SEGNAL & SYSTEMS

Continuous and Discrete

Lecture Notes

prepared by :

Dr . Ashraf AL-Rimawi

2018

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Fundamantal of Signals

1.1 : Introduction :

what is a signal ?

A signal is aquantitive description of aphysical phenomenon , event or process, Some Common examples include :

1- Electrical current or voltage in circuit.
2- Daily closing value of a share of stock last week
3- Audio signals continious-time in it's original form, or discrete-time when stored on CD.

More precisely, asignal is afunction. usually of one variable in time, How ever, in general, signals Can be functions of more than one variable, e.g image signal.

In this class we are interested in two types of signals

1. Continuous - time signal $x(t)$, where t is areal-valued variable denoting time, i.e, $t \in R$. We Use

paranthesis (.) to denote a continuous - time signal.

2- Discrete - time signal $x[n]$, where n is an integer... valued variable denoting the discrete samples of time, i.e $n \in Z$, we use square brackets [.] denote a discrete - time signals under the definition of a discrete - time signal, $x[1.5]$ is not defined, for example.

1.2 periodic and a periodic signals:

A signal $x(t)$ is periodic if and only if:

$$x(t + T_0) = x(t) \qquad -\infty < t < \infty$$

Example 1:

For the following signals:

1. $x_1(t) = A \sin(2\pi f_0 t + \theta)$.
2. $x_2(t) = 3 \sin(15t)$.
3. $x_3(t) = A + B \cos(2\pi f_0 t)$.

Check if it is periodic signal or not? Justify your answer

Ans :

1. $x_1(t) = A\sin(2\pi f_0 t + \theta)$

we have to check if $x(t+T_0) = x(t)$ or not, where $T_0 = \frac{1}{f_0}$, so :

$$x_1(t+T_0) = A\sin(2\pi f_0(t+T_0) + \theta)$$

$$= A\sin(2\pi f_0 t + 2\pi f_0 T_0 + \theta)$$

$$= A\sin((2\pi f_0 t + \theta) + 2\pi f_0 T_0)$$

since

$$\sin(A \mp B) = \sin(A)\cos(B) \mp \cos(A)\sin(B)$$

Then

$x_1(t+T) = A\sin(2\pi f_0 t + \theta)\cos(2\pi f_0 T_0) + A\cos(2\pi f_0 t + \theta)\sin(2\pi f_0 T_0)$

when $f_0 = \frac{1}{T_0}$, then

$\cos(2\pi \frac{1}{T_0} \cdot T_0) = 1$ and $\sin(2\pi \frac{1}{T_0} \cdot T_0) = 0$

$\Longrightarrow$

$$x_1(t+T_0) = A\sin(2\pi f_0 t + \theta) = x(t)$$

There fore, $x_1(t)$ is periodic signal.

2. $x_2(t) = 3\sin(15t)$

$$x_2(t+T_0) = 3\sin(15t + 15T_0)$$

$$= 3\sin(15t)\cos(15T_0) + 3\cos(15t)\sin(15T_0)$$

where $15\cancel{T_0} = 2\pi f_0 \cancel{T_0} \Rightarrow \frac{15}{2\pi} = f_0 \Rightarrow T_0 = \frac{1}{f_0} = \frac{2\pi}{15}$

$\Rightarrow$

$$x_2(t+T_0) = 3\sin(15t)\cos(2\pi) + 3\cos(15t)\sin(2\pi)$$

$$= 3\sin(15t) = x_2(t)$$ is periodic signal

3. $x_3(t) = A + B\cos(2\pi f_0 t)$

$$x_3(t+T_0) = A + B\cos(2\pi f_0 (t+T_0))$$

$$= A + B\cos(2\pi f_0 t + 2\pi f_0 T_0)$$

since

$$\cos(A \mp B) = \cos(A)\cos(B) \pm \sin(A)\sin(B)$$

Then

$$x_3(t+T_0) = A + [B\cos(2\pi f_0 t)\cos(2\pi f_0 T_0) - B\sin(2\pi f_0 t)\sin(2\pi f_0 T_0)]$$

when $f_0 = \frac{1}{T_0}$

$\Rightarrow x_3(t+T_0) = A + B\cos(2\pi f_0 t) = x_3(t)$ is

Periodic signal.

Fundamental frequency of Continuous signals

To identify the period T_0, The frequency $f_0 = 1/T_0$, or The angular frequency $\omega_0 = 2\pi f_0$ of a given or Complex exponential signal, it is always

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helpful to write it in any of the following forms

$$\sin(\omega_0 t) = \sin(2\pi f_0 t) = \sin(2\pi t / T_0)$$

The fundamental frequency of a signal is the greatest common divisor (GCD) of all the frequency components contained in a signal, and, equivalently the fundamental period is the least common multiple (LCM) of all indiuitual periods of the components.

Example 2 :

Find the fundamental frequency of the following Continuous signals :

1- $x_1(t) = \cos\left(\frac{10\pi}{3}t\right) + \sin\left(\frac{5\pi}{4}t\right)$

2. $x_2(t) = \sin\left(\frac{5\pi}{6}t\right) + \cos\left(\frac{3\pi}{4}t\right) + \sin\left(\frac{\pi}{3}t\right)$

3. $x_3(t) = \cos\left(\frac{10}{3}t\right) + \sin\left(\frac{5\pi}{4}t\right)$

Ans : 1. $x_1(t) = \cos\left(\frac{10\pi}{3}t\right) + \sin\left(\frac{5\pi}{4}t\right)$

The frequencies and periods of the two terms are respectively

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$\omega_1 = \frac{10\pi}{3}$, $f_1 = \frac{5}{3}$, $T_1 = \frac{3}{5}$, $\omega_2 = \frac{5\pi}{4}$, $f_2 = \frac{5}{8}$, $T_2 = \frac{8}{5}$

The fundamental frequency f_0 is the GCD of $f_1 = \frac{5}{3}$ and $f_2 = \frac{5}{8}$

$$f_0 = GCD\left(\frac{5}{3}, \frac{5}{8}\right) = GCD\left(\frac{40}{24}, \frac{15}{24}\right) = \frac{5}{24}$$

Alternatively, the period of the fundamental (T_0) is the LCM of $T_1 = \frac{3}{5}$ and $T_2 = \frac{8}{5}$:

$$T_0 = LCM\left(\frac{3}{5}, \frac{8}{5}\right) = \frac{24}{5}$$

Now we get $\omega_0 = 2\pi f_0 = 2\pi/T_0 = 5\pi/12$ and the signal can be written as :

$$x(t) = \cos\left(8 \frac{5\pi}{12} t\right) + \sin\left(3 \frac{5\pi}{12} t\right)$$

$$= \cos(8\omega_0 t) + \sin(3\omega_0 t)$$

i.e the two terms are the 3rd and 8th harmonic of the fundamental frequency was respectively

2. $x_2(t) = \sin\left(\frac{5\pi}{6} t\right) + \cos\left(\frac{3\pi}{4} t\right) + \sin\left(\frac{\pi}{3} t\right)$

The frequencies and periods of the three terms are respectively.

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$w_1 = \frac{5\pi}{6}$, $f_1 = \frac{5}{12}$, $T_1 = \frac{12}{5}$

$w_2 = \frac{3\pi}{4}$, $f_3 = \frac{3}{8}$, $T_2 = \frac{8}{3}$

$w_3 = \frac{\pi}{3}$, $f_3 = \frac{1}{6}$, $T_3 = 6$

The fundamental frequency f_0 is the GCD of f_1, f_2 and f_3 :

$$f_0 = GCD\left(\frac{5}{12}, \frac{3}{8}, \frac{1}{6}\right) = GCD\left(\frac{10}{24}, \frac{9}{24}, \frac{4}{24}\right) = \frac{1}{24}$$

Alternatively, the period of the fundamental T_0 is the LCM of T_1, T_2 and T_3 :

$$T_0 = LCM\left(\frac{12}{5}, \frac{8}{3}, 6\right) = LCM\left(\frac{36}{15}, \frac{40}{15}, \frac{90}{15}\right)$$

The signal can be written as :

$$x(t) = \sin\left(\frac{10\pi}{12}t\right) + \cos\left(\frac{9\pi}{12}t\right) + \sin\left(\frac{4\pi}{12}t\right)$$

i.e, the fundamental frequency is $w_0 = \pi/12$, The fundamental period is $T_0 = \frac{2\pi}{w_0} = 24$ and the three terms are the 10^{th}, 9^{th}, and 4^{th} harmonic of w_0, respectively

3. $x_3(t) = \cos\left(\frac{10}{3}t\right) + \sin\left(\frac{5\pi}{4}t\right)$

Here the angular frequencies of the two are, respectively

$\omega_1 = \frac{10}{3}$, $\omega_2 = \frac{5\pi}{4}$

The fundamental frequency ω_0 should be the GCD of ω_1 & ω_2

$\omega_0 = GCD\left(\frac{10}{3}, \frac{5\pi}{4}\right)$

Which does not exist as π is an irrational number which can not be expressed as a ratio of two integers, therefore the two frequencies can not be multiplies of the same fundamental frequency, In other words, the signal as the sum of the two terms is not periodic signal.

$\Rightarrow$

From the above example, it can be concluded that the sum of two sinusoids is periodic if the ratio of their respective periods can be expressed as a rational number.

One the other hand, for a discrete complex exponential $x[n] = e^{j\omega_1 n}$ to be periodic with period N, it has to satisfy

$e^{j\omega_1(n+N)} = e^{j\omega_1 n}$, i.e., $e^{j\omega_1 N} = 1 = e^{j2\pi k}$

That is, $\omega_1 N$ has to be a multiple of 2π :

$$\omega_1 N = 2\pi K, \quad \text{i.e}, \quad \frac{\omega_1}{2\pi} = \frac{K}{N}.$$

As it is an $\omega_1/2\pi$ has to be a rational number (a ratio of two integers). In order for the period

$$N = K\frac{2\pi}{\omega_1}$$

to be the fundamental period, K has to be the smallest integer that makes N an integer, and the fundamental angular frequency is :

$$\omega_0 = \frac{2\pi}{N} = \frac{\omega_1}{}$$

The original Signal can now be written as :

$$x[n] = e^{j\omega n} = e^{jK\omega_0 n} = e^{jK\frac{2\pi}{N}n}$$

Example 3 : Show that a discrete signal

$$x[n] = e^{jm(2\pi/N)n}$$

has fundamental period.

$$N_0 = N / GCD(N, m)$$

According to the discussion above, the fundamental period N_0 should satisfy

$$m \frac{2\pi}{N} N_0 = K 2\pi, \quad \text{or} \quad N_0 = \frac{KN}{m} = \frac{N}{m/K}.$$

we see that for N_0 to be an integer, $L \triangleq m/K$ has to divide N. but since $K = m/L$ is an integer, L also has to divide m, moreover, since K needs to be smallest integer satisfying the above equation, $L = m/K$ has to be the greatest common divisor of both N and m, i.e. $L = m/k$ has to be the greatest common divisor of both N and m, i.e, $L = \gcd(N, m)$, and the fundamental period can be written as:

$$N_0 = N/(m/K) = N/GCD(N, m).$$

1.3: phasor signals and spectra

Although physical systems always interact with real signals it is often mathematically convenient to represent real signals interms of complex quantities.

A complex sinusoid can be viwed as arotating phasor

$$\tilde{x}(t) = A e^{j(\omega_0 t + \theta)} \qquad -\infty < t < \infty$$

From this equation, it can be noted that the signal has three parameters, amplitude A, frequency f_0, and phase θ the fixed phasor portion is $Ae^{j\theta}$ while the rotating portion is $e^{j\omega_0 t}$, Therefore, as shown in Fig. 1(a) the real sinusoid signal $x(t)$ can be obtain from $\tilde{x}(t)$ where

$$x(t) = \mathrm{Re}\left\{\tilde{x}(t)\right\}$$

$$= \mathrm{Re}\left\{Ae^{j(\omega_0 t + \theta)}\right\}$$

By using Euler's theorem, $x(t)$ can be expressed as

$$x(t) = \mathrm{Re}\left\{A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta)\right\}$$

$$= A\cos(\omega_0 t + \theta)$$

We can also turn this around using the inverse Euler formula as shown in Fig 1 [b] where:

$$x(t) = A\cos(\omega_0 t + \theta)$$

$$= \frac{1}{2}\tilde{x}(t) + \frac{1}{2}\tilde{x}^*(t)$$

$$= \frac{1}{2}A\,e^{j(\omega_0 t + \theta)} + \frac{1}{2}A\,e^{-j(\omega_0 t + \theta)}$$

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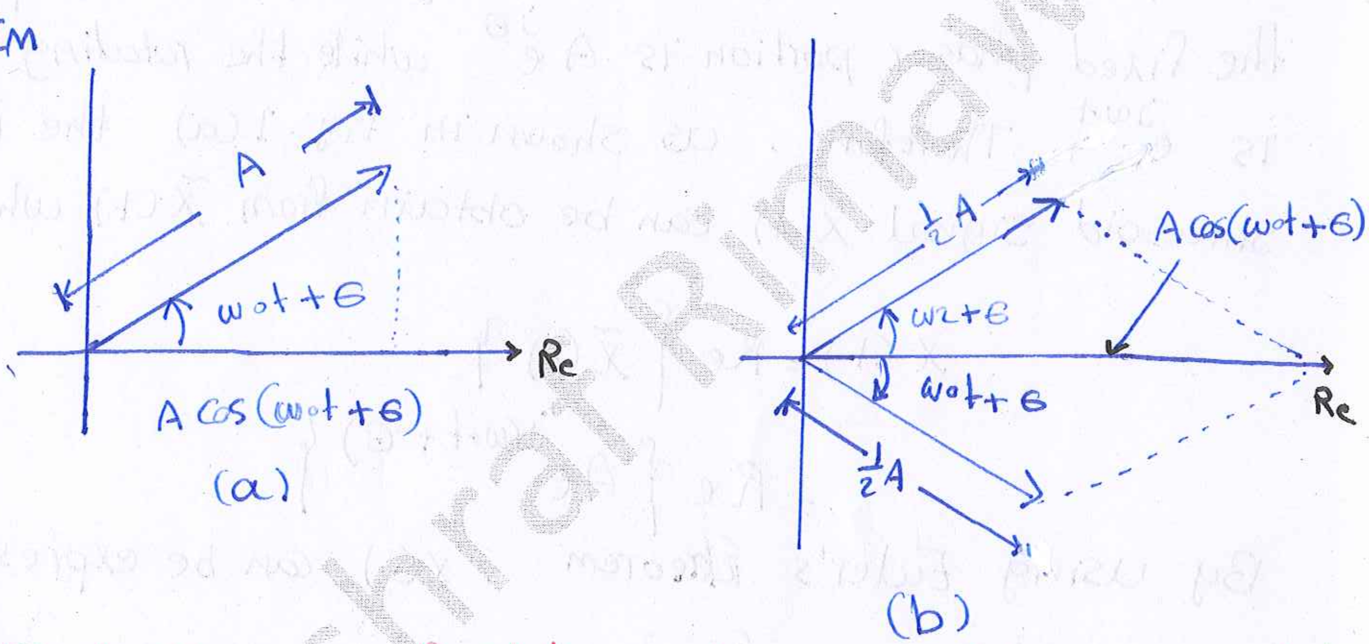

Fig 1: Two ways of relating a phasor signal to a sinusoidal signal. (a) obtain $x(t)$ from $\tilde{x}(t)$. (b) obtain $x(t)$ from $\tilde{x}(t)$ and $\tilde{x}^*(t)$.

An alternative representation for $x(t)$ is provided in frequency domain where the amplitude of the signal and it's phase is studied with respect to the value of frequency f_0.

The frequency domain takes two forms of plots as shown in fig 2.

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a. single-sided line spectra (Amplitude and phase)

b. double-sided line spectra (Amplitude and phase)

Both the single-sided and double-sided line spectra correspond to the real signal $x(t)$.

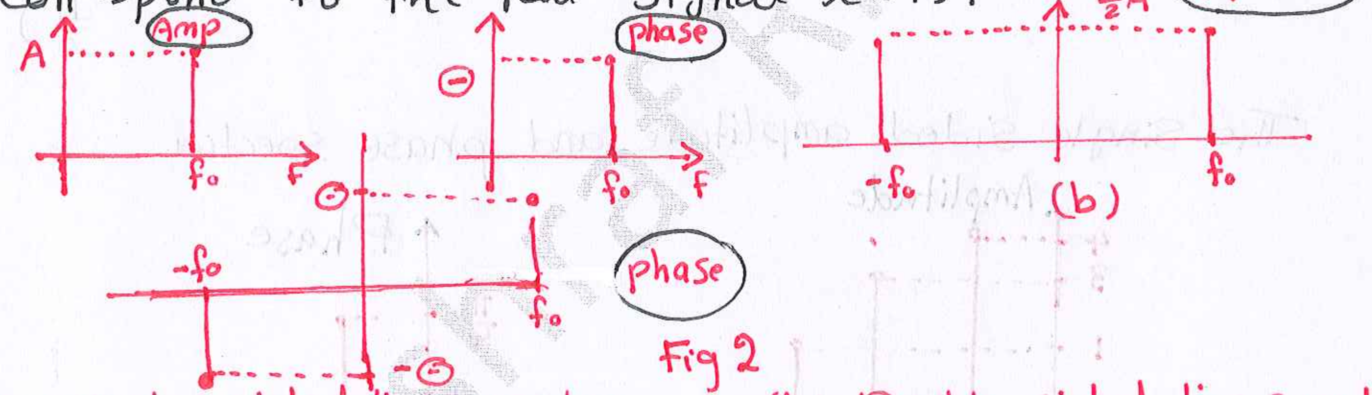

Fig 2

(a) single-sided line spectra (b) Double-sided line spectra

Example 4: Given the signal:

$$x(t) = 4\cos\left(20\pi t + \frac{\pi}{4}\right) + 3\cos\left(60\pi t - \frac{\pi}{6}\right) + \sin\left(80\pi t + \frac{\pi}{8}\right)$$

a. sketch its signal-sided amplitude and phase spectra

b. sketch its double-sided amptitude and phase spectra

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Ans :

(a) $x(t) = 4\cos(20\pi t + \frac{\pi}{4}) + 3\cos(60\pi t - \frac{\pi}{6}) + \cos(80\pi t + \frac{\pi}{6} - \frac{\pi}{2})$ .... where $\sin(80\pi t + \frac{\pi}{6}) = \cos(80\pi t + \frac{\pi}{6} - \frac{\pi}{2})$

$\Longrightarrow x(t) = 4\cos(20\pi t + \frac{\pi}{4}) + 3\cos(60\pi t - \frac{\pi}{6}) + \cos(80\pi t - \frac{\pi}{3})$

The single sided amplitude and phase spectra

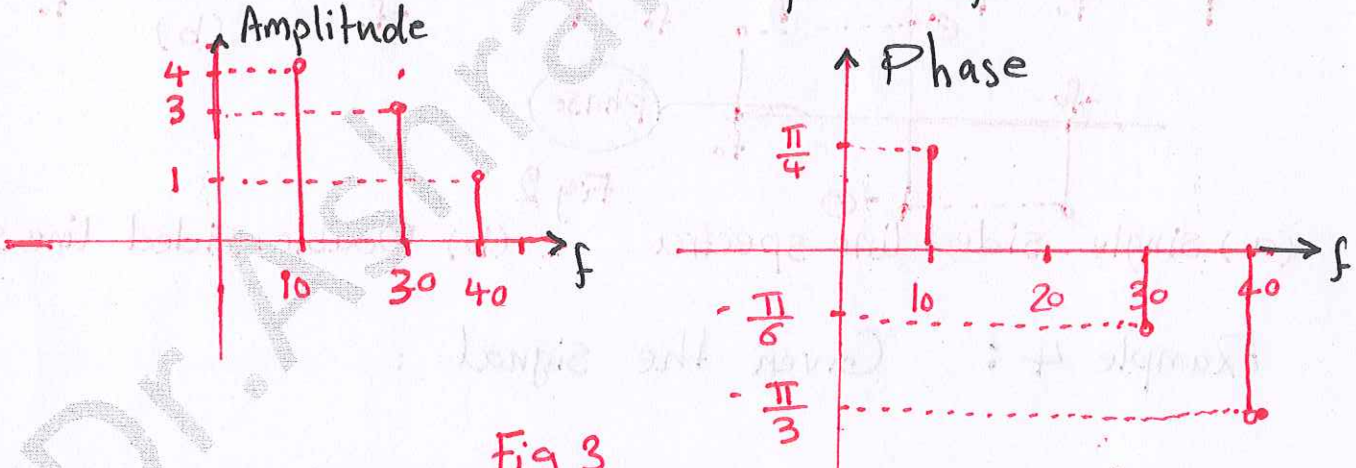

Fig 3

The double sided amplitude and phase spectra.

$x(t) = 4\cos(20\pi t + \frac{\pi}{4}) + 3\cos(60\pi t - \frac{\pi}{6}) + \cos(80\pi t - \frac{\pi}{3})$

$= \frac{4}{2}\left[e^{j(20\pi t - \frac{\pi}{3})} + e^{-j(20\pi t + \frac{\pi}{4})}\right] + \frac{3}{2}\left[e^{j(60\pi t - \frac{\pi}{6})} + e^{-j(60\pi t - \frac{\pi}{6})}\right]$

$+ \frac{1}{2}\left[e^{j(80\pi t - \frac{\pi}{3})} + e^{-j(80\pi t - \frac{\pi}{3})}\right]$

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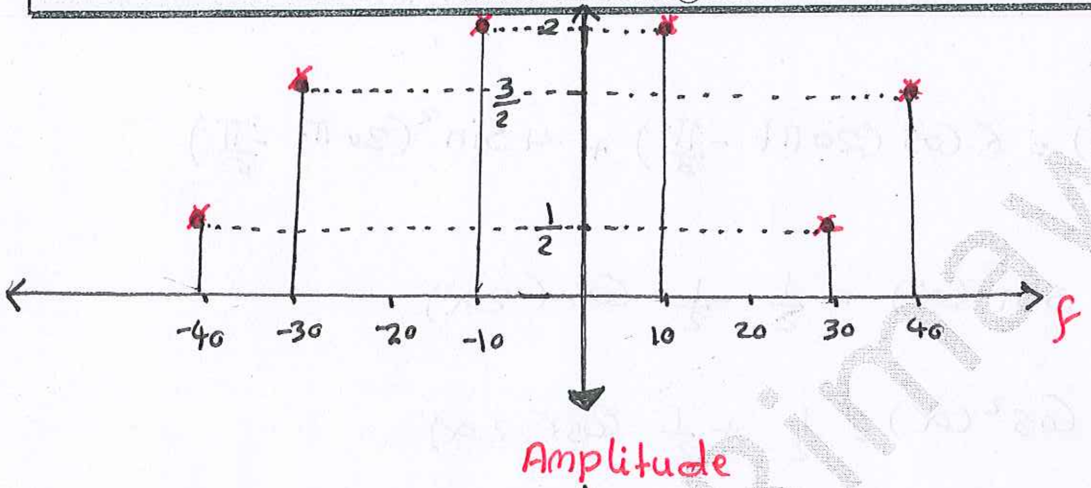

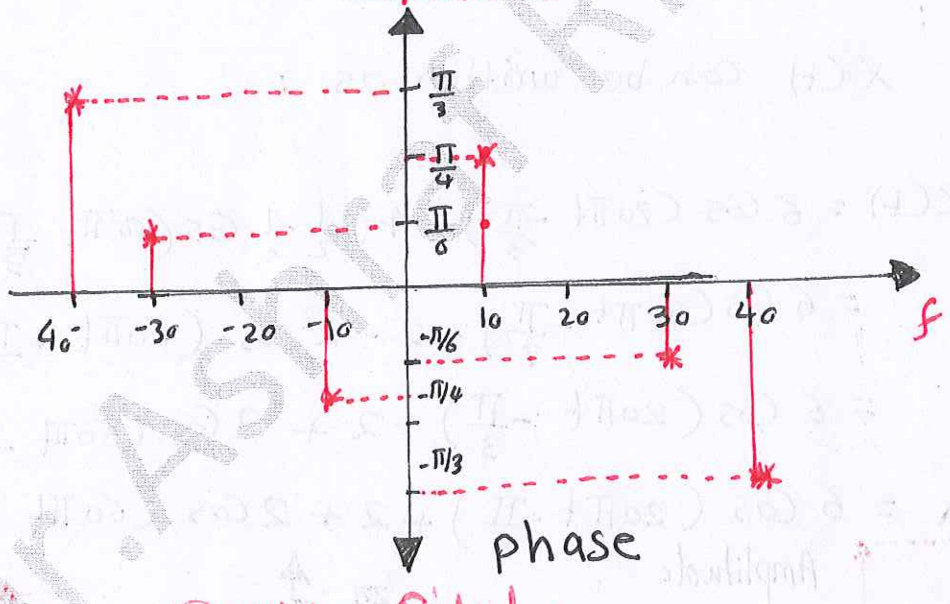

Fig(4) Double - Sided .

Example (5) : Given the signal

$$x(t) = 6\cos(20\pi t - \frac{\pi}{3}) + 4\sin^2(30\pi t - \frac{\pi}{6})$$

(a) sketch its single - sided amplitude and phase spectra.

(b) sketch its double - sided amplitude and phase spectra

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Ans :

a) $x(t) = 6\cos(20\pi t - \frac{\pi}{3}) + 4\sin^2(30\pi t - \frac{\pi}{6})$

Since

$$\sin^2(\alpha) = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$$

and

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$$

Then :

$x(t)$ can be written as :

$$x(t) = 6\cos(20\pi t - \frac{\pi}{3}) + 4\left(\frac{1}{2} - \frac{1}{2}\cos(60\pi t - \frac{\pi}{3})\right)$$

$$= 6\cos(20\pi t - \frac{\pi}{3}) + 2 - 2\cos(60\pi t - \frac{\pi}{3})$$

$$= 6\cos(20\pi t - \frac{\pi}{3}) + 2 + 2\cos(60\pi t - \frac{\pi}{3} + \pi)$$

$$= 6\cos(20\pi t - \frac{\pi}{3}) + 2 + 2\cos(60\pi t + \frac{2\pi}{3})$$

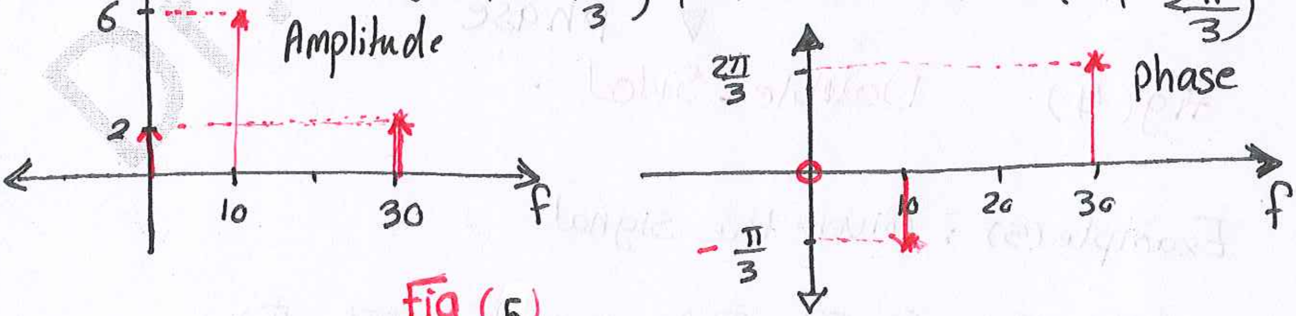

Fig (5)

b) $x(t) = 6\cos(20\pi t - \frac{\pi}{3}) + 2 + 2\cos(60\pi t + \frac{2\pi}{3})$

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$$x(t) = \frac{6}{2}\left[e^{j(20\pi t - \pi/3)} + e^{-j(20\pi t - \frac{\pi}{3})}\right] + 2 + \frac{2}{2}\left[e^{j(60\pi t + \frac{2\pi}{3})} + e^{-j(60\pi t + \frac{2\pi}{3})}\right]$$

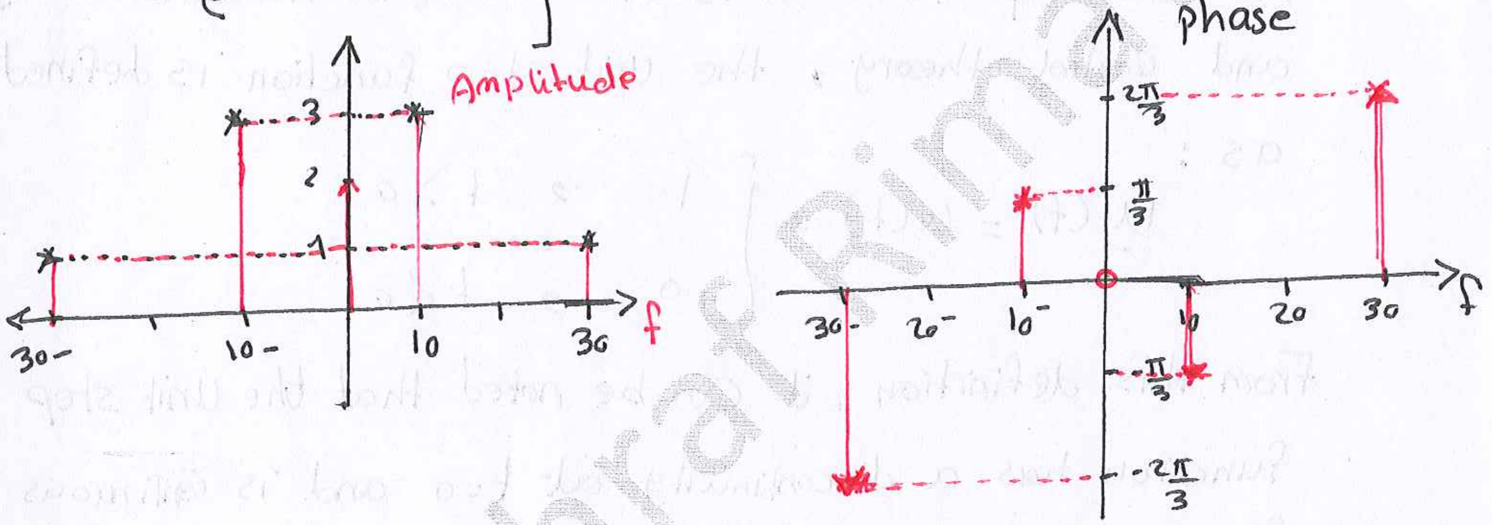

fig (6)

1.4 Singularity functions

Singularity functions are discontinuous functions or their derivatives are discontinuous the Commonly used singularity functions are :

* Step function .
* Ramp function .
* Impulse function .

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1.4.1 Unit-step function.

The unit step function is used widely in network theory and control theory, the unit step function is defined as:

$$u_{-1}(t) = u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

From this definition, it can be noted that the unit step function has a discontinuity at $t=0$ and is continuous for all other values of t

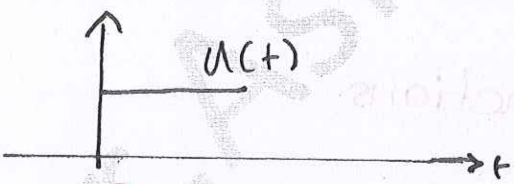

fig 7: plot of unit step function.

