Example

$$x(t) \longrightarrow \boxed{50\, e^{-3t}\, u(t)} \longrightarrow y(t)\ ?$$

let $x(t) = 10\, e^{-2t}\, u(t)$

$h(t) = 50\, e^{-3t}\, u(t)$

Find $y(t)$?

Sol. $y(t)$ can be obtained by evaluating the convolution integral

$$y(t) = x(t) * h(t)$$

OR $y(t) = \mathcal{F}^{-1}\{Y(f)\}$

$$Y(f) = H(f)\, X(f)$$

From table: $e^{-\alpha t} u(t), \alpha > 0 \longleftrightarrow \dfrac{1}{\alpha + j2\pi f}$

hence, $H(f) = \dfrac{50}{3 + j2\pi f}$, $X(f) = \dfrac{10}{2 + j2\pi f}$

$$\Rightarrow Y(f) = \frac{500}{(3 + j2\pi f)(2 + j2\pi f)}$$

using Partial Fraction Expansion (PFE)

$$Y(f) = \frac{A}{3 + j2\pi f} + \frac{B}{2 + j2\pi f}$$

$$\Rightarrow A\Big|_{j2\pi f = -3} = \frac{500}{2 - 3} = -500$$

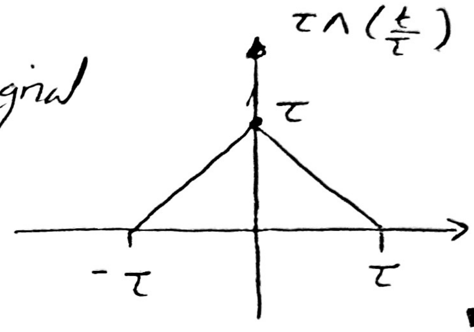
$$B\Big|_{j2\pi f = -2} = \frac{500}{3 - 2} = 500$$

$$\Rightarrow \quad Y(f) = 500\left(\frac{1}{2+j2\pi f} - \frac{1}{3+j2\pi f}\right)$$

$$y(t) = 500\left(e^{-2t} - e^{-3t}\right)u(t) = \mathcal{F}^{-1}\{Y(f)\}$$

Remark: Finding $y(t)$ using FT is much easier than evaluating convolution integral.

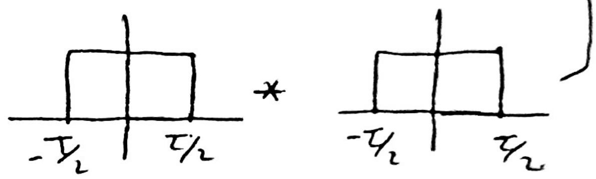
Example Consider a triangular signal



$$\tau\,\Lambda\left(\frac{t}{\tau}\right)$$

evaluate its Fourier transform.

Sol. $$\Pi\left(\frac{t}{\tau}\right) * \Pi\left(\frac{t}{\tau}\right)$$



$$= \tau\Lambda\left(\frac{t}{\tau}\right)$$

$\Rightarrow$ according to the convolution theorem.

$$\mathcal{F}\left\{\tau\Lambda\left(\frac{t}{\tau}\right)\right\} = \mathcal{F}\left\{\Pi\left(\frac{t}{\tau}\right)\right\}^2$$

$$= \tau^2 \operatorname{sinc}^2(f\tau)$$

$$\Longrightarrow \boxed{\tau\,\Lambda\left(\frac{t}{\tau}\right) \longleftrightarrow \tau^2 \operatorname{sinc}^2(f\tau)} \quad \text{Fourier Pair}$$

Example (Hilbert Transform)

The Hilbert transform, $\hat{x}(t)$, of a signal, $x(t)$, is obtained by convolving $x(t)$ with $\frac{1}{\pi t}$.

