3.6 Parseval's Theorem

- In ch. 1, we have shown that the average normalized power of aperiodic signal can be expressed as :-

$$P_{av} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad \text{(In general)}$$

$$P_{av} = \frac{1}{T_0} \int_{\langle T_0 \rangle} |x(t)|^2 dt \qquad \text{(For periodic signal)}$$

* we can express P_{av} in terms of Fourier Coeffecients of $x(t)$ such that :-

$$P_{av} = \frac{1}{T_0} \int x(t) x^*(t) \, dt$$

$$x(t) = \sum_{-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$\Rightarrow P_{av} = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) \left(\sum_{-\infty}^{\infty} X_n^* e^{-jn\omega_0 t} \right) dt$$

$$= \sum_{-\infty}^{\infty} X_n^* \left(\frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jn\omega_0 t} dt \right)$$

$$= \sum_{-\infty}^{\infty} X_n^* X_n$$

$$= \sum_{-\infty}^{\infty} |X_n|^2 = X_0^2 + |X_1|^2 + |X_2|^2 + \cdots + |X_{-1}|^2 + |X_{-2}|^2 + \cdots$$

$$= X_0^2 + 2 \sum_{n=1}^{\infty} |X_n|^2$$

⇒ P_{av} Evaluation for periodic signal $x(t)$:-

① In time domain :-

$$P_{av} = \frac{1}{T_0} \int_{\langle T_0 \rangle} |x(t)|^2 dt$$

② In Frequency domain

$$P_{av} = \sum_{n=-\infty}^{\infty} |X_n|^2 \quad \text{(In general)}$$

$$= X_0^2 + 2 \sum_{n=1}^{\infty} |X_n|^2 \quad \text{(For real signal)}$$

⇒ Average Power of periodic signal is the sum of powers ~~in the~~ in the phasor components of its FS

⇒ Equivelently, the Average power in a periodic signal is the sum of the power in its dc component plus the powers in its Harmonics.

Example Find P_{av} for $x(t) = 4\sin(50\pi t)$

Sol. From time-domain expression:-

$$P_{av} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

for periodic signal. $P_{av} = \frac{1}{T_0}\int_{\langle T_0\rangle} |x(t)|^2 dt$

$$\Rightarrow P_{av} = \frac{1}{T_0}\int_0^{T_0} 16\sin^2(50\pi t)\, dt \quad , \quad \omega_0 = 50\pi \;, \; T_0 = \frac{2\pi}{\omega_0} = \frac{1}{25}$$

$$= \frac{16}{T_0}\int_0^{T_0}\left(\frac{1}{2} - \frac{1}{2}\cos(100\pi t)\right) dt$$

$$= \frac{8}{T_0}\left(t - \frac{1}{100\pi}\sin(100\pi t)\right|_0^{T_0}$$

$$= \frac{8}{T_0}\left[T_0 - \frac{1}{100\pi}\sin(100\pi T_0) - 0\right]$$

$$= 8\times 25\left[\frac{1}{25} - \frac{1}{100\pi}\cancel{\sin(4\pi)}\right] = 8\,W$$

In general, for $x(t) = A\sin(\omega_0 t) \rightarrow P_{av} = \frac{A^2}{2}$

· We can find P_{av} also using Parseval's theorem.

$$P_{av} = X_0^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

where $X_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$

⋮

or $\quad x(t) = 4\sin 50\pi t$

$$= 4\sin \omega t \quad , \quad \omega = 50\pi$$

$$= 4\left[\frac{e^{j\omega t} - e^{-j\omega t}}{j2}\right]$$

$$= -j2\, e^{j\omega t} + j2\, e^{-j\omega t}$$

in comparison to $x(t) = \sum X_n e^{jn\omega_0 t}$

$$= X_0 + X_1 e^{j\omega_0 t} + X_2 e^{j2\omega_0 t} + \cdots$$
$$+ X_{-1} e^{-j\omega_0 t} + X_{-2} e^{-j2\omega_0 t} + \cdots$$

