Fourier Transforms in the Limit

- The conditions of existance of $X(f)$ are similar to those of the FS representation
- The sufficient conditions are called Dirichelet conditions :-

① $x(t)$ is absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \quad . \text{ (finite)}$$

② $x(t)$ has a finite no. of discontinuities on any finite interval

③ $x(t)$ has a finite no. of maxima & minima on any finite interval

- Some useful signals may not ~~poss~~ satisfy these conditions (not absolutely integrable) but have finite Energy
$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

For such signals, we define what is called the Fourier Transform in the limit.

Examples :-

$x(t) = \cos w_0 t \qquad -\infty < t < \infty$

$x(t) = \delta(t)$

$x(t) = A$, A const $\rightarrow$ (FT Does not exist)

$$X(f) = \int_{-\infty}^{\infty} A\, e^{-j 2\pi f t} dt = \infty \; !!$$

⇒ Derichelet conditions are suffecient and not necessary conditions !

Examples :-

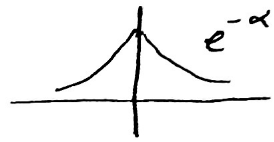
① To define the FT of $x(t) = \cos w_o t$, $-\infty < t < \infty$,

* we may first obtain the FT of

$$y(t) = e^{-\alpha|t|} \cos(w_o t) \quad , \quad \alpha > 0$$

which is abs. integrable

$e^{-\alpha|t|}$



* The FT of $x(t)$ then is defined as the limit of the FT of $y(t)$ as $\alpha \to 0$

since $\lim_{\alpha \to 0} e^{-\alpha|t|} = 1$, $-\infty < t < \infty$

$$\Rightarrow X(f) = \lim_{\alpha \to 0} Y(f)$$

② Similarly , the FT of $\delta(t)$ can be obtained by finding FT of $\delta_\epsilon(t) = \frac{1}{2\epsilon} \Pi\left(\frac{t}{2\epsilon}\right)$ then letting $\epsilon \to 0$

$$\mathcal{F}[\delta_\epsilon(t)] = \text{sinc}(2\epsilon f)$$

and as $\epsilon \to 0$

$\text{sinc}(2\epsilon f) \longrightarrow 1$ which is the FT of $\delta(t)$!

$$\boxed{\delta(t) \xleftrightarrow{\mathcal{F}} 1}$$ Fourier pair

Remarks :-

- Any useful signal $f(t)$ that meets the condition
$$E = \int |f(t)|^2 dt < \infty \quad \text{(Energy signal)}$$
is absolutely integrable.

- Energy signals generally include non-periodic signals that have finite time duration, (such as Rectangular function) and signals that asymptotically approach zero ($\lim_{t \to \infty} f(t) = 0$)

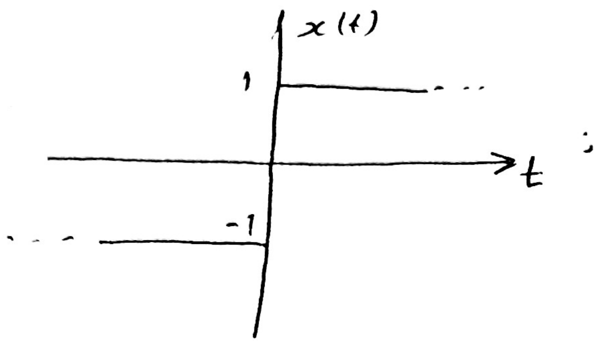
- Example $f(t) = e^{-t}$ Doesn't meet Dirichlet condition of absolutely integrable, yet $y(t) = e^{-t} u(t)$ does ~~meet~~ meet the Dirichlet condition & does have FT.

- Some other signals which are power signals such as (unit step, signum, and all periodic functions) do have valid FT (in the limit). (they meet DC's except abs. integrable).

- we will see that the FT that we derive for power signals contain impulse functions in the Frequency domain, and this is a general characteristic of power signals!