That is,

$$x(t) \longrightarrow \boxed{\frac{1}{\pi t}} \longrightarrow \hat{x}(t)$$

$$\hat{x}(t) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int \frac{x(\lambda)}{t-\lambda} \, d\lambda$$

From the following pair:

$$\operatorname{sgn}(t) \longleftrightarrow \frac{1}{j\pi f}$$

using Duality

$$\frac{1}{j\pi t} \longleftrightarrow \operatorname{sgn}(-f) = -\operatorname{sgn}(f)$$

$$\Rightarrow \quad \frac{1}{\pi t} \longleftrightarrow -j \operatorname{sgn}(f)$$

$$\Rightarrow \mathcal{F}\{\hat{x}(t)\} = \mathcal{F}\{\tfrac{1}{\pi t}\} \mathcal{F}\{x(t)\} = -j \operatorname{sgn}(f) X(f)$$

$$\hat{X}(f) = -j \operatorname{sgn}(f) X(f)$$

$$|\hat{X}(f)| = |X(f)| \qquad \theta_{\hat{x}}(f) = \begin{cases} \theta_x(f) - 90, & f > 0 \\ \theta_x(f) + 90, & f < 0 \end{cases}$$

* A Hilbert transform operation therefore multiplies all positive-spectral components by $(-j)$, and all negative-frequency spectral " by $(+j)$
* That's mean, the amplitude spectrum of the signal is left unchanged by Hilbert transform operation
* The phase is shifted by $\frac{\pi}{2}$ rad.

Properties of Hilbert transform:-

1) it is a linear transform. (Also, it has an inverse)

2) A signal $x(t)$ and its Hilbert transform $\hat{x}(t)$ have the same autocorrelation.

3) A signal $x(t)$ and its Hilbert transform $\hat{x}(t)$ have the same energy density spectrum

4) A signal $x(t)$ and its $\hat{x}(t)$ are mutually orthogonal

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = 0$$

Inverse Hilbert transform:-

$$x(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{x}(\lambda)}{t-\lambda}\, d\lambda$$

[6] Frequency Shifting (Modulation Property)

* it is a dual representation of the time-delay theorem.

* proof :-

let $x(t)$ be a low freq. signal, and mult. it with a high freq. signal such that

$$y(t) = x(t)\, e^{j\omega_0 t} \longrightarrow Y(f) = ?$$

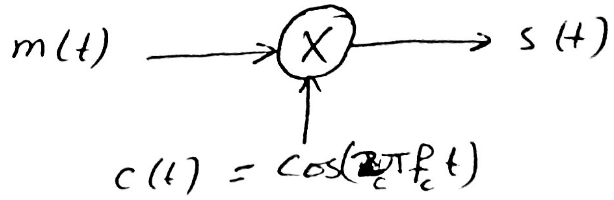
$$Y(f) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt$$

