System Analysis with the FT

[A] Signal Transmission through LTI systems

In Ch. 2, time-domain evaluation of the output is conducted using Convolution Integral

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

In Ch. 4,

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow y(t) = \mathcal{F}^{-1}\{H(f)X(f)\}$$

$h(t)$ is the impulse response

$H(f)$ is the frequency response (Transfer function)

Either $h(t)$ or $H(f)$ are equally good characterizations of the system.

since $H(f)$, is in general, a complex quantity,

$$\Rightarrow \quad H(f) = |H(f)|\, e^{j\angle H(f)}$$

$|H(f)|$: amplitude Response fun.

$\angle H(f)$: phase " "

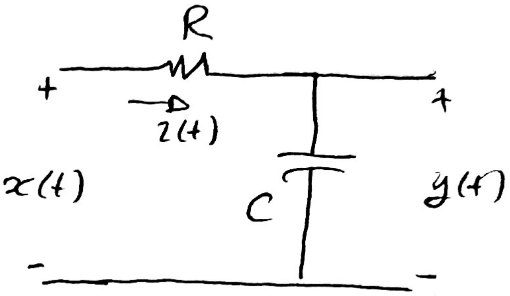
$$|Y(f)| = |X(f)|\,|H(f)|$$

$$\angle Y(f) = \angle H(f) + \angle X(f)$$

Three ways to obtain $H(f)$:

① Fourier - Transforming the differential equation relating input and output →

- apply KVL :-



$$x(t) = Ri(t) + y(t)$$

$$i(t) = C\frac{dy(t)}{dt}$$

$$\Rightarrow RC\frac{dy(t)}{dt} + y(t) = x(t)$$

- Assuming the system to be initially at rest, the diff equation becom in freq. domain:-

$$RC\{j2\pi f\, Y(f)\} + Y(f) = X(f)$$

$$(j2\pi f RC + 1)\, Y(f) = X(f)$$

$$\Rightarrow H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1 + j(2\pi RC)f} = \frac{1}{1 + j\left(\frac{f}{f_3}\right)}$$

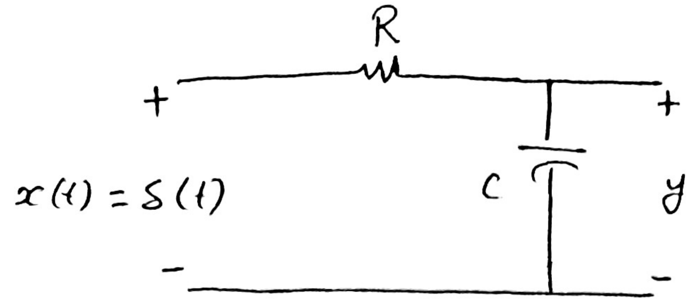
$$= \frac{1}{\sqrt{1 + \left(\frac{f}{f_3}\right)^2}}\, e^{-j\tan^{-1}\left(\frac{f}{f_3}\right)}$$

where $f_3 = \frac{1}{2\pi RC}$ is called the 3-dB or Half-power frequency.

(The Half-power freq. of a two-port system is defined as a freq. of an input sine wave which results in a steady-state sinusoidal output of amplitude $\left(\frac{1}{\sqrt{2}}\right)$ of the max. possible amplitude of the output sinusoid.
The Power of the output is then $\frac{1}{2}$ of the max. possible output power. ~~the actual power~~ $\left(\frac{P_{out}}{P_{max}}\right)_{dB} = 10\log_{10}\frac{1}{2} = -3\,dB$)

• equivalently, we can use Laplace Transform theory by replacing $j2\pi f$ by s.

② Second method, obtaining the impulse response of the system and finding its FT.



