Properties of The Fourier Transform

- The FT has a number of attractive properties that can be used to simplify computations of FT of some signals without computing the integrals.

A Linearity

if
$$\boxed{\begin{array}{l} x(t) \longrightarrow X(f) \\ y(t) \longrightarrow Y(f) \end{array}}$$

then
$$\boxed{\alpha x(t) + \beta y(t) \longrightarrow \alpha X(f) + \beta Y(f)}$$
for any $\alpha, \beta, x(t), y(t)$

Example $x(t) = A\cos(\omega_0 t)$

Using Euler $\Rightarrow$ $x(t) = \frac{A}{2} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j2\pi f_0 t}$

$$X(f) = \frac{A}{2} \mathcal{F}\{e^{j2\pi f_0 t}\} + \frac{A}{2} \mathcal{F}\{e^{-j2\pi f_0 t}\}$$

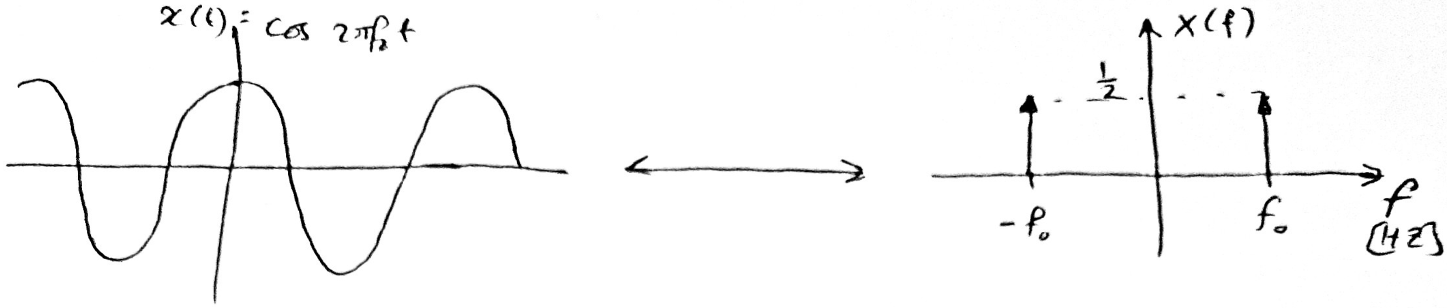
we have shown earlier

$$\mathcal{F}\{e^{j2\pi f_0 t}\} = \delta(f - f_0) \qquad \text{Freq. Shifting}$$

Hence, $X(f) = \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0)$

$$\Rightarrow \boxed{\cos 2\pi f_0 t \longleftrightarrow \frac{1}{2}\left[\delta(f - f_0) + \delta(f - f_0)\right]} \quad \text{Pair}$$

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right]$$

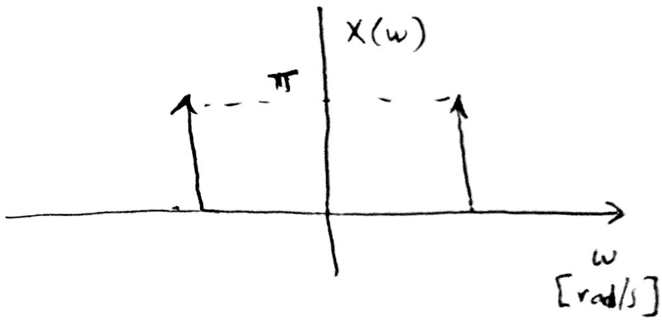


Time Domain

Frequency domain

OR $\cos(\omega_0 t) \longleftrightarrow \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$

w



Some textbooks use w instead of f
in doing so, every delta ~~will be scaled by~~ $\cancel{\frac{1}{2\pi}}$
$\delta(f)$ will be scaled by 2π
to maintain the area of delta
function to be equal to 1.

~~Hence~~ ~~Rephrase~~ 20

B Time Scaling property

$$\boxed{\begin{array}{l} x(t) \longleftrightarrow X(f) \\ x(at) \longleftrightarrow \dfrac{1}{|a|} X\left(\dfrac{f}{a}\right) \end{array}}$$

- Proof:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

change of variable

$\lambda = at$

$d\lambda = a\, dt$

$\Rightarrow dt = \frac{1}{a} d\lambda$

$$\mathcal{F}\{x(at)\} = \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi\left(\frac{f}{a}\right)\lambda}\, d\lambda$$

for $a > 0$ $\quad = \frac{1}{a} X\left(\frac{f}{a}\right)$

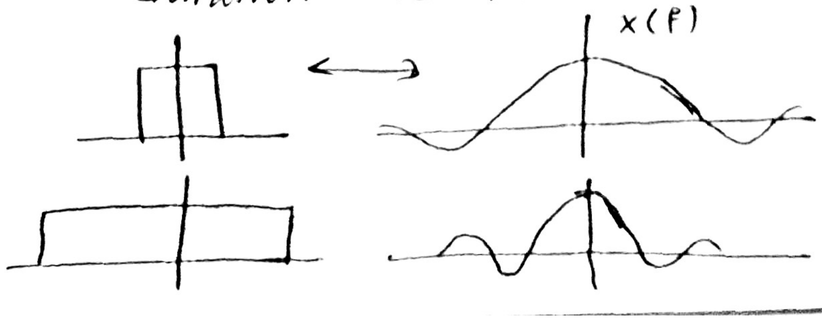
Similarly, for $a < 0$, $\quad x(at) \longleftrightarrow \frac{1}{-a} X\left(\frac{f}{a}\right)$

Hence $\quad \boxed{x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)}$

- Significance of the scaling property:-

Time compression of a signal results in its spectral expanssion and vise versa.