$$= \int x(t)\, e^{-j2\pi(f-f_0)t}\, dt$$

$$= X(f-f_0)$$

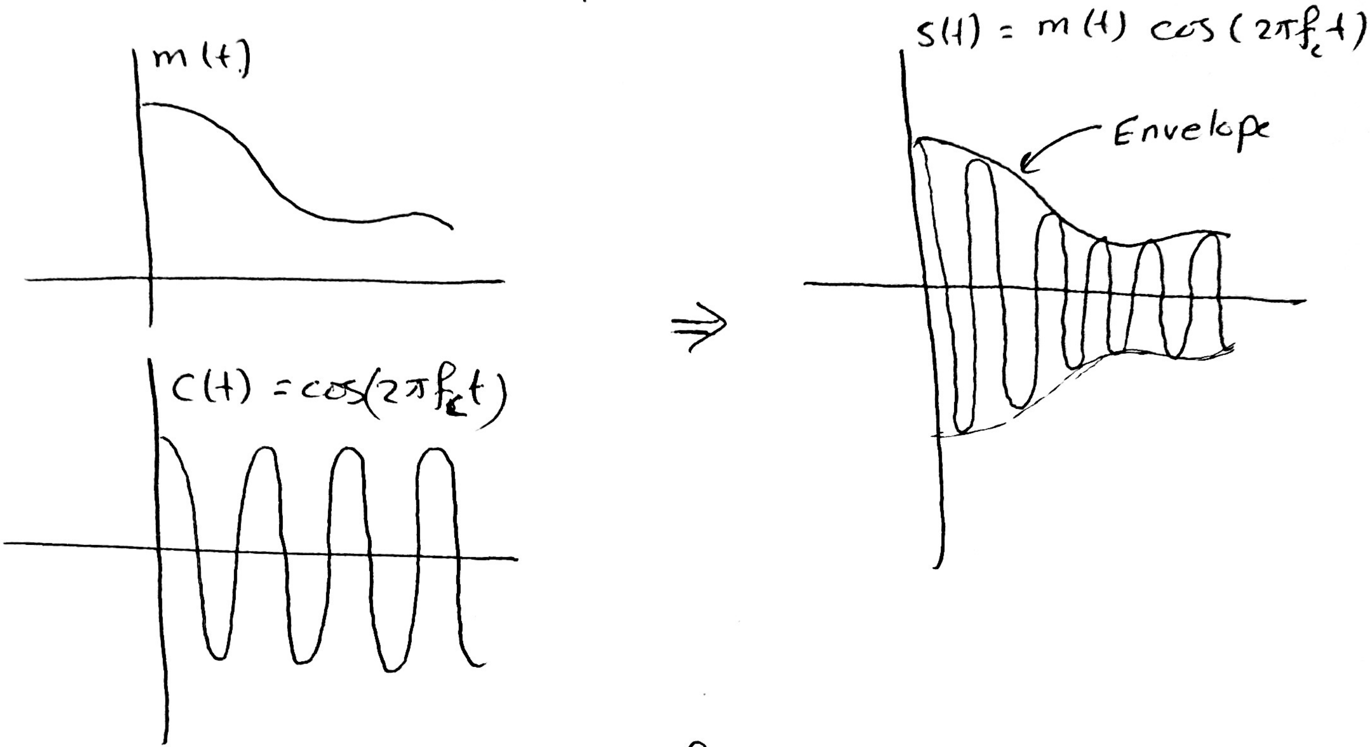
$$\Rightarrow \boxed{\begin{array}{ccc} x(t) & \longleftrightarrow & X(f) \\ \cancel{x(t)\,e^{j2\pi f t}} & & \\ x(t)\, e^{j2\pi f_0 t} & \longleftrightarrow & X(f-f_0) \end{array}}$$

* In communication, to send low freq. signals over long distances, we transfer these signals into high freq. signals by performing modulation.



- the mult. of the low freq. signal (message) $m(t)$ with high frequency signal (carrier) $c(t)$ is called modulation.

* In time domain :-

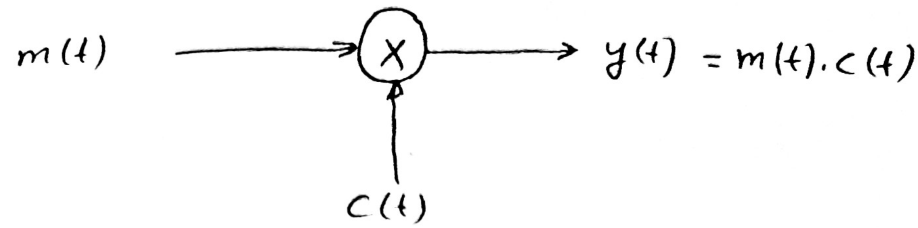


- The modulation theorem follows ~~results~~ from the freq. shifting theorem by using Euler's theorem to write ~~cos~~

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

and then apply superposition.

Example

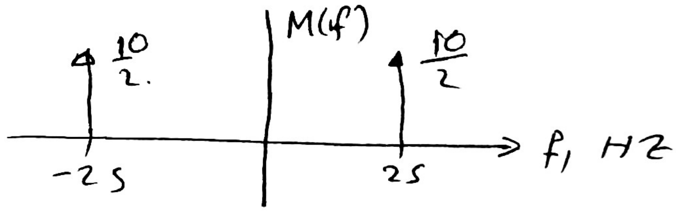


$m(t) = 10 \cos(50\pi t)$ message / modulating signal

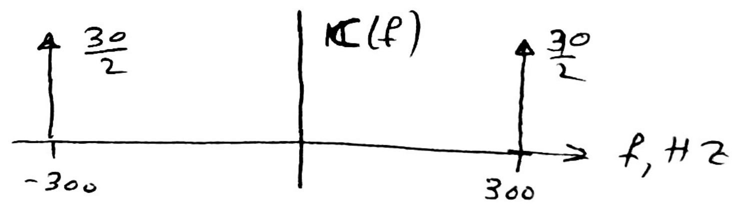
$c(t) = 30 \cos(600\pi t)$ carrier signal

