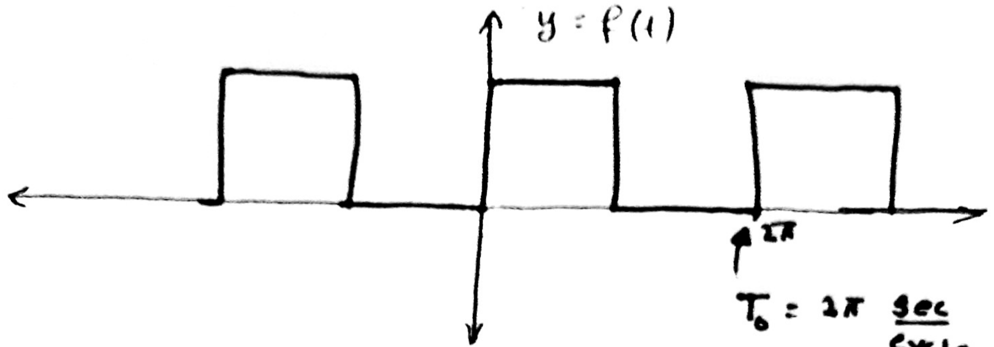
CH. Three

Fourier Series

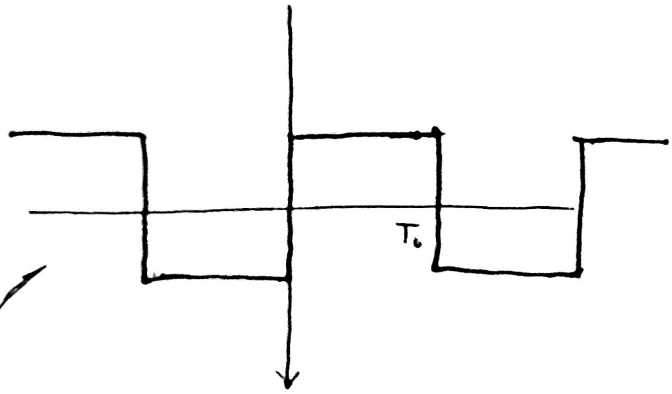


$T_0 = 2\pi \ \frac{sec}{cycle}$

$f_0 = \frac{1}{2\pi} \ \frac{cycle}{sec} \rightarrow \omega_0 = 1 \ rad/sec$

$$f(t) = a_0 + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + \cdots$$
$$+ b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \cdots$$

=



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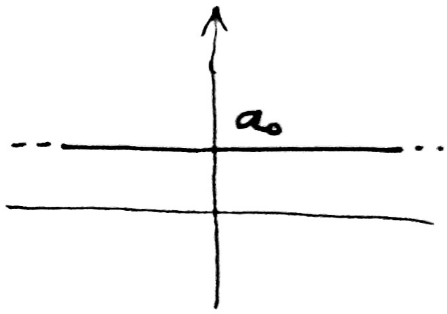
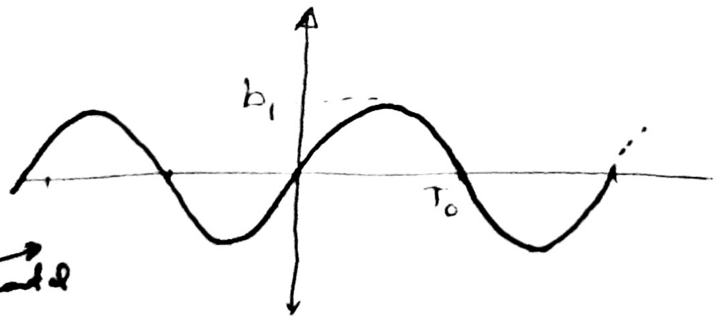


= fundamental periodic signals + Harmonic + DC (Average) value.



Fundamental signal.