Reflection operation on the unit step function

It is easy to visualize how $u(t)$ would be, this function $u(t)$ is reflected version of $u(t)$ and shown in fig 9

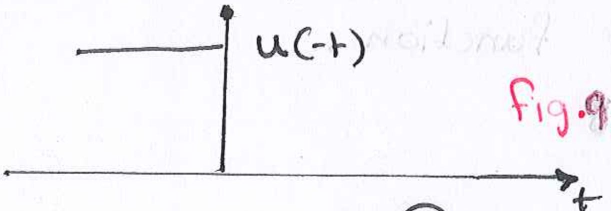

fig.9: plot of $u(-t)$

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Another example using the unit step function is shown in fig 10, this function is called the signum function and it is written as $sgn(t)$

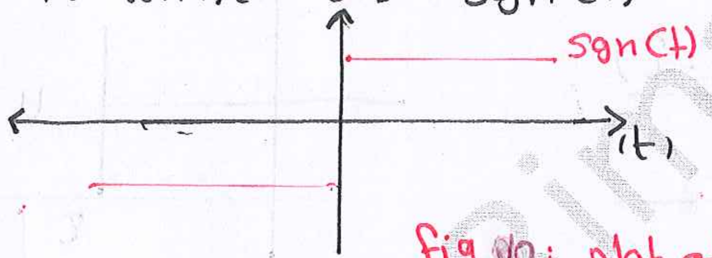

fig 10: plot of Signum function

Where the $sgn(t)$ can be expressed as

$$sgn(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

The signum function is often not used in the network theory, but it is used in communication and control theory. It is expressed in terms of unit step functions as indicated below

$$sgn(t) = -1 + 2u(t)$$

or

$$sgn(t) = u(t) - u(-t)$$

Shifting operation on the Unit Step function

The shifting operation on the unit step function shown in fig 11, and it can be expressed as

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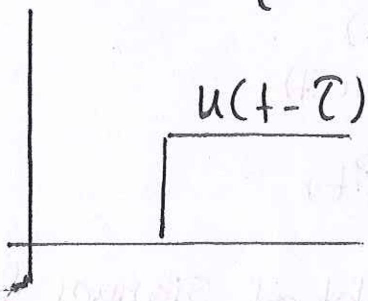
$$u(t-\tau) = \begin{cases} 1, & t > \tau \\ 0, & t < \tau \end{cases}$$



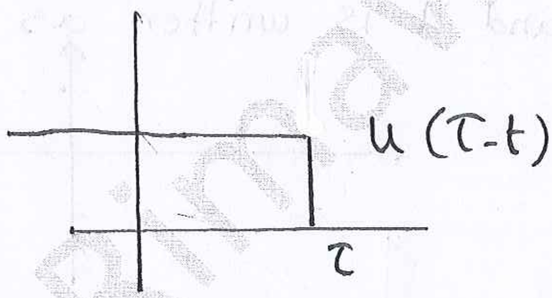


Fig 11: Shifted unit step function

1.4.2 Ramp Function

The ramp function shown in fig (12) can be expressed as

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

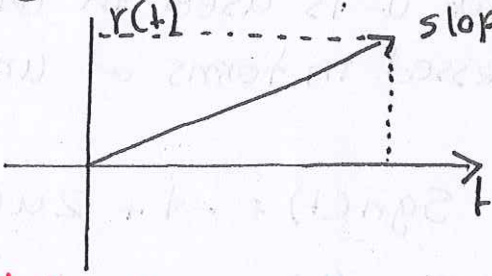


Fig (12): plot of ramp function r(t)

We can define the unit step function, as the derivative of the ramp function. Alternatively, we can state that the ramp function is the integral of the unit-step function, where

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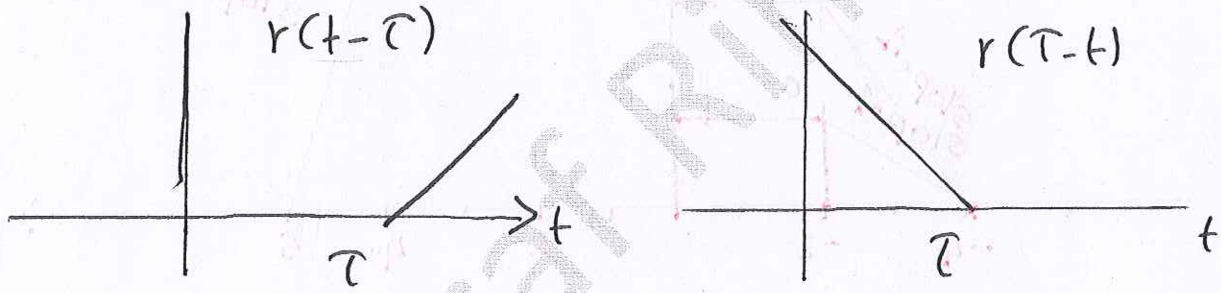
$$u(t) = \frac{dr(t)}{dt} \Rightarrow r(t) = \int_{-\infty}^{t} u(t')\,dt' = \int_{0}^{t} 1 \cdot dt = t\,u(t)$$

In addition, the plot of the shifted ramp function and the reflected ramp function are displayed in Fig '13'.



a) shifted ramp function b) shifted and reflected ramp function

Fig 13. Operation on ramp function

The ramp function is a signal generated by some electronic circuits with electric circuitry, it is possible to generate saw-tooth wave form displayed in Fig '14', such a signal is used in a Cathode-ray oscilloscop (CRO) as the timing signal, such a signal is used in a TV also for horizontal and vertical scanning

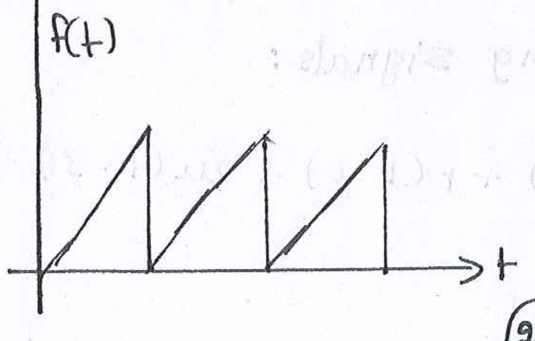


Fig 14. Saw-tooth waveform, used as sweep signal CROS

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Example : For the signals shown below, write an expression in terms of singularity function

a -

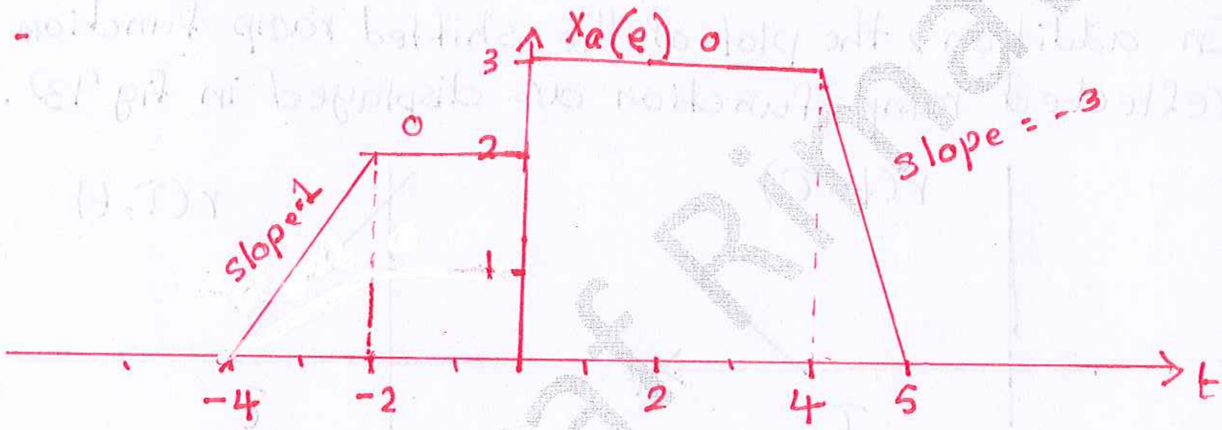

Fig (15)

Ans : By using slope method :

$x_a(t) = r(t+4) - r(t+2) + u(t) - 3r(t-4) + 3r(t-5)$

b -

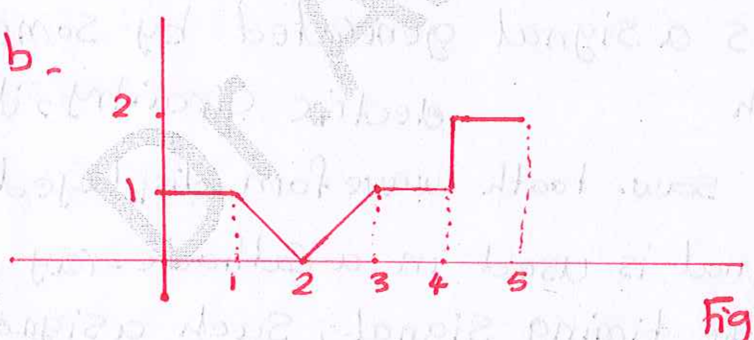

Fig. 16

$u(t) - r(t-1) + 2r(t-2) - r(t-3) + u(t-4) - 2u(t-5)$

Example : Sketch the following signals :

1- $x_1(t) = r(t+2) - 2r(t+1) + r(t-1) + 2u(t-3) - u(t-4)$

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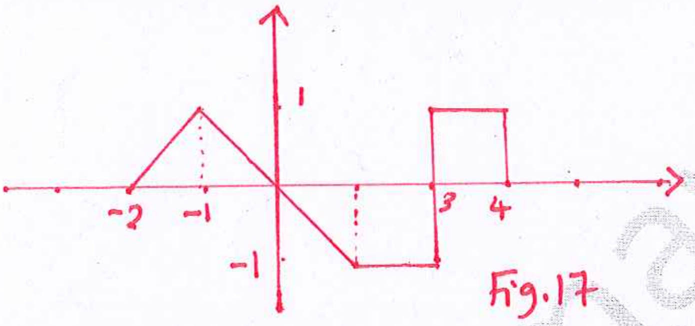

Fig.17

2 - $X_2(t) = 2u(t) - 2u(t-2) + u(t-3) - u(t-4)$

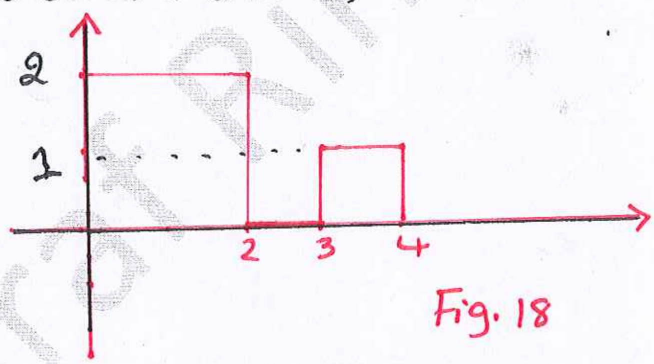

Fig. 18

3 - $X3(t) = u(t) + r(t-1) - 2r(t-2) + r(t-3) + u(t-4) - 2u(t-5)$

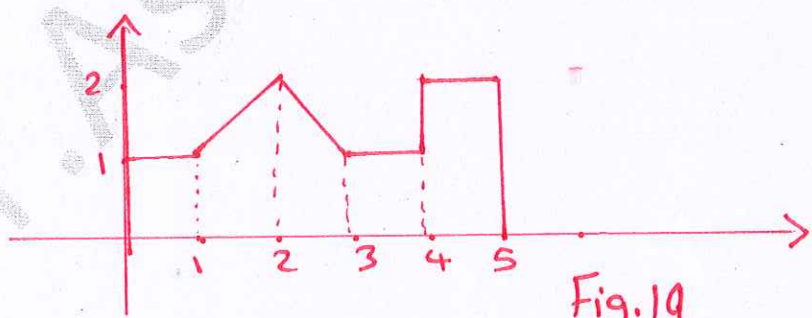

Fig.19

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1-1.4-3 Unit pulse function

The unit pulse function can be represented as:

$$\pi(t) = \begin{cases} 1 & , -\frac{1}{2} < t \le \frac{1}{2} \\ 0 & , \text{o.w} \end{cases}$$

Example : Express unit pulse function in terms of unit step function

Ans :

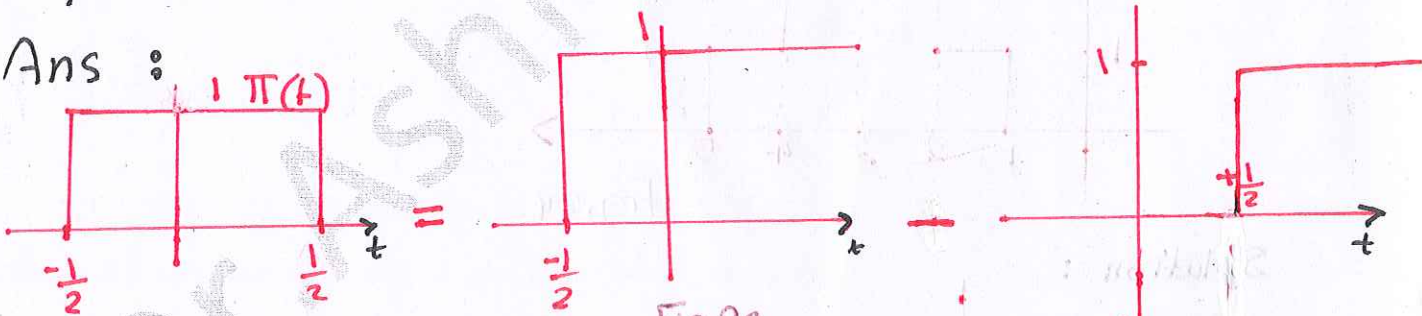

Fig 20

$$\pi(t) = u\left(t+\frac{1}{2}\right) - u\left(t-\frac{1}{2}\right)$$

Example : sketch the following signal

$$x_1(t) = A\pi(\alpha t - B) \quad \text{where} \quad B, \alpha > 0$$

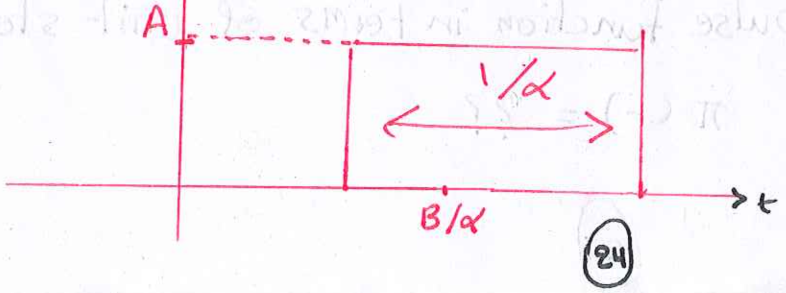

Fig 21

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Example : sketch $x(t) = 3\pi\left(\frac{1}{2}t - 2\right)$

$$= 3\pi\left(\frac{1}{2}(t-4)\right)$$

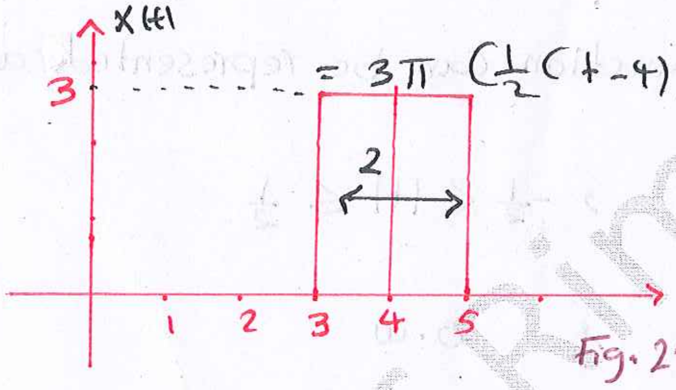

Fig. 23

Example : Express x(t) in terms of pulse function

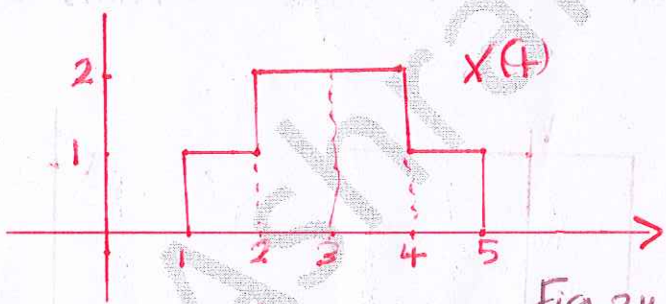

Fig. 24

Solution :

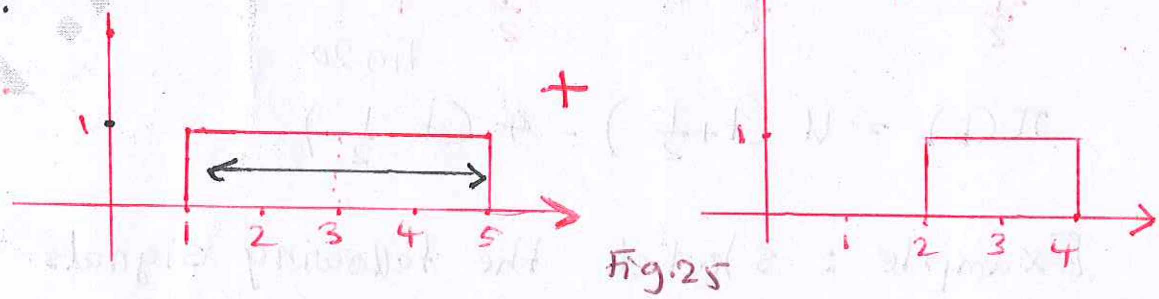

Fig. 25

$$x(t) = \pi\left(\frac{1}{4}(t-3)\right) + \pi\left(\frac{1}{2}(t-3)\right)$$

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1-1-4-4 Unit impulse function $\delta(t)$

The unit impulse function, designated $\delta(t)$, is abo called the Dirac delta function, it is used in network theory, control theory and signal theory. it is important because of it's properties and the insight it offers about the network to which it is applied.

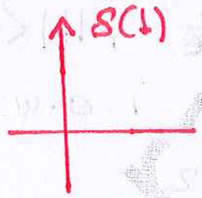

Fig26: impulse or Dirac delta function

The unit impulse function has the following properties

1- $\delta(at) = \delta(t)/|a|$ (change of variables)
2- $\delta(-t) = \delta(t)$ (even function)
3- sifting property

$$\int_{t_1}^{t_2} x(t)\,\delta(t-t_0)\,dt = \begin{cases} x(t_0), & t_1 < t_0 < t_2 \\ 0, & \text{o.w} \end{cases}$$

4- Sampling property

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

for continuous $x(t)$

5- Derivative property

$$\int_{t_1}^{t_2} x(t)\,\delta^{(n)}(t-t_0)\,dt = (-1)^n x^n(t_0) = (-1)^n \left.\frac{d^n x(t)}{dt^n}\right|_{t=t_0}$$

A test function for the unit impulse function helps in problem solving, there for two functions of interest are

$$\delta_\epsilon(t) = \frac{1}{2\epsilon}\,\Pi\left(\frac{t}{2\epsilon}\right) = \begin{cases} \frac{1}{2\epsilon}, & |t|<\epsilon \\ 0, & o.w \end{cases}$$

$$\delta_{1\epsilon}(t) = \epsilon\left(\frac{1}{\pi t}\sin\frac{\pi(t)}{\epsilon}\right)^2$$

$\epsilon = \frac{1}{4}$

$\epsilon = \frac{1}{2}$

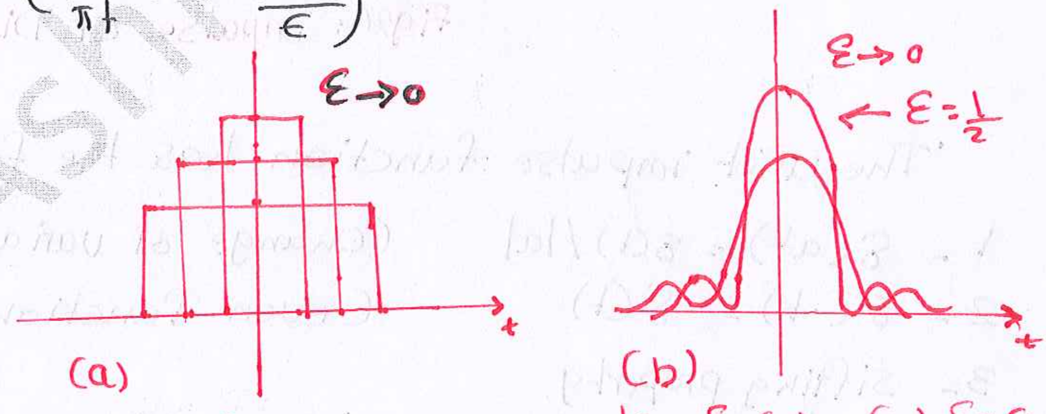

Fig 27: Test functions for the unit impulse $\delta(t)$: (a) $\delta_\epsilon(t)$ (b) $\delta_{1\epsilon}(t)$

$*$ $\delta(t) = \dfrac{du(t)}{dt}$

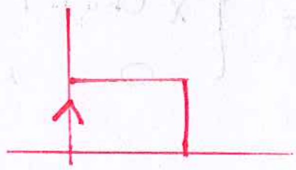

Fig. 28

Example: show that $\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$

Since $\delta(t) = \frac{du(t)}{dt}$

$$\int_{-\infty}^{\infty} x(t) \frac{du(t)}{dt} dt$$

$v = x(t) \qquad dr = du(t)$

$dv = x'(t) \qquad r = u(t)$

$$x(t)u(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x'(t)u(t)$$

$$x(\infty)u(\infty) - x(-\infty)u(-\infty) - [x(\infty) - x(0)]$$

$$= x(\infty) - x(\infty) + x(0) = x(0)$$

Example : Evaluate the following integrals

a) $\int_{5}^{10} \cos(2\pi t)\, \delta(t-2)\, dt = 0$

b) $\int_{0}^{5} \cos(2\pi t)\, \delta(t-2)\, dt = 1$

c) $\int_{-\infty}^{\infty} [e^{-3t} + \cos(2\pi t)]\, \dot{\delta}(t)\, dt$

$$= (-1)\left[-3e^{-3t} - (2\pi)\sin(2\pi t)\right]\Big|_{t=0}$$

$$= (-1)\left[-3e^{-3(0)} - 0\right] = -3e^{-3(0)} = 3$$

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d) $\int_{-\infty}^{\infty} e^{3t}\,\ddot{\delta}(t-2)\,dt = (-1)^2 (3)(3e^6)$

$= 9e^6$

Example : find the unspecified constants, denoted as $c_1, c_2,$ .... in the expressions :

a) $10\,\delta(t) + c_1\dot{\delta}(t) + (2 + c_2)\ddot{\delta}(t) = (3 + c_3)\,\delta(t) + 5\,\dot{\delta}(t) + 6\,\ddot{\delta}(t)$

$10 = 3 + c_3 \Rightarrow \boxed{c_3 = 7}$

$\boxed{c_1 = 5}$

$2 + c_2 = 6 \Rightarrow \boxed{c_2 = 4}$

Example : sketch the following signals :

a) $x_1(t) = 2u(t) + \delta(t-2)$

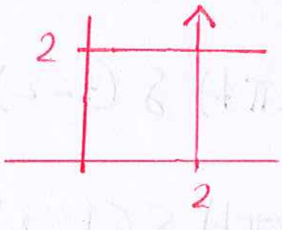

Fig.29

b) $x_2(t) = 2u(t)\,\delta(t-2)$

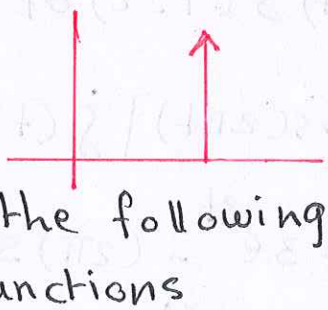

Fig.30

Example : plot accurately the following signals defined in terms of singularity functions

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a) $x_1(t) = \sum_{n=0}^{\infty} x_a(t-2n)$ [plot for $0 \le t \le 6$]

where $x_a(t) = r(t)u(2-t)$

b) $x_2(t) = \sum_{n=0}^{\infty} x_b(t-3n)$ [plot for $0 \le t \le 6$]

where $x_b(t) = r(t) - r(t-1) - r(t-2) + r(t-3)$

Ans : a) $x_a(t) = r(t)\,u(2-t)$

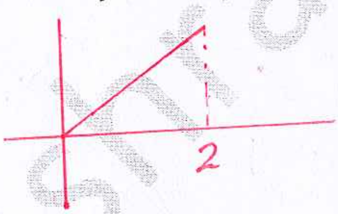

Fig. 31

$$x_1(t) = \sum_{n=0}^{\infty} r(t-2n)\,u(2-t+2n)$$

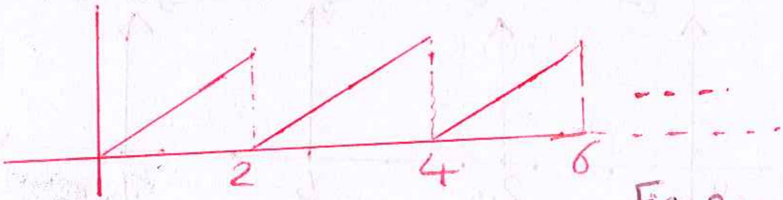

Fig. 32

$x_b(t) = r(t) - r(t-1) - r(t-2) + r(t-3)$

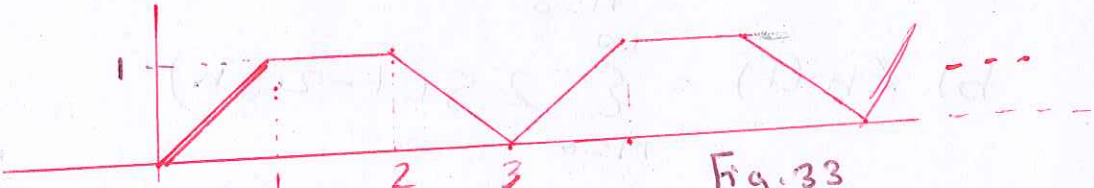

Fig. 33

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$$x_2(t) = \sum_{n=0}^{\infty} x_b(t-3n)$$

Example :

a) Sketch the signal $y(t) = \sum_{n=0}^{\infty} u(t-2n)\, u(1+2n-t)$

Ans :

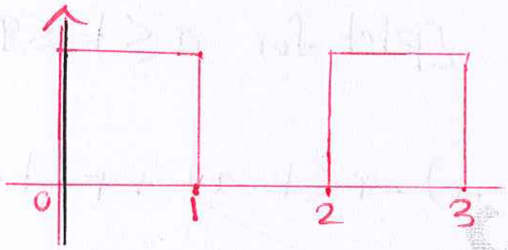

Fig.34

Example : Express the signal shown in terms of singularity functions .

a)

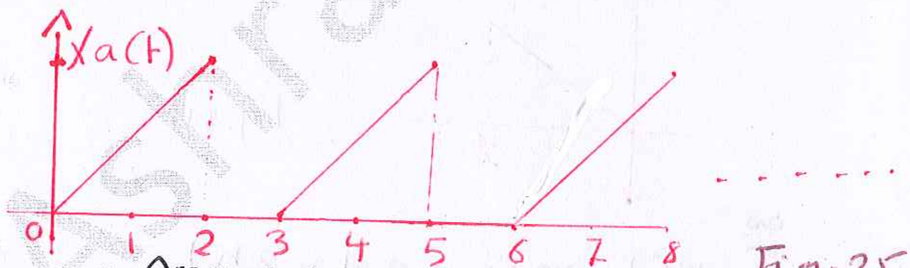

Fig.35

b)

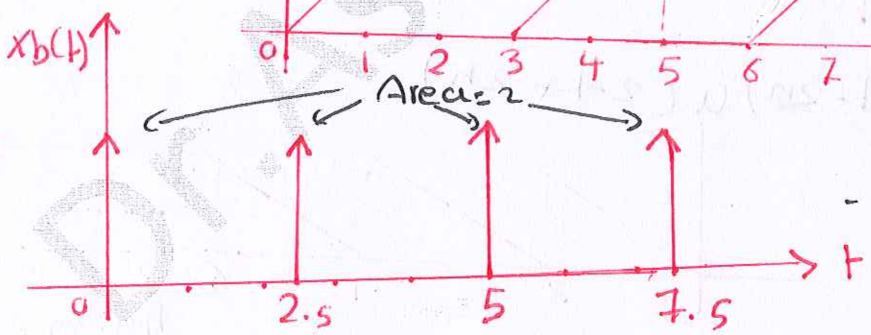

Fig.36

Ans : a) $x_a(t) = \sum_{n=0}^{\infty} r(t-3n)\, u(2+3n-t)$

b) $x_b(t) = \sum_{n=0}^{\infty} 2\,\delta(t-2.5n)$

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Example: plot the following signal using the elementary signals.

$$x(t) = 2\pi\left(\frac{t-4}{4}\right) + r\left(\frac{t-6}{6}\right) - r\left(\frac{t-9}{6}\right) + 2u(1-t)$$

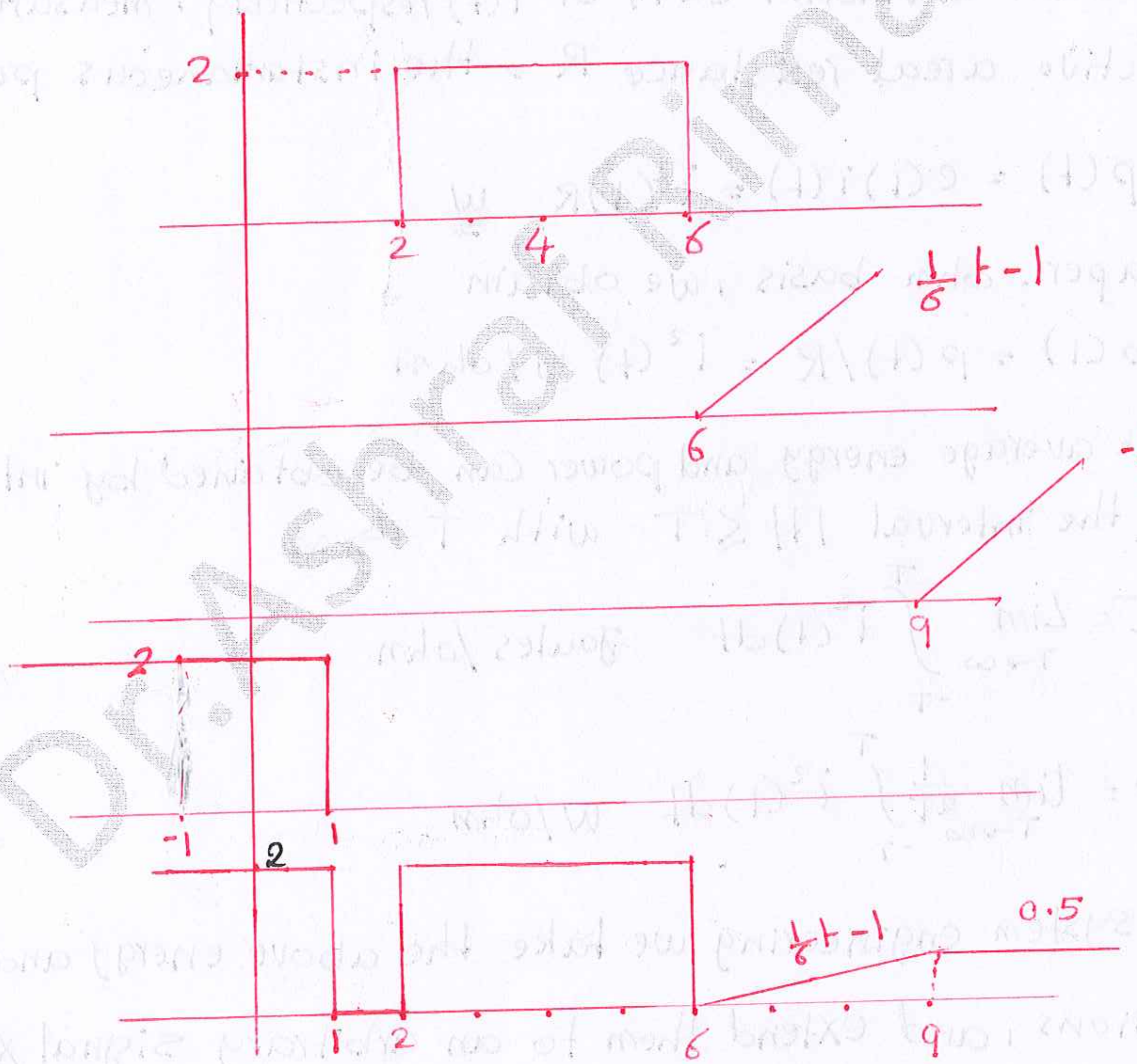

Fig. 37

Energy and power Signals :

From circuits and systems we know that areal voltage or current waveform, e(t) or i(t) respectively, measured with respective areal resistance R, the instantaneous power is

$$p(t) = e(t)i(t) = i^2(t)R \quad \underline{\underline{W}}$$

on aper-ohm basis, we obtain

$$p(t) = p(t)/R = i^2(t) \ w/ohm$$

The average energy and power can be obtained by integrating over the interval $|t| \leq T$ with $T \to \infty$

$$E = \lim_{T \to \infty} \int_{-T}^{T} i^2(t)\,dt \quad \text{Joules/ohm}$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)\,dt \quad w/ohm$$

In system engineering we take the above energy and power definitions, and extend them to an arbitrary signal $x(t)$, possibly complex, and define the normalized energy (e.g 1 ohm system) as :

$$E \triangleq \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

Signal Classes :

1- $x(t)$ is an energy signal if and only if $0 < E < \infty$ so that $P = 0$.

2- $x(t)$ is a power signal if and only if $0 < P < \infty$ which implies that $E \to \infty$.

Example : check if the following signal

$$x(t) = A e^{-\alpha t} u(t)$$

is power signal or energy signal? Justify your answer?

Ans :

$$E = \int_0^\infty (A e^{-\alpha t})^2 \, dt = \int_0^\infty A^2 e^{-2\alpha t} \, dt = \frac{-A^2}{2\alpha} e^{-2\alpha t} \Big|_0^\infty$$

$$= \frac{A^2}{2\alpha}$$

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Another way

$$E = \int_0^T (Ae^{-\alpha t})^2 dt = \int_0^T A^2 e^{-2\alpha t} dt = \lim_{T \to \infty} \frac{A^2}{-2\alpha} e^{-2\alpha t} \Big|_0^T$$

$$= \frac{A^2}{2\alpha}$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_0^T A^2 e^{-2\alpha t} dt = \lim_{T \to \infty} \frac{A^2}{2T} \cdot \frac{1}{2\alpha} \left[e^{-2\alpha T} - 1 \right]$$

$$= 0$$

Example : which of the following signals are power signals and which are enegy signals , Justify your answer

a- $u(t) + 5u(t-1) - 2u(t-2)$

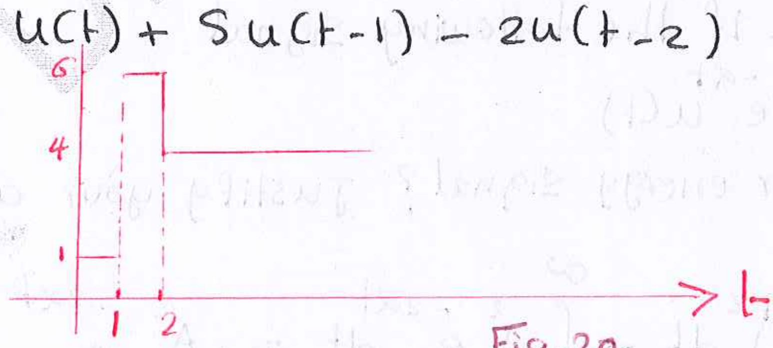

Fig.39

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

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$$= \lim_{T\to\infty} \left[\int_0^1 (1)^2 dt + \int_1^2 (6)^2 dt + \int_2^T (4)^2 dt \right]$$

$$= \lim_{T\to\infty} \left[1 + 36 + 16T - 32 \right] = \infty$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \frac{16}{2} < \infty$$

$\Rightarrow$ power signal

(b) $u(t) + 5u(t-1) - 6u(t-2)$

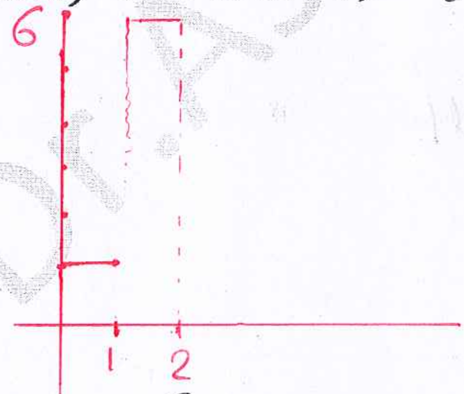

Fig. 39

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \left[\int_0^1 (1)^2 dt + \int_1^2 (6)^2 dt + \int_2^T (0)\, dt \right]$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = 0 \Rightarrow \text{Energy signal}$$

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Example :

c) $y(t) = 20\, r(t)\, \Pi(6-t) + \Pi(0.5t+6)$

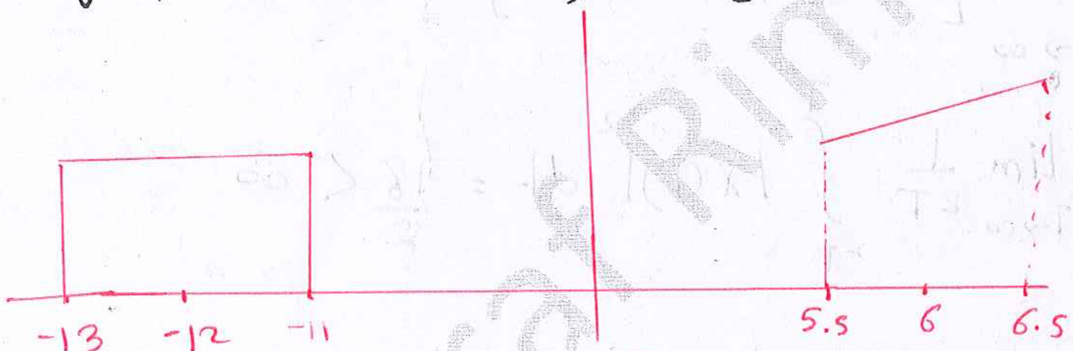

Fig. 40

Since the signal is bound and time limited
→ energy signal

$$E = \int_{-13}^{-11} (1)^2\, dt + \int_{5.5}^{6.5} (20t)^2\, dt$$

$$= 14.44 \text{ KJouls}$$

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Suggested Problems

Problem#1: Plot the following signals using the elementary signals:

ⓐ $x_a(t) = 2\,\Pi\left(\frac{5-t}{4}\right) + \Pi\left(\frac{t+3}{2}\right) - r\left(\frac{t-12}{2}\right) + 2u(t-16)$

ⓑ $x_b(t) = 2\,\Pi\left(\frac{t-4}{6}\right) - r\left(\frac{t-6}{2}\right) + r(-t+6)$

Problem #2: Given $x(t) = 10\sin^2\left(\pi t + \frac{\pi}{2}\right)$, compute

$$\int_{-\infty}^{\infty} x(t)\,\overset{\circ}{\delta}\left(t - \frac{\pi}{2}\right) dt$$

Problem #3: Determine if the signal $y(t) = 5\sin(10\pi t)\cdot\Pi\left(s(t-0.5)\right) + 4e^{2t}\,\Pi\left(\frac{t-5}{4}\right)$ is power/energy. In addition, in case it is power or energy determine the energy/the average power of the signal

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Problem#4: The signal $x(t)$ is composed as

$$x(t) = \cos(50\pi t) + 20\sin(19t).$$

a. Determine if the signal is periodic, in case it is, determine it's fundamental period.

b. Plot single-sided and double-sided for both phase and amplitude spectra.

