Fourier Representation :-

1. Trigonometric FS (TFS)
2. Complex FS
3. Compact TFS

Complex FS

- The TFS representation can be modified & made more compact using complex coeffecients (also called Exponential coefficients).

- Recall that $\cos(n\omega_0 t)$ and $\sin(n\omega_0 t)$ can be written, using Euler identity, as

$$\cos(n\omega_0 t) = \frac{1}{2}\left[e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right]$$

$$\sin(n\omega_0 t) = \frac{1}{2j}\left[e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right]$$

- by substituting into the TFS expansion :-

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$= a_0 + \frac{1}{2}\sum_{n=1}^{\infty} a_n \left[e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right] + \frac{1}{j2}\sum_{n=1}^{\infty} b_n\left[e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\frac{1}{2}a_n - j\frac{b_n}{2}\right]e^{jn\omega_0 t} + \underbrace{\sum_{n=1}^{\infty}\left[\frac{1}{2}a_n + j\frac{b_n}{2}\right]e^{-jn\omega_0 t}}_{\mathrm{I}}$$

by using change of variable in expression I,
so that we (n) by $(-n)$, hence

$$I = \sum_{n=-\infty}^{-1} \left[\frac{1}{2} a_{-n} + j\frac{b_{-n}}{2}\right] e^{jn\omega_0 t}$$

$$\Rightarrow \boxed{x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{jn\omega_0 t}}$$ Complex Exponential Representation

where

$$X_n = \frac{1}{2}(a_n - j b_n) \quad \text{for } n \geq 1$$

$$X_n = \frac{1}{2}(a_{-n} + j b_{-n}) \quad \text{for } n \leq -1$$

Complex or Exponential FS coefficients

- The above is called Complex FS representation.

Now, $X_n = \frac{1}{2}(a_n - j b_n) \qquad n \geq 1$

$$\Rightarrow X_n = \frac{1}{2} \cdot \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t)\cos(n\omega_0 t)\, dt - j\frac{1}{2} \cdot \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t)\sin(n\omega_0 t)\, dt$$

$$= \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t)\left[\cos(n\omega_0 t) - j\sin(n\omega_0 t)\right] dt$$

$$\boxed{X_n = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t)\, e^{-jn\omega_0 t}\, dt \qquad n = -\infty, \dots, \infty}$$

$\Rightarrow$ Complex FS :.

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{jnw_0 t}$$

$$X_n = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) \, e^{-jnw_0 t} \, dt$$

- Note that the above applies for any value of n

$$\boxed{X_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) \, dt \;=\; a_0}$$

is called DC-value or average value.

- Let us consider the case of a real signal $x(t)$, then

$$X_n^* = \frac{1}{T_0} \int_{\langle T_0 \rangle} \left(x(t) \, e^{-jnw_0 t} \right)^* dt$$

in real signals $x^*(t) = x(t)$

$$= \frac{1}{T_0} \int_{\langle T_0 \rangle} x^*(t) \, e^{jnw_0 t} \, dt$$

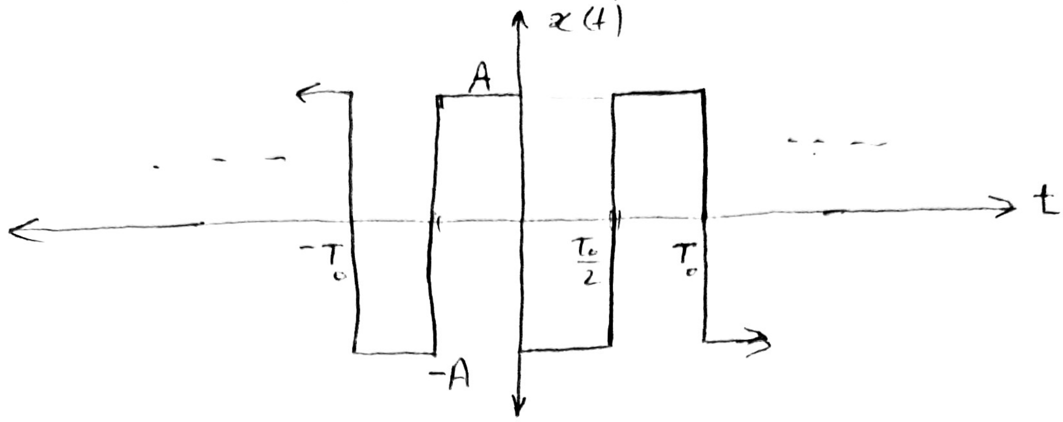
$$= \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) \, e^{jnw_0 t} \, dt \;=\; X_{-n}$$

- From the above, we can deduce the following Results (for real signals) :.

magnitude: $|X_n| = |X_{-n}| = |X_n^*|$ $\Rightarrow$ Magnitude is an Even Function

phase: $\angle -X_n = \angle X_{-n}$ $\Rightarrow$ phase is an odd function of n.