Example FT in the limit

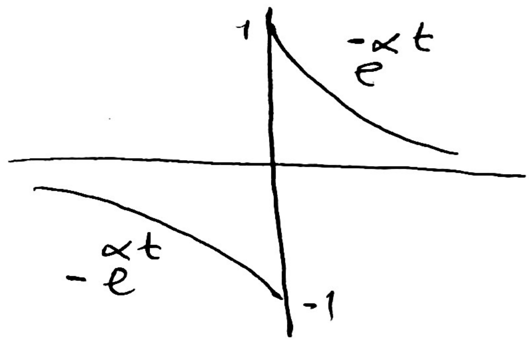
Determine FT of sgn(t).



(Note that, this signal is a power signal $P=1$, $E=\infty$)

- Direct evaluation of $\mathcal{F}\{x(t)\}$ is not possible which will give ∞.

- hence, one way is find the FT in the limit $x(t)$ can be seen as an approximation of $y(t)$ as $\alpha \to 0$



$y(t)$ is absolutely integrable hence $Y(f)$ can be evaluated

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} -e^{\alpha t} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} dt$$

$$= \frac{-e^{(\alpha-j\omega)t}}{(\alpha-j\omega)}\Bigg|_{-\infty}^{0} + \frac{-e^{-(\alpha+j\omega)t}}{(\alpha+j\omega)}\Bigg|_{0}^{\infty}$$

$$= \frac{-1}{\alpha-j\omega}[1-0] + \frac{-1}{\alpha+j\omega}[0-1]$$

$$= \frac{-1}{\alpha - jw} + \frac{1}{\alpha + jw} = \frac{-\alpha - jw + \alpha - jw}{\alpha^2 + w^2}$$

$$= -\frac{2jw}{\alpha^2 + w^2}$$

Now $X(w) = \lim_{\alpha \to 0} Y(w) = \boxed{\frac{2}{jw}}$ very specil function.

or $w = 2\pi f$

$$\Rightarrow X(f) = \frac{2}{j2\pi f} = (j\pi f)^{-1}$$

hence $\boxed{sgn(t) \longleftrightarrow \frac{1}{j\pi f}}$ Fourier pair

- ~~third~~ Another way of evaluating $X(f)$ of $x(t) = sgn(t)$ by using Fourier transform characteristics which will be studied later!

<u>Example</u> Evaluate $X(f)$ for $z(t) = u(t)$

<u>Sol.</u> Recall that $u(t) = \frac{1}{2}[sgn(t) + 1]$

~~then~~. Also $sgn(t) \overset{F}{\longleftrightarrow} \frac{1}{j\pi f}$

then $Z(f) = F\left\{\frac{1}{2}(sgn(t) + 1)\right\}$

$$= \frac{1}{2}\left(\frac{1}{j\pi f} + F\{1\}\right)$$

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* What about $\mathcal{F}\{1\}$?

to evaluate $\mathcal{F}\{1\}$, we will use the

Duality Theorem:-

$$x(t) \longrightarrow X(f)$$

then $X(t) \longrightarrow x(-f)$

* let us start with $x(t) = \delta(t)$

$$X(f) = \mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega(0)} = 1$$

hence $\boxed{\delta(t) \xleftrightarrow{\mathcal{F}} 1}$ Fourier Pair

* Now, using Duality Theorem:-

$x(t) = \delta(t) \longrightarrow X(f) = 1$

$X(t) = 1 \longrightarrow x(-f) = \delta(-f)$

$= \delta(f)$

$\Rightarrow \boxed{1 \xleftrightarrow{\mathcal{F}} \delta(f)}$ Fourier Pair

* Finally,

$$\mathcal{F}\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\mathcal{F}\{1\}$$

$$\boxed{u(t) \longleftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)}$$ Fourier pair

Conclusions (Relations between FS & FT)

[1] if we have a non-periodic signal and we wish to find its FS representation (spectrum) we assume it to be periodic with period T_0 and then let $T_0 \to \infty$, then we converge to FT.

[2] if we have periodic signal, we can find it FS representation by using the concept of FT. (How ?)

we find the FT for a single period and then samples the FT as follows

$$X_n = f_0 X(k\cancel{f_0}) = f_0 X(f)\Big|_{f = kf_0}$$

[3] if we have periodic signal, we can find its FT which is defined as impulse train with amplitude (Area) proportional to its FS coeffecient (This provides mechanisim to relate FS & FT with each other)

$$\underset{\text{FT}}{X(f)} = \sum_{n=-\infty}^{\infty} \underset{\text{FS coeffecent}}{X_n} \delta(f - nf_0)$$

$$= f_0 \sum_n X(nf_0)\, \delta(f - nf_0)$$

FT of Periodic Signals

- Periodic signals is another special class
~~which~~
- Periodic signals are not abs. integrable and are power signals.