Suggested Problems from text-book

Please try to solve the following problems from our text-book

1-16, 1-19, 1-20, 1-31, 1-33, 1-38, 1-39, 1-41, 1-43.

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Chapter Two: System Modeling in the Time Domain

what is a model? why do we need one?

We use the term model to refer to a set of a mathematical equations used to represent a physical system, relating the system's output signal to its input signal.

A model is required in order to:-

1. Understand system behavior (analysis)
2. Design a controller (synthesis)

A system is a quantitive description of a physical process which transforms signals (at its "input") to signals (at its "output").

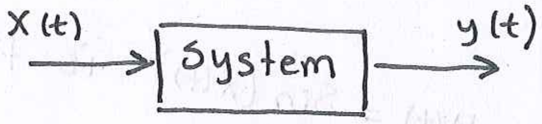

Fig 2.1

Properties of systems:

1. Continuous-Time and Discrete-Time Systems:

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If the signals processed by a system are continuous-time signals, the system itself is referred to as continuous-time system. If, on the other hand, the system process signals that exist only at discrete times, it is called a discrete-time system

2. Fixed and Time-variant system

A system is time-invariant if a time-shift of the input signal results in the same time-shift of the output signal. That is, if

$$x(t) \longrightarrow y(t)$$

then the system is time-invariant if

$$x(t-t_0) \longrightarrow y(t-t_0)$$

For any $t_0 \in \mathbb{R}$.

Example 2.1:

The system $y(t) = \sin(x(t))$ is time-invariant (Fixed)

whereas, the system $y(t) = x(t^2)$ is time-variant

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3. Causal and Non-causal Systems

A system is causal if the output at time t depends only on inputs at time $s \leq t$ (i.e, s defines the present and past time).

Example 2.2:

The system $3y(t) + \int_{-\infty}^{t} y(\lambda)\, d\lambda = x(t)$ is causal

whereas, the system $y(t) = x(t^2)$ and $y(t) = 10x(t+2) + 5$ are non-causal.

4. Dynamic and Instantaneous Systems :-

A system for which the output is a function of the input at the present time only is said to be instantaneous (or memoryless, or zero memory). A dynamic system, or one which is not instantaneous is one whose output depends on past or future values of the input in addition to present time. If the system is also causal it is dynamic system.

Example 2.3:

The system $y(t) = x(t)$ is instantaneous

whereas,

$$2\frac{dy(t)}{dt} + 3y(t) = \frac{d^2x(t)}{dt^2} + x(t)$$ is dynamic

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5. Linear and Non-linear System

A system is linear if it is additive and scalable. That is,

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

for all $\alpha_1, \alpha_2 \in \mathbb{C}$.

Example 2.4:

The system $y(t) = 2\pi\, x(t)$ is linear

Since

$$\alpha_1 y_1(t) = 2\pi\, \alpha_1 x_1(t)$$

$$\alpha_2 y_2(t) = 2\pi\, \alpha_2 x_2(t)$$

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = 2\pi\left(\alpha_1 x_1(t) + \alpha_2 x_2(t)\right)$$

$$\alpha_3 y_3(t) = 2\pi\, \alpha_3 x_3(t)$$

whereas,

$$\frac{dy(t)}{dt} + 10\, y(t) + 5 = x(t) \quad \text{is non-linear}$$

Since

$$\alpha_1 \frac{dy_1(t)}{dt} + 10\alpha_1 y_1(t) + 5\alpha_1 = \alpha_1 x_1(t)$$

$$\alpha_2 \frac{dy_2(t)}{dt} + 10\, \alpha_2 y_2(t) + 5\, \alpha_2 = \alpha_2 x_2(t)$$

$$\alpha_1 \frac{dy_1(t)}{dt} + \alpha_2 \frac{dy_2(t)}{dt} + 10\left[\alpha_1 y_1(t) + \alpha_2 y_2(t)\right] + 5(\alpha_1+\alpha_2) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow (1)$$

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If we assume

$$\alpha_3 y_3(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

$$\alpha_3 x_3(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

$$\Rightarrow \alpha_3 \frac{dy_3(t)}{dt} + 10\alpha_3 y_3(t) + 5\alpha_3 = \alpha_3 x_3(t) \longrightarrow (2)$$

But $\alpha_3 \neq \alpha_1 + \alpha_2$

$\Rightarrow$ Eq (1) $\neq$ Eq (2) $\Rightarrow$ non-linear System

Example 2.5: Which one of the following signals

1. $y(t) = x(t-2) + x(2-t)$
2. $y(t) = [\cos(3t)]\, x(t)$
3. $y(t) = \int_{-\infty}^{2t} x(\tau)\, d\tau$
4. $y(t) = x\left(\frac{t}{3}\right)$
5. $y(t) = \begin{cases} 0 & t < 0 \\ x(t) + x(t-2) & t \geq 0 \end{cases}$
6. $y(t) = \begin{cases} 0 & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$

is Linear, causal, time-invariant, and dynamic, Justify your answer?

Answer:

1. $y(t) = x(t-2) + x(2-t)$

ⓐ To check the Linearity; let us

$\alpha_1 y_1(t) = \alpha_1 x_1(t-2) + \alpha_1 x_1(2-t)$

$\alpha_2 y_2(t) = \alpha_2 x_2(t-2) + \alpha_2 x_2(2-t)$

$$\underbrace{\alpha_1 y_1(t) + \alpha_2 y_2(t)}_{} = \underbrace{\alpha_1 x_1(t-2) + \alpha_2 x_2(t-2)}_{} + \underbrace{\alpha_1 x_1(2-t) + \alpha_2 x_2(2-t)}_{}$$

$$\alpha_3 y_3(t) = \alpha_3 x_3(t-2) + \alpha_3 x_3(2-t)$$

$\Rightarrow$ Linear System.

ⓑ To check the causality, substitute any value for t, and then compare between input and output.

Assume $t = 0$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{Present}}}{y(0)} = \underset{\substack{\uparrow \\ \text{Previous}}}{x(-2)} + \underset{\substack{\uparrow \\ \text{Future}}}{x(2)}$$

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It can be noted that the output depends on the Future value.

$\Rightarrow$ Non-causal System.

(c) To check if the system is time-invariant or time-variant, we have to compare between "time-delay" result and "function-delay" result; where

$$y_1(t-t_0) = x_1(2-(t-t_0)) + x_1(t-t_0-2) \quad \text{"Time-Delay"}$$

$$y_2(t-t_0) = x_2(2-t-t_0) + x_2(t-t_0-2) \quad \text{"Function-Delay"}$$

Since $y_1(t-t_0) \neq y_2(t-t_0)$

$\Rightarrow$ the system is time-variant

(d) To check if the system is dynamic or instantaneous;

Since the output depends on past and future values of the input $\Rightarrow$ The system is dynamic.

The same procedure can be used in the rest; where

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Q	Linearity	Causality	Time-invariant	Dynamic
2	Linear	causal	time-variant	Memoryless
3	Linear	Non-causal	time-variant	Dynamic
4	Linear	Non-causal	time-variant	Dynamic
5	Linear	Causal	time-variant	Dynamic
6	Non-Linear	Causal	time-invariant	Dynamic

2.1 : The Superposition Integral for Fixed, Linear System

In this section we will show that the response of the system to a unit impulse applied at $t=0$ is $h(t)$ in which $h(t)$ is referred to as the impulse response of the system.

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For the system diagram shown in Fig. 2.1, if the system is Linear-time invariant (LTI) then

$$y(t) = x(t) \circledast h(t)$$

where $\circledast$ defines the convolution operation.

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

Example 2.6:- For the LTI System if

$$x(t) = 2\,\Pi\left(\frac{t-5}{2}\right) \quad \text{and} \quad h(t) = \Pi\left(\frac{t-2}{4}\right)$$

Find $y(t)$.

Answer: Since the system is LTI, then

$$y(t) = x(t) \circledast h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda = \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda$$

To do that, please follow the following procedure:-

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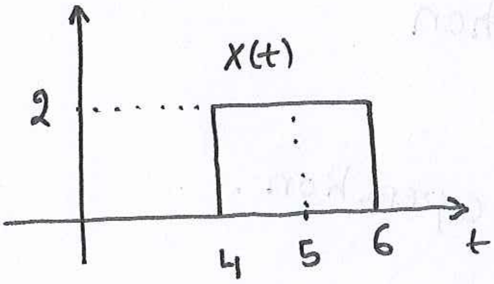
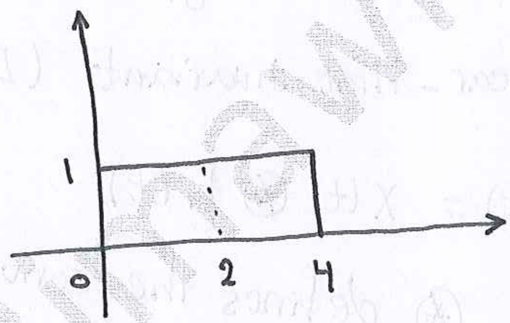
1. Plot $x(t)$ and $h(t)$





2. Specify the interval $y(t)$, this can be obtained from the intervals of $x(t)$ and $h(t)$

For $x(t) \xrightarrow{\text{interval}} [4, 6]$

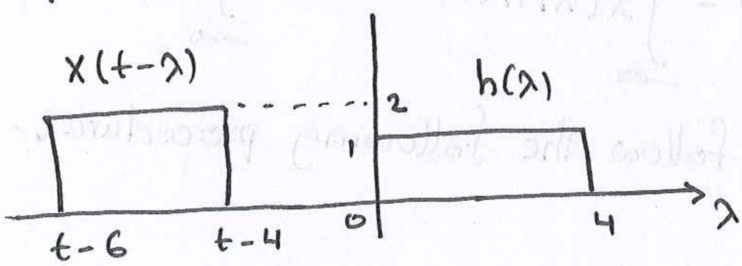
For $h(t) \xrightarrow{\text{interval}} [0, 4]$

$\Rightarrow$

For $y(t) \xrightarrow{\text{"}} [4, 6, 8, 10] \equiv [0+4, 0+6, 4+4, 4+6]$

3. Shift one of these signals, $x(t)$ or $h(t)$

In this example, we do the shift for $x(t)$



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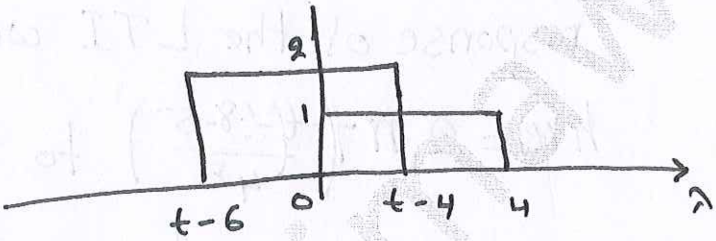
when $t<4$

$y(t)=0$

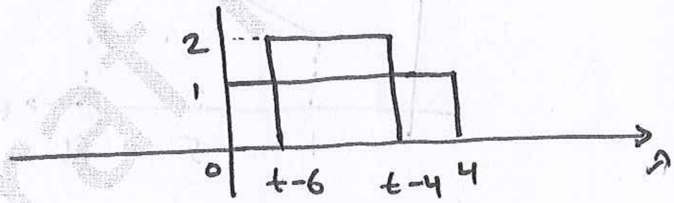
when $4<t<6$



$$y(t)=\int_0^{t-4}(2)(1)\,dt$$

$$=2(t-4)$$

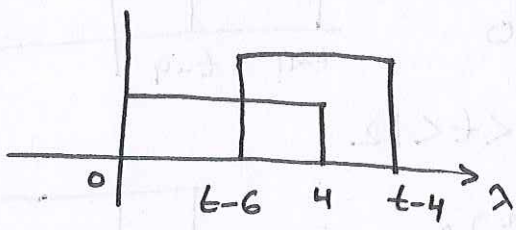
when $6<t<8$



$$y(t)=\int_{t-6}^{t-4}(1)(2)\,dt$$

$$=4$$

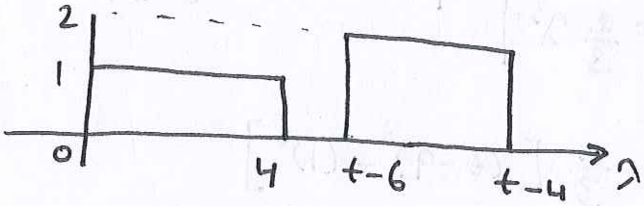
when $8<t<10$



$$y(t)=\int_{t-6}^{4}(1)(2)\,dt = 2(-t+6)$$

when $t>10$

$y(t)=0$



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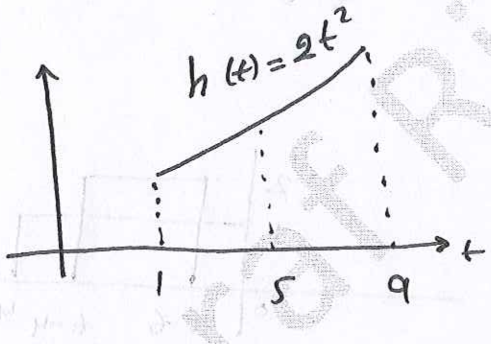
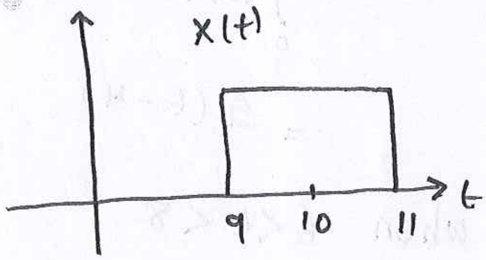
Example 2.7: Compute, using the convolution integral, the response of the LTI with impulse response $h(t) = 2t^2\,\Pi\left(\frac{t-5}{8}\right)$ to the input $x(t) = \Pi\left(\frac{t-10}{2}\right)$

Answer :-



$[1, 9]$

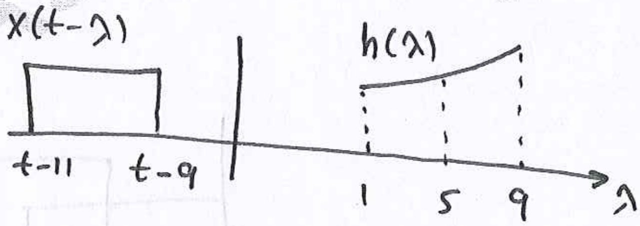


$[9, 11]$

$\Rightarrow$ The final interval is $[10, 12, 18, 20]$

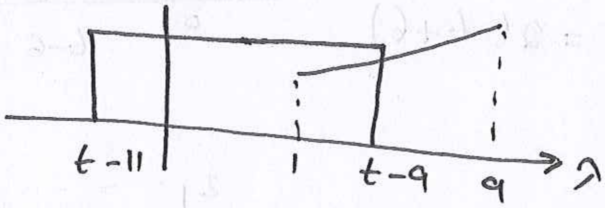
when $t < 10$

$y(t) = 0$



when $10 < t < 12$

$$y(t) = \int_1^{t-9} 2\lambda^2\, d\lambda$$



$$= \frac{2}{3}\lambda^3 \Big|_1^{t-9}$$

$$= \frac{2}{3}\left[(t-9)^3 - (1)^3\right]$$

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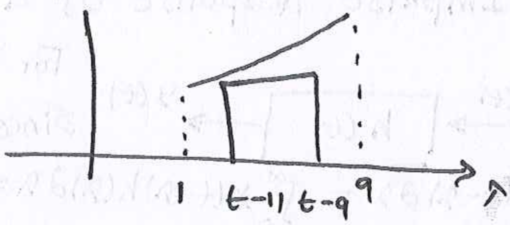
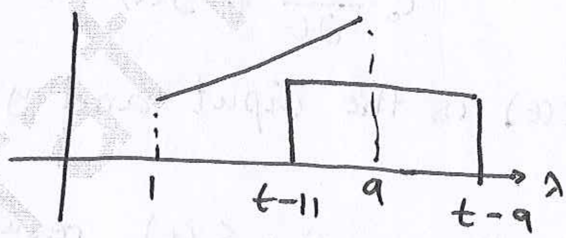
when $12 < t < 18$



$$y(t) = \int_{t-11}^{t-9} 2\lambda^2 \, d\lambda$$

$$= \frac{2}{3}\lambda^3 \Big|_{t-11}^{t-9} = \frac{2}{3}\left[(t-9)^3 - (t-11)^3\right]$$

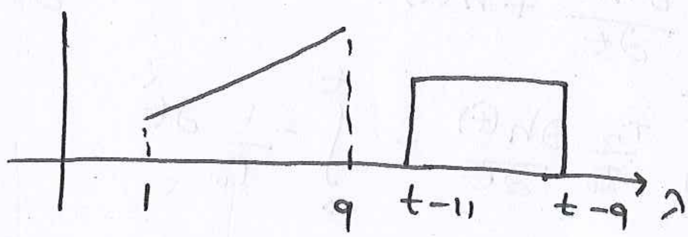
when $18 < t < 20$



$$y(t) = \int_{t-11}^{9} 2\lambda^2 \, d\lambda$$

$$= \frac{2}{3}\lambda^3 \Big|_{t-11}^{9} = \frac{2}{3}\left[(9)^3 - (t-11)^3\right]$$

when $t > 20$



$y(t) = 0$

Exercise: For the following signals:

$x(t) = u(t) - u(t-1)$ and $g(t) = \frac{t}{2}[u(t) - u(t-2)] + [u(t-2) - u(t-4)] + (-\frac{t}{2}+3)[u(t-4) - u(t-6)]$.

Find $y(t) = x(t) \circledast g(t)$

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2.2 Impulse Response of a Fixed, linear System

For LTI $\Rightarrow y(t) = x(t) \circledast h(t)$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Since $x(t) = \delta(t)$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, \partial\lambda = \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, \partial\lambda = \int_{-\infty}^{\infty} \delta(t-\lambda)\, h(\lambda)\, \partial\lambda = h(t)$$

Example 2.8: Find the impulse response of a system modeled by the differential equation.

$$\tau_0 \frac{\partial y(t)}{\partial t} + y(t) = x(t) \qquad -\infty < t < \infty$$

where $x(t)$ is the input and $y(t)$ is the output

Answer: Setting $x(t) = \delta(t)$ results is the response $y(t) = h(t)$

for $t > 0 \Rightarrow x(t) = 0$

$$\tau_0 \frac{\partial h(t)}{\partial t} + h(t) = 0 \Rightarrow \tau_0 \frac{\partial h(t)}{\partial t} = -h(t)$$

$$\int_0^t \frac{\tau_0}{\tau_0} \frac{\partial h(\grave{t})}{\partial \grave{t}} = \int_0^t -\frac{1}{\tau_0}\, \partial \grave{t}$$

$$\ln h(t) - \ln(h(0)) = -\frac{t}{\tau_0}$$

$$h(t) = h(0)\, e^{-t/\tau_0}$$

To find the initial value $h(0)$

$$\int_{-0}^{+0} \tau_0 \frac{\partial h(t)}{\partial t} + \int_{-0}^{+0} h(t)\,\partial t = \int_{-0}^{+0} \delta(t)\,dt$$

$$\tau_0 \left[h(0^+) - h(0^-) \right] + 0 = 1$$

$$\tau_0\, h(0^+) = 1 \Longrightarrow h(0^+) = 1/\tau_0$$

2.3 Superposition Integrals in Terms of Step Response

$$y(t) = \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, \partial\lambda$$

Let $u = x(t-\lambda)$ $\qquad dv = h(\lambda)\,\partial\lambda$

$du = -\dot{x}(t-\lambda)\,\partial\lambda \qquad v = -\int_{-\infty}^{\lambda} h(\lambda')\,\partial\lambda' = a(\lambda)$

$$x(t-\lambda)\,a(\lambda)\Big|_{-\infty}^{\lambda} + \int_{-\infty}^{\infty} \dot{x}(t-\lambda)\,a(\lambda)\,\partial\lambda$$

"The system is initially unexcited, so that $a(-\infty) = 0$ and $x(t-\lambda)\Big|_{\lambda=\infty} = 0$"

$$y(t) = \int_{-\infty}^{\infty} \dot{x}(t-\lambda)\,a(\lambda)\,\partial\lambda$$

" Duhamel's Integrals"

Example 2.9: Consider a system with a ramp input for which

$$x(t) = t\,u(t)$$
$$\dot{x}(t) = u(t)$$

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$$y_R(t) = \int_{-\infty}^{\infty} u(t-\lambda)\, a(\lambda)\, \partial\lambda$$

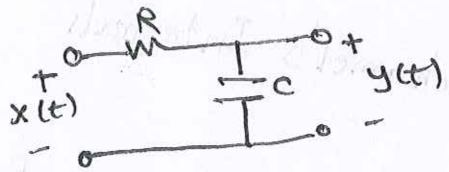
$$= \int_{-\infty}^{t} a(\lambda)\, d\lambda$$

Note that: the response of a system to a unit ramp, which is the integral of the unit step.

Generalizing, we conclude that for a fixed, linear system, any linear operation on the input produces the same linear operation on the output.

Example 2.10: Find the response of the RC circuit shown below to the triangle signal

$$x_\Delta(t) = r(t) - 2r(t-1) + r(t-2)$$



Answer: $-x(t) + R\, i(t) + y(t) = 0$

$$-x(t) + RC\frac{dy(t)}{dt} + y(t) = 0$$

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

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For the impulse response

$$h(t) = \frac{1}{RC} e^{-t/RC} \qquad t \geq 0$$

$$a(\lambda) = \int_0^{\lambda} \frac{1}{RC} e^{-t'/RC} \, \partial t' = 1 - e^{-t/RC}$$

$$y_R(t) = \int_{-\infty}^{t} [1 - e^{-t'/RC}] u(t') \, dt'$$

$$= r(t) - RC \left[1 - \exp\left(\frac{-t}{RC}\right)\right] u(t)$$

$$y_\Delta(t) = y_R(t) - 2\, y_R(t-1) + y_R(t-2)$$

In this result, $RC \ll 1$, the output closely approximates the input, whereas if $RC = 1$ the output does not resemble the input.

Example 2.11 : Determine the response of the following linear time invariant system (LTI) for a Dirac impulse input

$$x(t) = \delta(t)$$

$$\frac{\partial^2 y(t)}{\partial t^2} + 6 \frac{\partial y(t)}{\partial t} + 8\, y(t) = 18\, \dot{x}(t-2)$$

Use

Answer: Use $x(t)$ and then apply time-invariant shift

Let

$y_1(t) = g(t)\,u(t)$; where $g(t) = A e^{-2t} + B e^{-4t}$

Since $\lambda_{1,2} = -2, -4$

$$y_1(t) = [A e^{-2t} + B e^{-4t}]\,u(t)$$

$$y_1'(t) = g'(t)\,u(t) + g(t)\,\delta(t)$$

$$= g'(t)\,u(t) + g(0)\,\delta(t)$$

$$y_1''(t) = g''(t)\,u(t) + g'(t)\,\delta(t) + g'(t)\,\delta(t) + g(t)\,\dot{\delta}(t)$$

$$= g''(t)\,u(t) + g'(0)\,\delta(t) + g'(0)\,\delta(t) + g(0)\,\dot{\delta}(t)$$

Since

$$g(t) = A e^{-2t} + B e^{-4t}$$

$$g(0) = A + B$$

$$g'(t) = -2A e^{-2t} - 4B e^{-4t}$$

$$g'(0) = -2A - 4B$$

$$g''(t) = 4A e^{-2t} + 16B e^{-4t}$$

$$g''(0) = 4A + 16B$$

$$\Rightarrow g''(t)\,u(t) + g'(0)\,\delta(t) + g'(0)\,\delta(t) + g(0)\,\dot{\delta}(t) + 6\,g'(t)\,u(t) + 6\,g(0)\,\delta(t)$$
$$+ 8\,g(t)\,u(t) = 18\,\dot{\delta}(t)$$

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From previous Equation, it can be note that :-

$2\dot{g}(0) + 6\,g(0) = 0$ ----- (1)

$g(0) = 18$ ------ (2)

$\Rightarrow$ $2\dot{g}(0) + (6)(18) = 0$

$\dot{g}(0) = -54$ ------ (3)

Since $g(0) = A + B$ and $\dot{g}(0) = -2A - 4B$

$\Rightarrow$ $A + B = 18$ ------- (4)

$-2A - 4B = -54$ ------ (5)

From (4) and (5), the constants A and B are 9, and 9 respectively.

$\Rightarrow y_1(t) = 9\left[e^{-2t} + e^{-4t}\right] u(t)$

$y(t) = y_1(t-2)$

$= 9\left[e^{-2(t-2)} + e^{-4(t-2)}\right] u(t-2)$

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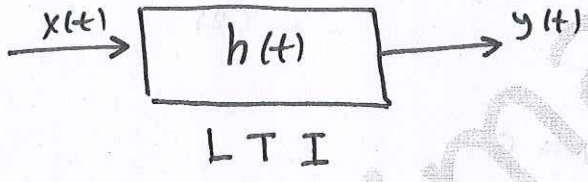
2.4 Frequency Response Function of a fixed, Linear System



If $x(t) = e^{j\omega t}$, then

$$y(t) = x(t) \circledast h(t) = \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda$$

$$y(t) = \int_{-\infty}^{\infty} e^{j\omega(t-\lambda)}\, h(\lambda)\, d\lambda$$

$$= e^{j\omega t} \int_{-\infty}^{\infty} e^{-j\omega\lambda}\, h(\lambda)\, d\lambda$$

$$= H(\omega)\, e^{j\omega t} \quad ; \quad H(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\lambda}\, h(\lambda)\, d\lambda$$

Later we shall see that $H(\omega)$ corresponds to the Fourier transform of the impulse response.

Example 2.12: Find the frequency response of RC circuit where $h(t) = \frac{1}{RC}\, e^{-t/RC}\, u(t)$

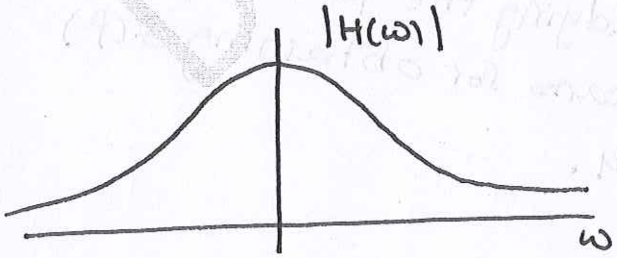
Answer: $$H(\omega) = \int_{-\infty}^{\infty} \frac{1}{RC} e^{-\frac{t}{RC}} e^{-j\omega t} u(t)\, dt$$

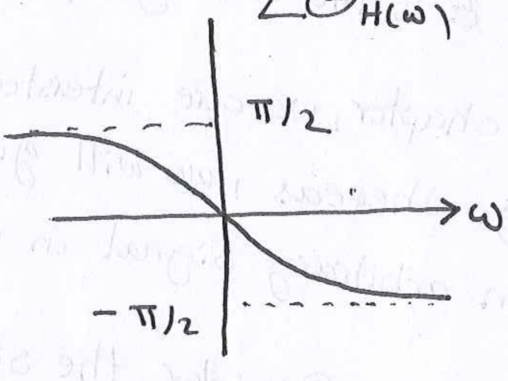
$$= \int_{0}^{\infty} \frac{1}{RC} e^{-\left(\frac{1}{RC} + j\omega\right)t} dt$$

$$= \frac{1}{1 + j\omega RC}$$

$$H(\omega) = \frac{1}{\sqrt{1+(\omega RC)^2}} e^{-j\tan^{-1}(\omega RC)}$$

$$= |H(\omega)| \, e^{j\angle\Theta_{H(\omega)}}$$





2.5: Energy and Power Spectral Density

The energy spectral density $(G(f))$ is defined as

$$E = \int_{-\infty}^{\infty} G(f)\, df$$

where E is the signal's total energy.

The power spectral density $(S(f))$ is defined as

$$P = \int_{-\infty}^{\infty} S(f)\, df$$

where P is the average power of the signal.

In this chapter, we are intersted in studying the power spectral density, whereas, we will give a means for obtaining $G(f)$ for an arbitrary signal in chapter 4.

Example 2.13: Consider the signal

$$x(t) = 10 \cos(10\pi t + \pi/7) + 4 \sin(30\pi t + \pi/8)$$

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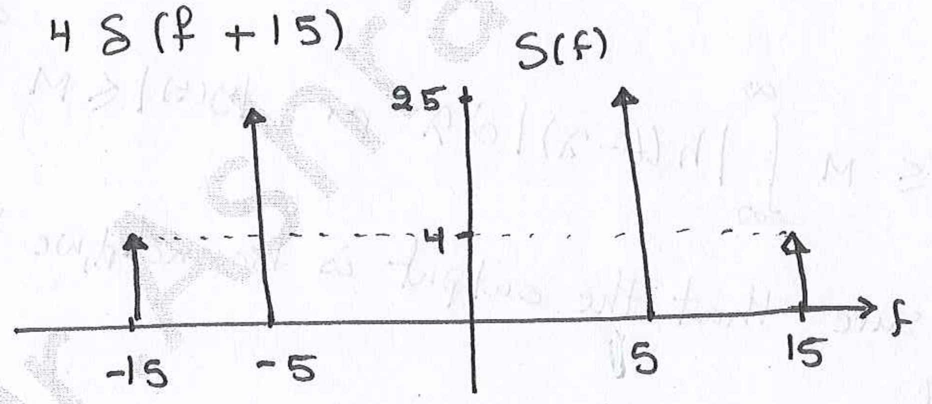
(a) Plot its power spectral density

(b) Compute the power lying within a frequency band from 10 Hz to 20 Hz.

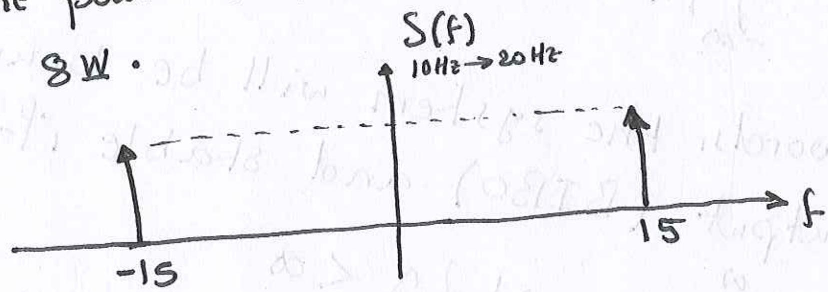
Answer:

(a) The power spectral density is

$$S(f) = 25\,\delta(f+5) + 25\,\delta(f-5) + 4\,\delta(f-15) + 4\,\delta(f+15)$$



(b) The power lying within a frequency band from 10 Hz to 20 Hz is 8 W.



whereas the total power is

$$P_{tot} = \int_{-\infty}^{\infty} S(f)\,df = 50 + 8 = 58 \text{ W}$$

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2.6 Stability of Linear Systems

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda \right| \leq \int_{-\infty}^{\infty} |x(\lambda)|\, |h(t-\lambda)|\, d\lambda$$

Since the input $x(t)$ is bounded, this means that

$|x(\lambda)| \leq M < \infty$; where M is constant.

Then

$$|y(t)| \leq M \int_{-\infty}^{\infty} |h(t-\lambda)|\, d\lambda \quad \text{or} \quad |y(t)| \leq M \int_{-\infty}^{\infty} h(\gamma)\, d\gamma$$

To make sure that the output is bounded, we have to check if

$$\int_{-\infty}^{\infty} |h(t-\lambda)|\, d\lambda < \infty$$

In other words, the system will be bounded input bounded output (BIBO) and stable if

$$\int_{-\infty}^{\infty} |h(t-\lambda)|\, d\lambda < \infty$$

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Example 2.14: For the following response system

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

check the stability of the system?

Answer: To check the stability, we have to check if $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ or not

Since

$$\int_{0}^{\infty} \frac{1}{RC} e^{-t/RC} dt = -\exp(-t) \Big|_{0}^{\infty} = 1 < \infty$$

which means that the system BIBO, then the system is stable.