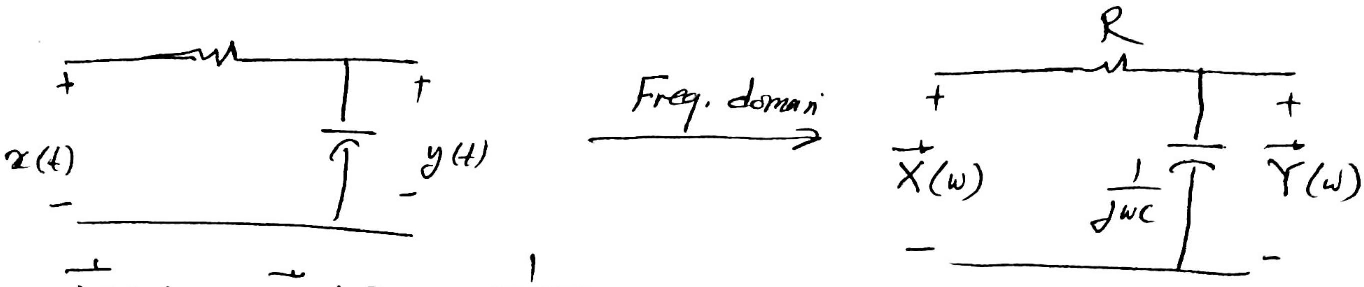
$y(t) = h(t)$
$= \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$

From FT table :-

$$e^{-\alpha t} u(t), \alpha > 0 \longleftrightarrow \frac{1}{\alpha + j2\pi f}$$

$$\Rightarrow H(f) = \frac{1/RC}{\frac{1}{RC} + j2\pi f} = \frac{1}{1 + j(2\pi RC) f} = \frac{1}{1 + j\left(\frac{f}{f_3}\right)}$$

③ A third method is to use ac sinusoidal steady-state analysis as shown in the figure, below, and find the ratio of output to input phasors $\frac{\vec{Y}}{\vec{X}}$



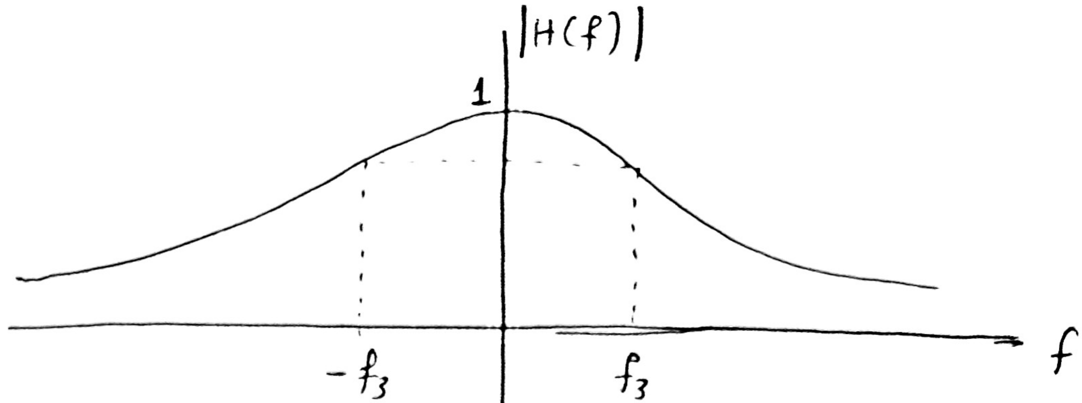
$$\vec{Y}(\omega) = \vec{X}(\omega) * \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \quad \text{(voltage Divider)}$$