1. Find and plot $M(f)$
2. " " " $C(f)$
3. " " " $Y(f)$

Sol. ① $M(f) = \frac{10}{2}\delta(f-25) + \frac{10}{2}\delta(f+25)$



② $C(f) = \frac{30}{2}\delta(f-300) + \frac{30}{2}\delta(f+300)$

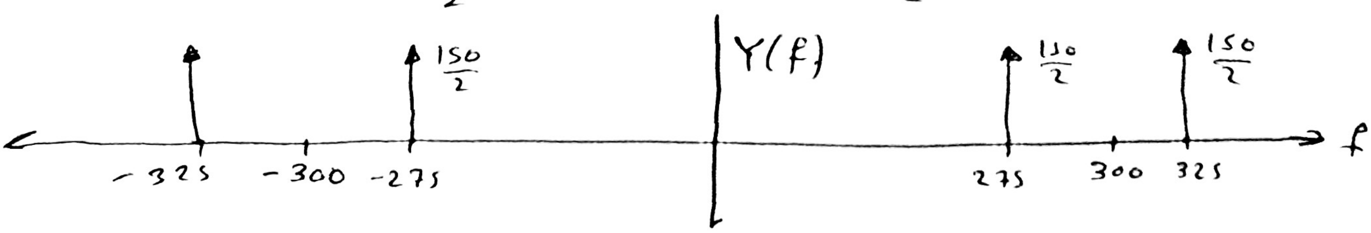


③ $y(t) = m(t).c(t)$

$= 300 \cos(50\pi t)\cos(600\pi t)$

$= 150 \cos(550\pi t) + 150 \cos(650\pi t)$

$$Y(f) = \frac{150}{2}\delta(f-275) + \frac{150}{2}\delta(f+275) + \frac{150}{2}\delta(f-325) + \frac{150}{2}\delta(f+325)$$



Or, Using Euler :-

$$y(t) = 300 \cos(50\pi t)\cos(600\pi t)$$

$$= 300 \cos(50\pi t)\left[\frac{e^{j600\pi t} + e^{-j600\pi t}}{2}\right]$$

$$= 150 \cos 50\pi t\, e^{j600\pi t} + 150 \cos(50\pi t)\, e^{-j600\pi t}$$

$$Y(f) = \frac{150}{2}\Big[\delta(f-25-300) + \delta(f+25-300) + \delta(f-25+300) + \delta(f+25+300)\Big]$$

$$= \frac{150}{2}\Big[\delta(f-325) + \delta(f-275) + \delta(f+275) + \delta(f+325)\Big]$$

H. Time Differentiation

if $x(t) \longrightarrow X(f)$

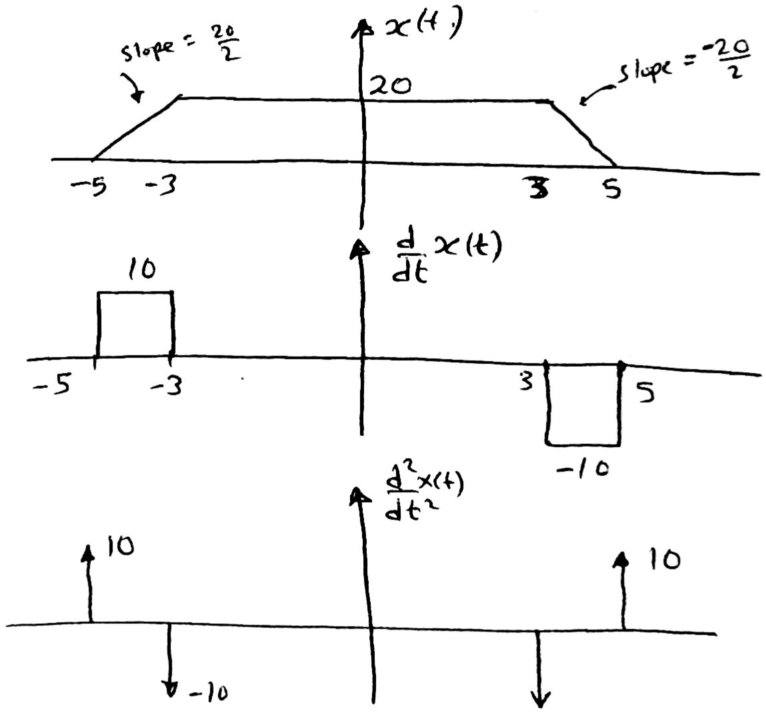
then $\frac{d}{dt} x(t) \longrightarrow j2\pi f\, X(f)$