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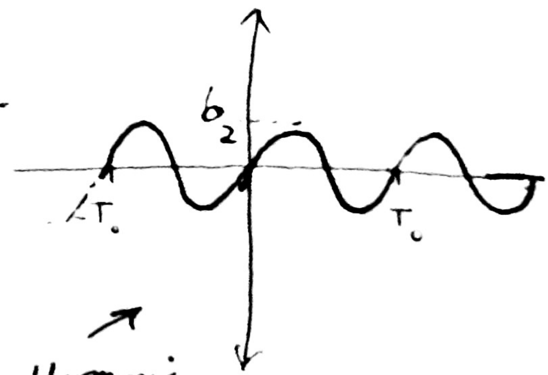

Harmonic

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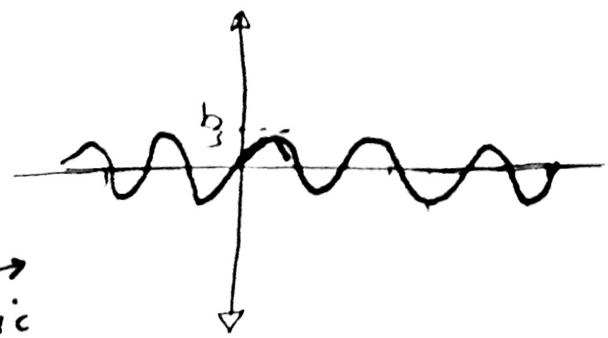


+ . . . . . .

Harmonic

→ Why $a_1, a_2, \ldots$ are decided to be zeros !

→ Why $b_1, b_2, \ldots$ are (maybe) not zeros !

→ How to evaluate such coefficients ?

→ Why are we expressing the periodic signal $f(t)$ as ~~a sum~~ a weighed sum of sinusoids ?

- The Answer to these and more Question's will the subject of this chapter.
  (Finding the Fourier Expansion of a cont.-time periodic signal)

- In this chapter, we are going to learn a new approach for decomposing signals.

  In particular, we will show that periodic signals can be decomposed into Harmonically Related sines and cosines.

  we will expand such analysis to decompose even nonperiodic continuous time signals (this is the topic of ch. 4)

Introduction :-

- In Engineering, we try to decompose complex problems into smaller & simpler problem.

- We have shown, that, one could express complicated signal into sum of basic signals such as unit step, ramp, impulse, pulse, - - -

- Also, we show that Complex systems can be expressed as a series or parallel simpler ~~system~~ subsystems.

- In Mathematics, we also try to simplify the analysis of complex functions using simpler ones!

- For example, for a given function $f(t)$, we can use Tayler Series to expand $f(t)$ into a constant plus a ramp plus a parabola, .. etc ...

$$f(t) = f(0) * \underbrace{1}_{\text{constant}} + \underbrace{f'(0)}_{\frac{d}{dt} f(t)\big|_{t=0}} \underbrace{t}_{\text{ramp}} + \underbrace{f''(0)}_{\frac{d^2}{dt^2} f(t)\big|_{t=0}} \underbrace{\frac{t^2}{2!}}_{\text{parabola}} + \cdots$$

- If we use the principle of <u>Linearity</u>, then solving any problem involving $f(t)$ can be done by considering its decompositions (constants, ramps, parabola, ...)

Approximating Periodic Functions

- we will start by discussing periodic cont.-time signals.

- three assumptions are made:-

  1. periodicity of the signal $x(t)$
     $$\Rightarrow x(t+T) = x(t) \quad \forall t$$

  2. $x(t)$ is periodic with period T, is also periodic with period nT (n: integer)
     $$\Rightarrow x(t+nT) = x(t) \quad \forall t$$