Exercise: check the stability for the system where it's response

$$h(t) = \sin(\omega_0 t).$$

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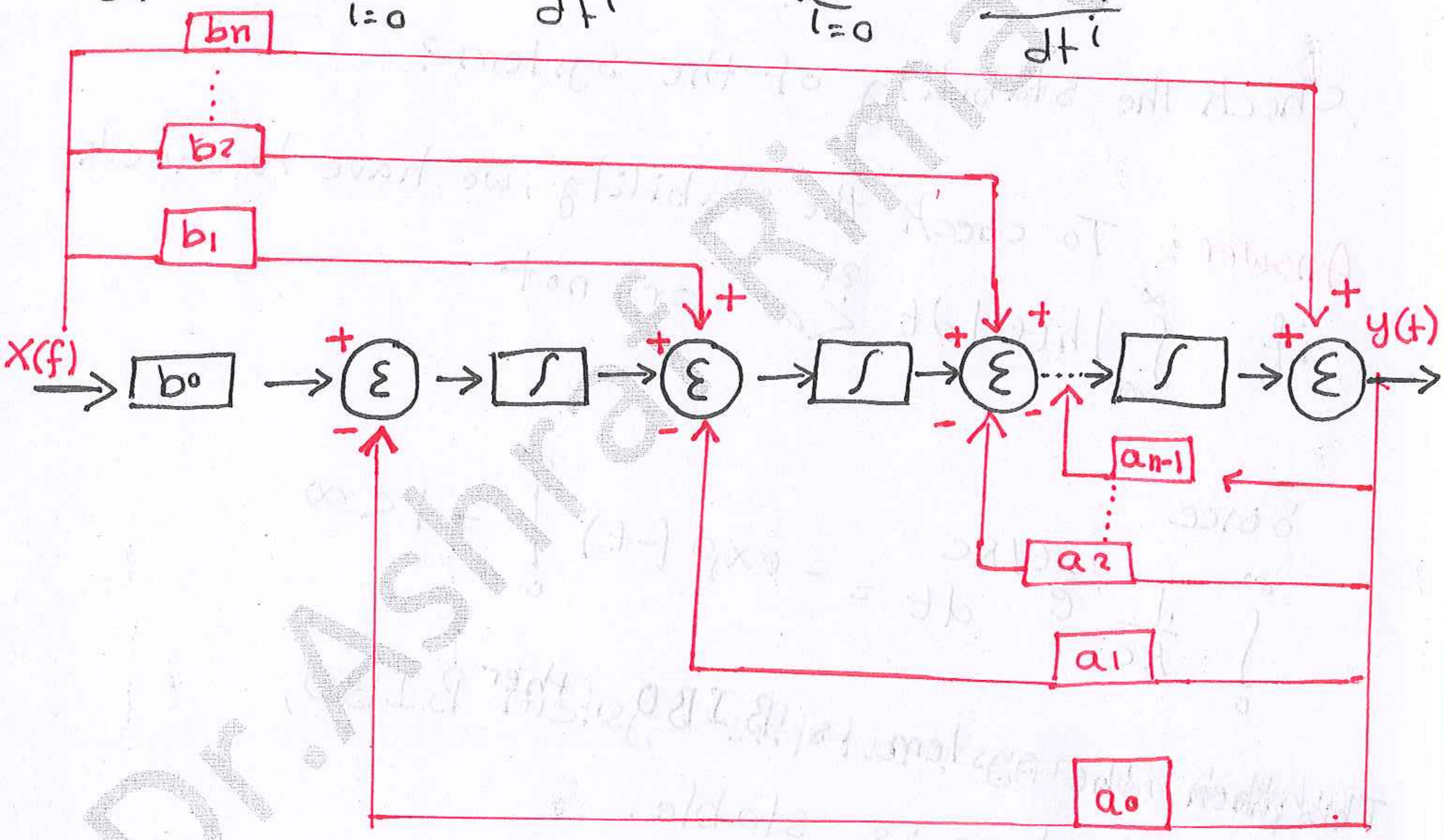
System Modeling and Simulation

$$\frac{d^n y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{n} b_i \frac{d^i x(t)}{dt^i}$$



Example : plot the simulink model for the following differential equation .

$$-x(t) + \frac{L}{R}\frac{dy(t)}{dt} + y(t) = 0$$

Answer : $\frac{L}{R}\frac{dy(t)}{dt} + y(t) = x(t) \Rightarrow \frac{dy(t)}{dt} + \frac{R}{L} y(t) = \frac{R}{L} x(t)$

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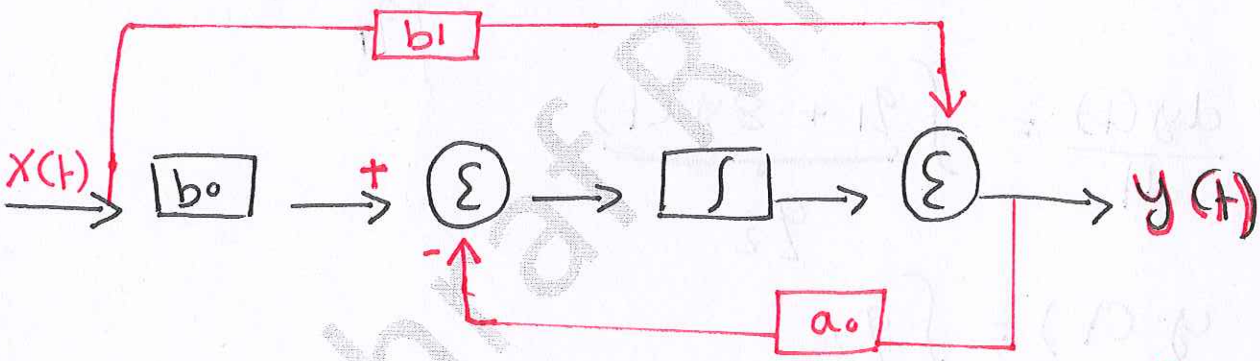
$$\frac{dy(t)}{dt} = \underbrace{\frac{R}{L}x(t) - \frac{R}{L}y(t)}_{q_0}$$

$$y(t) = \int q_0$$



Example : plot the simulink model for the fallowing differential equation

$$2\frac{d^3y(t)}{dt^3} - 8\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 2y(t) = 4\frac{dx(t)}{dt} + 2x(t)$$

Answer : $\frac{d^3y(t)}{dt^3} - 4\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 2\frac{dx(t)}{dt} + x(t)$

$$\frac{d^3y(t)}{dt^3} - 8\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 2y(t) = 4\frac{dx(t)}{dt} + 2x(t)$$

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$$\frac{d^3y(t)}{dt^3} - \frac{8d^2y(t)}{dt^2} + \frac{4dy(t)}{dt} - \frac{4dx(t)}{dt} = \underbrace{2x(t) - 2y(t)}_{q_0}$$

$$\frac{d^2y(t)}{dt^2} - \frac{8dy(t)}{dt} = \int q_0 \underbrace{- 4y(t) + 4x(t)}_{q_1}$$

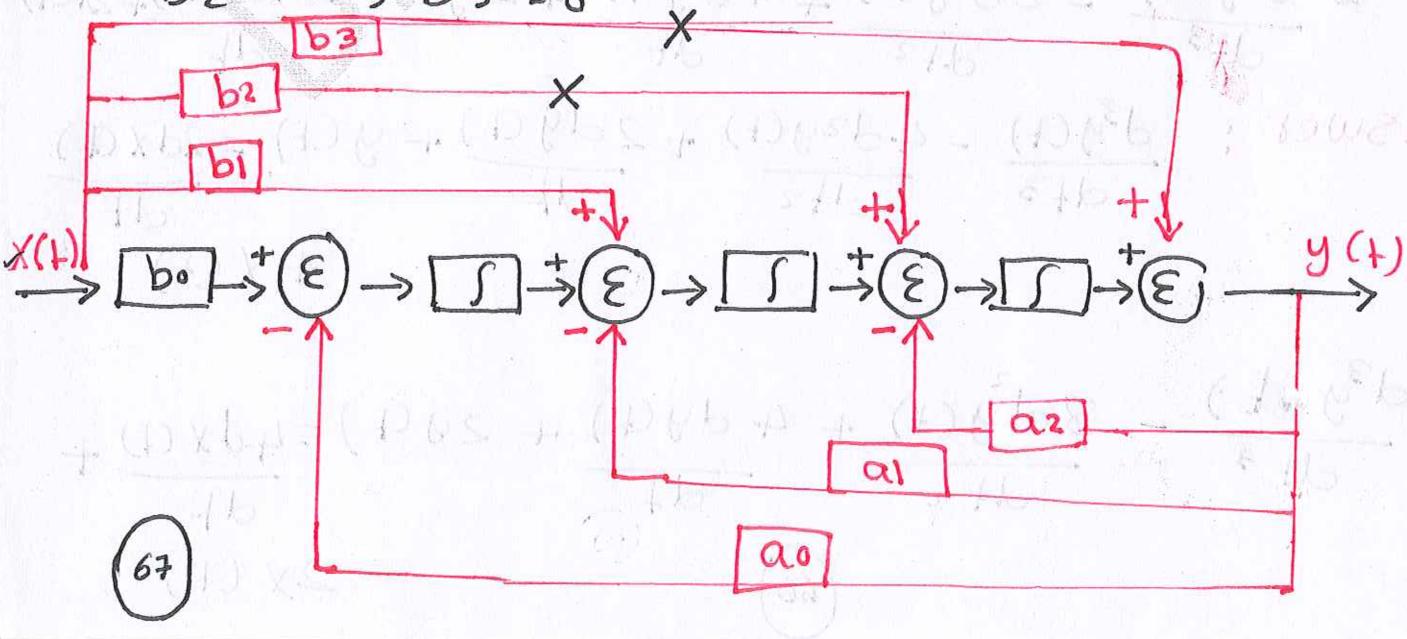
$$\frac{dy(t)}{dt} = \int \underbrace{q_1 + 8y(t)}_{q_2}$$

$$y(t) = \int q_2$$

From the equations it can be noted that:

$n=3$, $a_0=1$, $a_1=2$, $a_2=-4$, $b_0=1$, $b_1=2$

$b_2=0$, $b_3=0$



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Suggested Problems

Problem #1: Determine if the following system is linear, fixed, dynamic, and causal:

$$y(t) = \sqrt{x(t^2)}$$

Problem #2: Determine, using the convolution integral, the response of the system with impulse response $h(t) = t\,\Pi\left(\frac{t-4}{4}\right)$ to the input $x(t) = \Pi\left(\frac{t-4}{8}\right)$

Problem #3: Determine the response of the following system for $x(t) = \delta(t)$

$$\frac{\partial^2 y(t)}{\partial t^2} + 6\frac{\partial y(t)}{\partial t} + 5y(t) = 18\ddot{x}(t)$$

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Problem # 4: Plot the simulation diagram of the following system showing the modeling procedure

$$\frac{\partial^4 y(t)}{\partial t^4} - 5\frac{\partial^3 y(t)}{\partial t^3} + 6\frac{\partial y(t)}{\partial t} + 7\,y(t) =$$

$$6\frac{\partial^3 x(t)}{\partial t^3} - 5\frac{\partial x(t)}{\partial t} + 15\,x(t)$$

Suggested Problems from text-book

Please try to solve the following problems from our text-book

2-2, 2-3, 2-4, 2-5, 2-6, 2-7, 2-11, 2-13, 2-14, 2-17

2-29

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Chapter Three: Fourier Series

In this chapter and chapter Four we consider procedures for resolving certain classes of signals into superpositions of sines and cosines or, equivalently, complex exponential signals of the form $\exp(j\omega t)$.

The advantage of Fourier Series and Fourier Transform representations for signals are two fold: First, in the analysis and design of system, it is often useful to characterize signals in terms of frequency domain parameters such as bandwidth or spectral content. Second, the superposition property of linear systems, and the fact that the steady-state response of a fixed, linear system to a sinusoid of a given frequency is itself a sinusoid of the same frequency.

Based on the above, the main question is why do we want to work in frequency domain?

* In some systems the convolution integral is difficult.

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3.1: Trigonometric Series

For LTI system, it can be noted that the sinusoidal and $e^{j\omega t}$ only changes amplitude and phase through linear system.

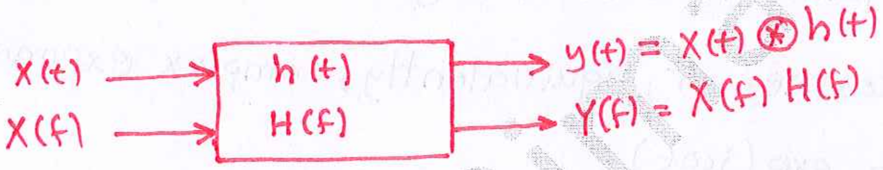

Fig 3.1

The main question is " Do you beleive that most signals are periodic signals? , and can be represented as sum of sinusoidal signal ? "

In general form, a trigonometric series for representing periodic signal is given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \quad \text{------- (3.1)}$$

The objective is

1. obtaining Trigonometric Fourier Series representation for periodic signal.
2. Find the trigonometric coefficient Fourier Series, a_0, a_n, b_n.

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In (3.1), if we take the integral on full period, then we obtain

$$\int_{T_0} x(t)\,dt = \int_{T_0} a_0\,dt + \int_{T_0} \sum_{n=1}^{\infty} a_n \underbrace{\cos(n\omega_0 t)\,dt}_{\to 0} + \int_{T_0} \sum_{n=1}^{\infty} b_n \underbrace{\sin(n\omega_0 t)\,dt}_{\to 0}$$

For periodic signal   For periodic signal

in which,

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

Now, to evaluate the coefficient "an", let us consider the following

$$\int_{T_0} x(t)\cos(m\omega_0 t)\,dt = \int_{T_0} a_0 \cos(m\omega_0 t)\,dt + \int_{T_0} \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt$$

$$+ \int_{T_0} \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt$$

Since

$$\cos(n\omega_0 t)\cos(m\omega_0 t) = \frac{1}{2}\cos((n+m)\omega_0 t) + \frac{1}{2}\cos((n-m)\omega_0 t)$$

Then, when $n = m$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt$$

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whereas, when $n \neq m$, a_n is not defined.

Finally, to evaluate the coefficient b_n, let us do the following

$$\int_{T_0} x(t)\sin(m\omega_0 t)\,dt = \int_{T_0} a_0 \sin(m\omega_0 t)\,dt + \int_{T_0} \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)\sin(m\omega_0 t)\,dt$$

$$+ \int_{T_0} \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt$$

By using the following

$$\sin(n\omega_0 t)\sin(m\omega_0 t) = \frac{1}{2}\cos((n-m)\omega_0 t) - \frac{1}{2}\cos((n+m)\omega_0 t)$$

then,

$$b_n = \begin{cases} \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt, & n = m \\ \text{not defined}, & n \neq m \end{cases}$$

Example 3.1: Consider the square wave defined by

$$x(t) = \begin{cases} A, & 0 < t < T_0/2 \\ -A, & \frac{T_0}{2} < t < T_0 \end{cases}$$

Find the trigonometric Fourier Series coefficients

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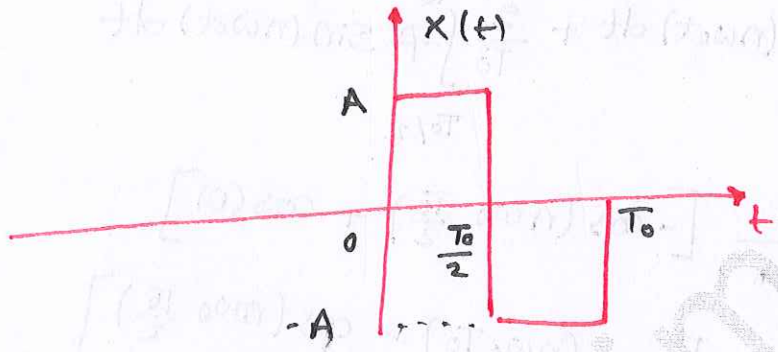
Answer:



$$a_0 = \frac{1}{T_0}\int_0^{T_0/2} A\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} -A\,dt = 0$$

$$a_n = \frac{2}{T_0}\int_0^{T_0/2} A\cos(n\omega_0 t)\,dt + \frac{2}{T_0}\int_{T_0/2}^{T_0} -A\cos(n\omega_0 t)\,dt$$

$$= \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\sin(n\omega_0 t)\Big|_0^{T_0/2} - \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\sin(n\omega_0 t)\Big|_{T_0/2}^{T_0}$$

Since $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

$\Rightarrow$

$$a_n = \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[\sin\left(n\cdot\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right) - \sin(0)\right]$$

$$-\frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[\sin\left(n\frac{2\pi}{T_0}\cdot T_0\right) - \sin\left(n\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right)\right]$$

$= 0$ $\quad$ n even or odd

$$b_n = \frac{2}{T_0}\int_0^{T_0/2} A \sin(n\omega_0 t)\,dt + \frac{2}{T_0}\int_{T_0/2}^{T_0} -A \sin(n\omega_0 t)\,dt$$

$$= \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[-\cos\left(n\omega_0 \frac{T_0}{2}\right) + \cos(0)\right]$$

$$+ \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[\cos(n\omega_0 T_0) - \cos\left(n\omega_0 \frac{T_0}{2}\right)\right]$$

$$= \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[-\cos\left(n\,\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right) + \cos(0)\right]$$

$$+ \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[\cos\left(n\,\frac{2\pi}{T_0}\cdot T_0\right) - \cos\left(n\,\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right)\right]$$

$$= \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[-\cos(n\pi) + 1\right] + \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[\cos(2\pi n) - \cos(n\pi)\right]$$

when n even

$\Rightarrow$

$$b_n = \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[-1+1\right] + \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[1-1\right] = 0$$

whereas, when n odd

$\Rightarrow$

$$b_n = \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[1+1\right] + \frac{2A}{T_0}\cdot\frac{1}{n\omega_0}\left[1+1\right]$$

$$= \frac{4A}{n\pi}$$

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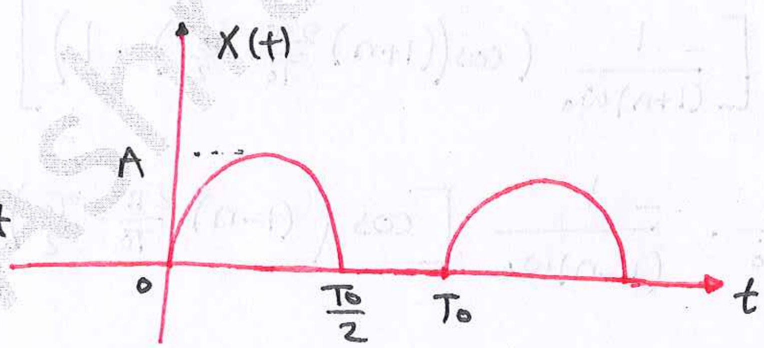
Example 3.2 : Find the coefficients of the trigonometric Fourier series for a half-rectified sine wave, defined by

$$x(t) = \begin{cases} A \sin(\omega_0 t), & 0 \le t \le T_0/2 \\ 0, & \frac{T_0}{2} \le t \le T_0 \end{cases}$$



Answer :

$$a_0 = \frac{1}{T_0}\int_0^{T_0/2} A\sin(\omega_0 t)\,dt$$

$$= \frac{A}{T_0}\left[-\frac{1}{\omega_0}\right]\cos(\omega_0 t)\Big|_0^{T_0/2}$$

$$= \frac{-A}{T_0 \cdot \frac{2\pi}{T_0}}\left[\cos\left(\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right) - 1\right]$$

$$= \frac{-A}{2\pi}\left[-2\right] = \frac{A}{\pi}$$

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$$a_n = \frac{2}{T_0}\int_0^{T_0/2} A\sin(\omega_0 t)\cos(n\omega_0 t)\,dt$$

Since

$$\sin(\omega_0 t)\cos(n\omega_0 t) = \frac{1}{2}\sin((1+n)\omega_0 t) + \frac{1}{2}\sin((1-n)\omega_0 t)$$

$\Rightarrow$

$$a_n = \frac{2A}{T_0(2)}\left[\int_0^{T_0/2}\sin((1+n)\omega_0 t)\,dt + \int_0^{T_0/2}\sin((1-n)\omega_0 t)\,dt\right]$$

$$= \frac{A}{T_0}\left[\frac{-1}{(1+n)\omega_0}\left(\cos\left((1+n)\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right)-1\right)\right]$$

$$+ \frac{A}{T_0}\cdot\frac{-1}{(1-n)\omega_0}\left[\cos\left((1-n)\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right)-1\right]$$

$\Rightarrow$

$$a_n = \frac{-A}{2(1+n)\pi}\left(\cos(\pi(1+n))-1\right) - \frac{A}{2(1-n)\pi}\left(\cos((1-n)\pi)-1\right)$$

if n even, then

$$a_n = \frac{-A}{2(1+n)\pi}(-2) - \frac{A}{2(1-n)\pi}(-2) = \frac{2A}{(1-n^2)\pi}$$

whereas, when n odd, then

$$a_n = \frac{+A}{(1+n)\pi}(0) + \frac{A}{(1-n)\pi}(0) = 0$$

Now, let us determine the coefficient b_n; that is,

$$b_n = \frac{2}{T_0}\int_0^{T_0/2} A\sin(\omega_0 t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2A}{2T_0}\left[\frac{1}{(1-n)\omega_0}\left[\sin\left((1-n)\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right) - 0\right]\right]$$

$$- \frac{A}{T_0}\left[\frac{1}{(1+n)\omega_0}\left[\sin\left((1+n)\frac{2\pi}{2}\right) - 0\right]\right]$$

$$= \frac{A}{T_0}\left[\frac{1}{(1-n)\omega_0}\left(\sin((1-n)\pi) - 0\right)\right]$$

$$- \frac{A}{T_0}\left[\frac{1}{(1+n)\omega_0}\left(\sin((1+n)\pi) - 0\right)\right]$$

when n even

$$b_n = \frac{A}{T_0}\left[\frac{1}{(1-n)\omega_0}(0-0)\right] - \frac{A}{T_0}\left[\frac{1}{(1+n)\omega_0}(0-0)\right]$$

$$= 0$$

when n odd

$b_n = 0$

Now let us determin the coefficients, an and bn at the values of n in which fcn. is undefined. In our example the values of n are 1, and -1

$a_1 = a_{-1} = 0$

whereas,

$$b_1 = \frac{2}{T_0}\int_0^{T_0/2} A \sin^2(1 \cdot \omega_0 t)\, dt = \frac{2}{T_0}\left[\int_0^{T_0/2} \frac{A}{2}\, dt - \int_0^{T_0/2} \frac{A}{2}\cos(2\omega_0 t)\, dt\right.$$

$$= \frac{\cancel{2}A}{T_0} \cdot \frac{1}{\cancel{2}} \cdot \frac{T_0}{2} - \left[\cancel{\frac{2A}{2T_0}\left(\frac{1}{2 \cdot \frac{2\pi}{T_0}}\left(\sin\left(2\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right) - 0\right)\right)}\right]^{0}$$

$$= \frac{A}{2} = -b_{-1}$$

3.3 The Complex Exponential Fourier Series

From Euler's Equation the $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ can be expressed as:

$$\sin(n\omega_0 t) = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{j2}$$

and

$$\cos(n\omega_0 t) = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

respectively.

By substituting in $x(t)$, where $x(t)$ is given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\left[e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right]$$

$$+ \sum_{n=1}^{\infty} b_n \left[\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{j2}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n - jb_n}{2}\right] e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \left[\frac{a_n + jb_n}{2}\right] e^{-jn\omega_0 t}$$

$$= X_0 + X_{-1} e^{-jw_0 t} + X_{-2} e^{-j2w_0 t} + X_1 e^{jw_0 t} + X_2 e^{j2w_0 t} + \cdots\cdots$$

$$= \sum_{n=-\infty}^{\infty} X_n e^{jnw_0 t}$$

$\Rightarrow$ From the equation above, it can be noted that the coefficients of the trigonometric fourier series and the complex coefficients are related by

$$X_n = \begin{cases} \frac{1}{2}(a_n - j b_n) & n > 0 \\ \frac{1}{2}(a_{-n} + j b_{-n}) & n < 0 \end{cases}$$

and

$$a_n = 2 \operatorname{Re}\{X_n\} \quad \text{and} \quad b_n = -2 \operatorname{Im}\{X_n\}$$

$X_n = X_{-n}^*$ where $X_n = |X_n| e^{j\theta_n}$

$$|X_n| = |X_{-n}| \text{ and } \theta_n = -\theta_{-n}$$

From previous example [half-wave rectified]

$$\Rightarrow X_n \begin{cases} \frac{A}{\pi(1-n^2)} & n = 0, \pm 2\ \ 2 \pm 4, \cdots\cdots \\ 0 & n \text{ odd and } n \neq \pm 1 \\ -\frac{1}{4} j n A & n = \pm 1 \end{cases}$$

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Question : How can we find the expression for X_n

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_0 t}$$

$$\int_{T_0} x(t) e^{-jnw_0 t} dt = \sum_{n=-\infty}^{\infty} X_n \int e^{jnw_0 t} e^{-jmw_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} X_n \int_{T_0} e^{j(n-m)w_0 t} dt$$

when $n = m$

$$\Rightarrow X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jnw_0 t} dt$$

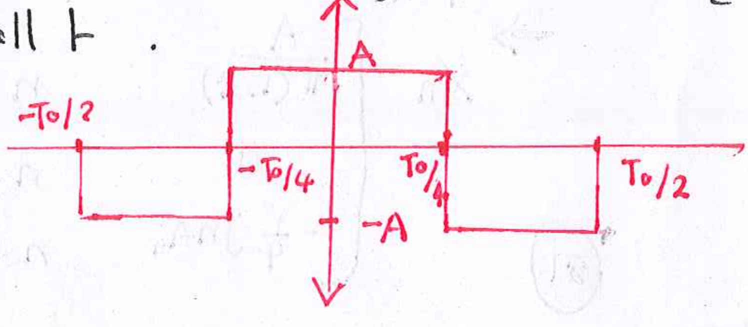
Example : Obtain the exponential fourier series of the square wave

$$x(t) = \begin{cases} A & -\frac{T_0}{4} < t \leq \frac{T_0}{4} \\ -A & -\frac{T_0}{2} < t \leq -\frac{T_0}{4} \text{ and } \frac{T_0}{4} < t \leq \frac{t_0}{2} \end{cases}$$

with $x(t) = x(t + T_0)$, all t .

Ans :

$$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jnw_0 t} dt$$



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$$= \frac{1}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt - j\frac{1}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{-T_0/4} -A\cos(n\omega_0 t)\,dt - j\frac{1}{T_0}\int_{-T_0/2}^{-T_0/4} -A\sin(n\omega_0 t)\,dt$$

$$+ \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\cos(n\omega_0 t)\,dt - j\frac{1}{T_0}\int_{-T_0/2}^{T_0/4} A\sin(n\omega_0 t)\,dt$$

$$+ \frac{1}{T_0}\int_{T_0/4}^{T_0/2} -A\cos(n\omega_0 t)\,dt - j\frac{1}{T_0}\int_{T_0/4}^{T_0/2} -A\sin(n\omega_0 t)\,dt$$

$$= \begin{cases} 0 & n \text{ even}. \\ (-1)^{(n-1)/2}\dfrac{2A}{n\pi}, & n \text{ odd}. \end{cases}$$

Example 2: Obtain the exponential fourier series of the sawtooth wave form defined by:

$$x(t) = At \quad , \quad -\frac{T_0}{2} \leq t < \frac{T_0}{2}$$

$$x(t) = x(t+T_0) \quad , \quad \text{all } t$$

$$= \sum_{n=-\infty}^{\infty} X_n X_n^*$$

$$= \sum_{n=-\infty}^{\infty} |X_n|^2$$

$$= X_0^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

Example : For the following signal

$$x(t) = 4 \sin(50\pi t)$$

Determine the average power .

Ans : Method $\underline{\underline{1}}$:

By using general formula to calculate the average power

$$P_{av} = \frac{1}{T_0}\int_{T_0} (4)^2 \sin^2(50\pi t)\, dt \quad ; \quad \cancel{2\pi} f_0 = \overset{25}{\cancel{50}}\, \cancel{\pi}$$

$$f_0 = 25 = \frac{1}{T_0}$$

$$= 25 \int_0^{0.04} 16 \sin^2(50\pi t)\, dt = \frac{16}{2} = 8\,W$$

Method $\underline{\underline{2}}$:

By using parseval's theorem :

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jnw_0 t}\, dt$$

$$= 25 \int_0^{0.04} 4\sin(50\pi t)\, e^{-jn\omega_0 t}\, dt$$

$$= 100 \int_0^{0.04} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2}\right) e^{-jn\omega_0 t}\, dt$$

$$= \frac{50}{j} \int_0^{0.04} e^{j(1-n)\omega_0 t}\, dt - \frac{50}{j} \int_0^{0.04} e^{-j(1+n)\omega_0 t}\, dt$$

$$= -j50 \left[\frac{1}{j(1-n)\omega_0} \left(e^{j(1-n)\omega_0 0.04} - 1\right) + \frac{1}{j(1+n)\omega_0} \left(e^{-j(1+n)\omega_0 \cdot 0.04} - 1\right) \right.$$

when n even or odd and $n \neq \pm 1$

$$X_n = 0$$

whereas, $n = 1$

$$X_1 = 25 \int_0^{0.04} 4\sin(50\pi t)\, e^{-j\omega_0 t}\, dt$$

$$= -j2$$

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when $n = -1$

$X_{-1} = +j2$

$\Rightarrow$

$$P_{av} = X_0^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

$$= 0 + 2(4) = 8\ \underline{\underline{W}}$$

Energy and power spectral densities:

It is useful for some applications to define functions of frequency that when integrated over all frequencies give total energy or total power, depending on whether the signal under consideration is respectively, an energy signal or a power signal, for an energy signal, a function of frequency when integrated that gives total energy is refered to as an energy spectral density.

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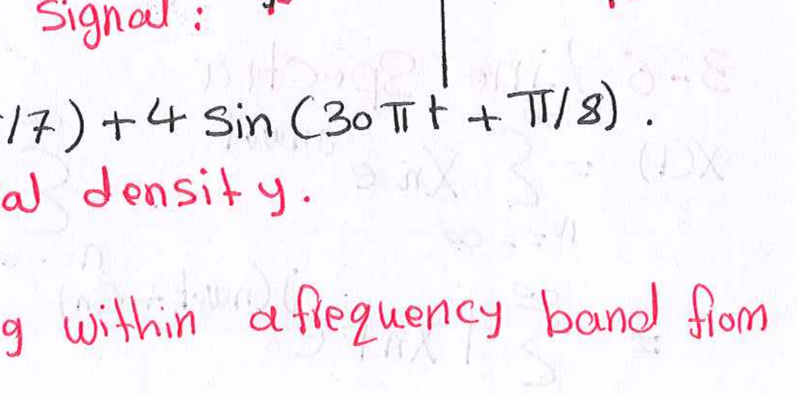
Denoting the energy spectral density of a signal $x(t)$ by $G(f)$ we have definition, that

$$E = \int_{-\infty}^{\infty} G(f)\, df$$

where E is the signal's total energy, we will give a mean for obtaining $G(f)$ for an arbitrary energy signal in chapter "4".

Denoting the power spectral density of a power signal $x(t)$ by $S(f)$, we have, by definition, that

$$P = \int_{-\infty}^{\infty} S(f)\, df$$



Example: Consider the signal:

$$x(t) = 10\cos(10\pi t + \pi/7) + 4\sin(30\pi t + \pi/8).$$

a) plot its power spectral density.

b) Compute the power lying within a frequency band from 10 HZ to 20 HZ.

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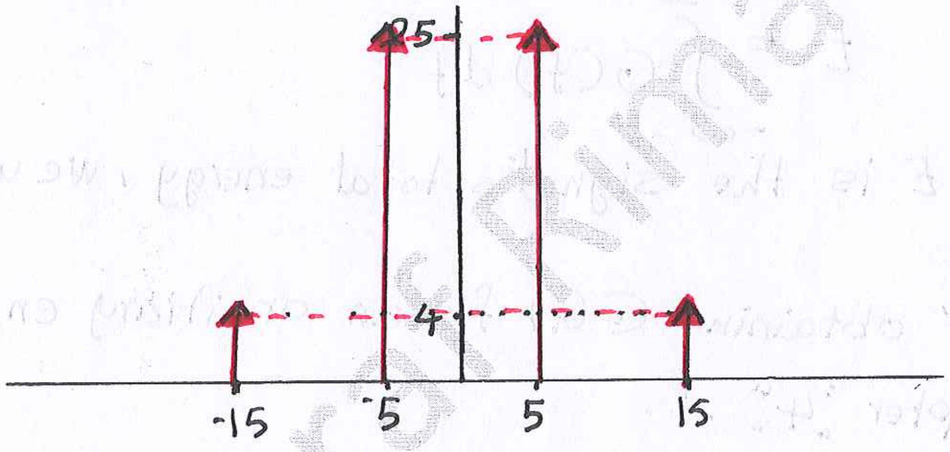
Answer : * The power spectral density is

$$S(f) = 25\,\delta(f+5) + 25\,\delta(f-5) + 4\,\delta(f-15) + 4\,\delta(f+15)$$



* The power lying within a frequency band from 10 HZ to 20 HZ is 8 W

whereas the total power of the signal

$$P = \int_{-\infty}^{\infty} S(f)\,df = 50 + 8 = 58\ \underline{\underline{W}}$$

3-6 Line Spectra

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_0t} = \sum_{n=-\infty}^{\infty} |X_n| e^{j\theta_{X_n}} e^{jnw_0t}$$

$$= \sum_{n=-\infty}^{\infty} |X_n| e^{j(nw_0t + \theta_n)}$$

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$$= \sum_{n=-\infty}^{-1} |X_n| e^{j(n\omega_0 t + \theta_n)} + X_0 + \sum_{n=1}^{\infty} |X_n| e^{j(n\omega_0 t + \theta_n)}$$

$$= X_0 + \sum_{n=1}^{\infty} 2|X_n| \frac{e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2}$$

$$= X_0 + 2 \sum_{n=1}^{\infty} |X_n| \cos(n\omega_0 t + \theta_n)$$

Example : The Complex exponential Fourier Series of a signal over an interval $0 \leq t \leq T_0$ is :

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{1+j\pi n} e^{j(3\pi n t/2)}$$

a) Determine the numerical value of T_0.

b) what is the average value of $x(t)$ over the interval $(0, T_0)$?

c) Determine the amplitude of the third-harmonic Component.

d) Determine the phase of the third-harmonic Component

e) Write down an expression for the third harmonic term in the fourier series.

Answer:

a) $n\omega_0 = (2\pi n f_0)$

$\frac{3\pi n}{2} = 2\pi n f_0 \Rightarrow f_0 = \frac{3}{4}\ Hz \Rightarrow T_0 = \frac{1}{f_0} = \frac{4}{3}$ sec

b) $X_0 = \frac{1}{1+j(0)} = 1$

c) $X_3 = \frac{1}{1+j3\pi}$ and $X_{-3} = \frac{1}{1-j3\pi}$

$\Rightarrow |X_3| = |X_{-3}| = \frac{1}{\sqrt{1+(3\pi)^2}}$

d) $\Theta_{X_3} = -\Theta_{X_{-3}}$

$\Rightarrow \Theta_{X_3} = -\tan^{-1}(3\pi)$

e) From the properties, it can be noted that

$a_3 = 2\,Re\{X_3\}$ and $b_3 = -2\,Im\{X_3\}$

and $X(t) = X_0 + 2\sum_{n=1}^{\infty} |X_n| \cos(n\omega_0 t + \Theta_n)$

So for third harmonic $\Rightarrow X(t) = \text{---} + 2\,|X_3| \cos(3\omega_0 t + \Theta_{X_3})$

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Exercise: Consider the periodic signal $x(t)$ given by the expression

$$x(t) = (2+j2)\, e^{-j30\pi t} - j3\, e^{-j20\pi t} + 5 + j3\, e^{j20\pi t} + (2-j2)\, e^{j30\pi t}$$

1. what is the average value of the signal $x(t)$.
2. Determine the expression of complex coefficient Fourier series.
3. Justify that $x(t)$ is a real signal and write the corresponding compact trigonometric fourier series representation.
4. Plot the two-sided (double-sided) amplitude and phase spectra for the signal $x(t)$.