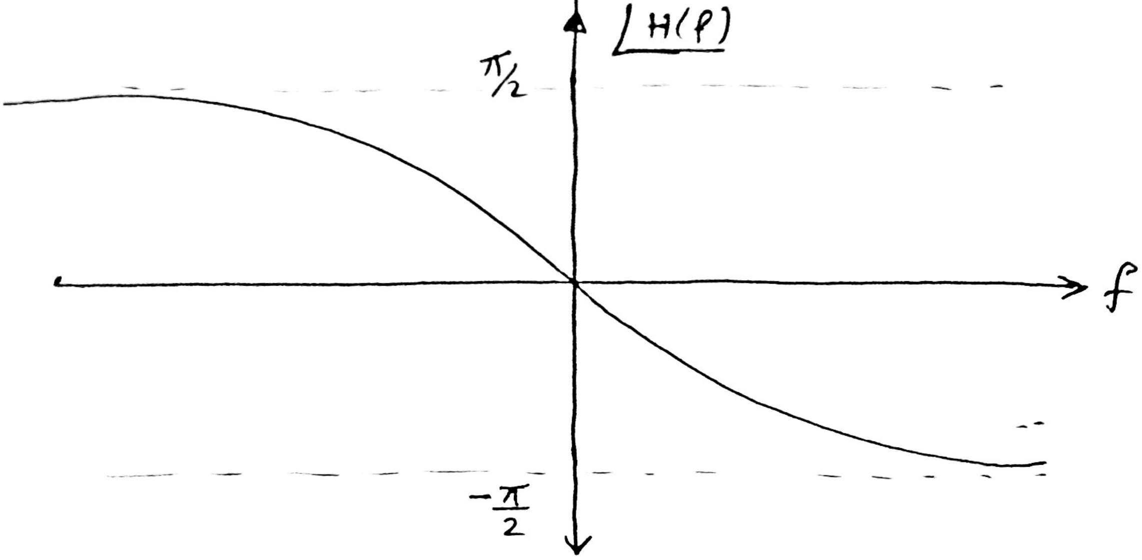
$$H(\omega) = \frac{\vec{Y}}{\vec{X}} = \frac{1/j\omega C}{R + \frac{1}{j\omega C}} \Rightarrow H(f) = H(\omega)\Big|_{\omega \to 2\pi f}$$

the amplitude and phase response for a low-pass RC filter

$$H(f) = \frac{1}{1 + j(f/f_3)} = \frac{1}{\sqrt{1 + (f/f_3)^2}}\, e^{-j \tan^{-1}\left(\frac{f}{f_3}\right)}$$

$$|H(f)| = \left[1 + \left(\frac{f}{f_3}\right)^2\right]^{-\frac{1}{2}} \quad \text{and} \quad \angle H(f) = -\tan^{-1}\left(\frac{f}{f_3}\right)$$





Distortion less Transmission

- Distortion less systems are loved by communications engineers.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = K\, x(t - t_d)$$

- The conditions for distortion less (in time-domain)
  1. fixed-amplitude scale (K)
  2. Constant delay (t_d)

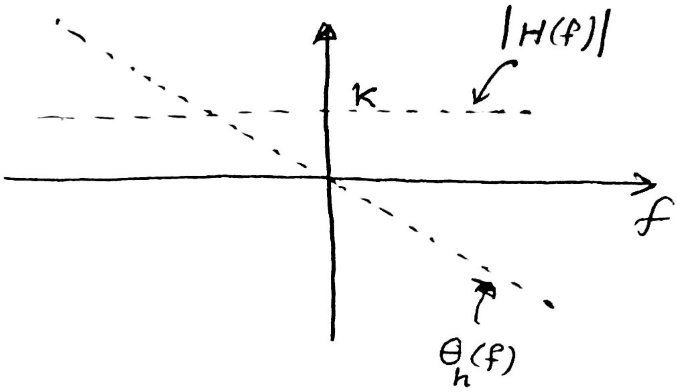
$$Y(f) = K\, X(f)\, e^{-j 2\pi f t_d}$$

$$\frac{Y(f)}{X(f)} = H(f) = K\, e^{-j 2\pi f t_d}$$

$$\Rightarrow |H(f)| = K \qquad \measuredangle H(f) = -2\pi f t_d = \theta_h(f)$$

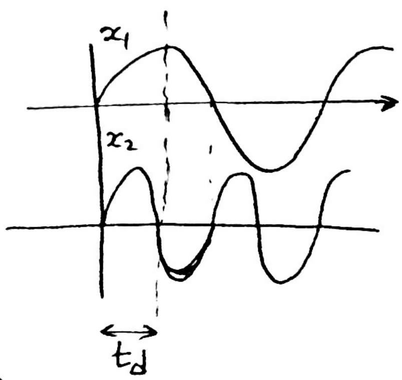
$$\Rightarrow t_d = -\frac{1}{2\pi} \frac{d\theta_h}{df}$$