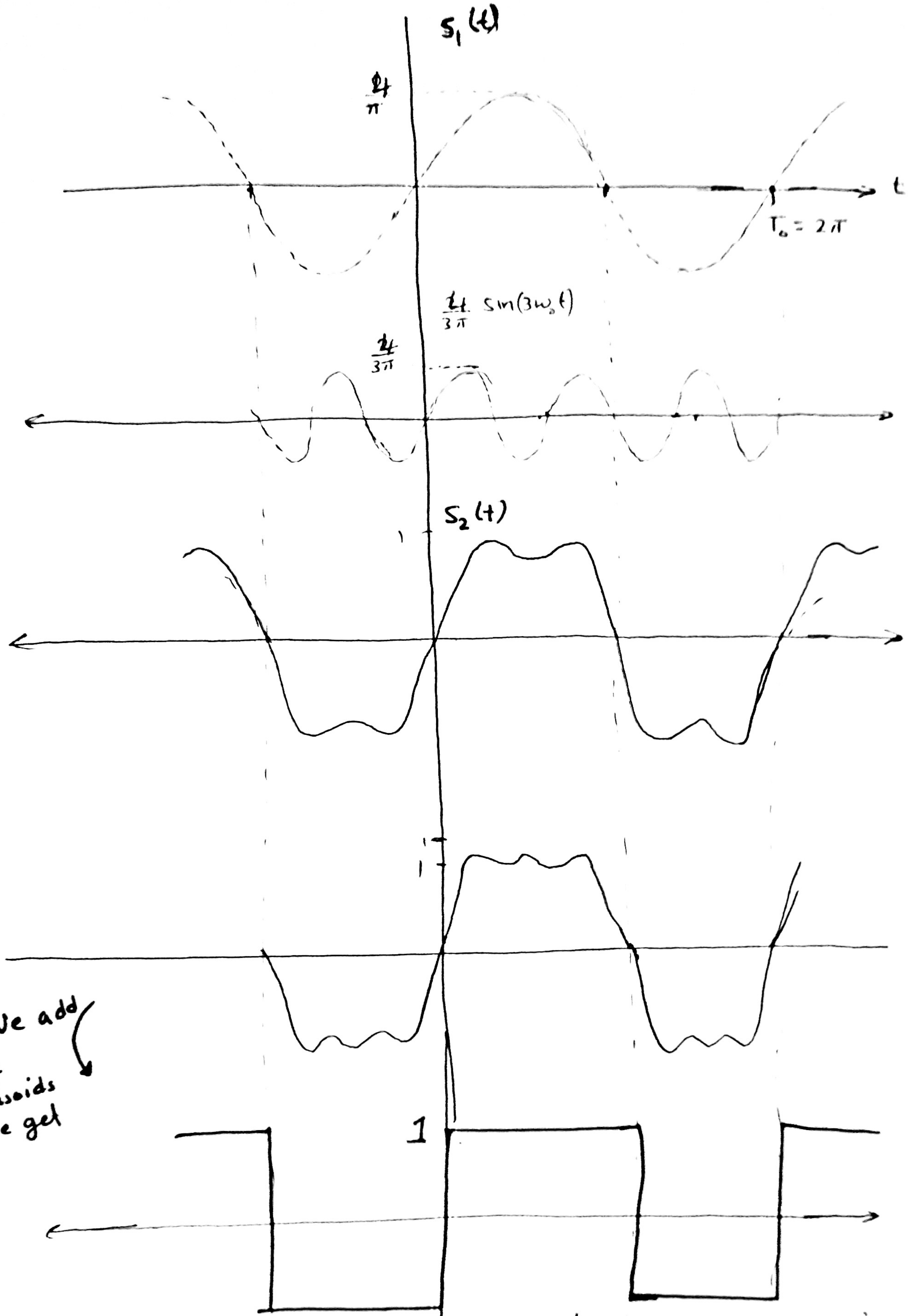
  3. We define the fundamental period T_0 as the minimum value of T satisfying $x(t+T) = x(t)$
     also
     $$f_0 = \frac{1}{T_0} \ \frac{\text{cycles}}{\text{sec.}}$$
     $$\omega_0 = 2\pi f_0 \ \text{rad/sec.}$$