Suggested Problems from the text-book

Please try to solve the following problems

3-12, 3-17, 3-18, 3-20

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Appendix I

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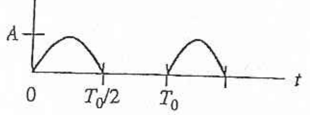
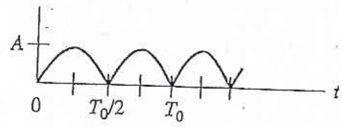
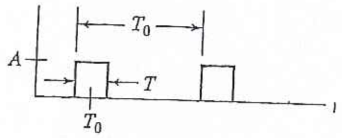
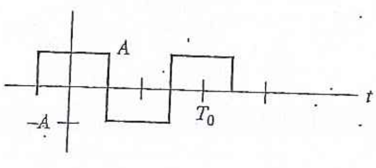
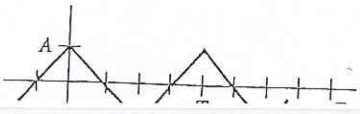
BLE 3-2
nmary of Fourier Series Properties[a]

Series	Coefficients[b]	Symmetry Properties
Trigonometric sine-cosine		
$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$	$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$ $a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt$ $b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt$	a_0 = Average value of $x(t)$ $a_n = 0$ for $x(t)$ odd, $b_n = 0$ for $x(t)$ even $a_n, b_n = 0$, n even, for $x(t)$ odd, half-wave symmetrical
omplex exponential		
$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$	$X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt$ $X_n = \begin{cases} \frac{1}{2}(a_n - jb_n), & n > 0 \\ \frac{1}{2}(a_{-n} + jb_{-n}), & n < 0 \end{cases}$ $X_n = X_{-n}^*$ for $x(t)$ real	X_0 = Average value of $x(t)$ X_n real for $x(t)$ even X_n imaginary for $x(t)$ odd $X_n = 0$, n even, for $x(t)$ odd half-wave symmetrical

even means that $x(t) = x(-t)$; $x(t)$ odd means that $x(t) = -x(-t)$; $x(t)$ odd half-wave symmetrical means that $x(t) = -x(t \pm T_0/2)$.
$\int_{T_0}(\cdot)\,dt$ means integration over any period T_0 of $x(t)$.

TABLE 3-1
Coefficients for the Complex Exponential Fourier Series of Several Signals

1. Half-rectified sine wave		$X_n = \begin{cases} \frac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n \text{ odd and } \neq \pm 1 \\ -\frac{1}{4}jnA, & n = \pm 1 \end{cases}$
2. Full-rectified sine wave*		$X_n = \begin{cases} \frac{2A}{\pi(1-n^2)}, & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$
3. Pulse-train signal		$X_n = \frac{A\tau}{T_0}\operatorname{sinc} nf_0\tau\, e^{-j2\pi nf_0 t_0}, \quad f_0 = T_0^{-1}$
4. Square wave		$X_n = \begin{cases} \frac{2A}{\lvert n\rvert\pi}, & n = \pm 1, \pm 5, \ldots \\ \frac{-2A}{\lvert n\rvert\pi}, & n = \pm 3, \pm 7, \ldots \\ 0, & n \text{ even} \end{cases}$
5. Triangular wave		$X_n = \begin{cases} \frac{4A}{\pi^2 n^2}, & n \text{ odd} \\ \ldots \end{cases}$

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Chapter 4: Fourier Transform

Fourier transform (FT) is a mathematical transformation employed to transform signals between time domain and frequency domain.

· We can obtain the fourier transform from fourier series when we assume that T_o is large enough so that the interval $[-T_o/2, T_o/2]$ and the index n approach infinity, then the product $n f_o$ approaches a continuous frequency variable f as shown in the following derivation.

$$x(t) = \sum_{nf_o=-\infty}^{\infty} \frac{X_n}{f_o} e^{j2\pi n f_o t} \Delta(nf_o) .$$

$$\tilde{X}(nf_o) \triangleq \frac{X_n}{f_o} = \int_{-1/(2f_o)}^{1/(2f_o)} x(t) e^{-j2\pi n f_o t} dt .$$

when $T_o \to \infty$, $f_o \to 0$ and when $n \to \infty$ then, $nf_o \to f$

$\Rightarrow$ $\frac{1}{T_o} = \Delta(nf_o) \to df$, and $\frac{X_n}{f_o} \longrightarrow \tilde{X}(f)$

So,

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

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$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

where $X(f)$ is the fourier transform of the signal $x(t)$, and it has magnitude and phase

$$X(f) = |X(f)| \angle \theta_{X(f)} .$$

where $|X(-f)| = |X(f)|$

and $\theta(-f) = -\theta(f)$

Example 1 : for the following signals, find $X(f)$

1 -

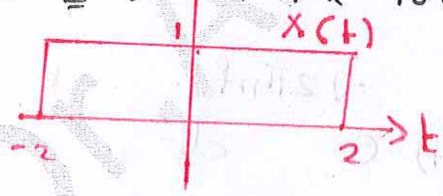

fig '1'

Ans : $X(f) = \int_{-2}^{2} (1)\, e^{-j2\pi f t}\, dt = -\frac{1}{j2\pi f} e^{-j2\pi f t}\Big|_{-2}^{2}$

$$= -\frac{1}{j2\pi f}\left(e^{-j2\pi f(2)} - e^{j2\pi f(2)}\right) = \frac{4}{4} \cdot \frac{\sin(4\pi f)}{\pi f}$$

$$= 4 \operatorname{sinc}(4f)$$

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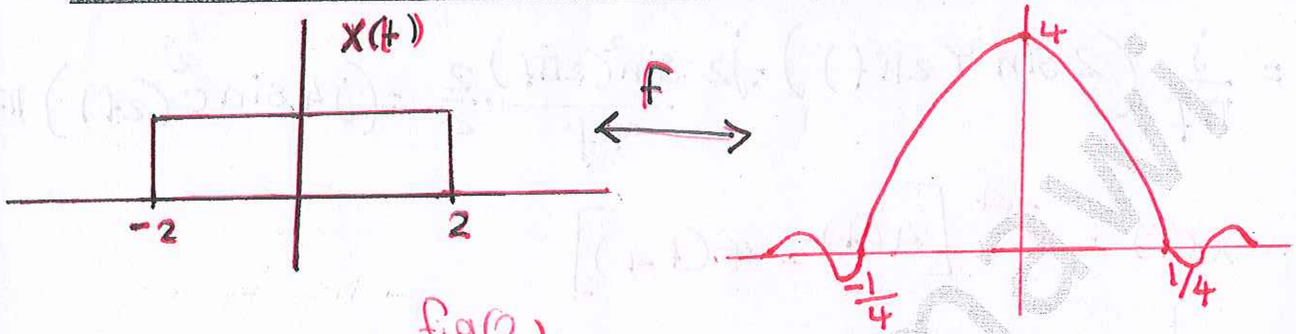
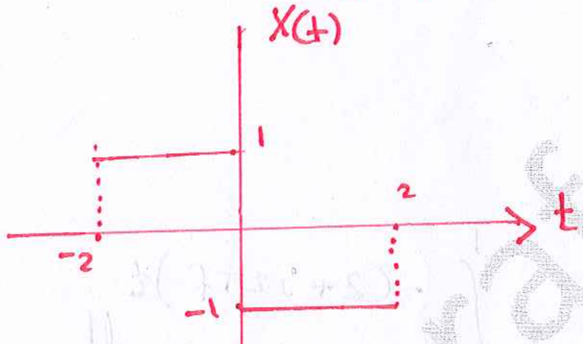


fig(2)

2 -



Fig(3)

Ans : $X(f) = \int_{-2}^{0} (1)\, e^{-j2\pi f t}\, dt + \int_{0}^{2} (-1)\, e^{-j2\pi f t}\, dt$

$$X(f) = \frac{-1}{j2\pi f} e^{-j2\pi f t}\Big|_{-2}^{0} + \frac{1}{j2\pi f} e^{-j2\pi f(t)}\Big|_{0}^{2}$$

$$= \frac{-1}{j2\pi f} + \frac{1}{j2\pi f} e^{j4\pi f} + \frac{1}{j2\pi f} e^{-j4\pi f} - \frac{1}{j2\pi f}$$

$$= \frac{-2}{j2\pi f} + \frac{1}{j\pi f} \cos(4\pi f)$$

$$= \frac{j}{\pi f} - \frac{j}{\pi f} \cos(4\pi f) = \frac{j}{\pi f}\left(1 - \cos(4\pi f)\right)$$

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$$= \frac{j}{\pi f}\left(2\sin^2(2\pi f)\right) - j2\,\frac{\sin^2(2\pi f)}{4f}\cdot\frac{2}{2} = \left(j4\,\mathrm{sinc}^2(2f)\right)\pi f$$

3. $x(t) = e^{-2t}\left[u(t) - u(t-1)\right]$

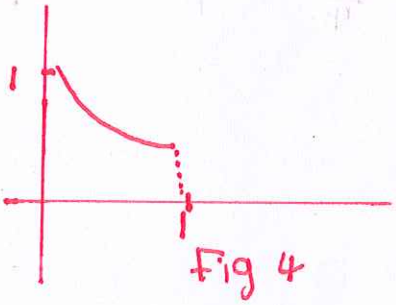


Fig 4

Ans : $X(f) = \int_0^1 e^{-2t} e^{-j2\pi f t}\,dt = \int_0^1 e^{-(2+j2\pi f)t}\,dt$

$$= \frac{-1}{2+j2\pi f}\left(e^{-(2+j2\pi f)} - 1\right)$$

Example 2 : For the following signal $x(t)$ shown in fig 5, find $X(f)$

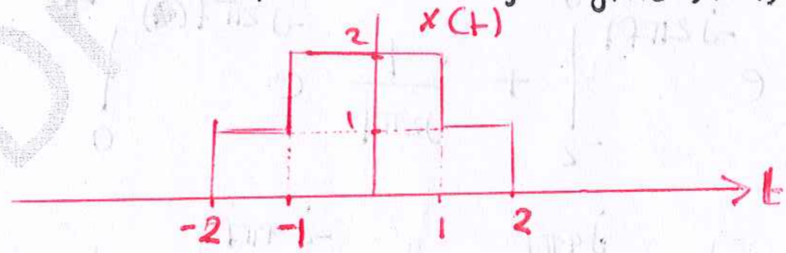
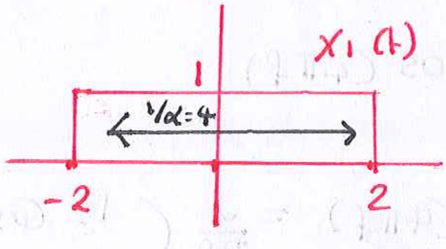


fig 5

Ans :



$x_1(t) = \Pi\left(\frac{t}{4}\right)$

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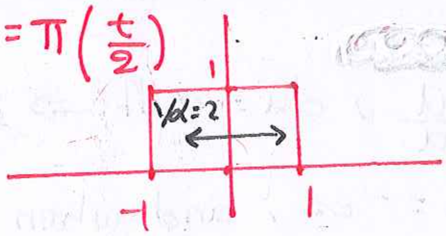
$x_2(t) = \Pi\left(\frac{t}{2}\right)$



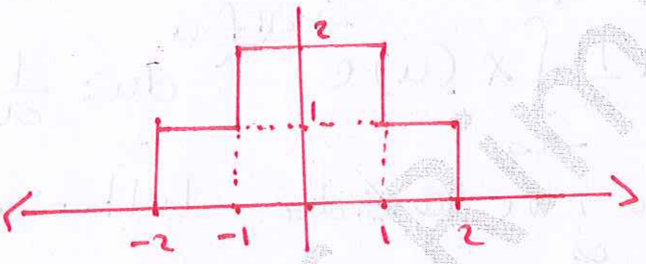


fig (6)

From figure 6 : we note that the signal $x(t)$ is expressed as the sum of two pulse, so

$$x(t) = x_1(t) + x_2(t) = \Pi\left(\frac{t}{4}\right) + \Pi\left(\frac{t}{2}\right)$$

Here, we use the following theorems: linearity theorem and scaling theorem, so $X(f)$ is written as :

1. Linearity (Superposition) theorem

$$F[x_1(t) + x_2(t)] = \int_{-\infty}^{\infty} (x_1(t) + x_2(t)) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi ft} dt + \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi ft} dt = X_1(f) + X_2(f)$$

2. scale change theorem

$$F[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt \qquad a > 0$$

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let $u = at \Rightarrow t = \frac{u}{a}$, $du = a\,dt \Rightarrow \frac{du}{a} = dt$

when $t = -\infty \Rightarrow u = -\infty$, and when $t = \infty \Rightarrow u = \infty$

$$F[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(u)\, e^{-j2\pi \frac{f}{a} u}\, du = \frac{1}{a} X(f/a)$$

but if $a < 0$, we consider $-|a|t = at$

$$F[x(at)] = \int_{-\infty}^{\infty} x(-|a|t)\, e^{-j2\pi f t}\, dt$$

let $u = -|a|t \Rightarrow t = -u/|a|$, $du = -|a|\,dt$

$$\Rightarrow F[x(at)] = \int_{-\infty}^{\infty} x(-|a|t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(u)\, e^{+j2\pi f \frac{u}{|a|}} \frac{dt}{|a|} = \frac{1}{|a|} X\left(\frac{f}{|a|}\right) = \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

since $-|a| = a$

$$X(f) = 4\,\mathrm{sinc}(4f) + 2\,\mathrm{sinc}(2f)$$

3- Time-delay theorem

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j2\pi f t}\, dt$$

let $u = t - t_0 \Rightarrow t = u + t_0$, $du = dt$

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$$\Rightarrow F[x(t-t_0)] = \int_{-\infty}^{\infty} x(u) e^{-j2\pi f(u+t_0)} du$$

$$= \left[\int_{-\infty}^{\infty} x(u) e^{-j2\pi f u} du \right] e^{-j2\pi f t_0}$$

$$= X(f) e^{-j2\pi f t_0}$$

Example 3: for the following signal $x(t)$: find $X(f)$

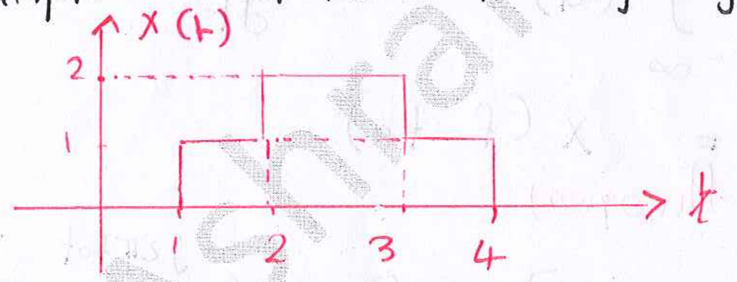

Fig '7'

Ans:

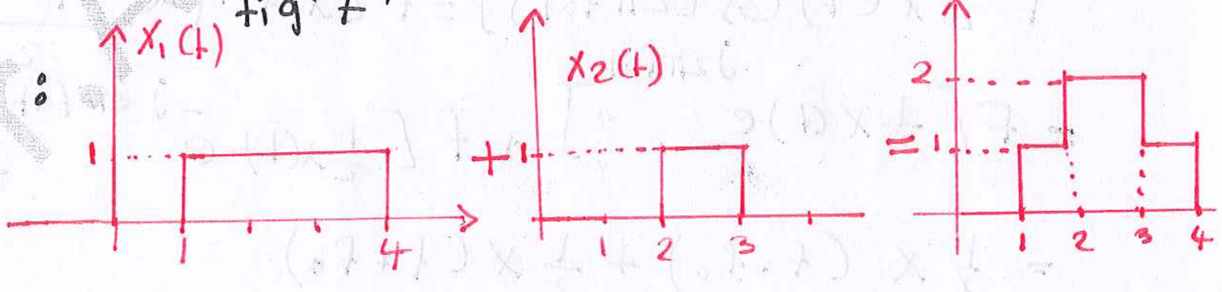

Fig '8'

From figure '8', $x(t)$ is written as

$$x(t) = x_1(t) + x_2(t) = \Pi\left(\frac{1}{3}(t-2.5)\right) + \Pi(t-2.5)$$

By using linearity, scaling and time-delay theorems then

$$X(f) = 3\operatorname{sinc}(3f)\, e^{-j2\pi f(2.5)} + \operatorname{sinc}(f)\, e^{-j2\pi f(2.5)}$$

$$= \left[3\operatorname{sinc}(3f) + \operatorname{sinc}(f)\right] e^{-j2\pi f(2.5)}$$

4_ Frequency translation theorem

$$F\left[x(t)\, e^{j2\pi f_0 t}\right] = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi (f-f_0)t}\, dt$$

$$= X(f-f_0)$$

5_ Modulation theorem

$$F\left[x(t)\cos(2\pi f_0 t)\right] = F\left[x(t)\cdot\left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right)\right]$$

$$= F\left[\tfrac{1}{2}x(t)\, e^{j2\pi f_0 t}\right] + F\left[\tfrac{1}{2}x(t)\, e^{-j2\pi f_0 t}\right]$$

$$= \tfrac{1}{2}X(f-f_0) + \tfrac{1}{2}X(f+f_0)$$

Example 4 : for the following Signals

1. $x_1(t) = \Pi\left(\frac{t}{3}\right)\cos(8\pi t)$

2 - $x_2(t) = \Lambda\left(\frac{t}{2}\right)\cos(10\pi t)$

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a. Find the fourier transform for each signal.

b. sketch the signals obtained in part a

Ans :

1. $x_1(t) = \Pi\left(\frac{t}{3}\right)\cos(8\pi t)$

$$x_1(t) = \frac{1}{2}\Pi\left(\frac{t}{3}\right)e^{j2\pi(4)t} + \frac{1}{2}\Pi\left(\frac{t}{3}\right)e^{-j2\pi(4)t}$$

$$X_1(f) = \frac{1}{2}\cdot 3\,\mathrm{Sinc}(3(f-4)) + \frac{1}{2}\cdot 3\,\mathrm{Sinc}(3(f+4))$$

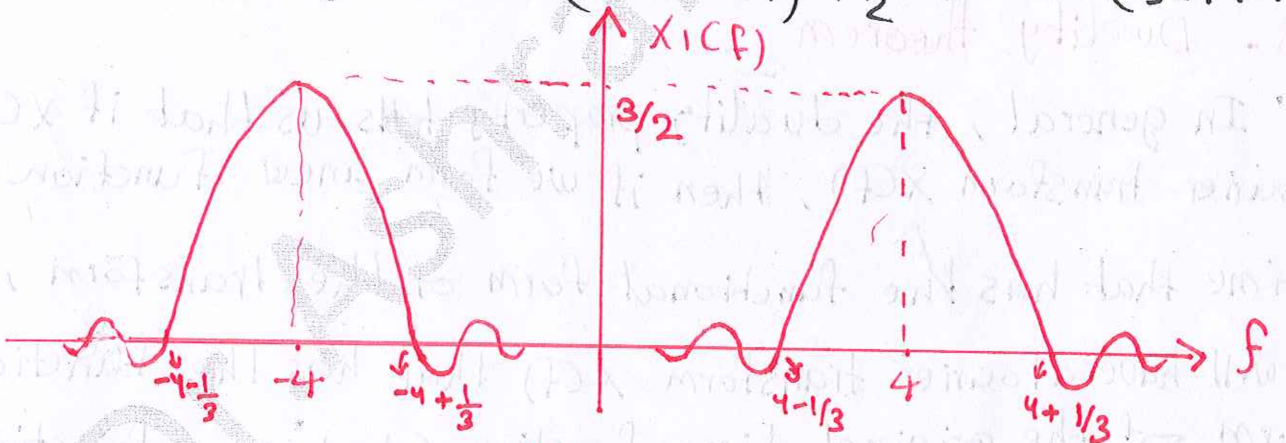


Fig.9

2. $x_2(t) = \Lambda\left(\frac{t}{2}\right)\cos(10\pi t)$

$$= \Lambda\left(\frac{t}{2}\right)\left[\frac{1}{2}e^{j10\pi t} + \frac{1}{2}e^{-j10\pi t}\right]$$

$$= \frac{1}{2}\Lambda\left(\frac{t}{2}\right)e^{j(2\pi(5)t)} + \frac{1}{2}\Lambda\left(\frac{t}{2}\right)e^{-j(2\pi(5)t)}$$

$$X_2(f) = \frac{1}{2}\cdot 2\,\mathrm{Sinc}^2(2(f-5)) + \frac{1}{2}\cdot 2\,\mathrm{Sinc}^2(2(f+5))$$

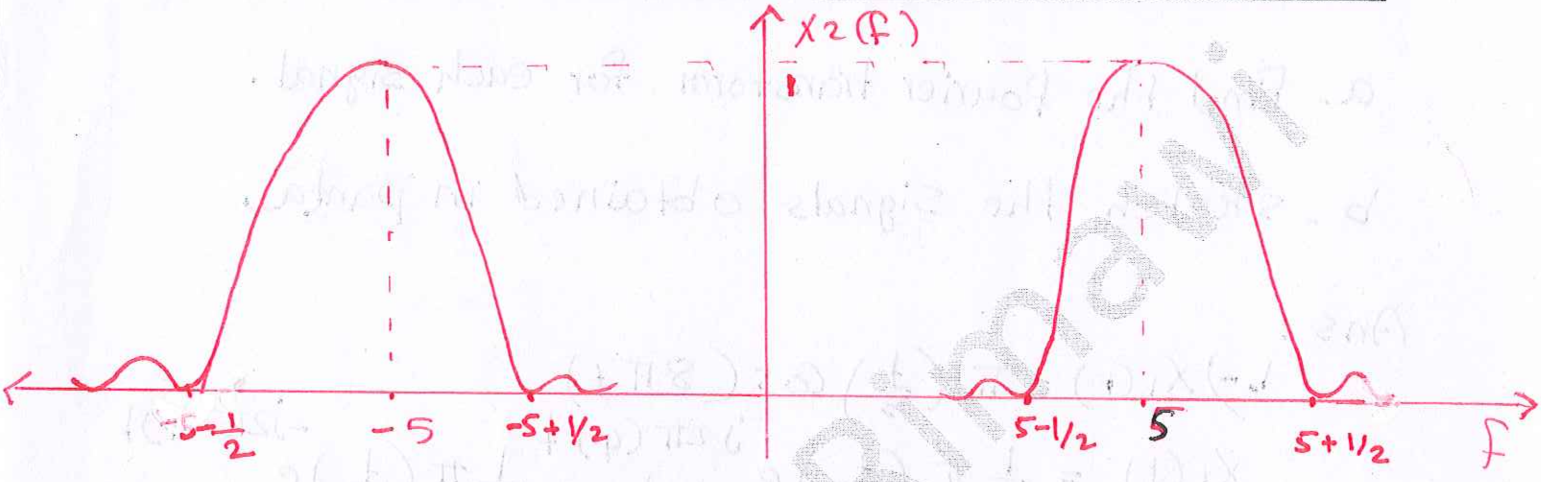


Fig "10"

6- Duality theorem

' In general, the duality property tells us that if $x(t)$ has a fourier transform $X(f)$, then if we form anew function of atime that has the functional form of the transform, $X(t)$, it will have afourier transform $x(f)$ that has the functional form of the original time function (but is a function of frequency).

Mathematically, we can write

$$x(t) \longleftrightarrow X(f)$$
$$X(t) \longleftrightarrow x(-f)$$

proof:

$$X(f) = \int_{-\infty}^{\infty} x(b)\, e^{-j2\pi f b}\, db$$

$$x(t) = \int_{-\infty}^{\infty} X(b) e^{j2\pi f b}\, db$$

$$X(-f) = \int_{-\infty}^{\infty} x(b) e^{-j2\pi f b}\, db = f\{x(b)\}$$

Example 5 : If $x(t) = 4\,\text{Sinc}(3(t-2))$, Find $X(f)$

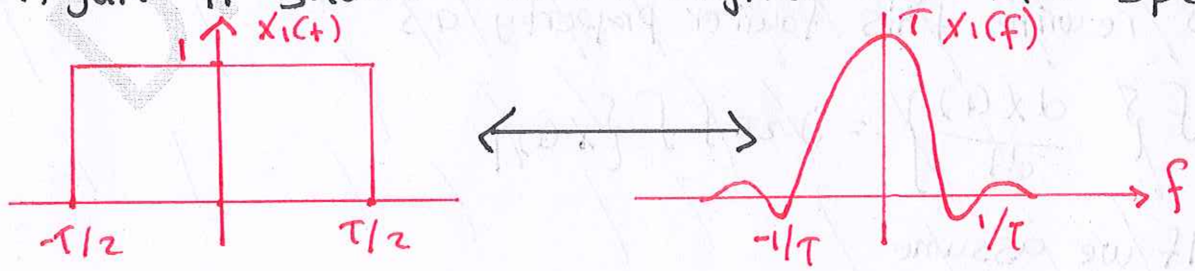
Ans : By using duality theorem , saling , and time delay

$$f\{x(t)\} = f\{4\,\text{sinc}(3(t-2))\}$$

$$= \frac{4}{3}\Pi\left(\frac{-f}{3}\right) e^{-j2\pi f(2)}$$

$$= \frac{4}{3}\Pi\left(\frac{f}{3}\right) e^{-j4\pi f}$$

Figure '11' shows various signals and their spectra



(a) square-pulse Signal

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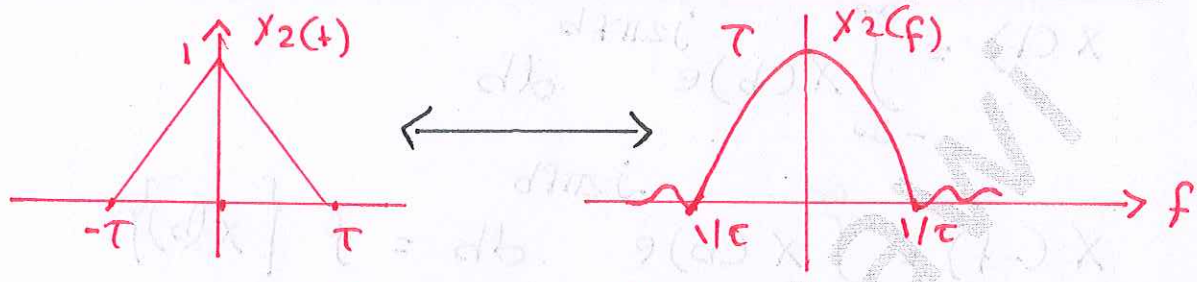
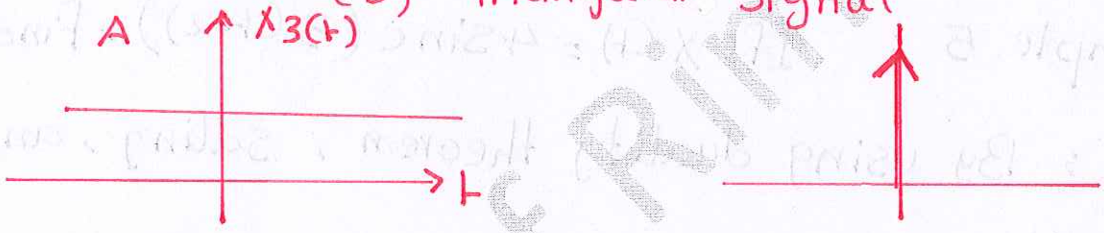


(b) Triangular signal



(c) impulse signal

Fig "11"

~~From the result obtain in , we can obtain the fourier transform of integrable function. Since~~

$$\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j2\pi f X(f)$$

~~let's rewrite this fourier property as~~

$$\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j2\pi f\, \mathcal{F}\{x(t)\}$$

~~If we assume~~

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From figure 11-c , we can conclude the following results :

1 - $A\,\delta(t) \longleftrightarrow A$

2 - $A\,\delta(t-t_0) \longleftrightarrow A e^{-j2\pi f t_0}$

3 - $A \longleftrightarrow A\,\delta(-f)$ since impulse function is even function then $\delta(-f) = \delta(f)$

4 - $A e^{j2\pi f_0 t} \longleftrightarrow A\,\delta(f-f_0)$

5 - $A\cos(2\pi f_0 t) \longleftrightarrow \frac{A}{2}\delta(f-f_0) + \frac{A}{2}\delta(f+f_0)$

7 - Differentiation & Integration theorems

a) $$F\left\{\frac{dx(t)}{dt}\right\} = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j2\pi f t}\, dt$$

let $u = e^{-j2\pi f t}$ $\qquad dv = \frac{dx(t)}{dt}$

$du = -j2\pi f e^{-j2\pi f t}$ $\qquad v = x(t)$

$$x(t) e^{-j2\pi f t}\Big|_{-\infty}^{\infty} + j2\pi f \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt$$

if $x(t)$ is absolutely integrable, $\lim_{t \to \pm\infty} |x(t)| = 0$ then

$$F\left\{\frac{dx(t)}{dt}\right\} = j2\pi f\, X(f)$$

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So, In general

$$\mathcal{F}\left\{\frac{d^n x(t)}{dt^n}\right\} = (j2\pi f)^n X(f)$$

From the result obtained in 7a we can obtain the fourier transform of integrable function, since

$$\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j2\pi f\, X(f)$$

let's rewrite this fourier property as

$$\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j2\pi f\, \mathcal{F}\{x(t)\}$$

if we assume $h(t) = \frac{dx(t)}{dt}$

$$\Longrightarrow \mathcal{F}\{h(t)\} = j2\pi f\, \mathcal{F}\left\{\int_{-\infty}^{t} h(\tau)d\tau\right\}$$

$$\mathcal{F}\left\{\int_{-\infty}^{t} h(\tau)d\tau\right\} = \frac{\mathcal{F}\{h(t)\}}{j2\pi f}$$

this result satisfied if

$$\int_{-\infty}^{\infty} x(\tau)d\tau = 0$$

but if the total integral at $x(t)$ is not zero, then there exists some constant c such that the total

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integral of $x(t) - c = 0$

$$\int_{-\infty}^{\infty} (x(\tau) - c)\, d\tau = 0$$

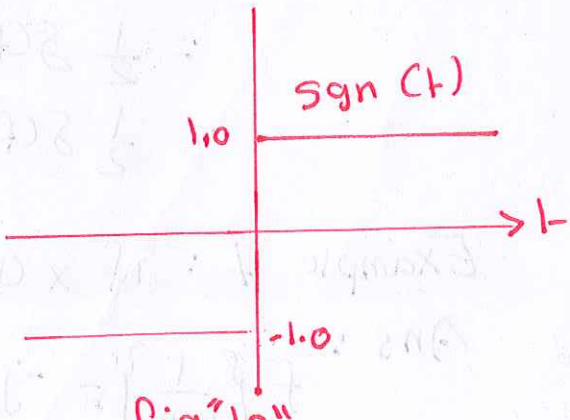
where c is the "average value" of the function $x(t)$, which is also often called the "dcterm" or the "Constant term", Using some math and the fourier transform of the impulse function we have the general formula for the fourier transform of the integral of a function

$$\mathcal{F}\left\{ \int_{-\infty}^{t} x(\tau)\, d\tau \right\} = \frac{\mathcal{F}\{x(t)\}}{j2\pi f} + c\,\delta(f)$$

$$= \frac{X(f)}{j2\pi f} + c\,\delta(f)$$

Example 6 : Find the fourier transform for the signum function which is defined as

$$\mathrm{sgn}(t) = \begin{cases} +1 , & t > 0 \\ 0 , & t = 0 \\ -1 , & t < 0 \end{cases}$$



fig"12"

and it may be expressed as

$$\mathrm{sgn}(t) = 2u(t) - 1$$

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Ans:

$$\operatorname{sgn}(t) = 2u(t) - 1$$

$$\frac{d}{dt}\left\{\operatorname{sgn}(t)\right\} = 2\delta(t)$$

$$\mathcal{F}\left[\frac{d}{dt}\operatorname{sgn}(t)\right] = 2\mathcal{F}\{\delta(t)\}$$

$$j2\pi f\, \mathcal{F}\{\operatorname{sgn}(t)\} = 2$$

$$\mathcal{F}\{\operatorname{sgn}(t)\} = \frac{1}{j\pi f}$$

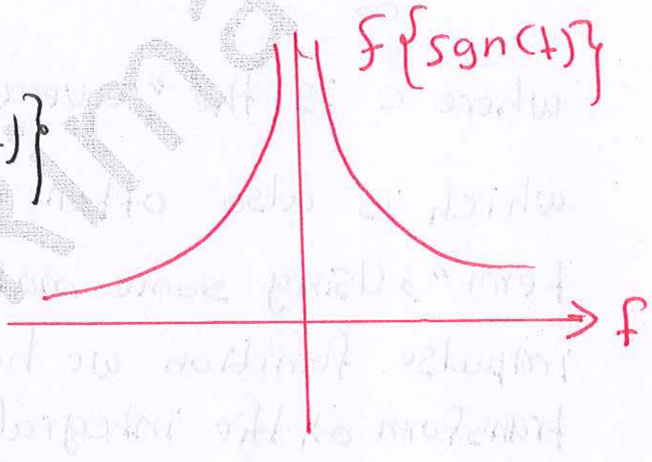

Fig 13

By using this result, we can evaluate the fourier transform of step function, since

$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$

$$\mathcal{F}\{u(t)\} = \mathcal{F}\left\{\frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)\right\}$$

$$= \frac{1}{2}\delta(f) + \frac{1}{2}\cdot\frac{1}{j\pi f}$$

$$= \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

Example 7: if $x(t) = \frac{1}{\pi t}$, find $X(f)$

Ans:

$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = -j\operatorname{sgn}(f)$, since the signum function is

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odd function

8. Convolution theorem

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda\right]e^{-j2\pi ft}\,dt$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(t-\lambda)e^{-j2\pi ft}\,dt\right]x_1(\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} X_2(f)x_1(\lambda)e^{-j2\pi f\lambda}\,d\lambda$$

$$= X_2(f)\int_{-\infty}^{\infty} x_1(\lambda)e^{-j2\pi f\lambda}\,d\lambda$$

$$= X_1(f)\,X_2(f)$$

Example 8: if $x(t) = \Pi\left(\frac{t}{3}\right) * \Pi\left(\frac{t}{3}\right)$, find $X(f)$

Ans:

$$\mathcal{F}\left\{\Pi\left(\tfrac{t}{3}\right) * \Pi\left(\tfrac{t}{3}\right)\right\} = \mathcal{F}\left\{\Pi\left(\tfrac{t}{3}\right)\right\}\mathcal{F}\left\{\Pi\left(\tfrac{t}{3}\right)\right\}$$

$$= (3\,\text{sinc}(3f))\,(3\,\text{sinc}(3f))$$

$$= 9\,\text{sinc}^2(3f)$$

$$= 3\mathcal{F}\left\{\Lambda\left(\tfrac{t}{3}\right)\right\}$$

So, we can conclude that

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$$\mathcal{F}\left\{\Pi\left(\frac{t}{\tau}\right) * \Pi\left(\frac{t}{\tau}\right)\right\} = \mathcal{F}\left\{\tau \Lambda\left(\frac{t}{\tau}\right)\right\}$$

Example 9: Find the fourier transform for the Hilbert-transform function which is defined as

$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\lambda)}{t-\lambda}\, d\lambda$$

Ans: From its definition, we note that the (HT) may be considered as the convolution of $x(t)$ with $\frac{1}{\pi t}$, so the fourier transform of $\hat{x}(t)$ is given as

$$\mathcal{F}\{\hat{x}(t)\} = \mathcal{F}\left\{\frac{1}{\pi t} * x(t)\right\}$$

$$\mathcal{F}\{\hat{x}(t)\} = \mathcal{F}\left\{\frac{1}{\pi t}\right\} \mathcal{F}\{x(t)\}$$

$$= -j\,\mathrm{sgn}(f)\, X(f)$$

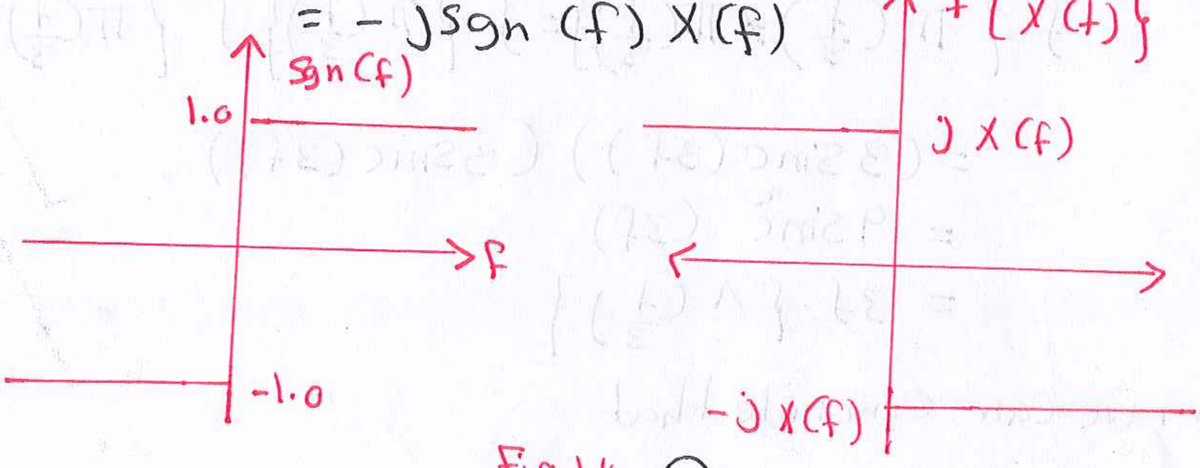

Fig 14

From figure 14, we note that positive frequencies are multiplied by -j, so the phase shift is -90, whereas for negative frequencies, the phase shift is 90, since it is multiplied by j.