and $\frac{d^n x(t)}{dt^n} \longrightarrow (j2\pi f)^n X(f)$

Differentiation results in a multiplication by (jw) in frequency domain.

Example Find $X(f)$ for the following signal $x(t)$

one can perform successive diff. till obtaining rectangle or impulses



$$z(t) = \frac{d}{dt} x(t) = 10\left[u(t+5) - u(t+3) - u(t-3) + u(t-5)\right]$$

$$y(t) = \frac{d^2}{dt^2} x(t) = 10\left[\delta(t+5) - \delta(t+3) - \delta(t-3) + \delta(t-5)\right]$$

$$Y(w) = (j2\pi f)^2 X(f) = 10\left[e^{j2\pi f(5)} - e^{j2\pi f(3)} - e^{+j2\pi f(-3)} + e^{j2\pi f(-5)}\right]$$

$$\Rightarrow (j2\pi f)^2 X(f) = 10 \cdot 2\left[\cos[2\pi f(5)] - \cos[2\pi f(3)]\right]$$

$$\Rightarrow X(f) = \frac{20}{4\pi^2 f^2}\left[\cos(10\pi f) - \cos(6\pi f)\right]$$

$\boxed{\text{I.}}$ Time Integration

if $x(t) \longrightarrow X(f)$

then $y(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda \longrightarrow Y(f) = \frac{X(f)}{j2\pi f} + \frac{1}{2} X(0)\,\delta(f)$

where $X(0) = X(f)\Big|_{f=0} = \int_{-\infty}^{\infty} x(t)\, dt$

if $x(t)$ has a non-zero time averaged value (DC) then

$X(0) \neq 0$

Example Consider finding the unit step response $a(t)$ of LTI system as a function of $h(t)$

$$a(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\lambda)\, \underbrace{u(t-\lambda)}_{=1 \text{ for } t \geq \lambda}\, d\lambda$$

$$= \int_{-\infty}^{t} h(\lambda)\, d\lambda$$ $\Rightarrow$ unit step response is just an integral of impulse response.

In freq. domain ?

$$A(f) = H(f) \cdot \mathcal{F}\{u(t)\} = H(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$$

$$= \frac{H(f)}{j2\pi f} + \frac{1}{2} H(0)\,\delta(f)$$

The Significance of time-differentiation & Integration

- The Diff. enhances the high-frequency content of a signal, which is reflected by the factor $j2\pi f$ in the transform of a derivative

- while integration smooths out time fluctuations or suppresses the high-frequency content of a signal, as indicated by $\frac{1}{j2\pi f}$ in the transform of an integral.

[J] Frequency Differentiation

if $x(t) \longrightarrow X(f)$

$$(-jt)^n x(t) \longrightarrow \left(\frac{1}{2\pi}\right)^n \frac{d^n X(f)}{df^n}$$

or

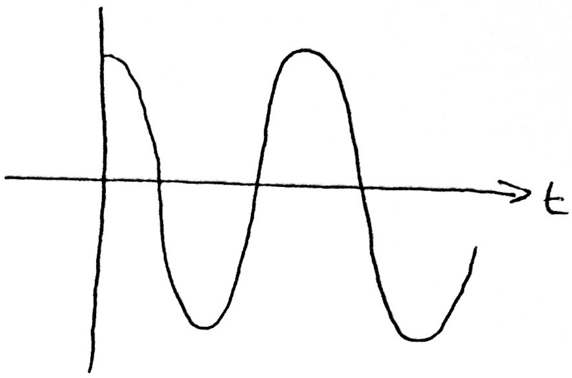
$$t^n x(t) \longrightarrow \left(\frac{j}{2\pi}\right)^n \frac{d^n X(f)}{df^n}.$$

Summary. the previous properties can be used to simplify computations of FT and also can be used in system analysis.