- Let us consider the following series

$$x(t) = \frac{4}{\pi}\sin \omega_0 t + \frac{4}{3\pi}\sin 3\omega_0 t + \frac{4}{5\pi}\sin(5\omega_0 t) + \cdots$$

let $\omega_0 = 1$ rad/sec $\Rightarrow T_0 = \frac{2\pi}{\omega_0} = 2\pi$

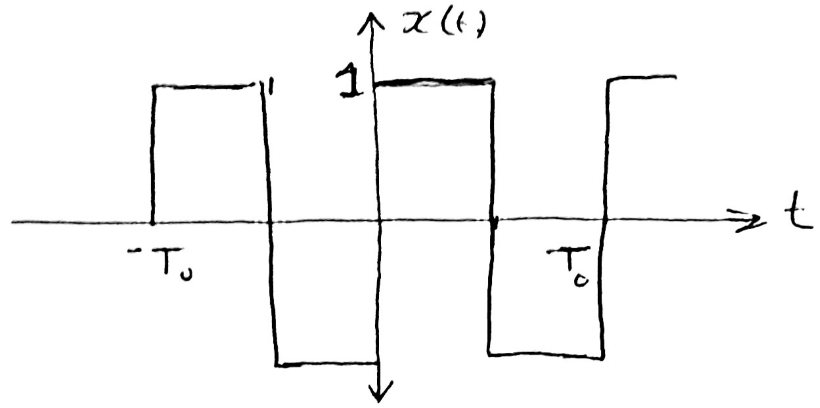
define
$$S_1(t) = \frac{4}{\pi}\sin \omega_0 t$$
$$S_2(t) = S_1(t) + \frac{4}{3\pi}\sin(3\omega_0 t)$$
$$S_3(t) = S_2(t) + \frac{4}{5\pi}\sin(5\omega_0 t)$$



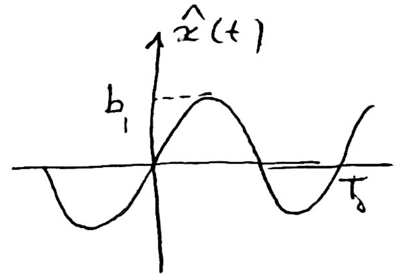
So, we see that as we add $\frac{4}{k\pi}\sin(k\omega_0 t)$ harmonics to the series, we converge more and more towards a square periodic signal!

Determining the Trigonometric FS Representation

- Let us consider the rectangular periodic signal $x(t)$,



- Let us assume that we need to approximate $x(t)$ with just one sinusoid.



indicates the approx. → $\hat{x}(t) = b_1 \sin \omega_0 t$

- The approximation error:

$$e(t) = x(t) - \hat{x}(t)$$

- To find b_1, we commonly ~~minimize~~ minimize a certain function of error $e(t)$.

- Let us ~~find~~ find b_1 by minimizing the mean-square error (MSE) which is defined as

$$J[e(t)] = \frac{1}{T_0} \int_0^{T_0} e^2(t)\,dt$$

$$J[e(t)] = \frac{1}{T_0}\int_0^{T_0} [x(t) - b_1 \sin w_0 t]^2 \, dt$$

* to find b_1, we differeniate over b_1:

$$\frac{d}{dt} J = \frac{1}{T_0}\int_0^{T_0} 2[x(t) - b_1 \sin w_0 t](-\sin w_0 t) \, dt$$

$$= 0$$

* solving the above, we get

$$\frac{2}{T_0}\int_0^{T_0} x(t) \sin w_0 t = \frac{2}{T_0}\int_0^{T_0} b_1 \sin^2 w_0 t \, dt$$

$$\int_0^{T_0} x(t) \sin w_0 t = b_1 \int_0^{T_0} \left[\frac{1}{2} - \frac{1}{2}\cos(2w_0 t)\right] dt$$

$$= b_1 \int_0^{T_0} \frac{1}{2} \, dt - \frac{b_1}{2}\int_0^{T_0} \cos(2w_0 t)$$