From the definition of (HT), we can conclude the following properties:

1- The signal and it's Hilber Transform are orthogonal, this is because, by rotating the signal 90° we have now made it orthogonal to the original signal, that being the definition of orthogonality

2- The signal and it's Hilbert Transform have identical energy because phase shift doesn't change the energy of the signal only amplitude changes can do that.

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Energy Spectral Density

The energy of a signal can be expressed in the frequency domain by proceeding as following:

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x^*(t) \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \cdot dt$$

$$= \int_{-\infty}^{\infty} X(f) \int_{-\infty}^{\infty} x^*(t) e^{j2\pi ft} dt\, df$$

$$= \int_{-\infty}^{\infty} X(f) X^*(f) df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

This is refered to as parseval's theorem for fuirer transforms.

Now, let us define the energy spectral density where

$$G(f) \triangleq |X(f)|^2$$

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Integration of $G(f)$ over all frequencies from $-\infty$ to ∞ yields the total (normalized) energy contained in a signal. Simillarly, integration of $G(f)$ over an infinite range of frequencies gives the energy contained in the signal within the range frequencies represented by the limits of integration.

Example : For the following signal :

$$x(t) = \exp(-\alpha t)\, u(t) \quad , \alpha > 0$$

a) Find the fourier transform of this signal, $X(f)$.

b) Find the energy spectral density of the signal.

Answer :

a) $$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-\alpha t}\, e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-(\alpha + j2\pi f)t}\, dt = \frac{-1}{\alpha + j2\pi f}\, e^{-(\alpha + j2\pi f)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{\alpha + j2\pi f}$$

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b) The energy spectral density is

$$G(f) = \frac{1}{\alpha^2 + (2\pi f)^2}$$

The energy contained in this signal in the frequency range

$$-B < f < B$$

$$E_B = \int_{-B}^{B} \frac{df}{\alpha^2 + (2\pi f)^2} = \frac{1}{\pi\alpha} \int_{0}^{2\pi B/\alpha} \frac{dv}{1+v^2}$$

$$= \frac{1}{\pi\alpha} \tan^{-1}\left(\frac{2\pi B}{\alpha}\right)$$

$$E = \lim_{B \to \infty} E_B = 1/2\alpha$$

System Analysis with the fourier transform

For the LTI system shown in fig 15

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Fig. 15

The output signal $y(t)$ is given by

$$y(t) = x(t) * h(t)$$

The fourier transform of $y(t)$

$$\mathcal{F}[y(t)] = \mathcal{F}[x(t) * h(t)]$$

$$Y(f) = X(f)\, H(f)$$

Since $H(f)$ is in general a complex quantity, we write it as:

$$H(f) = |H(f)| \angle H(f) = |H(f)|\, e^{j \angle H(f)}$$

where $|H(f)|$ is the amplitude-response function and $\angle H(f)$ is the phase-response function of the network. In addition,

$$|H(f)| = |H(-f)|$$

and

$$\angle H(f) = -\angle H(-f)$$

Example: For the RC circuit shown in fig.16

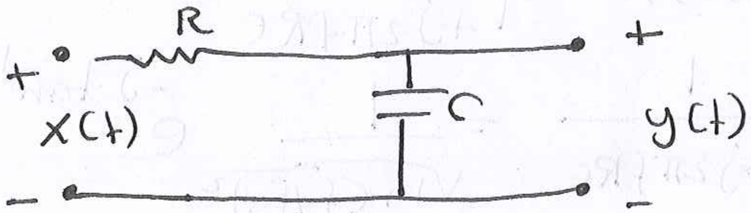

Fig.16

a) find the amplitude and phase responses of this system

b) plot the amplitude and phase responses

Answer :

a) The differential equation of the system (as discussed and derived in the previous) is

$$RC\frac{dy}{dt} + y(t) = x(t) \qquad -\infty < t < \infty$$

The fourier transform of the system is

$$RC\,\mathcal{F}\left[\frac{dy}{dt}\right] + \mathcal{F}\left[y(t)\right] = \mathcal{F}\left[x(t)\right]$$

$$j2\pi fRC\,Y(f) + Y(f) = X(f)$$

$$\left[1 + j2\pi fRC\right]Y(f) = X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1+j2\pi fRC}$$

$$\Rightarrow H(f) = \frac{1}{1+j2\pi fRC} = \frac{1}{\sqrt{1+(f/f_3)^2}}\,e^{-j\tan^{-1}(f/f_3)}$$

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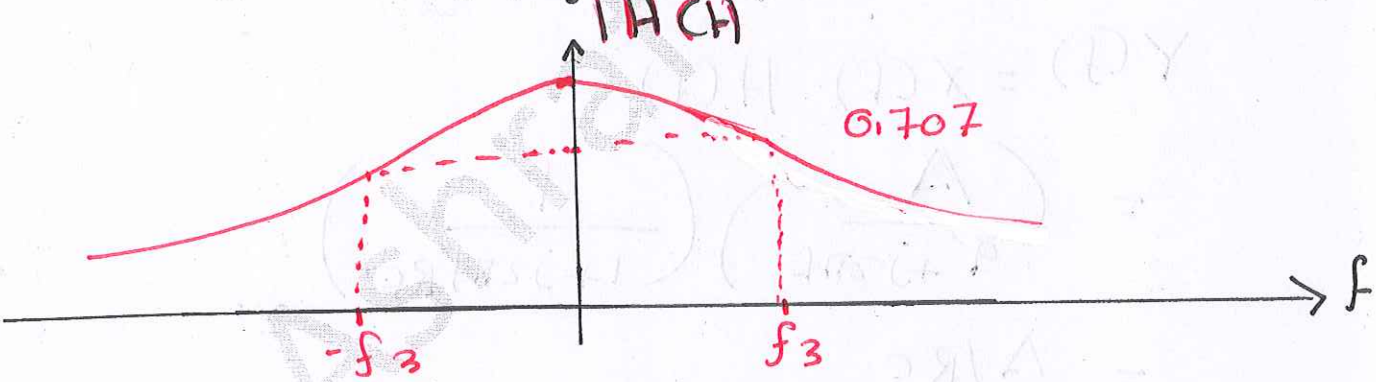
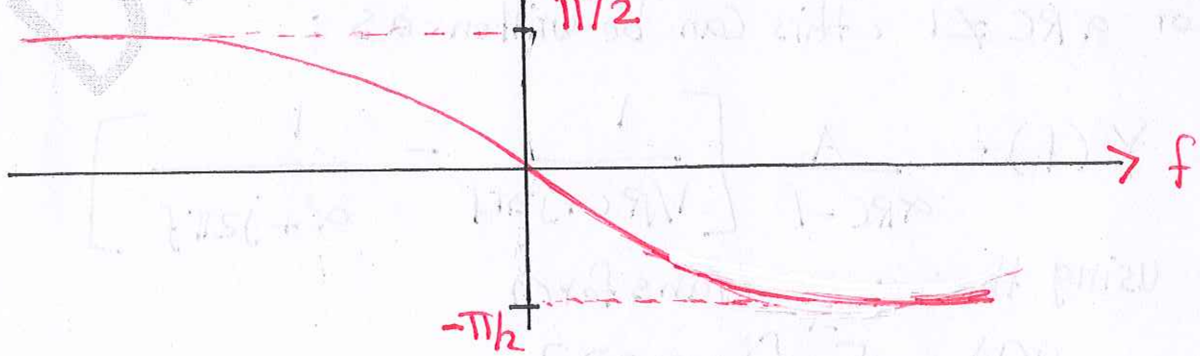
where $f_3 = 1/2\pi RC$ is the 3-dB or half power frequency.

b) The amplitude and phase responses of the system is:

$$|H(f)| = \left[1 + \left(\frac{f}{f_3}\right)^2\right]^{-\frac{1}{2}} \quad \text{and} \quad \angle H(f) = -\tan^{-1}\left(\frac{f}{f_3}\right)$$



(a) Amplitude Response



(b) phase Response

Fig.17

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Now let us consider $x(t) = Ae^{-\alpha t}u(t)$, $\alpha > 0$ for the LTI system shown in fig16, then the output signal $y(t)$ can be expressed as

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(t)$$

$$y(t) = x(t) * h(t)$$

$$\mathcal{F}[y(t)] = \mathcal{F}[x(t) * h(t)]$$

$$Y(f) = X(f)\, H(f)$$

$$= \left(\frac{A}{\alpha + j2\pi f}\right)\left(\frac{1}{1 + j2\pi f RC}\right)$$

$$= \frac{A/RC}{(\alpha + j2\pi f)(1/RC + j2\pi f)}$$

For $\alpha RC \neq 1$, this can be written as :

$$Y(f) = \frac{A}{\alpha RC - 1}\left[\frac{1}{1/RC + j2\pi f} - \frac{1}{\alpha + j2\pi f}\right]$$

By using the inverse transform

$$y(t) = \mathcal{F}^{-1}\{Y(f)\}$$

$$= \frac{A}{\alpha RC - 1}\left[\exp\left(\frac{-t}{RC}\right) - \exp(-\alpha t)\right]u(t)$$

if $\alpha RC \to 1$, then

$$y(t) = A\left(\frac{t}{RC}\right)\exp\left(\frac{-t}{RC}\right)u(t), \quad \alpha = \frac{1}{RC}$$

Example : Again, we consider the system shown in Fig but with input

$$x(t) = A\Pi\left(\frac{t - T/2}{T}\right) = A\left[u(t) - u(t - T)\right]$$

and the step response

$$a_s(t) = \left(1 - e^{-t/RC}\right)u(t)$$

Answer : Noting that $x(t)$ Consists of the difference of two steps and using superposition , we find output to be :

$$y(t) = \begin{cases} 0, & t < 0 \\ A\left(1 - e^{-t/RC}\right), & 0 \leq t \leq T \\ A\left(e^{-(t-T)/RC} - e^{-t/RC}\right), & t > T \end{cases}$$

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From the figure, it can be noted that the input is essentially passed undistorted by the system when the filter bandwidth is large compared with the spectral width of the input pulse whereas, the system distorts the input spectrum and the output does not resemble the input when $2\pi f_3 / T^{-1} << 1$.

Since th energy spectral density of a signal is proportional to the magnitude of its fourier transform squared, it follows that

$$G_Y(f) = |H(f)|^2 G_X(f)$$

where $G_X(f)$ and $G_Y(f)$ are the energy spectral densities of the system input and output, respectively.

Steady - State System Response to Sinusoidal Inputs by Means of the Fourier Transform.

$$y(t) = x(t) * h(t)$$

$$\mathcal{F}[y(t)] = X(f)\, H(f)$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

$$X(f) = \sum_{n=-\infty}^{\infty} X_n\, \delta(f - n f_0)$$

$$Y(f) = \sum_{n=-\infty}^{\infty} X_n H(f)\, \delta(f - n f_0)$$

$$= \sum_{n=-\infty}^{\infty} X_n\, H(nf_0)\, \delta(f - n f_0)$$

$$= \sum_{n=-\infty}^{\infty} (|X_n| \angle X_n)(|H(nf_0)| \angle H(nf_0))\, \delta(f - n f_0)$$

$$= \sum_{n=-\infty}^{\infty} (|X_n|\, |H(n f_0)| \angle (X_n + H(nf_0)))\, \delta(f - nf_0)$$

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n| \, |H(n f_0)| \, e^{[j(2\pi n f_0 t + \angle X_n + \angle H(n f_0))]}$$

Example : Consider asystem with amplitude - and phase-response functions gives by

$$|H(f)| = K\Pi\left(\frac{f}{2B}\right) = \begin{cases} K, & |f| \leq B \\ 0, & \text{o.w} \end{cases}$$

and

$$\angle H(f) = -2\pi t_0 f$$

if $x(t) = A\cos(2\pi f_0 t + \theta_0)$, find $y(t)$

Answer :

$$x(t) = \frac{A}{2} e^{j\theta_0} \cdot e^{j(2\pi f_0)t} + \frac{A}{2} e^{-j(\theta_0)} e^{j/(2\pi f_0)t}$$

$$X_1 = \frac{1}{2} A e^{j\theta_0} = X_{-1}^*$$

and $X_n = 0$ for other

$$y(t) \begin{cases} 0, & f_0 > B \\ KA\cos[2\pi f_0(t - t_0) + \theta_0], & f_0 \leq B \end{cases}$$

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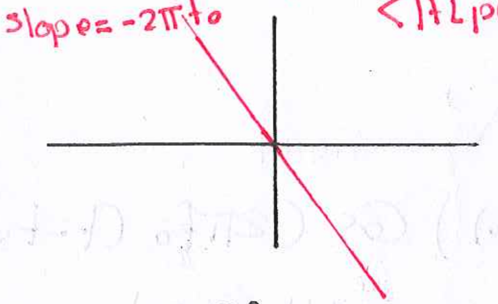
Ideal filter

1. Low pass filter (L.p.f)



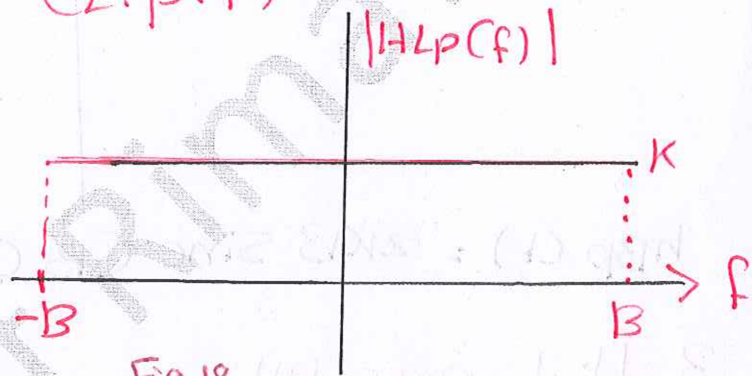


Fig.18

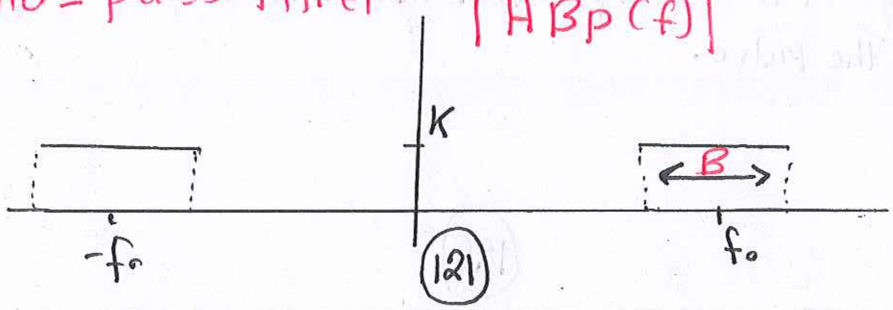
$$h_{Lp}(t) = \int_{-\infty}^{\infty} H_{Lp}(f)\, e^{j2\pi f t}\, df$$

$$= \int_{-\beta}^{\beta} K e^{-j2\pi f t_0} e^{j2\pi f t}\, df$$

$$= \int_{-\beta}^{\beta} K e^{j2\pi f (t-t_0)}\, df$$

$$= 2BK\, \text{Sinc}\left(2B(t-t_0)\right)$$

2. Band-pass filter



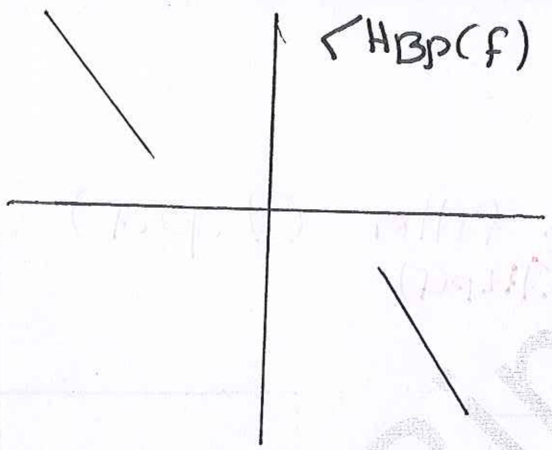


Fig.19

$$h_{Bp}(t) = 2KB \operatorname{Sinc}(B(t-t_0)) \cos(2\pi f_0 (t-t_0))$$

3-High-pass filter

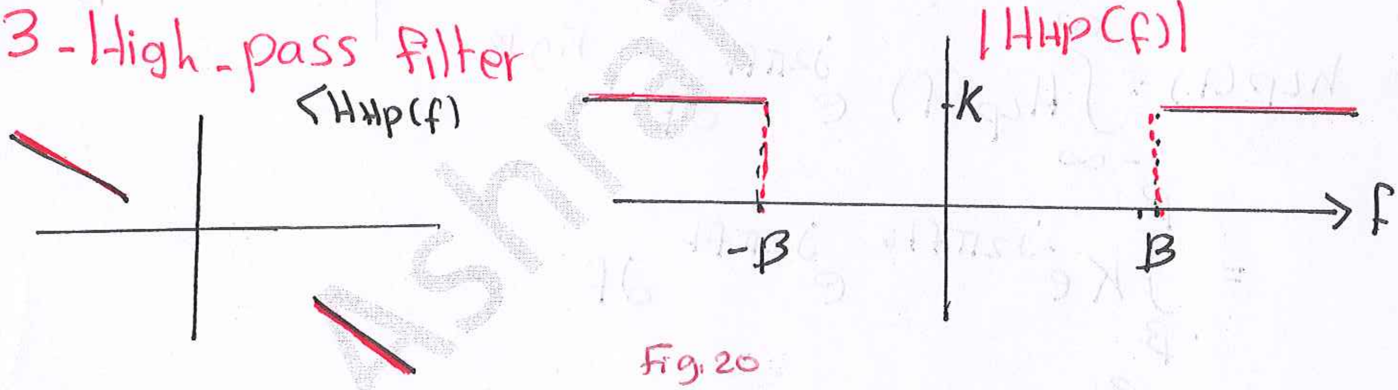


Fig. 20

$$h_{Hp}(t) = K\,\delta(t-t_0) - 2BK \operatorname{Sinc}(2B(t-t_0))$$

Band width and Rise time

The rise time of a pulse is the amount of time that it takes in going from a prespecified minimum value, say 10% of the final value of the pulse, to a perspecified maximum value, say 90% of the final value of the pulse.

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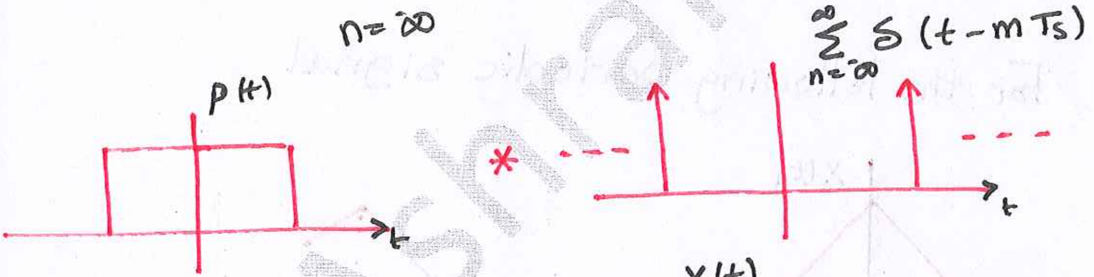
Fourier Transform for a periodic signal

From Fourier series, $x(t)$ can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

where $x(t)$ can be rewritten as

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - mT_s) * p(t) = \sum_{m=-\infty}^{\infty} p(t - mT_s)$$



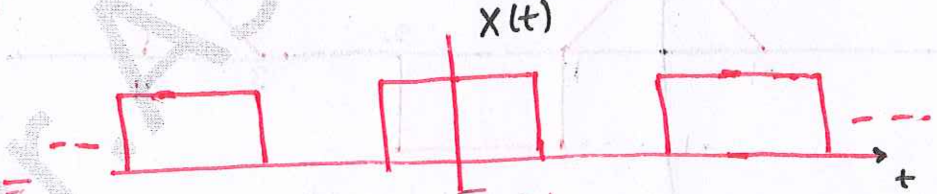


Fig. 21

$$\mathcal{F}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s) * p(t)\right] = \mathcal{F}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right] P(f)$$

Since $\mathcal{F}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right] = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} Y_n e^{jn\omega_0 t}\right]$

$$= \mathcal{F}\left[\sum_{n=-\infty}^{\infty} f_0 e^{jn\omega_0 t}\right] = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

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$$\Rightarrow X(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f-nf_0)\, P(f)$$

From sampling theorem

$$\Rightarrow X(f) = f_0 \sum_{n=-\infty}^{\infty} P(nf_0)\, \delta(f-nf_0)$$

and

$$x(t) = f_0 \sum_{n=-\infty}^{\infty} P(nf_0)\, e^{j2\pi n f_0 t}$$

Example : For the following periodic signal

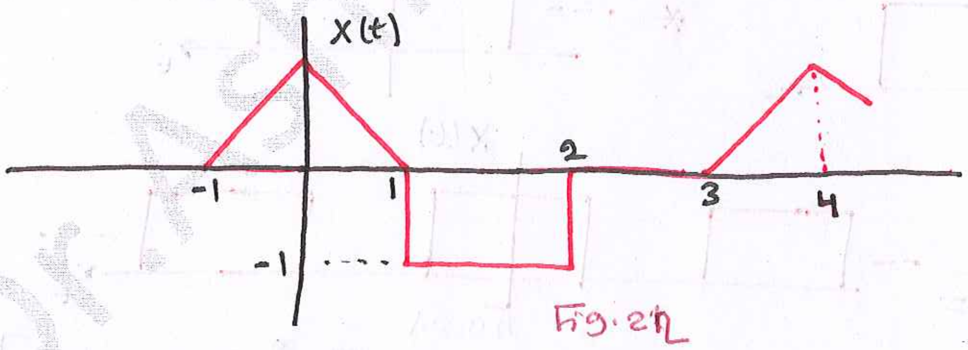

Fig. 22

Find the fourier transform, $X(f)$

Answer : $p(t) = \Lambda(t) - \Pi(t-1.5)$

$$P(f) = \operatorname{sinc}^2(f) - \operatorname{sinc}(f)\, e^{-j2\pi f(1.5)}$$

$$P(nf_0) = \operatorname{sinc}^2(n(0.25)) - \operatorname{sinc}(n(0.25))\, e^{-j2\pi (n(0.25)(1.5))}$$

$f_0 = \frac{1}{4}$ since $T_0 = 4$

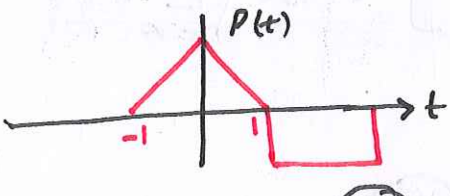

Fig. 23

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$$\Rightarrow X(f) = f_0 \sum_{n=-\infty}^{\infty} p(nf_0)\, \delta(f - nf_0)$$

and

$$x(t) = f_0 \sum_{n=\infty}^{\infty} p(nf_0)\, e^{j2\pi n f_0 t}$$

Example : For the following periodic signal

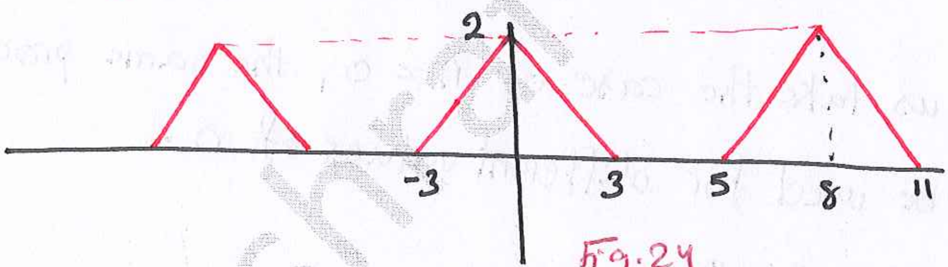

Fig. 24

Find the fourier transform, $X(f)$

Answer : From figure shown above, $p(t)$ can be expressed as

$$p(t) = 2 \Lambda\left(\frac{t}{3}\right)$$

$$\Rightarrow P(f) = 6 \operatorname{sinc}^2(3f)\ ;$$

$$P(nf_0) = 6 \operatorname{sinc}^2(3(n)f_0)\ ; \quad f_0 = \frac{1}{8}\ \text{Hz}$$

$$\Rightarrow P(nf_0) = 6 \operatorname{sinc}^2\left(\frac{3}{8} n\right)$$

and

$$X(f) = \frac{1}{8} \sum_{n=-\infty}^{\infty} 6 \operatorname{sinc}^2\left(\frac{3}{8} n\right) \delta(f - n/8)$$

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Example : Obtain the fourier transform of the periodic raised-cosine pulse train.

$$x(t) = \frac{1}{2}A\sum_{n=-\infty}^{\infty}\left[1+\cos\left(20\pi(t-nT_0)\right)\right]\Pi\left(\frac{t-nT_0}{\tau}\right)$$

where $T_0 \geq \tau$, sketch the wave-form and the amplitude spectrum for the case $\tau = T_0$

Answer : Let us take the case of $n=0$, the same procedure can be used for different values of n.

So, when $n=0$

$$p(t) = \frac{1}{2}A\left[1+\cos(20\pi t)\right]\Pi\left(\frac{t}{\tau}\right)$$

$$P(f) = \frac{1}{2}A\tau\,\text{sinc}(\tau f) + \frac{A}{4}\tau\,\text{sinc}(\tau(f-10)) + \frac{A}{4}\tau\,\text{sinc}(\tau(f+10))$$

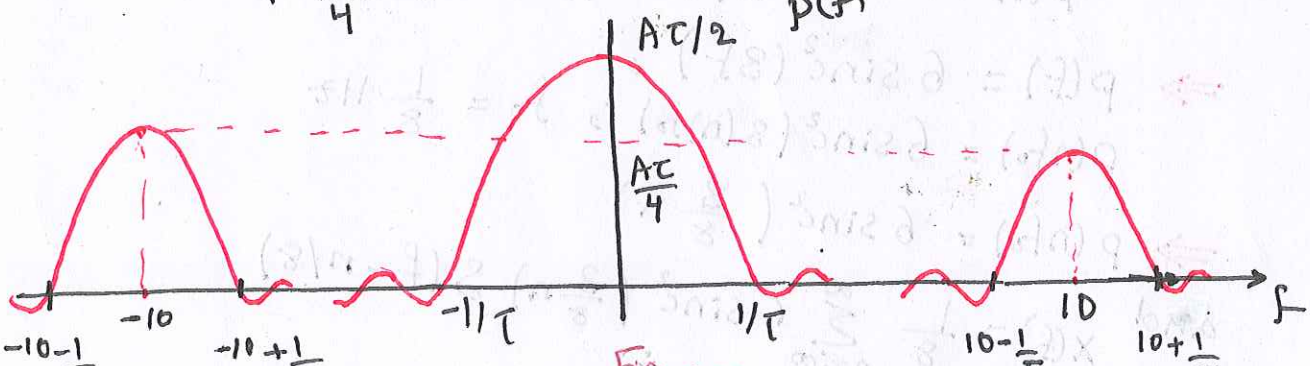

Fig. 25

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Example:

A signal $x(t) = \cos(2\pi(400)t)$ modulates the amplitude of the carrier signal $c(t) = 100\cos(2\pi * 10^4 t)$

a. Plot the double sided spectral representation of the signal and the carrier

b. Determine and plot the spectral representation of the modulated signal $s(t)$ in which $s(t)$ is expressed in the following figure

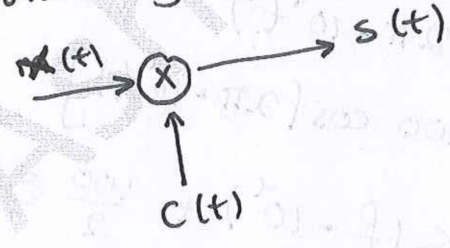

Fig: 26

c. Determine and plot the spectral representation of the Hilbert transformed of the carrier : $c^H(t)$.

