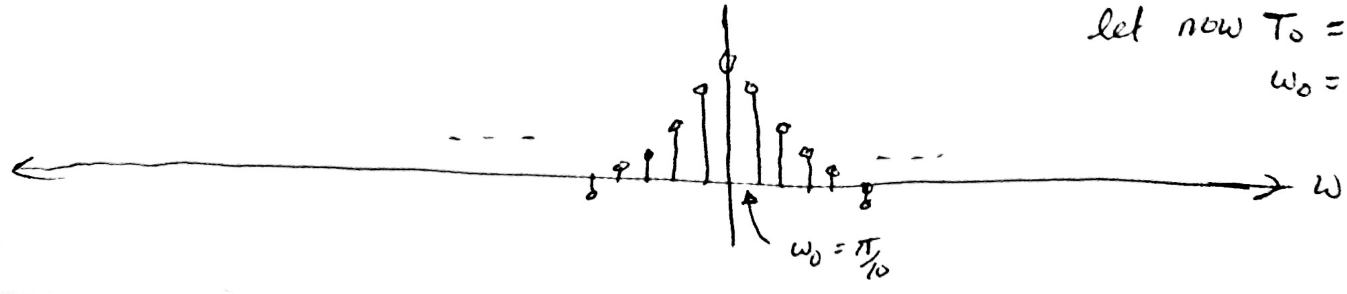
⟶ Hence, we just obtain a non-periodic signal.

• As (T_0) increases, the X_n's become closer to each others since the distance is ω_0



consider $T_0 = 4$
$\omega_0 = \frac{\pi}{2}$

let now $T_0 = 20$
$\omega_0 = \frac{\pi}{10}$



Moving From FS to FT:-

- Let $\omega_0 = \Delta\omega = (n+1)\,\omega_0 - n\omega_0$ Spacing between two spectrum lines

- For aperiodic signal, Let $T_0 \to 0$
  $\Rightarrow \omega_0 \to 0$
  $\Rightarrow \Delta\omega \to d\omega$ ($\Delta\omega$ becomes a freq. differential).

that is to say:-

$$\lim_{T_0 \to \infty} \omega_0 = \lim_{T_0 \to \infty} \frac{2\pi}{T_0} = \lim_{T_0 \to \infty} \Delta\omega = d\omega$$

$$\Rightarrow \boxed{\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega}$$

$$X_n = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t)\, e^{-jn\omega_0 t}\, dt$$

as $T_0 \to \infty \Rightarrow X_{n\infty} = \lim_{T_0 \to \infty} \frac{2\pi}{2\pi} \cdot \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{T_0/2} x(t)\, e^{-jn\frac{2\pi}{T_0}t}\, dt$ — (*)

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} x(t)\, e^{-jnd\omega t}\, dt \right] d\omega$$

$$X_{n\infty} = \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right] d\omega \Rightarrow \boxed{X_{n\infty} = \frac{1}{2\pi} X(\omega)\, d\omega}$$

- Note that as $\omega_0 \to d\omega$, $n\omega_0 \to \omega$ (becomes variable)

- What is in the brackets is what we denote as Fourier Transform of $x(t)$

$$\boxed{\mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = X(\omega)}$$

From ⊛ one can show that

$$X_n = \frac{1}{2\pi} X(n\omega_o)\,\omega_o \quad , \quad \omega_o = \frac{2\pi}{T_o}$$

$$\rightarrow X_n = \frac{1}{T_o} X(k\omega_o)$$

hence, $$\boxed{X_n = \frac{1}{T_o} X(\omega)\Big|_{\omega = k\omega_o}}$$

this means that, one can obtain the FS representation by ~~Finding~~ considering the periodic signal to be temporarly non periodic, then Finding its FT $X(\omega)$, after that multiply $X(\omega)$ by $\frac{1}{T_o}$ & sample $\omega = k\omega_o$.

(we can get the FS coeffecients from FT by sampling the envelop of the FT at $n\omega_o$)

* To obtain the inverse FT

* Recall that FS representation $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$

$\Rightarrow$ as $T \to \infty$ (aperiodic)

$$x(t) = \sum_{-\infty}^{\infty} X_{n\infty} e^{jn\omega_0 t} \quad , \quad X_{n\infty} = \frac{1}{2\pi} X(\omega) d\omega$$

$$= \frac{1}{2\pi} \sum X(\omega) e^{jn\omega_0 t} d\omega$$

- since $\lim_{T_0 \to \infty} n\omega_0 = \omega$

- the summation can now be replaced with an integral

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega} d\omega = \mathcal{F}^{-1}\{X(\omega)\}}$$