$$= \frac{b_1 T_0}{2} - \frac{b_1}{2}\int_0^{T} \frac{2w_0 \cos(2w_0 t)}{2w_0}$$

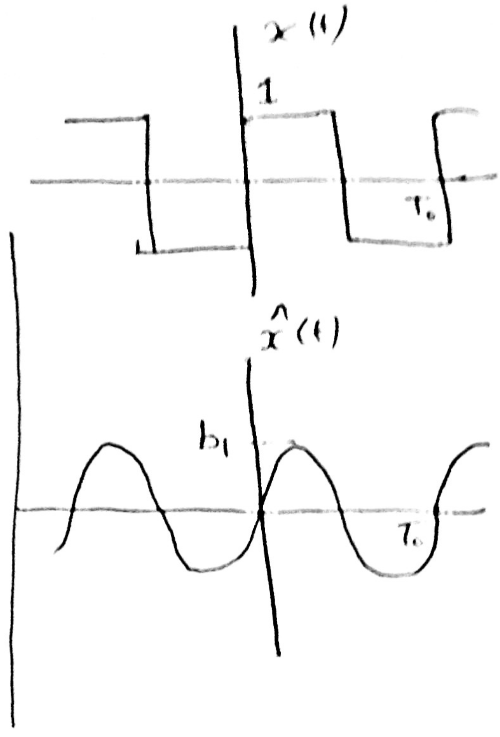
$$= \frac{b_1 T_0}{2} - \frac{b_1}{4w_0} \sin(2w_0 t) \Big|_0^{T_0}$$

$$= \frac{b_1 T_0}{2} - \frac{b_1}{4w_0}\left[\cancel{\sin\left(2 \cdot \frac{2\pi}{T_0} T_0\right)} - \cancel{\sin 0}\right]$$

$$\Rightarrow \boxed{b_1 = \frac{2}{T_0}\int_0^{T_0} x(t) \sin w_0 t}$$

we can verify that the second derivative is +ve, hence the MSE is minimized.

- For the case of a square signal



$$b_1 = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\left[\int_0^{T_0/2} 1 \sin\omega_0 t\,dt + \int_{T_0/2}^{T_0}(-1)\sin\omega_0 t\right]$$

$$= \frac{2}{T_0}\int_0^{T_0/2} \frac{-\omega_0}{-\omega_0}\sin\omega_0 t\,dt + \frac{2}{T_0}\int_{T_0/2}^{T_0}\frac{-\omega_0}{\omega_0}\sin\omega_0 t$$

$$= \frac{-2}{T_0\omega_0}\cos\omega_0 t\Big|_0^{T_0/2} + \frac{2}{T_0\omega_0}\cos\omega_0 t\Big|_{T_0/2}^{T_0}$$

$$= \frac{-2}{T_0\cdot\frac{2\pi}{T_0}}\left[\cos\frac{2\pi}{T_0}\cdot\frac{T_0}{2} - \cos(0)\right] + \frac{2}{T_0\cdot\frac{2\pi}{T_0}}\left[\cos\frac{2\pi}{T_0}\cdot T_0 - \cos\left(\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right)\right]$$

$$= -\frac{1}{\pi}\left[\overset{-1}{\cancel{\cos\pi}} - \overset{1}{\cancel{\cos(0)}}\right] + \frac{1}{\pi}\left[\overset{1}{\cancel{\cos 2\pi}} - \overset{-1}{\cancel{\cos\pi}}\right]$$

$$= +\frac{2}{\pi} + \frac{2}{\pi} = \frac{4}{\pi}$$

or not $\boxed{T_0\omega_0 = 2\pi}$

$$\Rightarrow \hat{x}(t) = s_1(t) = \frac{4}{\pi}\sin(\omega_0 t)$$

- Now, let us generalize the approximation & define the Trigonometric Fourier Series (TFS) given by:-

$$\boxed{\begin{aligned} x(t) &= a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \cdots \\ &\quad + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \cdots \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \end{aligned}}$$

- by minimizing the mean-square error (MSE) over all the parameters, we can show that:-

$$a_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t)\, dt \quad \leftarrow \text{Average value (DC component)}$$

$$a_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \cos(n\omega_0 t)\, dt \qquad n = 1, 2, \cdots$$

$$b_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(n\omega_0 t)\, dt \qquad n = 1, 2, \cdots$$

a_0 : called average value (DC value)

a_0, a_n, b_n : FS coeffecients

$\langle T_0 \rangle$ denotes integration over one period:

for example $[-\frac{T_0}{3}, \frac{2}{3}T_0]$, $[0, T_0]$, $[-\frac{T_0}{2}, \frac{T_0}{2}]$, ...

Choose whichever is more ~~convient~~ convenient to simplify the integration.