Example:- periodic square signal



Find the Complex FS coeffecients ?

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_o t} \quad , \quad \text{periodic with period } T_o$$
$$\Rightarrow \omega_o = \frac{2\pi}{T_o}$$

$$X_n = \frac{1}{T_o} \int_{\langle T_o \rangle} x(t)\, e^{-jn\omega_o t}\, dt$$

$$= \frac{1}{T_o} \int_0^{T_o/2} (-A)\, e^{-jn\omega_o t}\, dt + \frac{1}{T_o} \int_{T_o/2}^{T_o} (A)\, e^{-jn\omega_o t}\, dt$$

$$= \frac{1}{T_o} \frac{A}{jn\omega_o} e^{-jn\omega_o t} \Big|_0^{T_o/2} + \frac{1}{T_o} \frac{A}{-jn\omega_o} e^{-jn\omega_o t} \Big|_{T_o/2}^{T_o}$$

$$= \frac{A}{j2\pi n} \left[e^{-jn\omega_o \frac{T_o}{2}} - e^{0} \right] + \frac{-A}{j2\pi n} \left[e^{-jn\omega_o T_o} - e^{-jn\omega_o \frac{T_o}{2}} \right]$$

replace $\omega_o T_o = 2\pi$

$$= \frac{A}{j2\pi n} \left[e^{-jn\pi} - 1 \right] + \frac{-A}{j2\pi n} \left[e^{-j2\pi n} - e^{-jn\pi} \right]$$

note that $e^{-j2\pi n} = \overset{1}{\cancel{\cos(2\pi n)}} - j\overset{0}{\cancel{\sin(2\pi n)}} = 1$

$$\Rightarrow X_n = \frac{A}{j2\pi n}\left[e^{-jn\pi} - 1\right] - \frac{A}{j2\pi n}\left[1 - e^{-jn\pi}\right]$$

$$= \frac{A}{j2\pi n}\left[e^{-jn\pi} - 1 - 1 + e^{-jn\pi}\right]$$

$$= \frac{A}{j2\pi n}\left[2e^{-jn\pi} - 2\right]$$

Note that $e^{-jn\pi} = \cos(n\pi) - j\,\overset{0}{\cancel{\sin(n\pi)}} = \cos(n\pi)$

$$X_n = \frac{A}{j\pi n}\left[\cos(n\pi) - 1\right]$$

$$\Rightarrow X_n = \begin{cases} 0 & n: \text{even} \quad 2, 4, 6 \ldots \\ -\dfrac{2A}{j\pi n} & n: \text{odd} \quad 1, 3, 5, \ldots \end{cases}$$

$X_0 = 0$ (No DC content or Average value = 0)

$$X_1 = \frac{-2A}{j\pi} = j\frac{2A}{\pi} \longrightarrow X_{-1} = X_1^* = -j\frac{2A}{\pi}$$

$$= \frac{2A}{\pi}\angle 90^\circ \qquad\qquad = \frac{2A}{\pi}\angle -90^\circ$$

$$X_2 = 0 \longrightarrow X_{-2} = 0$$

$$X_3 = +j\frac{2A}{3\pi} = \frac{2A}{3\pi}\angle 90^\circ \longrightarrow X_{-3} = \frac{2A}{3\pi}\angle -90^\circ$$

$\vdots \qquad\qquad\qquad \vdots$

$X_0 = 0$

$X_1 = \frac{2A}{\pi} \angle 90^\circ$ $\qquad$ $X_{-1} = \frac{2A}{\pi} \angle -90$

$X_2 = 0$ $\qquad$ $X_{-2} = 0$

$X_3 = \frac{2}{3\pi} A \angle 90^\circ$ $\qquad$ $X_{-3} = \frac{2}{3\pi} A \angle -90$