- Consider $x(t)$ to be periodic signals, hence,

$$x(t) = \sum_{-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

- Let us determine the FT of $x(t)$

$$X(f) = \mathcal{F}\left\{\sum X_n e^{jn\omega_0 t}\right\}$$

$$= \sum_{n=-\infty}^{\infty} X_n \mathcal{F}\left\{e^{jn\omega_0 t}\right\} \quad (*)$$

- $\mathcal{F}\left\{e^{jn\omega_0 t}\right\} = ??$

- let us determine $\mathcal{F}^{-1}\left\{Z(f) = \delta(f-f_0)\right\}$

$$z(t) = \int_{-\infty}^{\infty} \delta(f-f_0)\, e^{j2\pi f t}\, df$$

$$= e^{j2\pi f_0 t}$$

$$\Rightarrow \boxed{e^{j2\pi f_0 t} \longleftrightarrow \delta(f-f_0)}$$ FT pair

Back into (*)

$$e^{j2\pi f_0 n t} \longrightarrow \delta(f - n f_0)$$

$$\Rightarrow \boxed{X(f) = \sum_{n=-\infty}^{\infty} X_n \, \delta(f - n f_0)} \quad (**)$$

- Because, we are forcing the concept of FT on periodic signals, we get impulse fun's (Discrete spectrum) in the freq. domain
- These impulses occurs at $f = n f_0$ (same as Complex FS).
- The above concept also applies for step & signum functions.
- Remember: one can evaluate X_n

by temprorly evaluating FT for one period and then ~~eval~~ Finding the samples values

$$X_n = \frac{1}{T_0} X(\omega)\Big|_{\omega = k\omega_0}$$

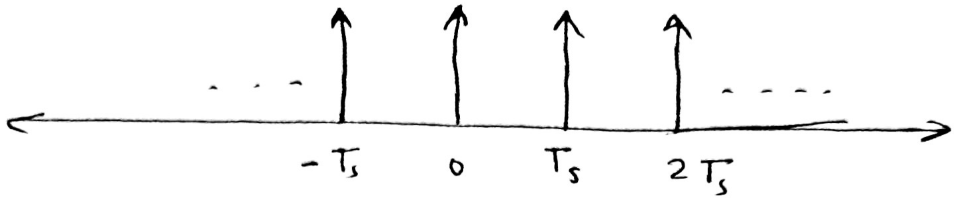
or $$X_n = f_0 \, X(f)\Big|_{f = k f_0}$$

In consequence ** becomes

$$\boxed{X(f) = f_0 \sum_{n=-\infty}^{\infty} X(nf_0)\,\delta(f - nf_0)}$$

Example consider the following periodic train of impulses

$$y(t) = \sum_{m=-\infty}^{\infty} \delta(t - m T_s)$$



① Evaluate the Complex FS coeffecients ?

$$y(t) = \sum_{n=-\infty}^{\infty} Y_n \, e^{j 2\pi f_s n t} \quad , \quad f_s = \frac{1}{T_s}$$

$$Y_n = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) \, e^{-j 2\pi f_s n t} \, dt$$