(There is a reciprocal relation between the duration and the bandwidth)



high data rate
more Bandwidth needed

Low data rate
less Bandwidth needed

Example show that $x(-t) \longleftrightarrow X(-f)$

from the scaling property

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$a = -1$

$$\Rightarrow \mathcal{F}\{x(-t)\} = \frac{1}{|-1|} X(-f)$$

$$\Rightarrow x(-t) \longleftrightarrow X(-f)$$

Example use the above result and the pair

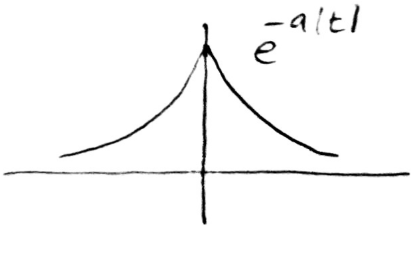
$$e^{-at} u(t) \longleftrightarrow \frac{1}{a + j2\pi f}$$

to find the FT of :-

① $e^{at} u(-t)$

using the scaling $\Rightarrow$ $e^{at} u(-t) \longleftrightarrow \dfrac{1}{a - j2\pi f}$

② $e^{-a|t|}$



solution:

$$e^{-a|t|} = e^{-at} u(t) + e^{at} u(-t)$$

$$\Rightarrow \mathcal{F}\{e^{-a|t|}\} = \mathcal{F}\{e^{-at} u(t)\} + \mathcal{F}\{e^{at} u(-t)\}$$

$$= \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2}$$

[C] Time - Shifting Property

$$x(t) \longleftrightarrow X(f)$$
$$x(t-t_0) \longleftrightarrow X(f)\, e^{-j2\pi f t_0}$$

- Proof:-

let $y(t)$ be a shifted version of $x(t)$, $y(t) = x(t-t_0)$
then

$$Y(f) = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j2\pi f t}\, dt$$

$$Y(f) = \int x(\lambda)\, e^{-j2\pi f(\lambda + t_0)}\, d\lambda$$

$$= e^{-j2\pi f t_0} \int x(\lambda)\, e^{-j2\pi f \lambda}\, d\lambda$$

$$= e^{-j2\pi f t_0}\, X(f)$$

change of variable
$t - t_0 = \lambda$
$dt = d\lambda$

- The significance of Time-shifting property:-

it shows that delaying a signal by t_0 seconds Does Not change the amplitude spectrum, while the phase spectrum is changed by a linear phase $(-2\pi f t_0)$

- Example :- ~~x(t) =~~

Example :-

$$x(t) = 10 \cos\left[2\pi * 50 * \left(t - \frac{1}{200}\right)\right]$$ delay by $t_o = \frac{1}{200}$ sec.

$$= 10 \cos\left[100\pi t - \frac{\pi}{2}\right]$$ phase shift by $\frac{\pi}{2}$

Find $X(f)$?

Using time-shifting property :-

$$X(f) = \mathcal{F}\{x(t)\} = 10\, \mathcal{F}\{\cos(100\pi t)\}\, e^{-j 2\pi f \cdot \frac{1}{200}}$$

$$\mathcal{F}\{\cos 2\pi f_o t\} = \frac{1}{2}\left[\delta(f-f_o) + \delta(f+f_o)\right]$$

$$X(f) = \frac{10}{2}\left[\delta(f-50) + \delta(f+50)\right] e^{-j\frac{\pi f}{100}}$$

$$= 5\,\delta(f-50)\, e^{-j\frac{\pi f}{100}} + 5\,\delta(f+50)\, e^{-j\frac{\pi f}{100}}$$

Note that $G(f)\,\delta(f-f_o) = G(f_o)\,\delta(f-f_o)$

$$\Rightarrow X(f) = 5\,\delta(f-50)\, e^{-j\frac{\pi}{2}} + 5\,\delta(f+50)\, e^{+j\frac{\pi}{2}}$$

$\boxed{D}$ Time Transformation (Time Scaling + Time shifting)

$$\boxed{\begin{array}{lcl} x(t) & \longleftrightarrow & X(f) \\ x(at - t_o) & \longleftrightarrow & \frac{1}{|a|} X\left(\frac{f}{a}\right) e^{-j 2\pi \frac{f}{a} t_o} \end{array}}$$

~~x(t-t₀) then~~ ~~x(at~~

$\boxed{E}$ Duality Property.