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Answer :

a. $x(t) = \cos(2\pi(400)t)$

$$\mathcal{F}[x(t)] = \mathcal{F}[\cos(2\pi(400)t)]$$

$$= \frac{1}{2}\delta(f-400) + \frac{1}{2}\delta(f+400)$$

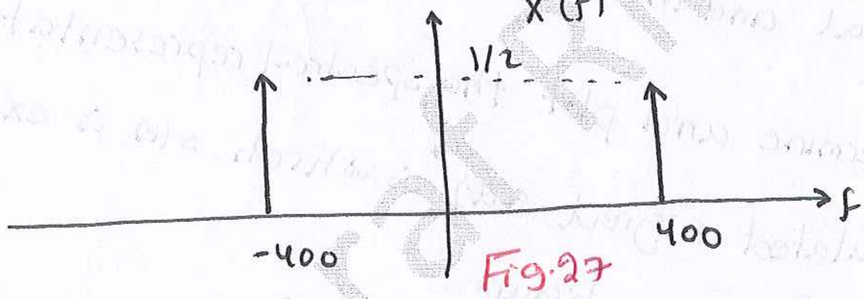

Fig.27

and

$$c(t) = 100\cos(2\pi * 10^4 t)$$

$$\mathcal{F}[c(t)] = \mathcal{F}[100\cos(2\pi * 10^4 t)]$$

$$= \frac{100}{2}\delta(f-10^4) + \frac{100}{2}\delta(f+10^4)$$

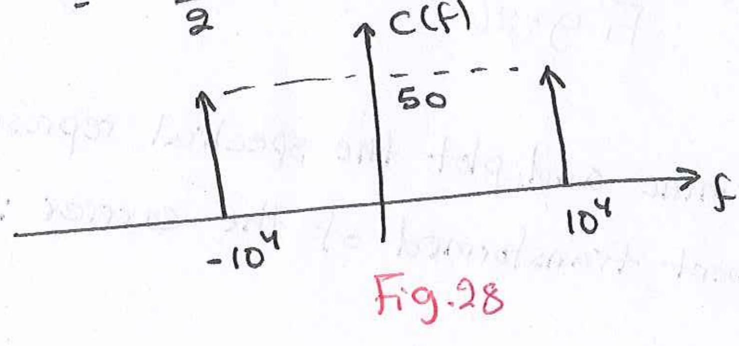

Fig.28

b. $s(t) = \left[\cos(2\pi(400)t)\right] \times \left[100\cos(2\pi(10^4)t)\right]$

$$s(t) = \frac{100}{2}\cos(2\pi(400+10^4)t) + \frac{100}{2}\cos(2\pi(10^4-400)t)$$

$$\mathcal{F}[s(t)] = \mathcal{F}\left[50\cos(2\pi(10^4+400)t)\right] + \mathcal{F}\left[50\cos(2\pi(10^4-400)t)\right]$$

$$= 25\,\delta(f-(10^4+400)) + 25\,\delta(f+(10^4+400)) + 25\,\delta(f+(10^4-400))$$

$$+ 25\,\delta(f-(10^4-400))$$

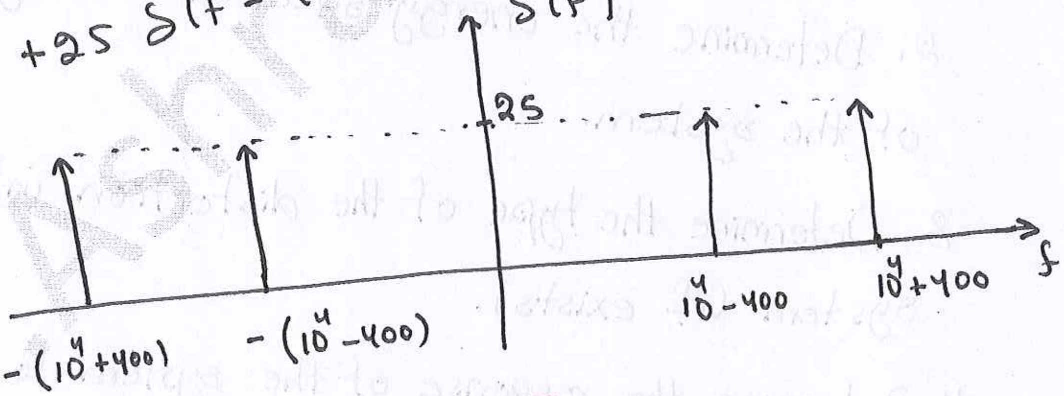

Fig. 29

c. $c^H(t) = \frac{1}{\pi t} * c(t)$

$$\mathcal{F}[c^H(t)] = -j\,\text{sgn}(f)\,C(f) = -j\,\text{sgn}(f)\left[50\,\delta(f-10^4) + \delta(f+10^4)\right]$$

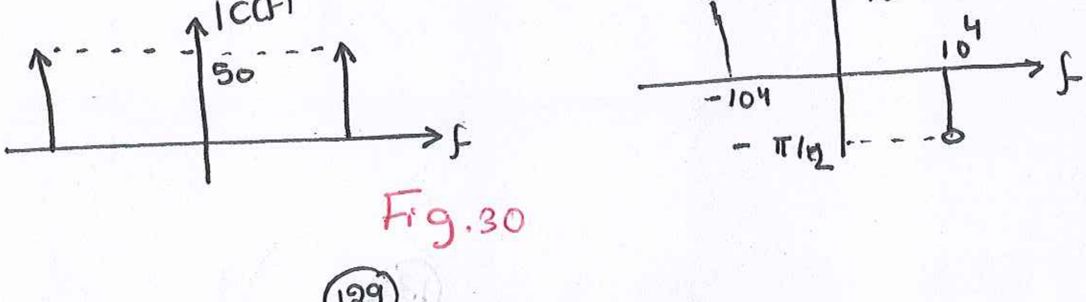

Fig. 30

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Example :

A Linear time invariant system is defined by the following differential equation

$$\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = x(t)$$

1. Determine the frequency response of the system.
2. Determine the energy spectral density representation of the system.
3. Determine the type of the distortion introduced by the system (if exists).
4. Determine the response of the system to the signal

$X(t) = 100\cos(2\pi t + \pi/3)$

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3. The type of distortion is

- Amplitude distortion, since $|H(f)|$ is not constant. (depends on frequency).

$$|H(f)| = \frac{1}{\sqrt{(4-(2\pi f)^2)^2 + (4\pi f)^2}}$$

- phase distortion, since the $\angle\Theta_{H(f)}$ is not Linear

$$\angle\Theta_{H(f)} = -\tan^{-1}\left(\frac{4\pi f}{4-(2\pi f)^2}\right)$$

4. $$y(t) = \sum_{n=-\infty}^{\infty} |X_n|\,|H(nf_0)|\, e^{j(\omega_0 t + \angle\Theta_{H(nf_0)} + \angle\Theta_{X_n})}$$

where

$$|H(nf_0)| = \frac{1}{\sqrt{(4-(2\pi(nf_0))^2)^2 + (4\pi(nf_0))^2}} \quad \text{and} \quad \angle\Theta_{H(nf_0)} = -\tan^{-1}\left(\frac{4\pi(nf_0)}{4-(2\pi nf_0)^2}\right)$$

$|X_1| = |X_{-1}| = 50$ and $\angle\Theta_{X_1} = -\angle\Theta_{X_{-1}} = \pi/3$

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Answer:

1. $\mathcal{F}\left[\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t)\right] = \mathcal{F}[x(t)]$

$$(j2\pi f)^2 Y(f) + 2(j2\pi f)Y(f) + 4Y(f) = X(f)$$

$$\left[4-(2\pi f)^2 + j4\pi f\right]Y(f) = X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{4-(2\pi f)^2 + j4\pi f}$$

$$= \frac{1\angle 0}{\sqrt{(4-(2\pi f)^2)^2 + (4\pi f)^2}\ \angle \tan^{-1}\left(\frac{4\pi f}{4-(2\pi f)^2}\right)}$$

$$= \frac{1}{\sqrt{(4-(2\pi f)^2)^2 + (4\pi f)^2}}\ \angle -\tan^{-1}\left(\frac{4\pi f}{4-(2\pi f)^2}\right)$$

$$= |H(f)| \angle \Theta_{H(f)}$$

2. The energy spectral density is

$$G(f) = |H(f)|^2$$

$$= \frac{1}{\sqrt{(4-(2\pi f)^2)^2 + (4\pi f)^2}}$$

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Chapter 8: Discrete-Time Signals and Systems

8.1 Introduction to Discrete-Time Signals and Systems

Signals in life can be analog or digital. The analog signal can be converted into digital signal by using analog-to-digital convertor (ADC) in which the stages of the analog-to-digital conversion could be summerized in Fig. 8.1

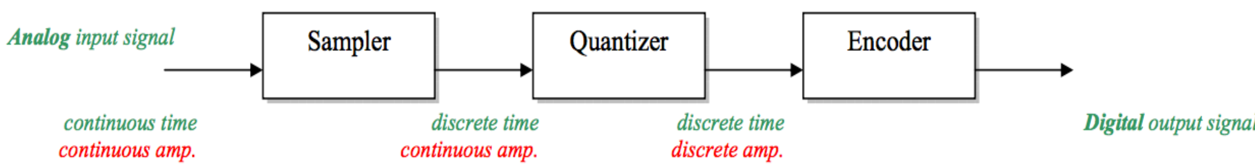


Fig 8.1: Block Diagram of Analog-To-Digital Convertor (ADC)

8.1.1 Sampling

The sampled signal, $x_s(t)$ can be generated by applying a switch to the input signal x(t) as shown in the figure:

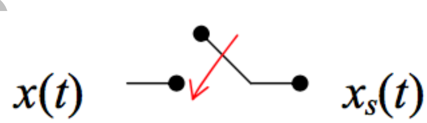
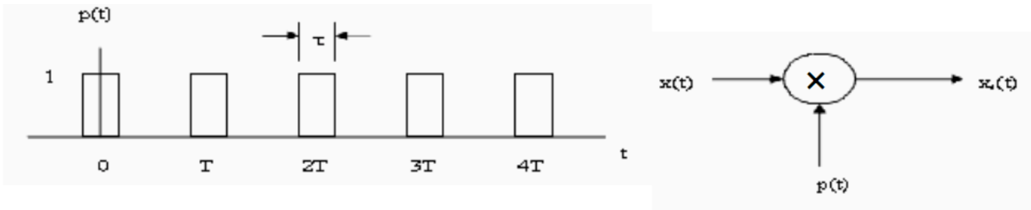


Fig 8.2 : Switch closes at t=nT

From Fig 8.2, in ideal case it can be noted that the switch passes the input signal to the output signa when it is closed whereas, nothing will pass to the output when the swich is opened. On the other hand, mathematically, this swich could be modeled as multiplier where the input signal is multiplying with another periodic signal, $p(t)$ which can take only two values 0 or 1 as shown in Fig 8.3.



(a) **(b)**

Fig 8.3 : The sampling operation, (a) Model of sampling device and (b) Sampling Function

In Fig. 8.3, it can be noted $T = \frac{1}{f_s}$, and τ is the sampling duration which is theoretically zero. In addition, the sampled frequency $x_s(t)$ can be expressed as

$$x_s(t) = x(t)p(t) \text{ ....(1)}$$

Since $p(t)$ is periodic signal, then $p(t)$ can be represented by exponent fourier series where

$$p(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_s t} \text{ ....(2)}$$

where $C_n = \frac{1}{T}\int_{-T/2}^{T/2} p(t)e^{-j2\pi f_s t}$

and f_s is the samling frequency or the frequency of the periodic signal of $p(t)$.

by substituting (2) into (1), then $x_s(t)$ can be expressed as

$$x_s(t) = \sum_{n=-\infty}^{\infty} C_n x(t)e^{j2\pi f_s t} \text{ ....(3)}$$

Now, by substituting (3) into (2) with interchanging the order of summation and integration, the result can be put in the following form

$$x_s(t) = \sum_{n=-\infty}^{\infty} C_n x(t)e^{j2\pi f_s t} \text{ . ...(4)}$$

<u>8.1.1.1 Spectrum of Sampled Signal</u>

The Fourier transform of $x_s(t)$ can be given by

$$X_s(f) = \int_{-\infty}^{\infty} x_s(t)\ e^{-j2\pi f t}dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_n x(t)e^{j2\pi n f_s t}e^{j2\pi f t}dt \text{.................(5)}$$

with interchanging summation and integration

$$x_s(t) = \sum_{n=-\infty}^{\infty} C_n \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-nf_s)}dt \text{..................(6)}$$

Hence, the fourier transform of the sampled signal, $x_s(t)$ as

$$X_s(f) = \sum_{n=-\infty}^{\infty} C_n X(f-nf_s) \text{.................(7)}$$

where $X(f - nf_s) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-nf_s)}dt$.

From (7), it can be concluded that the spectrum of the sampled continuous-time signal $x(t)$ is composed of the spectrum of $x(t)$ translated to each harmonic of the sampling frequency. Moreover, from

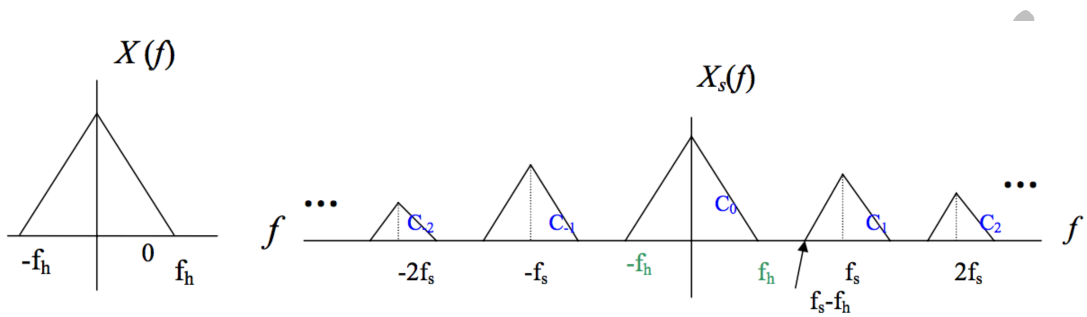


Fig 8.4 : Spectrum of Sampled Signal

Sampling Theorem :

From Fig. 8.4, it can be noted that

the original signal can be completely reconstructed by using low pass filter. Further, it can be noted that the constant scaling factor C_0 can be easily accounted by using an amplifier with gain equal to $\frac{1}{C_0}$.

<u>8.1.1.2 Ideal Sampling: Impulse-Train Sampling Model</u>

Consider $p(t)$ is composed of an infinite train of impulse function of period T. Thus,

$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ .... (8)

which is the sampling function illustrated in Fig. 8.5.

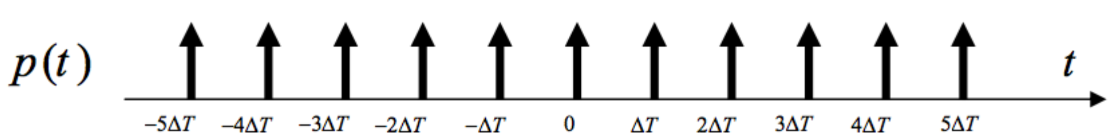


Fig 8.5 : Impulse train Function

Since $p(t)$ is periodic signal, then the values of C_n can be expressed as

$C_n = \frac{1}{T}\int_{-\infty}^{\infty} p(t)e^{-j2\pi n f_s t}dt$ …………… (9)

By using sifting property, C_n results

$C_n = T = \frac{1}{f_s}$ ………………. (10)

8.1.1.3 Ideal Sampling: Impulse –Train Sampling Model

By substituting (10) in (7), then the spectrum of sampled signal $x_s(t)$ can be given be

$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$ …………… (11)

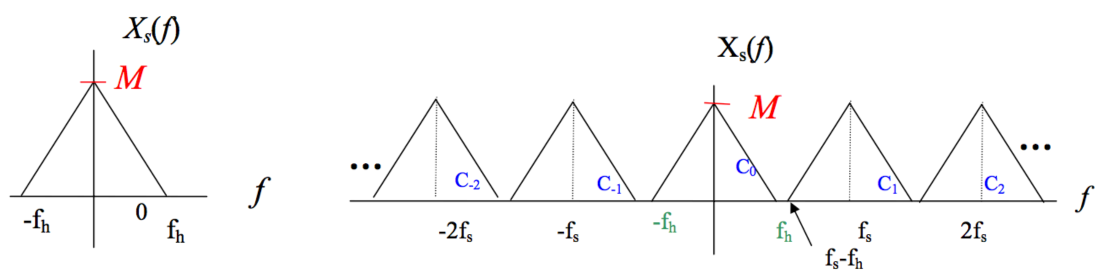


Fig 8.6 : Spectrum of Sampled Signal

8.1.2 Data Reconstruction

As shown in Fig 6.7, the original signal can be perfectly reconstructed using a low-pass filter with cut-off frequency equals to $f_{s/2}$ provided that the original signal was sampled at a frequency above $2\,f_h$. In other words, the original signal can be completely reconstructed by using low pass filter. Further, it can be noted that the constant scaling factor C_0 can be easily accounted by using an amplifier with gain equal to $\frac{1}{C_0}$.

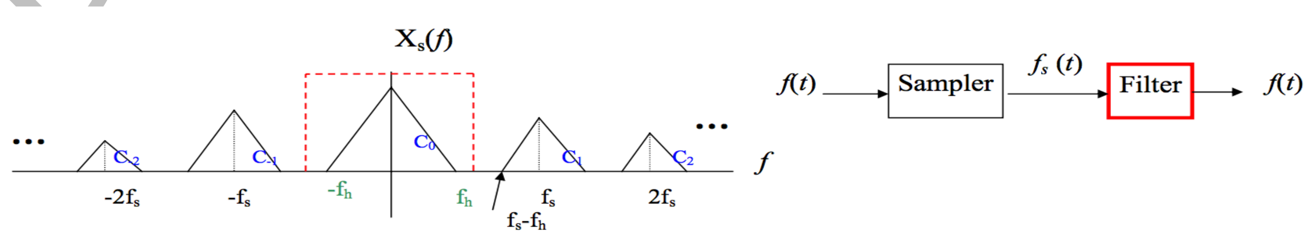


Fig 8.7: Data Reconstruction

Aliasing

Whereas, if the original signal is sampled at a rate less than twice the highest frequency then the translated spectrums will overlap and the original signal will not be reconstructed properly. This effect is know aliasing and it is illustrated in Fig. 8.8,

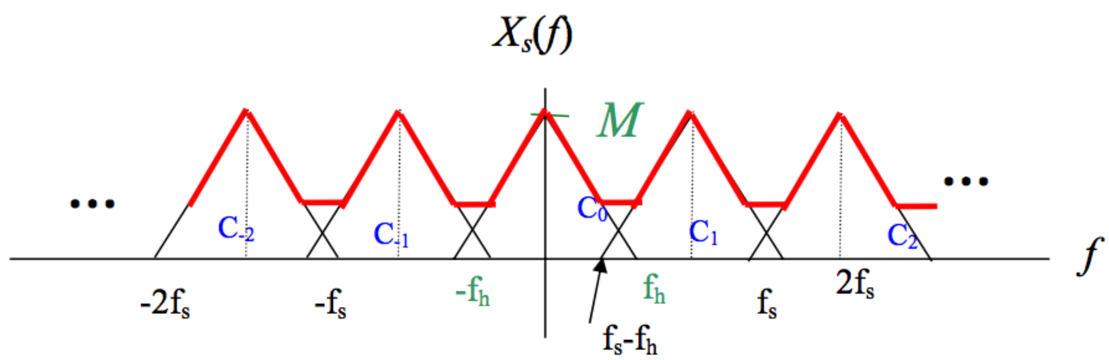


Fig 8.8: Illustration of sampled signal for $f_s < 2\, f_h$

<u>8.2.1 Ideal Reconstruction Filter</u>

An ideal low-pass filter can be used to reconstruct the data. It has the following transfer function

$$H(f) = \begin{cases} T & |f| < 0.5\, f_s \\ 0 & o.w \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

By using Inverse Fourier Transform, then $h(t)$ can be expressed as

$$h(t) = \frac{\sin(\pi f_s t)}{\pi f_s t} = sinc(f_s t) \ \ldots\ldots\ldots\ldots (13)$$

From (13) it can be noted that the impulse response is not time limited and non-causal.

In addition, from Fig 8.8 it can be noted that the constructed signal could be obtained by using the convolution theorem between $x_s(t)$ and $h(t)$ where the final result can be given by

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT) sinc(\tfrac{t}{T} - k) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

If a value is to be interpolated between nT and $nT + T$ as shown in Fig. 8.9, and l samples each side of the value to be interpolated, then we have

$$x(t) = \sum_{k=n-l+1}^{n+1} x(kT) sinc(\frac{t}{T} - k) \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

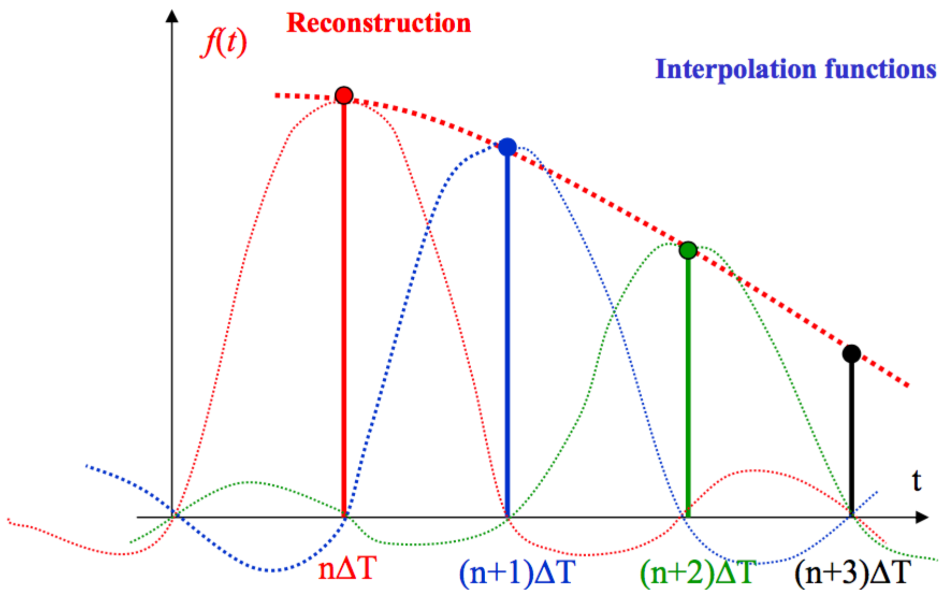


Fig 8.9: Time-domain equivalent

Example 8.1: The signal

$$x(t) = 6\cos(10\pi t)$$

sampled at 7 Hz and 14 Hz. For each sampled frequency

A. Plot the spectrum of $x(t)$.

B. Plot the spectrum of sampled signal

C. Plot the output of reconstruction filter.

Answer

In this example we are interest to see the effect of sampling a signal at both a frequency less and greater than twice the highest frequency where the highest frequency (the only frequency in this case) is 5 Hz.

A. By using Fourier Transform, $X(f)$ can be expressed as

$X(f) = 3\delta(f-5) + 3\delta(f+5)$ ……………………(16)

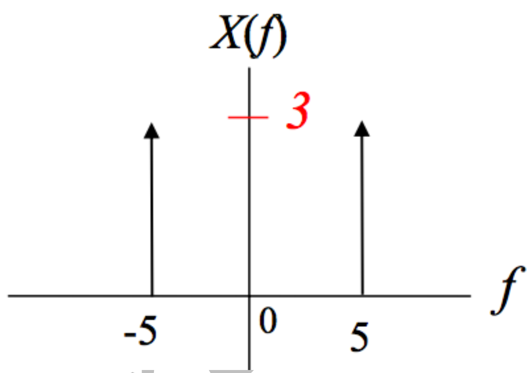


Fig 8.10 : Spectrum of x(t)

B. The spectrum of the sampled signal can be easily found by using (11) where

$X_s(f) = 3f_s \sum_{n=-\infty}^{\infty}[\delta(f-5-nf_s) + \delta(f+5-nf_s)$ ………………(17)

For the case of $f_s = 7Hz$

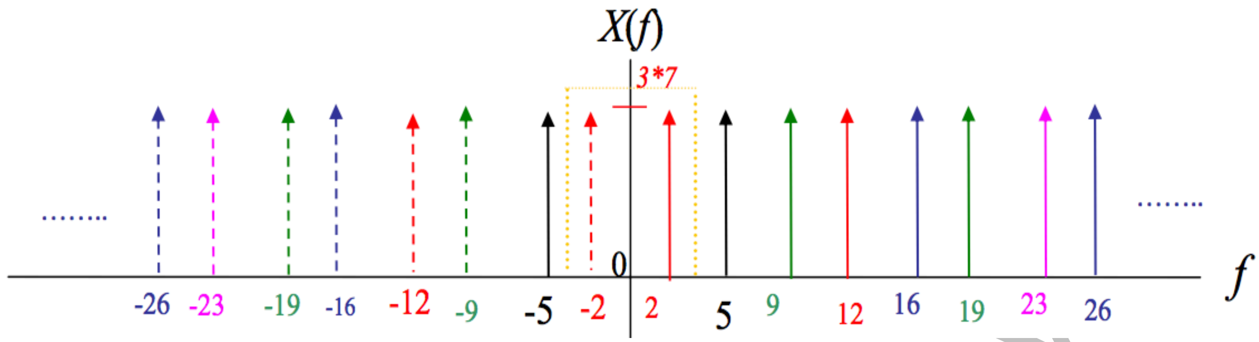


Fig 8.11: Spectrum of sampled signal with $\boldsymbol{f_s = 7Hz}$

A low-pass filter with cut-off frequency $\frac{f_s}{2} = \frac{7}{2} = 3.5$ Hz is used. The amplitude of the filter in the low-pass region should be $\frac{1}{f_s} = \frac{1}{7}$.

C. The reconstructed spectrum is shown

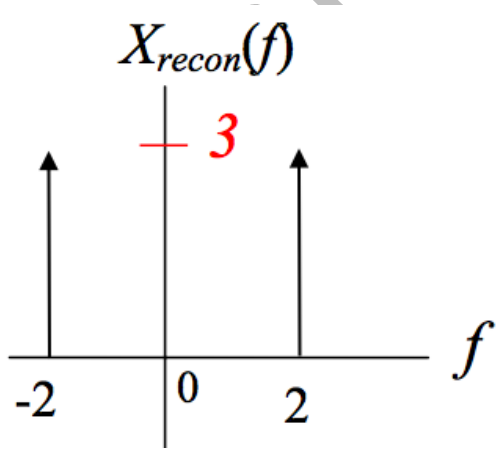


Fig 8.12: Output of reconstruction filter with $\boldsymbol{f_s = 7\ Hz}$.

This is equivalent in the time domain to

$x(t) = 6\cos(4\pi t) = 6\cos\,(2\pi(2)t)$ ……………..(18)

Because the original signal was sampled below Nyquist rate it could not be reconstructed properly. Note that the reconstructed signal is similar to the original one with lower frequency as a result of aliasing.

Now, let the sampling frequency be 14Hz which above Nyquist rate. The spectrum of the sampled signal becomes

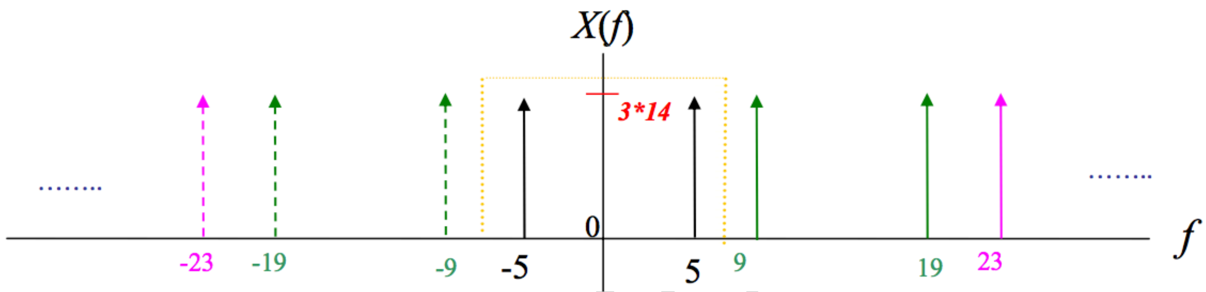


Fig 8.13: Spectrum of sampled signal with $f_s = 14\ Hz$

Now, a low-pass filter with cut-off frequency =fs/2=7/2=7 Hz. The amplitude of the filter in the low- pass region should be 1/fs=1/14. The reconstructed spectrum is exactly like the original signal.

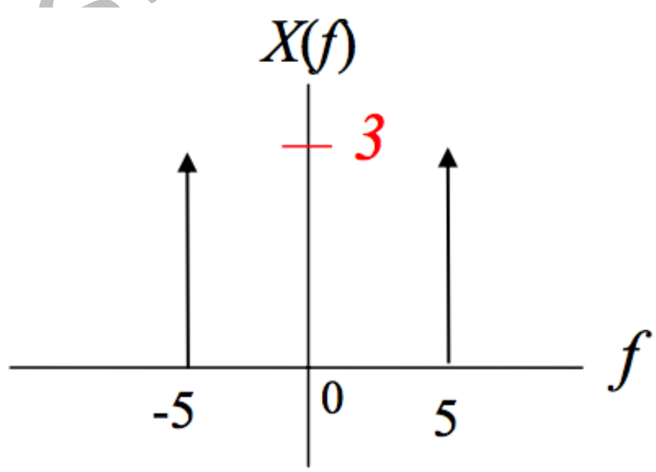


Fig 8.13: Output of reconstruction filter with $f_s = 14\ Hz$

Example 8.2: Consider the following signal,

$x(t) = 4\cos(8\pi t) + 6\cos(6\pi t)$ ………………. (19)

A) What is the minimum required sampling frequency to avoid aliasing?

B) If the signal is sampled at a rate of 10 samples/second, What are the possible bandwidths of the low-pass filter required to reconstruct $x(t)$ from $x_s(t)$?

C) sketch the spectrum of $x(t)$ and the spectrum of $x_s(t)$?

Answer:

A) Greater than twice the highest frequency=2*4=8 Hz.

B) If we sketch the spectrum of the sampled signal. It is easy to see that the bandwidth should be between 4 & 6 Hz.

C) This part is left for you ☺ ☺ ☺

<u>8.2.2 Practical reconstruction</u>

There are other different methods to reconstruct the signals which are not exact:

* In the time-domain one may use linear interpolation between the points. Other averaging techniques are also possible.

- In frequency-domain, RC circuit might be used to approximate low-pass filter.

Finally, as shown in the figure below the reconstructed spectrum may suffer from variation in the amplitude in the pass-band region in addition to non-zero amplitude in the stop-band region.

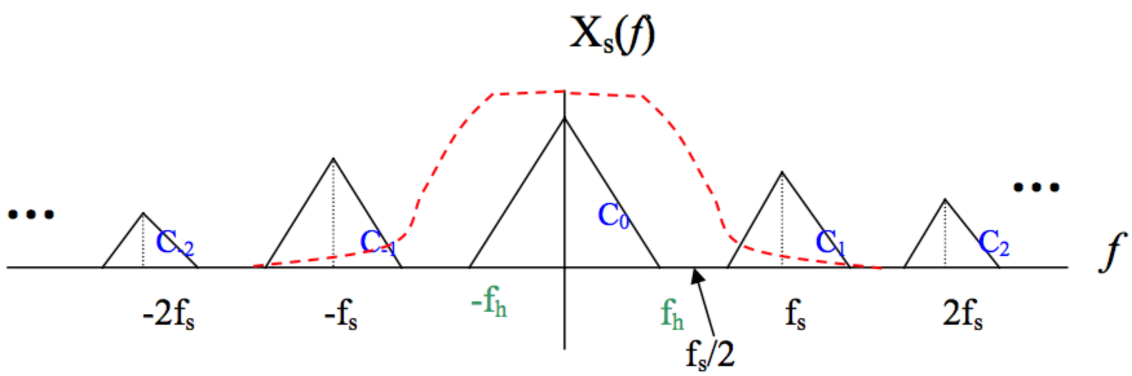


Fig 8.14 Simple first-order low pass reconstruction filter

8.2 The Z-Transform

The z-transform is the basic tool for the analysis and synthesis of discrete-time systems in which it is defined as

$X(z) = \sum_{n=-\infty}^{\infty} x(nT)Z^{-n}$ .... (20)

where the coefficient $x(nT)$ denote the sample value and Z^{-n} denotes that the sample occurs n sample periods after the t=0 reference.

8.2.1 Derivation of the Z-transform

The sampled signal may be written as

$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$ .. .. (21)

since $\delta(t - nT) = 0$ for all t except at $t = nT$, $x(t)$ can be replaced by $x(nT)$. Assuming $x(t) = 0$ for $t < 0$. Then

$x_s(t) = \sum_{n=0}^{\infty} x(nT)\delta(t - nT)$ .... (22)

Taking Laplace transform yields

$$X_s(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x(nT)\delta\,(t - nT)e^{-st}dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

By sifting property of the delta function

$$X_s(S) = \sum_{n=0}^{\infty} x(nT)e^{-snT} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Now, let us define the complex variable z as the laplace time-shift operator

$$z = e^{sT} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

By substituting (25) in (24), $X(z)$ can be expressed as

$$X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n} \quad \ldots\ldots\ldots\ldots\ldots (26)$$

In additon to, from (25) it can be noted that the left-half plane correspond to $\sigma < 0$ is mapped to $|z| < 1$ in the z-plane which is the region inside the unit circle as shown in Fig 8.15.

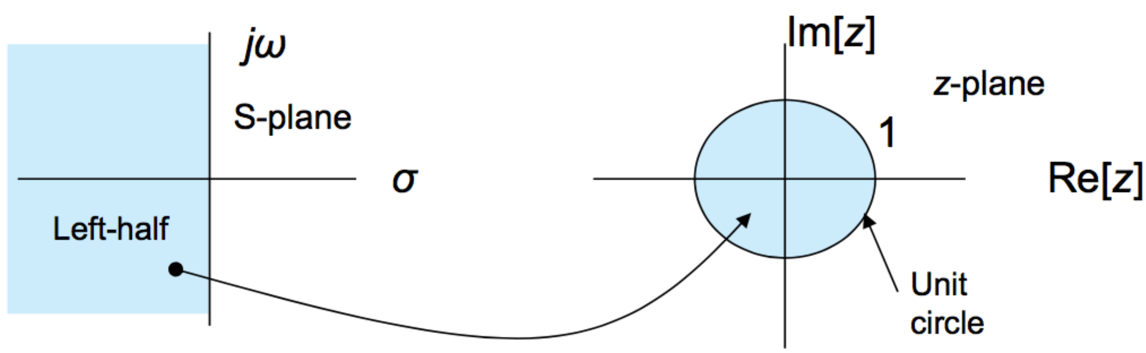


Fig 8.15

Example 8.3: The unit pulse sequence is defined by the sample values:

$$x(nT) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases} \triangleq \delta(n)$$

Determine the z-transform $X(z)$.

Ans: $$X(z) = \sum_{n=0}^{\infty} x(nT)\, z^{-n}$$

$$= 1 + 0 \cdot z^{-1} + 0 \cdot z^{-2} + \cdots$$

$$X(z) = 1$$

Example 8.4: The unit step sample sequence is defined by the sample values

$$x(nT) = 1, \quad n \geq 0$$

Determine the z-transform $X(z)$.

Ans: $$X(z) = \sum_{n=0}^{\infty} x(nT)\, z^{-n}$$

we note that for $|x| < 1$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

Example 8.5: The unit exponential sequence is defined by the sample values:

$$x(nT) = e^{-\alpha nT}, \quad \alpha > 0, \ n \geq 0$$

Determine the Z-transform $X(z)$

Ans:
$$X(z) = \sum_{n=0}^{\infty} x(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(z\, e^{\alpha T} \right)^{-n}$$

$$= \frac{1}{1 - z^{-1} e^{-\alpha T}}, \quad |z| > e^{-\alpha T}$$

Example 8.6: For

$$x(nT) = a^{n} \cos\left(\frac{n\pi}{2}\right)$$

Find $X(z)$

Ans:
$$X(z) = \sum_{n=0}^{\infty} x(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{n} \cos\left(\frac{n\pi}{2}\right) z^{-n}$$

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & , \ n \text{ odd} \\ \pm 1 & , \ n \text{ even} \end{cases}$$

$$\Rightarrow X(z) = \sum_{k=0}^{\infty} a^{2k} (-1)^{k} z^{-2k}$$

$$= \sum_{k=0}^{\infty} (-a^2 z^{-2})^k$$

$$= \frac{1}{1+a^2 z^{-2}}$$

Example 8.7: Determine the z-transform of the signal

$$x[n] = 0.5^n u[n]$$

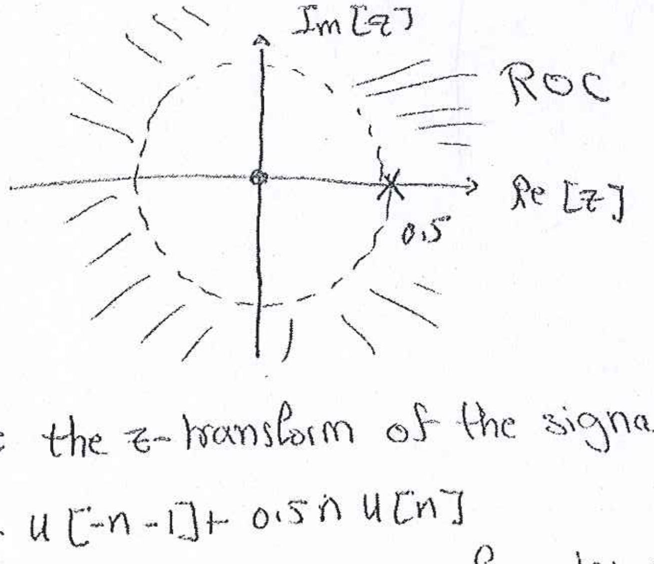
Depict the ROC and the locations of poles and zeros of $X(z)$ in the z-plane.

Ans: $X(z) = \sum_{n=0}^{\infty} (0.5)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n$

$$= \frac{1}{1-0.5z^{-1}}, \quad |z| > 0.5$$

$$= \frac{z}{z-0.5}, \quad |z| > 0.5$$

~~poles~~ zero at $z = 0$, ~~zero~~ pole at $z = 0.5$, ROC is the $|z| > 0.5$ as shown in Fig



Example 8.8: Determine the z-transform of the signal

$$x[n] = -u[-n-1] + 0.5^n u[n]$$

Depict the ROC and locations of poles and zeros of $X(z)$ in the z-plane

Ans: $$X(z) = \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^{n} - \sum_{n=-\infty}^{-1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^{-n} + 1 - \sum_{k=0}^{\infty} z^{k}$$

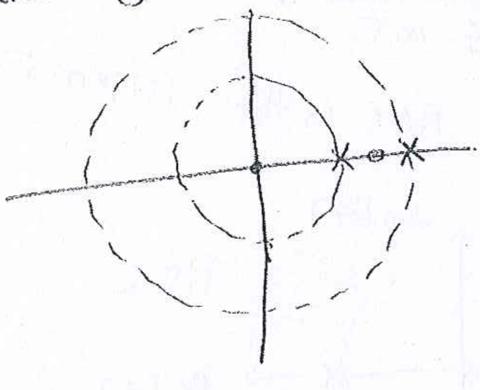
The sum converges provided that $|z| > 0.5$ and $|z| < 1$

$$X(z) = \frac{1}{1 - 0.5 z^{-1}} + 1 - \frac{1}{1 - z}, \quad 0.5 < |z| < 1$$

$$= \frac{z(2z - 1.5)}{(z - 0.5)(z - 1)}, \quad 0.5 < |z| < 1$$

Poles at $z = 0.5, 1$, zeros at $z = 0, 0.75$

ROC is the region in between



8.2.2 Properties of the Z-transform

Now, let us investigate some of the z-transform properties:

1. Linearity.
2. Time-shift property.
3. Initial and final value theorems.

8.2.2.1 Linearity

The z-transformation is a linear operation. In other words, if A and B are constants,

$$\sum_{n=0}^{\infty} [A x_1(nT) + B x_2(nT)] z^{-n} = A \cdot X_1(z) + B X_2(z) \quad \text{------ (27)}$$

where $X_1(z)$ and $X_2(z)$ are the z-transforms of $x_1(nT)$ and $x_2(nT)$, respectively. This is easily seen by recognizing that the left-hand side of (27) can be written

$$\sum_{n=0}^{\infty} [A x_1(nT) + B x_2(nT)] z^{-n} = A \sum_{n=0}^{\infty} x_1(nT) z^{-n} + B \sum_{n=0}^{\infty} x_2(nT) z^{-n} \quad \text{------ (28)}$$

By definition, the two sums on the right-hand side of (28) are $X_1(z)$ and $X_2(z)$.