- Hence, we define the FT pair

$$\boxed{x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} X(\omega)}$$

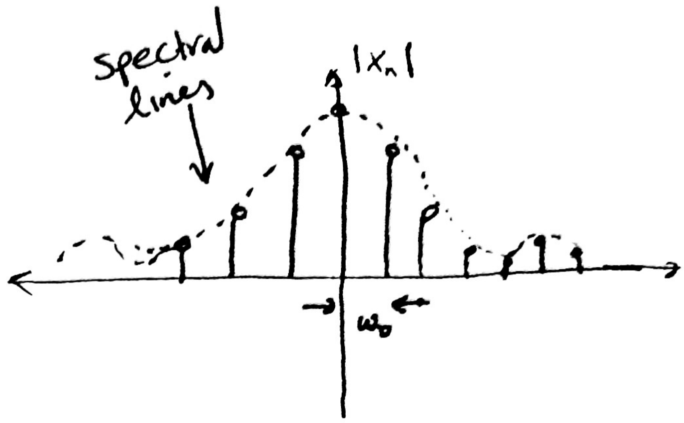
$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad , \quad \text{FT of } x(t)$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad , \quad \text{Inverse FT of } X(\omega)$$

- One can also use f as frequency variable

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \qquad x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

* Let us Relate Fourier Transform to Fourier series



- Complex FS

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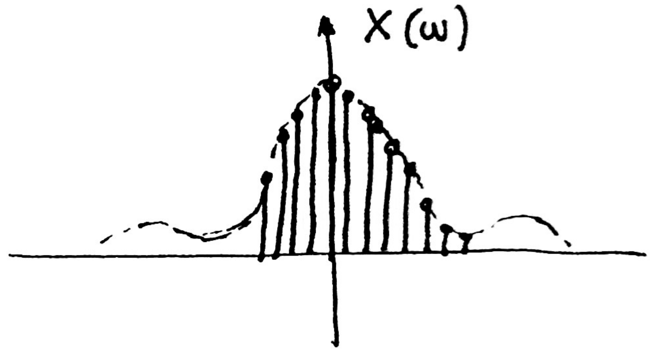

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⇒ Aperiodic signal Representation by Fourier transform

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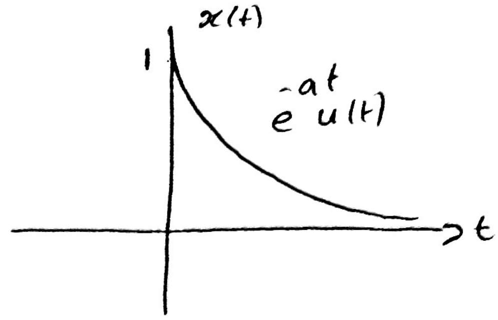
while the Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Example : Find the Fourier transform of $x(t) = e^{-at} u(t)$



$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

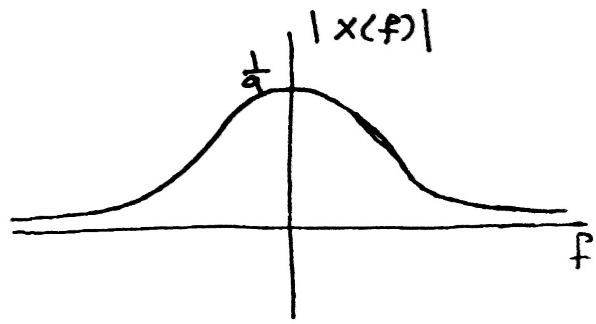
$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{-1}{a+j\omega}\, e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a+j\omega} \,, \quad a > 0$$

$$= \frac{1}{a+j2\pi f} \,, \quad a > 0$$

Amplitud Response

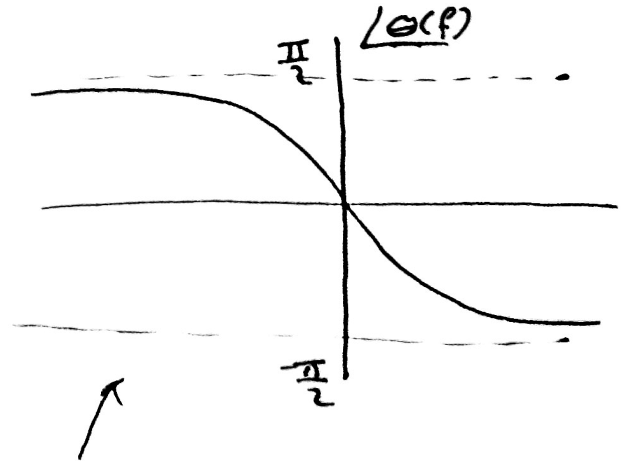


<u>in polar form</u>

$$X(\omega) = \frac{1}{\sqrt{a^2+\omega^2}}\, e^{-j\tan^{-1}\left(\frac{\omega}{a}\right)}$$

$$|X(f)| = \frac{1}{\sqrt{a^2+(2\pi f)^2}}$$

$$\theta(f) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$



phase Response.

* referring to the previous example, one notice that :

$$X(f) = |X(f)|\, e^{j\theta(f)}$$

when $x(t)$ is real valued signal, then :

$$|X(f)| = |X(-f)| \quad \text{even symmetry}$$

$$\theta(f) = -\theta(-f) \quad \text{odd symmetry}$$

$|X(f)|$ generates Amplitude spectrum.

$\theta(f)$ = phase spectrum