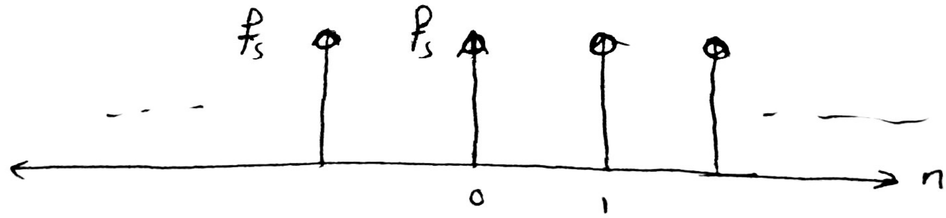
$$= \frac{1}{T_s} e^{0} = f_s .$$

hence,

$$y(t) = f_s \sum e^{j 2\pi n f_s t}$$

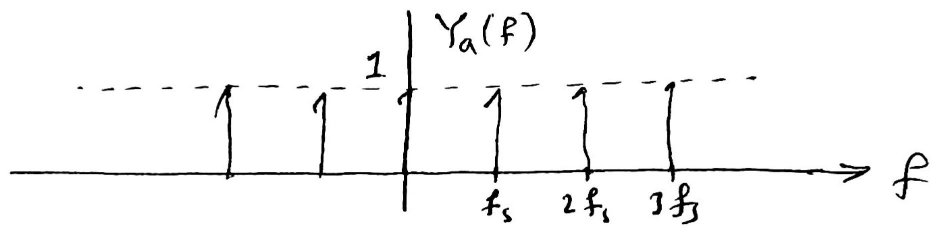
Complex FS Form

Amplitud spectrum



Another way of finding Y_n is by evaluating $Y_a(f)$ of aperiodic impulse given by $y_a(t) = \delta(t)$

$$y_a(t) = \delta(t) \longrightarrow Y_a(f) = 1$$



then samples $Y_a(f)$ at nf_s such that

$$Y_n = f_s \, Y_a(f)\Big|_{nf_s} = f_s$$

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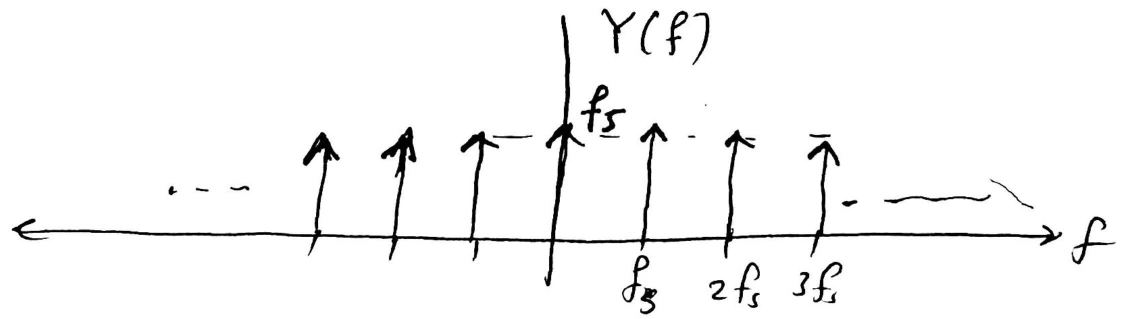
(2) Find the FT of $y(t)$

$$Y(f) = f_s \sum_{n=-\infty}^{\infty} Y_a(nf_s)\, \delta(f - nf_s)$$

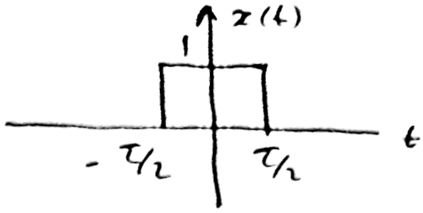
Since $Y_a(nf_s) = 1$

$$\Rightarrow Y(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$



Example

Consider $x(t) = \Pi\left(\frac{t}{\tau}\right)$



Evaluate $X(f)$?

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2}$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right] = \frac{1}{j\omega}\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right]$$

$$= \frac{1}{j\omega} \cdot j2 \sin(\omega\tau/2)$$

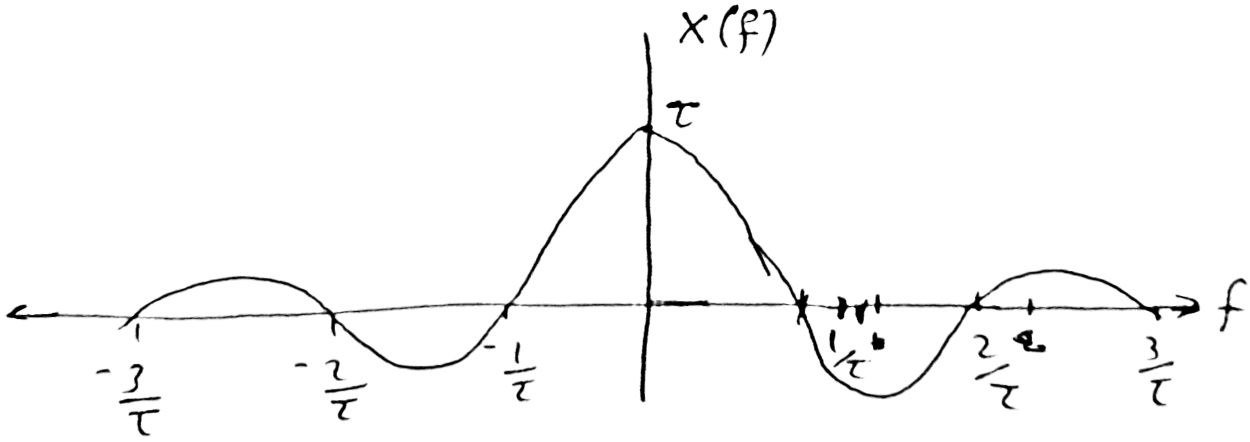
$$= \frac{2}{\omega} \sin(\omega\tau/2) = \frac{2}{2\pi f} \sin(2\pi f\,\tau/2)$$