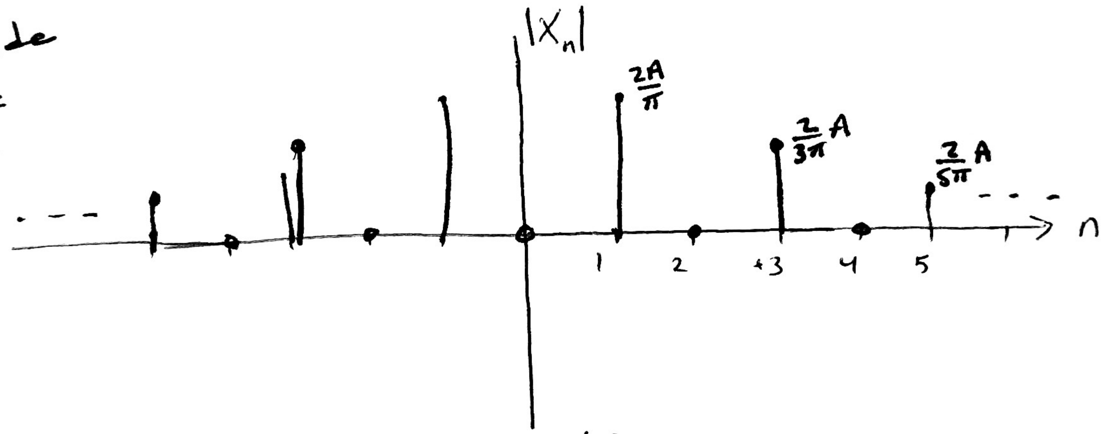
$X_4 = 0$ $\qquad$ $X_{-4} = 0$

$X_5 = \frac{2}{5\pi} A \angle 90$ $\qquad$ $X_{-5} = \frac{2}{5\pi} \angle -90$

$\vdots$ $\qquad$ $\vdots$

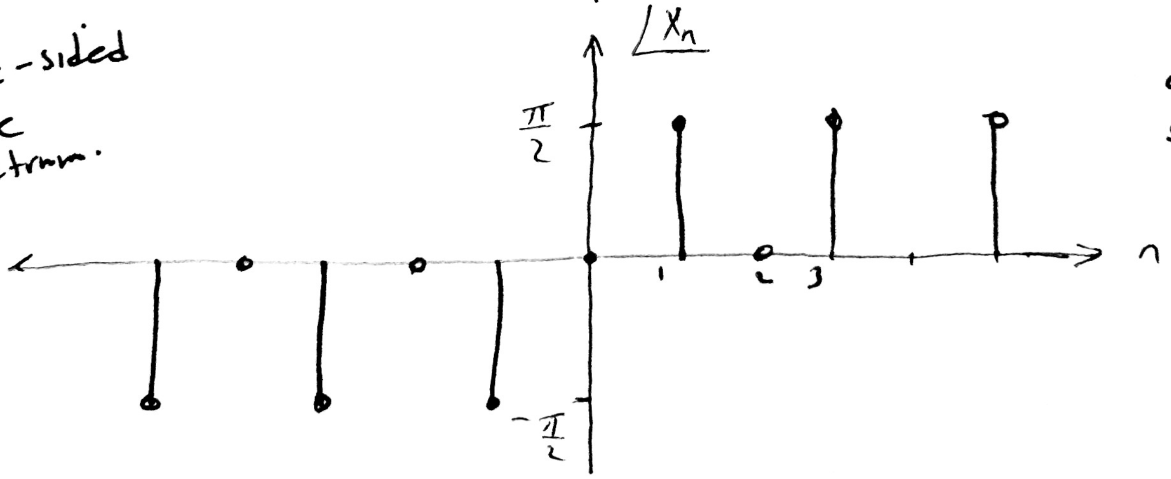
From Complex Exponential FS, we can draw Both the amplitud & phase spectra (Double-Sided)

Double-side Amplitude spectrum



Even Symmetry

Double-sided phase spectrum.



odd Symmetry

Relation between the TFS & Complex FS :-

a_n and b_n is related to X_n for real Signals:

$$\boxed{\begin{aligned} X_n &= \tfrac{1}{2}a_n - j\tfrac{1}{2}b_n \\ X_0 &= a_0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} a_n &= 2\,Re(X_n) \\ b_n &= -2\,I_m(X_n) \end{aligned}}$$

$-\infty < n < \infty$ $\qquad\qquad$ $n \geq 1$

For the previous example. (Square pulse)
Find the TFS coeffecients ?

we already found Complex FS Coeffecients

$$X_n = \begin{cases} 0 & n: \text{even} \\ \dfrac{-2A}{j\pi n} & n: \text{odd} \end{cases} = \begin{cases} 0, & n: \text{even} \\ j\dfrac{2}{\pi n}A, & n: \text{odd} \end{cases}$$

$X_0 = 0$

hence,

$a_0 = X_0 = 0$

$a_n = 2\,Re\{X_n\} = 2 \times 0 = 0 \qquad n = 1, 2, \cdots$

$$b_n = -2\,Im\{X_n\} = \begin{cases} -2 \cdot +\dfrac{2A}{\pi n} = \dfrac{-4A}{\pi n}, & n: \text{odd} \\ 0, & n: \text{even} \end{cases}$$