- This translates in frequency-domain to
  1. Constant amplitude response $|H(f)|$
  2. Linear phase $\theta_H(f)$



- Why linear phase requirments? (physical Explanation)

let $x(t) = x_1 + x_2$



if we shift by t_d amplitude wise $\rightarrow$ no problem

phase wise $\theta_{x_1} = 90^\circ$, $\theta_{x_2} = 180 \longrightarrow$ doubled since f is double.

- There are two forms of distortion :-

1. Amplitude Distortion
   Audio applications are sensitive to amplitude distortion

2. Phase Distortion
   Video applications are sensitive to phase change

(Read text p. 94-95 Lathi 4th ed.)

Example

Consider the system characterized by $H(f)$.
find the response of the system to the input

$$x(t) = A e^{-\alpha t} u(t), \quad \alpha > 0, \quad H(f) = \frac{1}{1 + j f/f_3}, \quad f_3 = 2\pi RC$$

Sol. using FT table:-

$$X(f) = \frac{A}{\alpha + j2\pi f}$$

$$\Rightarrow Y(f) = H(f)\, X(f) = \frac{A/RC}{(\alpha + j2\pi f)(\frac{1}{RC} + j2\pi f)}$$

• let $\alpha \neq \frac{1}{RC}$ (distinict Roots)

using Partial Fraction Expansion (PFE):-

$$Y(f) = \frac{M_1}{\alpha + j2\pi f} + \frac{M_2}{\frac{1}{RC} + j2\pi f}$$

$$M_1 \Big|_{j2\pi f = -\alpha} = \frac{A/RC}{\frac{1}{RC} - \alpha} = \frac{-A}{\alpha RC - 1}$$

$$M_2 \Big|_{j2\pi f = -\frac{1}{RC}} = \frac{A/RC}{\alpha - \frac{1}{RC}} = \frac{A}{\alpha RC - 1}$$

$$\Rightarrow Y(f) = \frac{A}{\alpha RC - 1}\left[\frac{1}{1/RC + j2\pi f} - \frac{1}{\alpha + j2\pi f}\right]$$

$$\Rightarrow y(t) = \frac{A}{(\alpha RC - 1)}\left[e^{-\frac{t}{RC}} - e^{-\alpha t}\right] u(t)$$

Exercise:

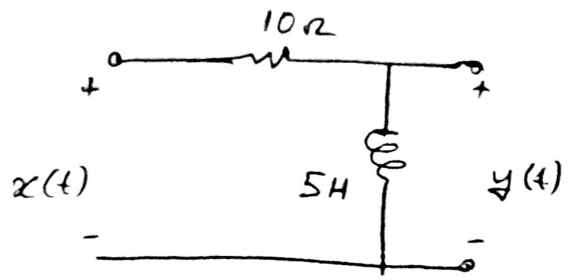
Find the response when $\alpha = \frac{1}{RC}$ (Repeated Roots)

then $$Y(f) = \frac{A/RC}{\left(\frac{1}{RC} + j\,2\pi f\right)^2}$$

Sol.

$$y(t) = \frac{A}{RC}\; t\; e^{-\frac{1}{RC}\cdot t}\; u(t).$$

Example



A) let $x(t) = 10\, u(t)$, find the step response.