$$= \frac{1}{\pi f} \sin(\pi f \tau)$$

$$= \frac{\tau}{\tau} \frac{\sin(\pi f \tau)}{\pi f} = \tau \frac{\sin \pi f \tau}{\pi f \tau}$$

$$= \tau \operatorname{sinc}(f\tau)$$, where $\operatorname{sinc} f = \frac{\sin \pi f}{\pi f}$

$\Rightarrow$ $\boxed{\Pi\left(\frac{t}{\tau}\right) \longleftrightarrow \tau \operatorname{sinc}(f\tau)}$ Fourier pair

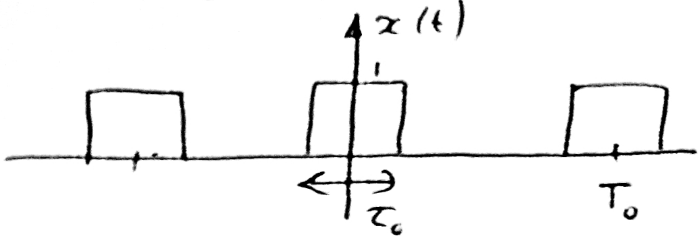


intersections : $\sin(\pi f \tau) = 0 \Rightarrow \pi f \tau = k\pi$

$\Rightarrow f = \frac{k}{\tau}$, k : integers

$\Rightarrow$ As $\tau \rightarrow 0 \Rightarrow X(f)$ spreads wider

Example: Physical Significance of FT



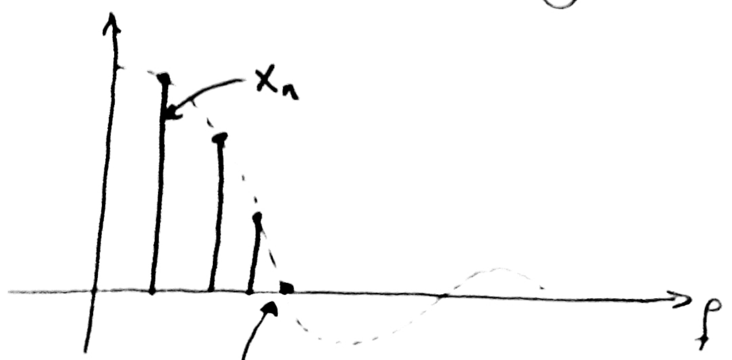
$x(t)$ is periodic signal with FS expansion:

$$x(t) = \sum X_n e^{jn\omega_0 t}, \quad X_n = \frac{1}{T_0} X_a(nf_0)$$

where $x_a = \Pi\left(\frac{t}{\tau}\right) \longrightarrow X_a(f) = \tau \,\text{sinc}(f\tau)$

$$\Rightarrow X_n = \frac{\tau}{T_0} \,\text{sinc}(nf_0\tau) = \frac{\tau}{T_0} \frac{\sin(nf_0\pi\tau)}{nf_0\pi\tau}$$

the corresponding Double-sided amplitude spectrum (right side is only considered)