Examples

Consider a cosine function turned on only for $t \geq 0$

$$x(t) = \cos(\omega_0 t)\, u(t)$$



Find $X(f)$?

$$x(t) = \left(\frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}\right) u(t)$$

we know that (from table)

$$u(t) \longleftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

using Modulation Property

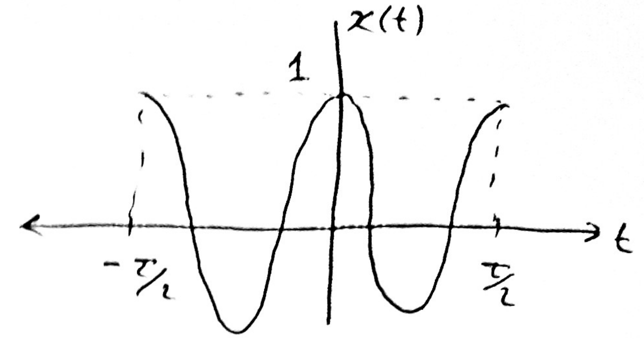
$$g(t)\, e^{j2\pi f_0 t} \longleftrightarrow G(f - f_0)$$

$$\Rightarrow X(f) = \frac{1}{2}\left[\frac{1}{j2\pi(f-f_0)} + \frac{1}{2}\delta(f-f_0)\right] + \frac{1}{2}\left[\frac{1}{j2\pi(f+f_0)} + \frac{1}{2}\delta(f+f_0)\right]$$

$$= \frac{1}{4}\left[\delta(f-f_0) + \delta(f+f_0)\right] + \frac{j2\pi f}{(2\pi f_0)^2 - (2\pi f)^2}$$

Example Pulsed Cosine

$$x(t) = \Pi\left(\frac{t}{\tau}\right) \cos(\omega_0 t)$$



Such signals are used in radar or sonar applications to know, for example, the existance and type of target or attack.

$$\Pi\left(\frac{t}{\tau}\right) \longrightarrow \tau \,\mathrm{sinc}(\tau f)$$

modulation property

$$g(t)\cos(\omega_0 t) \longleftrightarrow \frac{1}{2} G(f-f_0) + \frac{1}{2} G(f+f_0)$$

$$\Rightarrow X(f) = \frac{\tau}{2}\,\mathrm{sinc}\left[\tau(f-f_0)\right] + \frac{\tau}{2}\,\mathrm{sinc}\left[\tau(f+f_0)\right]$$