Sol. $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{j\omega L}{j\omega L + R} = \dfrac{jw}{jw + R/L}$

$$H(f) = H(\omega)\Big|_{\omega = 2\pi f} = \frac{j2\pi f}{j2\pi f + 2}$$

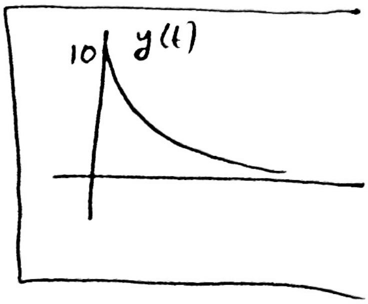
$$Y(f) = H(f) . X(f)$$

$$= \frac{j2\pi f}{j2\pi f + 2} \cdot 10 \left[\frac{1}{j2\pi f} + \frac{\delta(f)}{2} \right]$$

$$= \frac{10 . j2\pi f}{j2\pi f + 2} \cdot \frac{1}{j2\pi f} + \frac{10 . j2\pi f}{j2\pi f + 2} \cdot \frac{\delta(f)}{2}$$

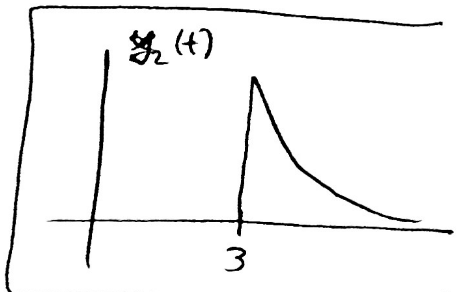
$$= \frac{10}{j2\pi f + 2} + 0$$

$$y(t) = \mathcal{F}^{-1}\left\{ \frac{10}{j2\pi f + 2} \right\} = 10\, e^{-2t} u(t)$$



Exercise show that when $x_2(t) = 10\, u(t-3)$, the output will be

$$y_2(t) = 10\, e^{-2(t-3)} u(t-3).$$



Example: Consider the RL system given by

$$H(f) = \frac{j2\pi f}{j2\pi f + 2}$$

Find the system response for $x(t) = 10\,\Pi\left(\frac{t - 3/2}{3}\right)$?

Sol. $\Pi\left(\frac{t-\frac{3}{2}}{3}\right) = \begin{cases} 1 & -\frac{1}{2} \le \frac{t-3/2}{3} \le \frac{1}{2} = 0 \le t \le 3 \\ 0 & \text{else} \end{cases}$

one can write $x(t) = 10\,[u(t) - u(t-3)]$

$$y(t) = \mathcal{F}^{-1}\{H(f)\,X(f)\}$$

$$= \mathcal{F}^{-1}\left\{\frac{j2\pi f \cdot 10}{j2\pi f + 2}\left[\left(\frac{1}{j2\pi f} + \frac{\delta(f)}{2}\right) - \left(\frac{1}{j2\pi f} + \frac{\delta(f)}{2}\right)e^{-j2\pi f \cdot 3}\right]\right\}$$

$$= 10\left[e^{-2t}u(t) - e^{-2(t-3)}u(t-3)\right]$$

$$= \begin{cases} 0 & , t < 0 \\ 10\,e^{-2t} & , 0 < t < 3 \\ 10\left[e^{-2t} - e^{-2(t-3)}\right] & , t > 3 \end{cases}$$

Summary:- Among many applications of the FT,

1) we define the freq. response of a system as the FT of the impulse Response (namely Transfer Function)

2) The Transfer function Describes the relationship between the input and the output signal of a system in freq. Domain.

Signal Transmission Through LTI system (periodic signals case).

- we have shown earlier that when the input is sinusoidal, the output is sinusoidal with the same frequency but could have different amplitude & different phase. (in tearms of phasors)
- In this section, we will use FT to investigate the relationship between the output and the input when the input is periodic signal.

- a periodic signal $x(t)$ can be represented in terms of FS as:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi f_0 t}$$

- using the FT pair $e^{j2\pi f_0 t} \longleftrightarrow \delta(f - n f_0)$
- The FT term by term yield

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \, \delta(f - n f_0)$$

$$x(t) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}} \longrightarrow y(t) = h(t) * x(t) = \mathcal{F}^{-1}\{H(f)\,X(f)\}$$

- in evaluating $Y(f)$, $H(f)\,\delta(f - n f_0) = H(n f_0)\,\delta(f - n f_0)$

$$\Rightarrow Y(f) = \sum X_n \, H(n f_0)\, \delta(f - n f_0)$$

$$= \sum |X_n|\,|H(n f_0)|\, e^{j(\angle X_n + \angle H(n f_0))} \delta(f - n f_0)$$