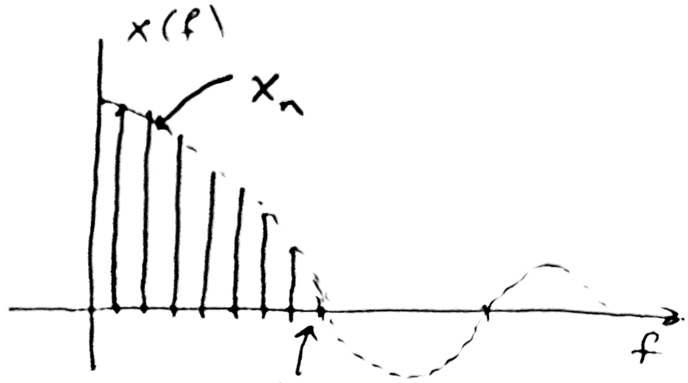
first zero

$\tau n f_0 \pi = \pi$

$\Rightarrow n = \frac{T_0}{\tau}$

let $\frac{T_0}{\tau} = 4$

$\Rightarrow n = 4$ spectrum lines



first zero

$n = \frac{T_0}{\tau}$

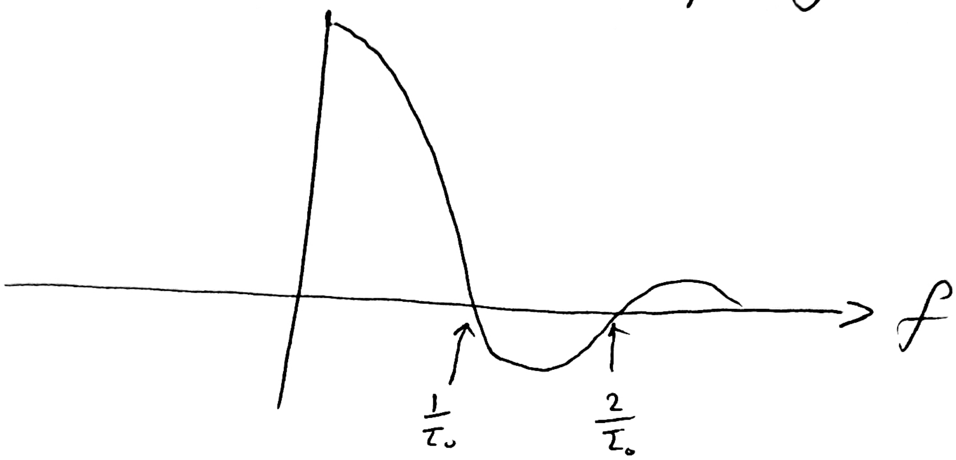
let $\frac{T_0}{\tau} = 8$

$\Rightarrow n = 8$ spectrum lines

As we keep τ constant and increase T_0, we get same shape spectrum, zero crossing at same frequencies ($f = k/\tau$) but more freq. components, hence, closer to each other.

$$\Delta f = \frac{1}{T_0}$$

and as $T_0 \longrightarrow \infty$, we obtain a smooth continuous function of frequency.



For negative freq. we will get same magnitude spectrum as for $f > 0$, but ~~the~~ phase will be odd symmetric to phase [$x(t)$ real signal].

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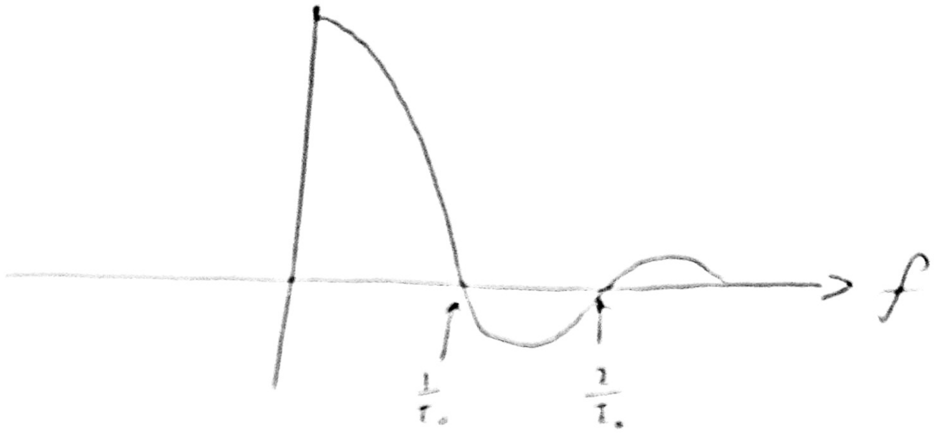

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