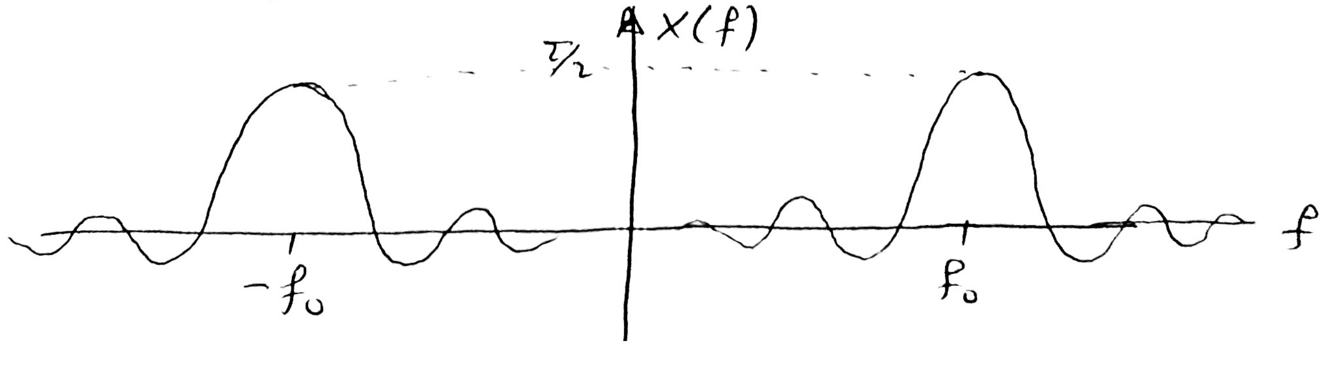
or using convolution property :-

$$z(t)\cdot h(t) \longleftrightarrow Z(f) * H(f)$$

$$X(f) = \mathcal{F}\left\{\Pi\left(\frac{t}{\tau}\right)\right\} * \mathcal{F}\left\{\cos\omega_0 t\right\}$$

$$= \tau\,\mathrm{sinc}(\tau f) * \left\{\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right\}$$

$$= \frac{\tau}{2}\,\mathrm{sinc}\left[\tau(f-f_0)\right] + \frac{\tau}{2}\,\mathrm{sinc}\left[\tau(f+f_0)\right]$$



Alternative view of Periodic Signals :-

Early we expressed periodic signals using FS :-

$$x(t) = \sum X_n e^{jn\omega_0 t}$$

$$X(f) = \sum X_n \mathcal{F}\{e^{jn\omega_0 t}\} = \sum X_n \, \delta(f - nf_0)$$

$$= f_0 \sum X(nf_0) \, \delta(f - nf_0)$$

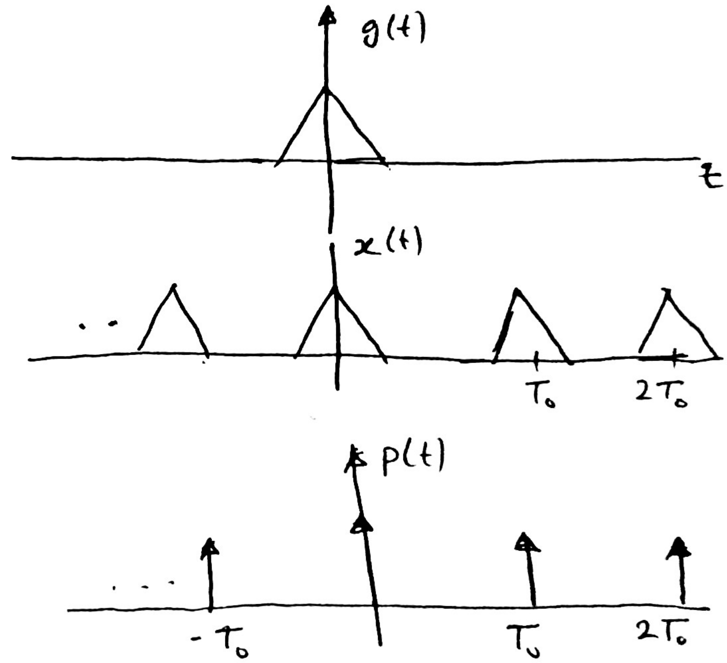
Periodic signals can be seen as a sum of shifted versions of a given signal (generating signal.)

$$x(t) = \sum_{-\infty}^{\infty} g(t - nT_0)$$

$$= \sum_{n=-\infty}^{\infty} g(t) * \delta(t - nT_0)$$

$$= g(t) * \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT_0)}_{p(t)}$$

$$= g(t) * p(t)$$



$p(t)$ is a periodic signal (train of impulses), hence using FS expansion :-

$$p(t) = \sum P_n e^{jn\omega_0 t}, \qquad P_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) \, e^{-jn\omega_0 t} = \frac{1}{T_0}$$

$$\Rightarrow p(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \xrightarrow{\mathcal{F}} \boxed{P(f) = \frac{1}{T_0} \sum \delta(f - nf_0)}$$