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n|\,|H(n f_0)|\, e^{j(2\pi n f_0 + \angle X_n + \angle H(n f_0))}$$

Now,

$$\begin{cases} x(t) = \sum_{n=-\infty}^{\infty} |X_n| \, e^{j\angle X_n} \, e^{j2\pi f_0 t} \\ y(t) = \sum_{n=-\infty}^{\infty} |X_n| \, |H(nf_0)| \, e^{j(\angle X_n + \angle H(nf_0))} \, e^{j2\pi f_0 t} \end{cases}$$

The significance of these two equations:-

- the n^{th} spectral component $|X_n|$ appears at the output with amplitude scaled by the amplitude response function $|H(nf_0)|$
- the phase ~~~~ $\angle X_n$ is also shifted by the system phase response $\angle H(nf_0)$

$$|Y_n| = |X_n| \, |H(nf_0)|$$

$$\angle Y_n = \angle X_n + \angle H(nf_0)$$

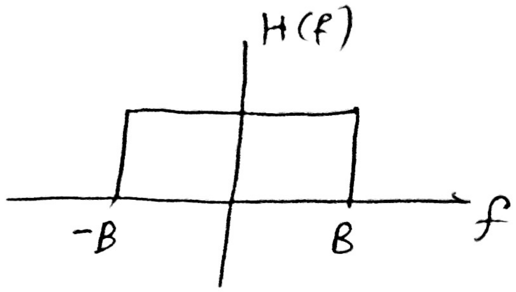
Exercise

Show that, when the input $x(t) = A\cos(\omega_0 t + \theta_x)$

the output $y(t) = A\,|H(f_0)|\cos(\omega_0 t + \theta_x + \angle H(f_0))$

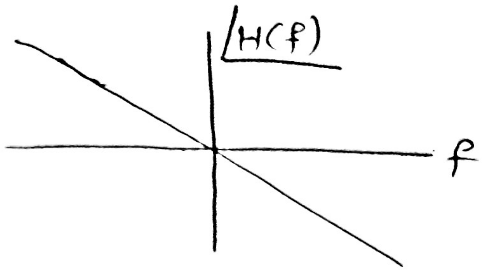
Example Consider a system with amplitude- and phase-response functions given by :-

$$|H(f)| = \begin{cases} K & |f| \le B \\ 0 & 0 \end{cases}$$



$$= K\,\Pi\left(\frac{f}{2B}\right)$$

$$\angle H(f) = -2\pi t_0 f$$



Note :
A filter with this transfer function is referred to as an Ideal Low Pass Filter (LPF) and it is Distortionless system.

if $x(t) = A\cos(2\pi f_0 t + \theta_0)$, find $y(t)$?

Sol. one can write $x(t) = \frac{A}{2} e^{j\theta_0} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j\theta_0} e^{-j2\pi f_0 t}$

~~and~~ $\Rightarrow$ $X_1 = \frac{A}{2} e^{j\theta_2} = X_{-1}^*$, also $X_n = 0 \;\; \forall n \neq 1, -1$

and use $y(t) = \sum |X_n|\,|H(nf_0)|\, e^{j(\angle X_n + \angle H(nf_0))} e^{j2\pi f_0 t}$

$\vdots$

OR $y(t) = A\,|H(f_0)| \cos(2\pi f_0 t + \theta_0 + \angle H(f_0))$

$$= AK \cos(2\pi f_0 t + \theta_0 - 2\pi t_0 f_0)$$

$$= \begin{cases} AK\cos(2\pi f_0 (t - t_0) + \theta_0) & f_0 \le B \\ 0 & f_0 \ge B \end{cases}$$

then, an ideal LPF completely rejects all spectral components with frequencies greater than some cutoff frequency (B), and passes all input spectral components below this cutoff frequency except that their amplitudes are multiplied by a constant K and they are phase shifted by $(-2\pi t_0 f_0)$ or delayed in time by t_0.