it is clear that

$$X_0 = 0$$

$$X_1 = -j2 \qquad X_2 = 0 \quad ----$$

$$X_{-1} = j2 \qquad X_{-2} = 0 \quad ---$$

Note that $\quad X_1 = X_{-1}^*$

$$X_1 = 2\angle{-\tfrac{\pi}{2}} \Rightarrow |X_1| = 2$$

$$|X_2| = |X_3| = \cdots = 0$$

$$\Rightarrow P_{av} = |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

$$= 0 + 2 * (2)^2 + 0 + \cdots$$

$$= 8 \text{ Watts}$$

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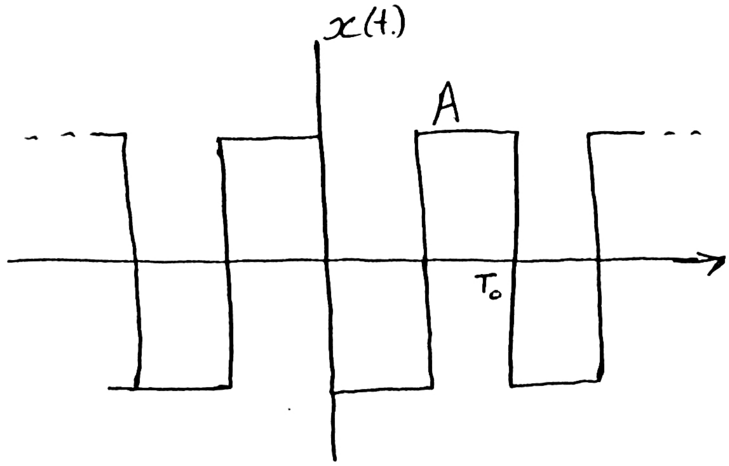
Now, we study the effect of selecting some harmonics to represent a certain signal in terms of average power :-

Example From the earlier example (periodic square)



we have shown earlier that

$$x(t) = \sum_{-\infty}^{\infty} X_n e^{j n w_0 t}$$

$$X_n = \begin{cases} j\dfrac{2A}{n\pi} & n: \text{odd } 1, 3, \ldots \\ 0 & n: \text{even } 2, 4, \ldots \end{cases}$$

$$X_0 = 0$$

we can find the TFS coeffecients

$$a_0 = X_0 = 0$$

$$a_n = 2\,\text{Real}(X_n) = 0 \quad \forall n$$

$$b_n = -2\,\text{Im}(X_n) = \begin{cases} -2 * \dfrac{2A}{n\pi} = \dfrac{-4A}{n\pi}, & n: \text{odd} \\ 0, & n: \text{even} \end{cases}$$

Hence,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \sum_{n = odd} \left(-\frac{4A}{n\pi}\right) \sin(n\omega_0 t)$$

$$= -\frac{4A}{\pi} \sum_{n: odd} \frac{1}{n} \sin(n\omega_0 t)$$

- Assume that we approximate (select) the signal $x(t)$ with one sinusoid (select only one component)

→ then, $\hat{x}_1 = b_1 \sin(1\omega_0 t)$

$$= -\frac{4A}{\pi} \sin \omega_0 t \quad (n = 1, \text{ only one Harmonic})$$

→ then, P_{av} for $x(t)$:

$$P_{av} = \frac{1}{T_0} \int_{\langle T_0 \rangle} |x(t)|^2 dt = \frac{1}{T_0} A^2 T_0 = A^2$$

P_{av} for $\hat{x}_1(t) \Rightarrow P_1 = \frac{\left(-\frac{4A}{\pi}\right)^2}{2} = \frac{8A^2}{\pi^2}$

→ the lost power in percentage :-

$$\frac{P_x - P_1}{P_x} = \frac{A^2 - \frac{8A^2}{\pi^2}}{A^2} * 100\% = 18.9\%$$

• Assume that we use two sinusoids (Remember $b_2 = 0$ ⇒ select b_3)

$$\Rightarrow \hat{x}_2 = -\frac{4A}{\pi} \sin \omega_0 t - \frac{4A}{3\pi} \sin 3\omega_0 t$$

then, the lost Power :-

$$\frac{P_x - P_2}{P_x} * 100\% = \frac{A^2 - \left(\frac{8A^2}{\pi^2} + \frac{8}{9\pi} A^2\right)}{A^2} * 100\%$$

$$= 9.9\% \text{ only.}$$

⇒ <u>Conclusion</u> :-

if we use bandwidth $1\omega_0$ ⟶ 18.9% Power lost
if we use bandwidth $3\omega_0$ ⟶ 9.9% Power lost

⇒ Spectrum can be used to decide the Bandwidth needed to <u>store</u> or <u>Transmit</u> $x(t)$.

⇒ More Harmonics Give better approximation But More Bandwidth.

Again one can find these percentages from the Complex FS and $\boxed{\text{parseval's theorem}}$:-

$$X_n = \begin{cases} j\dfrac{2A}{n\pi} & n: \text{odd} \\ 0 & n: \text{even.} \end{cases}$$

$$\Rightarrow |X_n| = \begin{cases} \dfrac{2A}{n\pi} & n: \text{odd} \\ 0 & n: \text{even.} \end{cases}$$

- Assume that we approximate $x(t)$ with the first harmonic

$$\hat{x}_1(t) = -j\frac{2A}{\pi}e^{-j\omega_0 t} + j\frac{2A}{\pi}e^{j\omega_0 t}$$

$\Rightarrow P_{av}$ for $\hat{x}_1(t) \Rightarrow P_1 = |X_0|^2 + 2|X_1|^2$

$$= 0 + 2\left(\frac{2A}{\pi}\right)^2 = \frac{8A^2}{\pi^2}$$

$$P_{av} \text{ for } x(t) = 2\sum_{\substack{n=1 \\ \text{odd}}}^{\infty}\left(\frac{2A}{n\pi}\right)^2 = A^2$$

$\rightarrow$ the lost power :-

$$\frac{A^2 - \frac{8A^2}{\pi^2}}{A^2} \times 100\% = 18.9\%$$