8.2.2.2 Initial value and final value theorems

The initial value theorem states that

$$X(0) = \lim_{z \to \infty} X(z) \quad \text{----- (29)}$$

This result is easily derived. By definition

$$X(z) = \sum_{n=0}^{\infty} X(nT) z^{-n} = X(0) + \sum_{n=1}^{\infty} X(nT) z^{-n} \quad \text{--- (30)}$$

As $z \rightarrow \infty$, the summation on the right vanishes and (29) results.

The final value theorem states that

$$x(\infty) = \lim_{z \rightarrow 1} (1 - z^{-1}) X(z) \quad \text{-----} \quad (31)$$

Several interesting proofs of the final value theorem are given in the literature.

Example 8.9: Find the initial and final values for the following signal expressed in its z-transform

$$F(z) = \frac{0.792 z^2}{(z-1)(z^2 - 0.416 z + 0.208)}$$

Ans: Initial-value $F(z \rightarrow \infty) = \dfrac{0.792 z^2}{z^3} = 0$

Final-value $f(n \rightarrow \infty) = \dfrac{0.792}{(1 - 0.416 + 0.208)} = 1$

8.2.2.3 Time-shift property

If $x[n] \overset{z}{\longleftrightarrow} X(z)$ with ROC $= R$, then

$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0} X(z)$ with ROC $= R$

Proof: $\mathcal{Z}\{x(nT-KT)\} = \sum_{n=0}^{\infty} x(nT-KT) z^{-n}$

Let

$m = n - K \Rightarrow$

$$z[x(nT-KT)] = \sum_{m=-K}^{\infty} x(mT) z^{-m-K}$$

$\Rightarrow Z[x(nT-KT)] = z^{-K} X(z)$

Example 8.10 : For the following input signal

$$x[n] = 7(1/3)^{n-2} u(n-2) - 6(1/2)^{n-1} u[n-1]$$

Find the z-transform $X(z)$

Ans: $X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n}$

$$= 7 z^{-2} \frac{1}{1-\frac{1}{3}z^{-1}} - 6 z^{-1} \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$= 7 \frac{1}{z^2 - \frac{1}{3}z} - 6 \frac{1}{z - 1/2}$$

8.3 : Inverse Z-transform :

The inverse operation for the z-transform may be accomplished by:

1. Long division
2. Partial fraction expansion.

Example 8.11: Find the inverse z-transform using both partial fraction expansion and long division

$$X(z) = \frac{z^2}{(z-1)(z-0.2)}$$

Ans: If we treat z^{-1} as the variable in the partial fraction expansion, we can write

$$X(z) = \frac{1}{(1-z^{-1})(1-0.2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.2z^{-1}}$$

where

$$A = (1 - z^{-1})\, X(z) = \frac{1}{1 - 0.2z^{-1}} = \frac{1}{0.8} = 1.25$$

$$B = (1 - 0.2z^{-1})\, X(z) = \frac{1}{1 - z^{-1}} = -0.25$$

$$\Rightarrow X(z) = \frac{1.25}{1 - z^{-1}} + \frac{-0.25}{1 - 0.2z^{-1}}$$

$$\Rightarrow x(nT) = \left(1.25 - 0.25(0.2)^n\right) u[n]$$

From which we may find that $x(0) = 1$, $x(T) = 1.2$, $x(2T) = 1.24$, $x(3T) = 1.248$.

The same result can by be obtained if we use long division. where:

$$X(z) = 1 + 1.2z^{-1} + 1.24z^{-2} + 1.248z^{-3} + \ldots$$

The solution is left for you .... ☺

Example 8.12: Find the inverse z-transform for the following $Y(z)$, where

$$Y(z) = \left[\frac{z^2}{z^2 - 1.2z + 0.2}\right] z^{-2}$$

Ans:

$$Y(z) = \left[\frac{z^2}{z^2 - 1.2z + 0.2}\right] z^{-2}$$

$$= X(z)\, z^{-2}$$

where $X(z) = \dfrac{z^2}{z^2 - 1.2z + 0.2} = \dfrac{z^2}{(z-1)(z-0.2)}$

$$\Rightarrow y(nT) = x(nT)\, z^{-2} = x((n-2)T)$$

$$= 1.25 - 0.25\,(0.2)^{n-2}, \quad n \geq 2$$

8.4 Differential Equations and Discrete-Time systems

8.4.1 properties of systems:

$$\xrightarrow{x(nT)} \boxed{h(nT)} \xrightarrow{y(nT)}$$

$$y(nT) = \mathcal{R}\left[x(nT)\right]$$

1. Shift-Invariant System:

A system is fixed or time invariant if the input-output relationship does not change with time.

$$\mathcal{R}\left[x(nT - n_0T)\right] = y(nT - n_0T)$$

for any finite value of n_0.

2. Causal and non-causal System:

A system is causal if its response to an input does not depend on future values of the input.

$$x_1(nT) = x_2(nT) \qquad \text{for } n \le n_0$$

Emplies the condition

$$\mathcal{R}\left[x_1(nT)\right] = \mathcal{R}\left[x_2(nT)\right] \qquad \text{for } n \le n_0$$

for any $x_1(nT)$, $x_2(nT)$ and n_0.

3. Linear Systems:

$$\mathcal{R}\left[\alpha_1 x_1(nT) + \alpha_2 x_2(nT)\right]$$

$$= \mathcal{R}\left[\alpha_1 x_1(nT)\right] + \mathcal{R}\left[\alpha_2 x_2(nT)\right]$$

$$= \alpha_1 \mathcal{R}[x_1(nT)] + \alpha_2 \mathcal{R}[x_2(nT)]$$

$$= \alpha_1 y_1(nT) + \alpha_2 y_2(nT)$$

Finally, the transfer function of a discrete time LTI system is the z-transform of the system's impulse response in which the convolution theorem is used [The proof in text book].

where

$$y(nT) = \sum_{k=0}^{n} h(kT)x(nT-kT)$$

$$= \sum_{k=0}^{n} x(kT)h(nT-kT)$$

Example 8.13: For the following LTI differential Equation, Find the transfer function $H(z) = Y(z)/X(z)$.

$$y[n] - 0.8\,y[n-1] = x[n]$$

Ans:
$$Z[y[n] - 0.8\,y[n-1]] = Z[x[n]]$$

$$Y(z) - 0.8\,Y(z)\,z^{-1} = X(z)$$

$$[1 - 0.8z^{-1}]\,Y(z) = X(z)$$

$$H(z) = \frac{1}{1-0.8z^{-1}}$$

$$h[n] = (0.8)^n u[n]$$

Example 8.15: If $x(nT) = \left(\frac{1}{2}\right)^n u(n)$ and

$$h(nT) = \left(\frac{1}{3}\right)^n u(n)$$

Find $y(nT) = x(nT) * h(nT)$

Ans: $$y(nT) = \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u(m) \left(\frac{1}{3}\right)^{n-m} u(n-m)$$

$$= \left(\frac{1}{3}\right)^n \sum_{m=0}^{n} \left(\frac{3}{2}\right)^m$$

By using formula

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}$$

$$\Rightarrow \quad y(nT) = \left(\frac{1}{3}\right)^n \frac{1-\left(\frac{3}{2}\right)^{n+1}}{1-\frac{3}{2}} \quad , \quad n \geq 0$$

4. Stable Systems: (BIBO)

A linear discrete-time system is BIBO stable if

$$|y(nT)| < \infty \quad , \quad \text{all } n$$

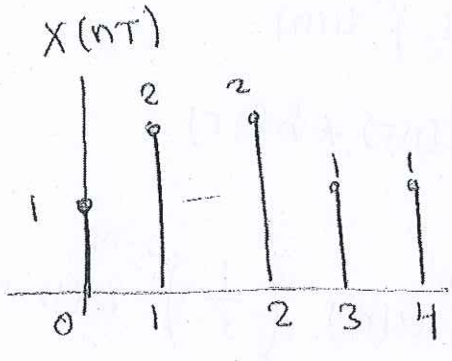
For all bounded inputs.

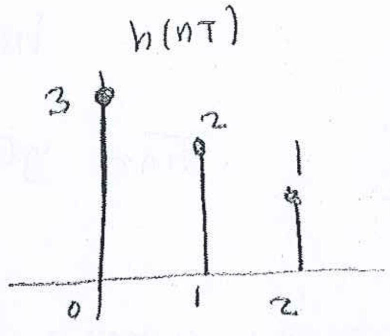
$$y(nT) = \sum_{k=-\infty}^{\infty} x(kT)\, h(nT-kT)$$

$$|y(nT)| = \left| \sum_{k=-\infty}^{\infty} x(kT)\, h(nT-kT) \right|$$

$$|y(nT)| \leq \sum_{k=-\infty}^{\infty} |x(kT)|\, |h(nT-kT)|$$

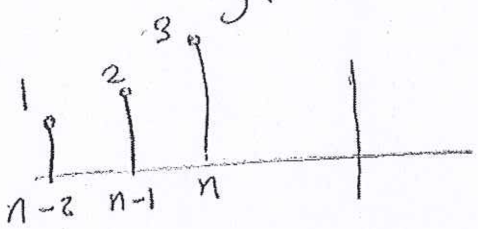
Example 8.14: Convolve the two functions shown in Fig.

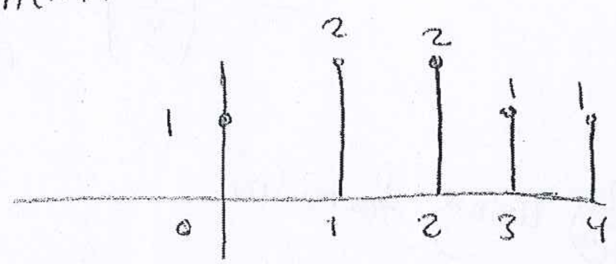




Ans: Let us define

$y(nT) = x(nT) * h(nT)$





when $n=0$

$y(0) = (3)(1) = 3$

when $n=1$

$y(T) = 3.2 + 1.2 = 8$

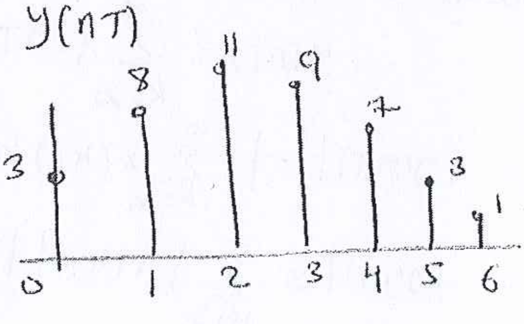
when $n=2$

$y(2T) = 1.1 + 2.2 + 2.3 = 11$

$\vdots$

when $n=6$

$y(6T) = 1.1 = 1$



Example 8.17: For the system described by the following differential equation

$$6\,y[n] - 5\,y[n-1] + y[n-2] = x[n]$$

Calculate the step response of the system

Ans : $H(z) = \dfrac{1}{6 - 5z^{-1} + z^{-2}} = \dfrac{1}{(2-z^{-1})(3-z^{-1})}$

for step response $X(z) = \dfrac{1}{1-z^{-1}}$ where $x[n] = u[n]$

$\Rightarrow \quad Y(z) = \dfrac{1}{(3-z^{-1})(2-z^{-1})(1-z^{-1})}$

$$= 0.5\,\frac{1}{(3-z^{-1})} - \frac{1}{(2-z^{-1})} + 0.5\,\frac{1}{(1-z^{-1})}$$

$$= 0.167\,\frac{1}{(1-\frac{1}{3}z^{-1})} - 0.5\,\frac{1}{(1-\frac{1}{2}z^{-1})} + 0.5\,\frac{1}{(1-z^{-1})}$$

$$y[n] = \left(0.167\left(\tfrac{1}{3}\right)^n - 0.5\left(\tfrac{1}{2}\right)^n + 0.5\right)u[n].$$

For bounded input

$$|x(nT)| \le M < \infty \, , \quad \text{all } n$$

$$\Rightarrow \quad |y(nT)| \le M \sum_{k=-\infty}^{\infty} |h(nT-kT)|$$

$$= M \sum_{n=-\infty}^{\infty} |h(nT)|$$

Thus the system output is bounded if

$$\sum_{n=-\infty}^{\infty} |h(nT)| < \infty$$

For causal system this is equivalent to the requirement that the system poles be inside the unit circle in the z-plane.

Example 8.16: For the system defined by

$$h(nT) = \left[4\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{4}\right)^n \right] u[n]$$

check the stability of the system.

ANS:

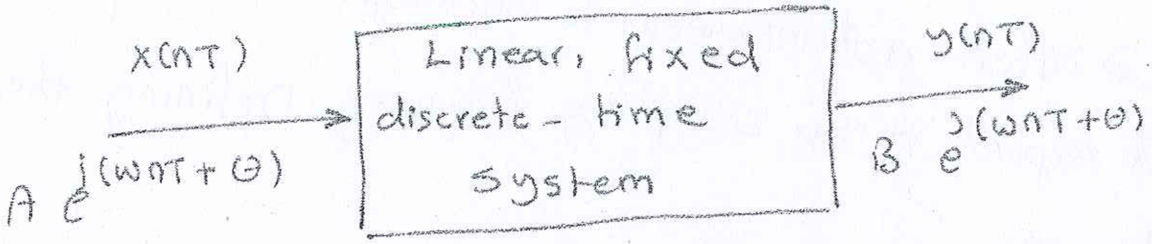
$$\sum_{n=-\infty}^{\infty} |h(nT)| = \sum_{n=0}^{\infty} 4\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{4}\right)^n$$

This yields

$$\sum_{n=-\infty}^{\infty} |h(nT)| = \frac{4}{1-\frac{1}{3}} - \frac{3}{1-\frac{1}{4}} = 2 < \infty$$

$\Rightarrow$ BIBO

8.4.2 Steady State Response of a linear Discrete-Time system.



Example 8.17: For the following system

$$y(nT) = x(nT) + x(nT-2T)$$

calculate the steady-state frequency response

Ans: $\mathcal{Z}[y(nT)] = \mathcal{Z}[x(nT)] + \mathcal{Z}[x(nT-2T)]$

$$Y(z) = X(z) + X(z)\,z^{-2}$$

$$Y(z) = [1+z^{-2}]\,X(z)$$

$$H(z) = \frac{1+z^{-2}}{[1]} = \frac{Y(z)}{X(z)}$$

$$= 1 + z^{-2}$$

$$H(e^{j\omega T}) = 1 + e^{-j2\omega T}$$

$$= (e^{j\omega T} + e^{-j\omega T})\,e^{-j\omega T}$$

$$= 2\cos(\omega T)\,e^{-j\omega T}$$

Note that:

Since $H(e^{j\omega T})$ is periodic in the sampling frequency, it is often advantageous to normalize the frequency variable with respect to the sampling frequency. Defining the frequency ratio as

$$r = \frac{\omega}{\omega_s}$$

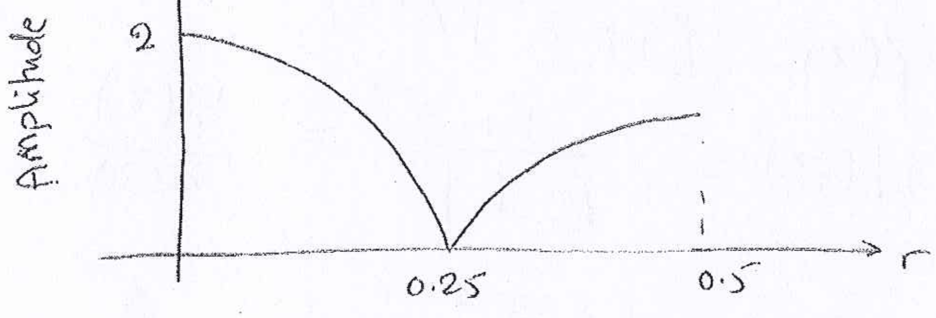
allows ωT to be replaced by

$$\omega T = r\,\omega_s T = 2\pi r$$

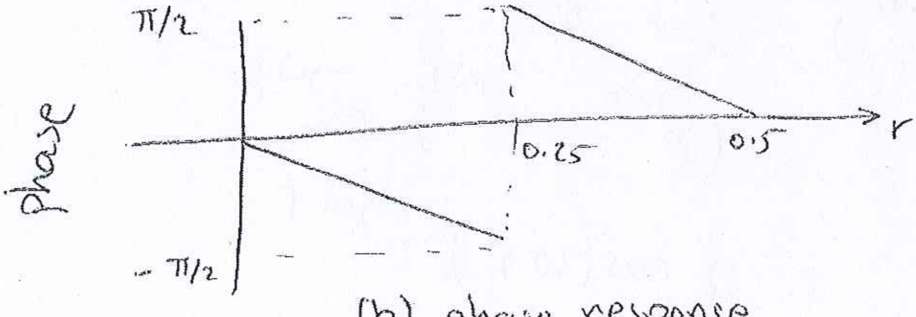
$\Rightarrow$ The steady state frequency response in terms of normalized frequency given by

$$H(e^{j2\pi r}) = 2\cos(2\pi r)\, e^{-j2\pi r}$$

where its Amplitude and phase are shown below



(a) Amplitude ~~Spectrum~~ response.



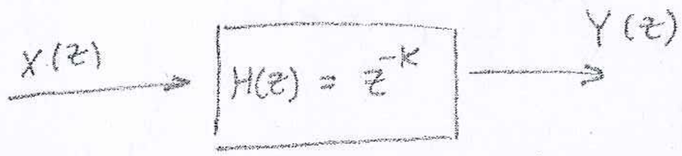
(b) phase response

Example 8.18: Plot the amplitude and phase response of a system that produces an output equal to the input delayed by K sample periods, where

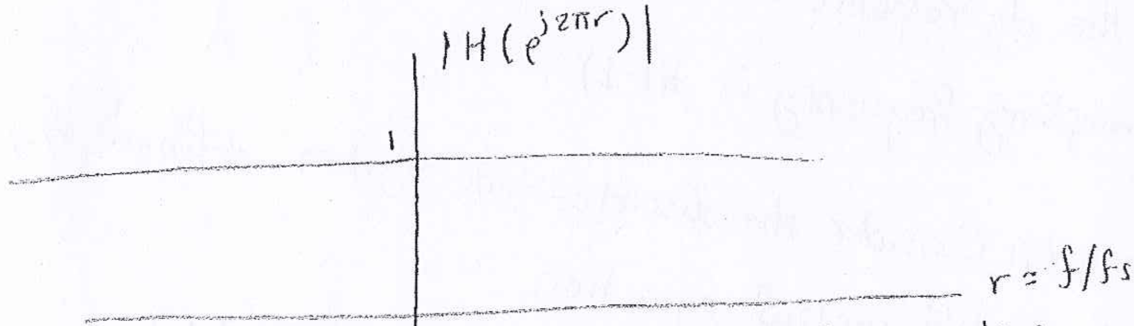
$$H(z) = z^{-K}$$

Ans: The sinusoidal steady-state frequency response is, in terms of normalized frequency, given by

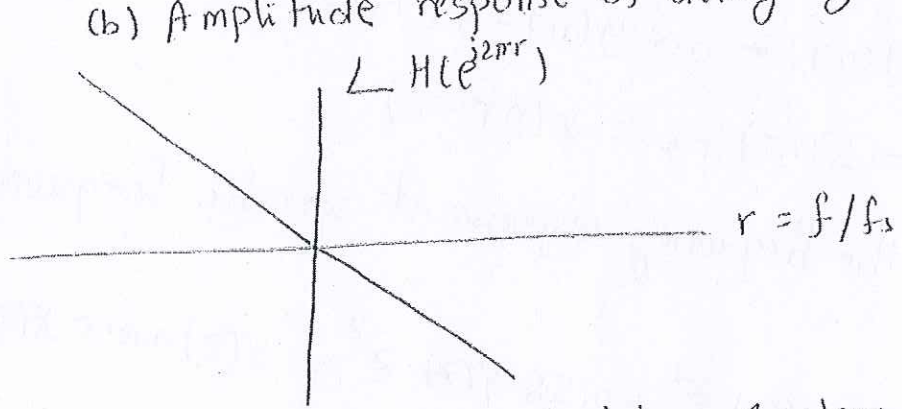
$$H(e^{j2\pi r}) = e^{-j2\pi K r}$$



(a) Discrete-time delay system



(b) Amplitude response of delay system



(c) phase response of delay-System

8.4.2.1 Frequency Response at $f = 0$ and $f = 0.5\, f_s$

As can be seen from the previous section, the expression for the frequency response of a discrete-time system of digital filter is often rather complicated. It is easy, however, to determine $H(e^{j2\pi fT})$ at $f=0$ and $f = 0.5\, f_s$ if we first recognize

that
$$e^{j2\pi fT}\Big|_{f=0} = e^{j0} = 1$$

and
$$e^{j2\pi fT}\Big|_{f=0.5f_s} = e^{j\pi f_s T} = e^{j\pi} = -1$$

Thus, the dc response is $H(1)$ and the response at one-half the sampling frequency is $H(-1)$.

Example 8.19: Consider the discrete-time system defined by the differential equation

$$y(nT) - 0.5\, y(nT-T) + 0.38\, y(nT-2T)$$
$$= x(nT) + 0.5\, x(nT-T)$$

Find the frequency response at specific frequency.

Ans:

$$Y(z) - 0.5\, Y(z)\, z^{-1} + 0.38\, Y(z)\, z^{-2} = X(z) + 0.5\, X(z)\, z^{-1}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.5\, z^{-1}}{1 - 0.5\, z^{-1} + 0.38\, z^{-2}}$$

The dc response is

$$H(1) = \frac{1+0.5}{1-0.5+0.38} = 1.70$$

and the response at $f = 0.5 f_s$ is

$$H(-1) = \frac{1-0.5}{1+0.5+0.38} = 0.27$$

Chapter 9:
Analysis and Design of Digital Filters

9.1: Structure of Digital Process Direct-Form Realization

In the previous chapter, we determined the general form of the pulse transfer function of a fixed-discrete-time system where,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{r} L_i z^{-i}}{1+\sum_{j=1}^{m} K_j z^{-j}}$$

$\Rightarrow$

$$\left[1+\sum_{j=1}^{m} K_j z^{-j}\right] Y(z) = \left[\sum_{i=0}^{r} L_i z^{-i}\right] X(z)$$

$$Y(z) + \sum_{j=1}^{m} K_j Y(z) z^{-j} = \sum_{i=0}^{r} L_i z^{-i} X(z)$$

This equation can be realized by the structure shown in Fig 1

Fig 1

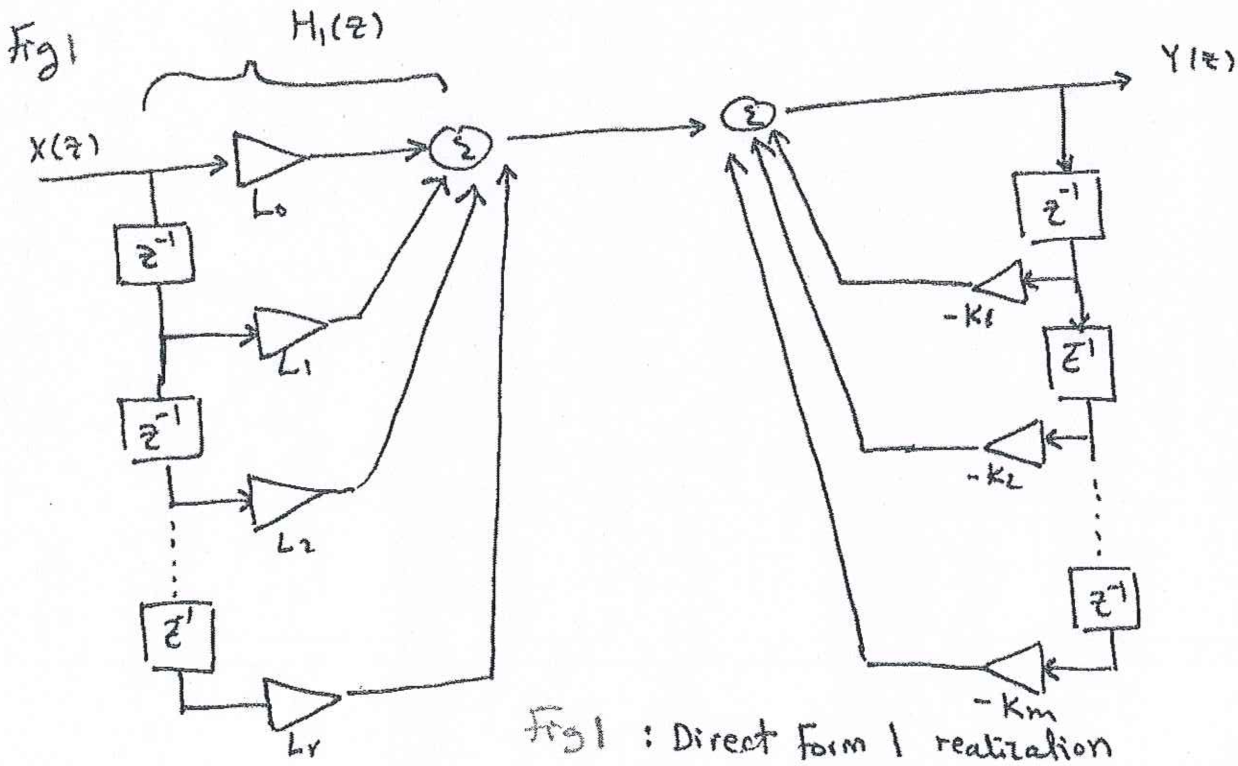


Fig 1 : Direct Form 1 realization

This structure is called Direct Form I realization.

9.2 Filtering and Algorithm

Digital filters are used in audio systems for attenuating or boosting the energy content of a sound wave at specific frequencies. The most common filter forms are high-Pass, low-pass, band-pass and notch. Any of these filters can be implemented in two ways. These are the finite impulse response (FIR) and the infinite impulse response filter (IIR), and they are often used as building blocks to more complicated filtering algorithms like parametric equalizer and graphic equalizers.

9.2.1 Finite Impulse Response (FIR) filter

The FIR filter's output is determined by the sum of the current and past input, each of which is first multiplied by a filter coefficient. The FIR summation equation, shown in Fig2, is also known as "convolution," one of the most important operations in signal processing. In this syntax, x is the input vector, y is the output vector, and h holds the filter coefficients.

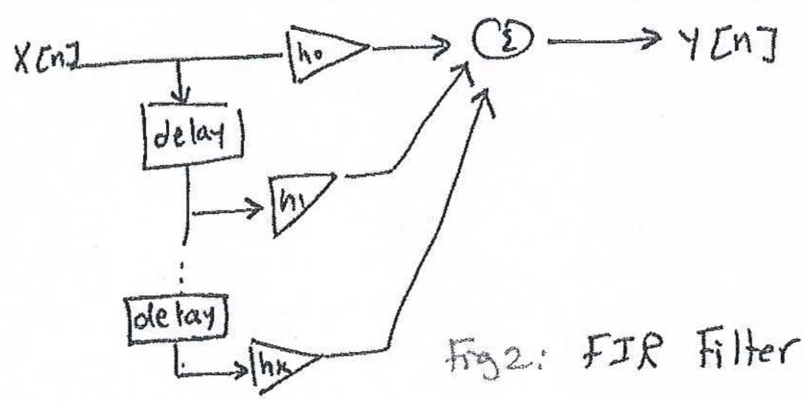


Fig2: FIR Filter; $y[n] = \sum_{k=0}^{K} h[k]\, x[n-k]$

9.2.2 Infinite Impulse Response (IIR) filter

Unlike the FIR, whose output depends only on inputs, the IIR filter realies on both inputs and past outputs. The basic equation for an IIR filter is a difference equation, as shown in Fig3 because of the current outputs dependence on past outputs, IIR filters are often referred to as "recursive filters".

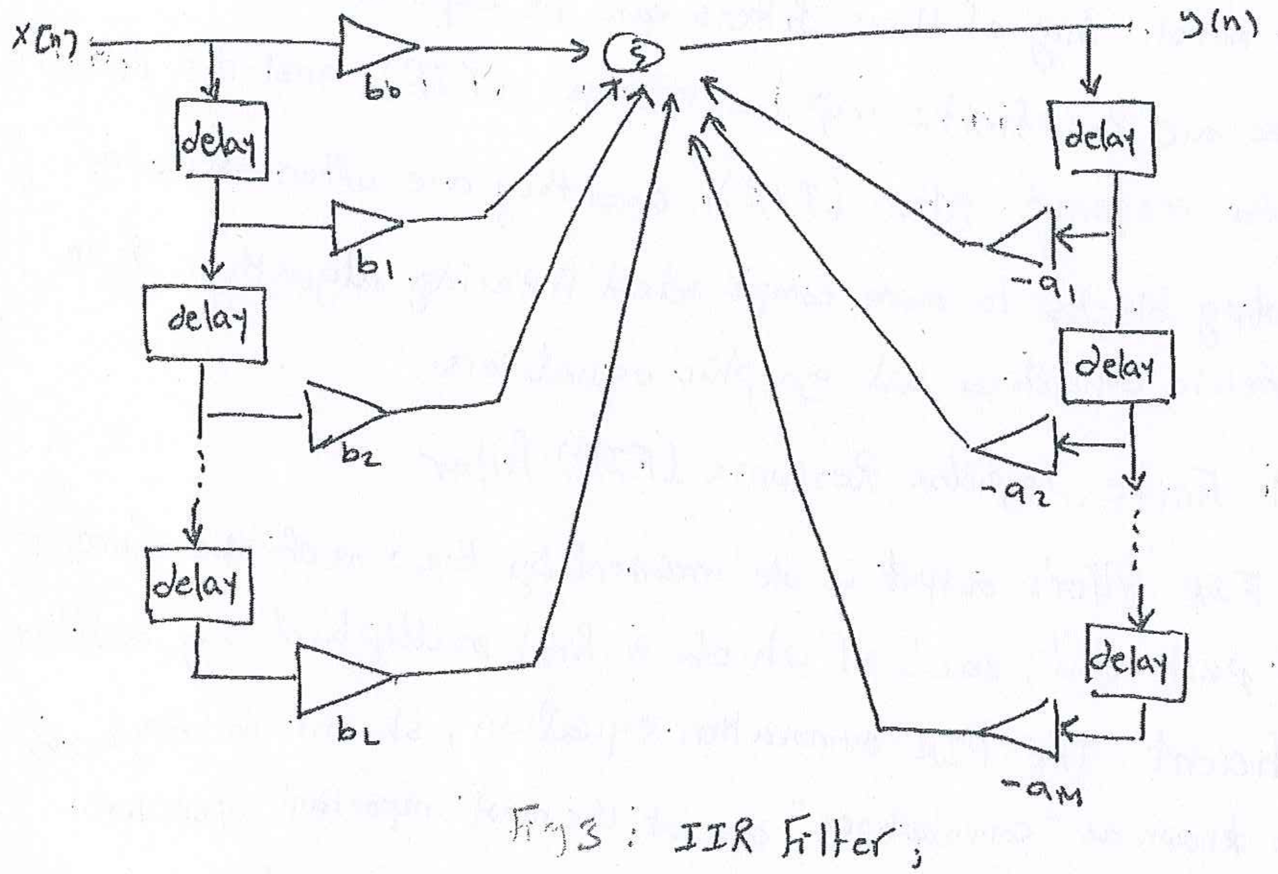


Fig3 : IIR Filter;

$$y[n] = \sum_{i=1}^{M} (-a_i\, y[n-i]) + \sum_{j=0}^{L} (b_i\, x[n-j])$$

Summary

This chapter basically considered two topics: the implementation of digital signal processors from the pulse transfer function, $H(z)$, and the design of digital signal processors to meet some performance specification. The four main implementations considered were Direct Form I, Direct Form II, cascade, and parallel.

The design or synthesis problem usually involves the development of a digital signal processor that meets some time-domain or frequency-domain specification. The impulse-invariant and step-invariant digital filters are based on a time-domain specification, while the bilinear z-transform digital filter is based on a frequency-domain specification. All of these filters are infinite-duration impulse response (IIR) digital filters.

The finite-duration impulse response (FIR) digital filter is based on a frequency-response specification, and the filter implementation is accomplished by taking the Fourier transform of the desired frequency-response specification.

There are advantages to the use of both IIR and FIR digital filters. The main advantages of IIR filters are as follows:

1. The design techniques for IIR digital filters are very easy to apply. The design is initiated with an analog prototype, and one who is familiar with analog filter theory will usually have a good feel for the performance of a given filter in a given application.
2. Hardware requirements for an IIR digital filter are usually less than the hardware requirements for a comparable FIR filter. However, with modern LSI and VLSI techniques, hardware considerations are becoming less important.

The main advantages of FIR digital filters are the following:

1. FIR filters can be designed that have perfectly linear phase. Therefore, phase distortion is eliminated.
2. Since FIR filters have no feedback, they have no poles and are therefore always stable.
3. The fast Fourier transform (FFT), which is the main topic covered in the next chapter, gives the filter designer a very simple and efficient tool for determining the filter weights.
4. Since no analog prototype is required in the synthesis procedure, digital filters can be designed that have no analog equivalent.

There are also disadvantages that are often important. The main disadvantages of the IIR synthesis techniques treated in this chapter are these:

1. Since the design procedure is initiated with an analog filter function, it is first necessary to determine an analog filter that meets the desired specifications.
2. Phase distortion is frequently a problem.

The main disadvantages of the FIR filter synthesis techniques discussed in this chapter are that:

1. If the digital filter is to have an extremely small bandwidth, a large number of filter weights may be necessary. The result will be a digital filter with a large group delay.
2. The selection of an appropriate window function may be difficult.

It should be clear that the designer of a digital filter often has many options available. Choosing the appropriate technique for an application requires a good understanding of digital filter theory and the requirements of the specific application of interest.