• Time - Frequency Duality

$$\begin{tikzcd} \boxed{x(t)} \arrow[rr, bend left, "\int x(t)\, e^{-j2\pi ft}\, dt"] & & \boxed{X(f)} \arrow[ll, bend left, "\int X(f)\, e^{j2\pi ft}\, df"] \end{tikzcd}$$

if $\quad x(t) \longleftrightarrow X(f)$

then $\quad X(t) \longleftrightarrow x(-f)$

Duality (Symmetry) property

This property states that if a function $x(t)$ has the Fourier Transform $X(f)$. and if you have a function of time $X(t)$ such that

$$X(t) = X(f)\Big|_{f=t}$$

$$\text{then} \quad F\{X(t)\} = x(-f) = x(t)\Big|_{t=-f}$$

note that

$$x(t) \longleftrightarrow X(\omega)$$

$$X(\omega) \longleftrightarrow 2\pi\, X(-\omega)$$

in case of using ω

Example $\quad x(t) = 10 \; \text{sinc}\,(30t)$, Find $X(f)$

Sol.

let us start with the FT of $\Pi(\frac{t}{\tau})$

$$\Pi\left(\frac{t}{\tau}\right) \longleftrightarrow \tau\,\text{sinc}(f\tau)$$

using Duality , then

$$\tau\,\text{sinc}\,(\tau t) \longleftrightarrow \Pi\left(-\frac{f}{\tau}\right)$$

since $\Pi(f) = \Pi(-f)$ even func.

then $\quad \tau\,\text{sinc}\,(\tau t) \longleftrightarrow \Pi\left(\frac{f}{\tau}\right)$

in the example

$\cancel{\tau = 30}$ $\quad x(t) = \frac{1}{3} \cdot 30 \; \text{sinc}\,(30t)$

hence $\quad \tau = 30$

$$X(f) = \frac{1}{3} \, \mathcal{F}\{30 \; \text{sinc}\,(30t)\}$$

$$= \frac{1}{3} \, \Pi\left(\frac{f}{30}\right)$$

$\boxed{F}$ Convolution

* Convolution in time results into multiplication in frequency domain.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$$

$$x(t) \longrightarrow X(f)$$
$$h(t) \longrightarrow H(f)$$

$$\boxed{x(t) * h(t) \longrightarrow X(f)H(f)}$$

- This property is very useful in analyzing LTI systems.

- proof :-

$$Y(f) = \int_t y(t)\, e^{-j2\pi f t} = \int_t \int_\lambda x(\lambda)\, h(t-\lambda)\, d\lambda \; e^{-j2\pi f t} dt$$

$$= \int_\lambda x(\lambda) \Big\{ \underbrace{\int_t h(t-\lambda)\, e^{-j2\pi f t} dt}_{\text{Change of variable}} \Big\} d\lambda$$

$$= \int_\lambda x(\lambda)\, H(f)\, e^{-j2\pi f \lambda} d\lambda = H(f) \int_\lambda x(\lambda)\, e^{-j2\pi f\lambda} d\lambda$$

$$= H(f)\, X(f)$$

- hence, Convolution in time results in just simple Mult. in freq. domain.

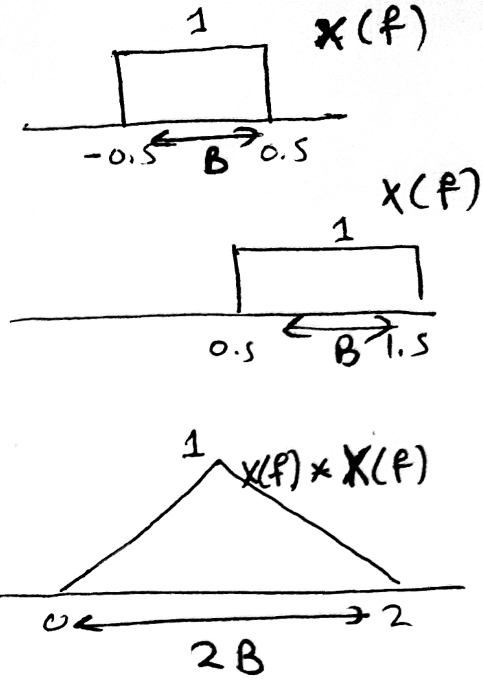
* Using the duality, we can also have frequency domain convolution

$$x(t)\, h(t) \longleftrightarrow X(f) * H(f)$$

* note that

$$x(t) \cdot x(t) \longleftrightarrow X(f) * X(f)$$

multiplication signals with same time period in time domain results into ~~Doupling~~ Doubling the Bandwidth



$\Rightarrow$ $\boxed{\text{Bandwidth of the product of two signals = Sum of bandwidth}}$

<u>Examples</u>

$\text{sinc}(t) \longrightarrow BW = B$

$\text{sinc}^2(t) \longrightarrow BW = 2B$

$\text{sinc}^3(t) \longrightarrow BW = 3B$