since $x(t) = g(t) * p(t)$

$$X(f) = G(f) \cdot P(f)$$

$$= G(f) \cdot \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - n f_0)$$

$$\boxed{X(f) = f_0 \sum_{n=-\infty}^{\infty} G(n f_0)\, \delta(f - n f_0)}$$

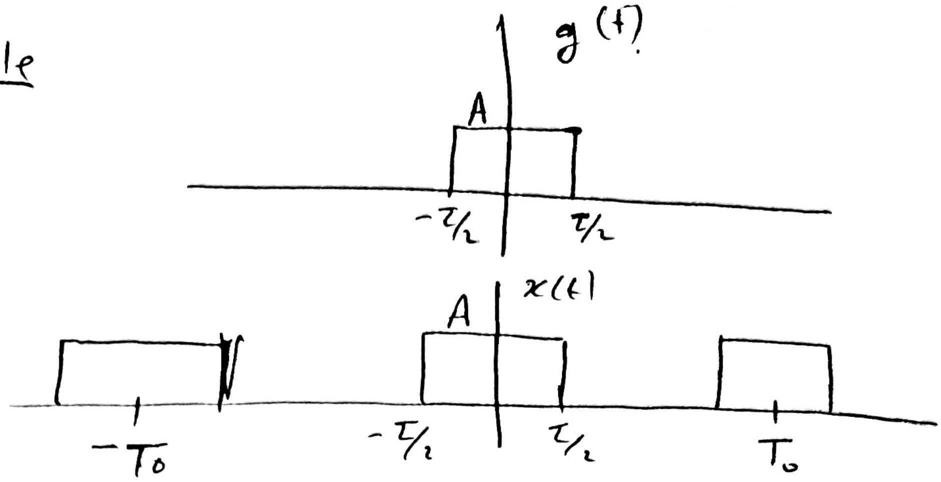
$F\{g(t)\}$: FT of the generating signal sampled at the harmonics $(n f_0)$

Hence, to obtain FT of periodic signal, we either find X_n using FS, or we find FT of one signal occurrence of signal, then we use shifted deltas.

$$X(f) = \sum X_n \, \delta(f - n f_0)$$

$$= f_0 \sum X(n f_0)\, \delta(f - n f_0)$$

Example



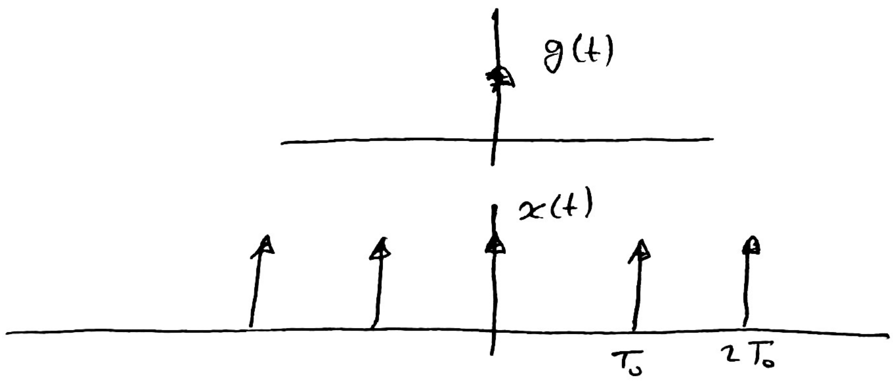
τ : pulse width
T_0 : fundamental period

$$g(t) = A\,\mathrm{rect}\left(\frac{t}{\tau}\right) \longleftrightarrow G(f) = A\tau\,\mathrm{sinc}(\tau f)$$

$$X(f) = f_0 \sum G(nf_0)\,\delta(f - nf_0)$$

$$= f_0 \sum_{n=-\infty}^{\infty} A\tau\,\mathrm{sinc}(\tau n f_0)\,\delta(f - nf_0)$$

Example



$$g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(f) = 1$$

$$X(f) = f_0 \sum G(nf_0)\,\delta(f - nf_0)$$

$$= f_0 \sum 1 \cdot \delta(f - nf_0)$$

$$= f_0 \sum \delta(f - nf_0)$$

which is also a train of delta fun.