Compact TFS

- This is the 3rd representation of FS.
- The FS representation / expression can be written interms of cosines only.
- Starting From Complex FS:

$$x(t) = \sum_{-\infty}^{\infty} X_n e^{jnw_0 t}$$

$$= X_0 + \underbrace{\sum_{-\infty}^{-1} X_n e^{jnw_0 t}}_{\substack{\text{Change of variable} \\ \text{replace } (n) \text{ by } (-n)}} + \sum_{1}^{\infty} X_n e^{jnw_0 t}$$

$$= X_0 + \sum_{n=1}^{\infty} \left(X_{-n} e^{-jnw_0 t} + X_n e^{jnw_0 t} \right) \quad \text{——} (1)$$

For a given n:-

$$X_{-n} e^{-jnw_0 t} + X_n e^{jnw_0 t} = |X_n| e^{-j\theta_n} e^{-jnw_0 t} + |X_n| e^{j\theta_n} e^{jnw_0 t}$$

$$= |X_n| \left(e^{j(nw_0 t + \theta_n)} + e^{-j(nw_0 t + \theta_n)} \right)$$

$$= 2|X_n| \cos(nw_0 t + \theta_n)$$

$$= \boxed{D_n \cos(nw_0 t + \theta_n)} \quad \text{——} (2)$$

Compact TFS applies only for real signals!

where $D_n = 2|X_n|$, called Compact TFS coeffecient

- Back to ① and substitute ②

$$\Rightarrow \boxed{x(t) = X_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n)}$$

$$\boxed{\text{where } D_n = 2|X_n| \qquad n: 1, \cdots, \infty}$$

- From Complex FS

$$X_n = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jn\omega_0 t} dt$$

$$X_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) dt$$

- Also, $|X_n| = \frac{1}{2}(a_n - j b_n) = |X_n| e^{j\theta_n}$

$$\Rightarrow X_n = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$\theta_n = +\tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

- Note that, we have shown that (earlier)

$$\int_{\langle T_0 \rangle} \sin(m\omega_0 t) \cos(n\omega_0 t) dt = 0 \quad \text{for any integers } m, n$$

$\Rightarrow$ the sin and cosine are <u>orthogonal functions</u>
(the set of functions to be orthogonal, their inner product should equal to zero)

- Since the set of Fourier series basis function (sines and cosines) are orthogonal basis

i.e their inner product is zero.

$$\int_{\langle T \rangle} \sin(m\omega t) . \cos(n\omega t)\, dt = 0 \qquad \forall\ m, n .$$

- Then From the TFS :-

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

for a specific n:-

$$a_n \cos(n\omega_0 t) \mp b_n \sin(n\omega_0 t) = D_n \cos(n\omega_0 t + \theta_n)$$

where $D_n = \sqrt{a_n^2 + b_n^2}$

$\theta_n = \tan^{-1} - \frac{b_n}{a_n}$

use Trignometric identities to show that

$$x(t) = a_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n)$$

Finally, ① Complex FS can be used to plot the Double-sided spectrum.

② Compact FS can be used to plot the Single sided spectrum.

Summary :-

- The FS are used to represent a given periodic signal $x(t)$ with period T_0 and frequency $\omega_0 = \frac{2\pi}{T_0}$ as a weighted sum of Harmonically related sinusoids
- $(n\omega_0)$ are called the n^{th} harmonic.

- There are 3 variants of FS :-

① TFS

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t)\, dt$$

$$a_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \cos(n\omega_0 t)\, dt \qquad n \geq 1$$

$$b_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(n\omega_0 t)\, dt \qquad n \geq 1$$

② Complex FS

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$X_0 = a_0$$

$$X_n = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jn\omega_0 t}, \quad n = -\infty, \infty$$

$$X_n = \frac{1}{2}(a_n - j b_n)$$

③ Compact TFS

$$x(t) = X_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n)$$

$$D_n = 2|X_n|$$