Network Functions

From the book:

The analysis and design of linear circuits

By: Thomas, Rosa and Toussaint

Chapter 11

DEFINITION OF A NETWORK FUNCTION

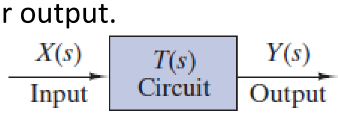
- A network function is defined as the ratio of a zero-state response (forced response) transform (output) to the excitation (input) transform.

$$\text{Network function} = \frac{\text{Zero-state response transform}}{\text{Input signal transform}}$$

  - this definition specifies zero initial conditions and implies only one input.
- To study the role of network functions in determining circuit responses, we write the s-domain input-output relationship as:

$$Y(s) = T(s)X(s)$$

  - where T(s) is a network function, X(s) is the input signal transform, and Y(s) is a zero-state response or output.



DEFINITION OF A NETWORK FUNCTION

- In an **analysis** problem, the circuit and input [X(s) or x(t)] are specified. We determine T(s) from the circuit, use $Y(s) = T(s)X(s)$ to find the response transform Y(s), and use the inverse transformation to obtain the response waveform y(t).
- In a **design** problem the circuit is unknown. The input and output are specified, or their ratio T(s) is given. The objective is to devise a circuit that realizes the specified input-output relationship.
- A linear circuit analysis problem has a unique solution, but a design problem may have one, many, or even no solutions.

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DEFINITION OF A NETWORK FUNCTION

$$Y(s) = T(s)X(s)$$

- the poles of the response Y(s) come from either the network function T(s) or the input signal X(s).
- When there are no repeated poles, the partial-fraction expansion of the right side of this equation takes the form:

$$Y(s) = \underbrace{\sum_{j=1}^{N} \frac{k_j}{s - p_j}}_{\substack{\text{natural} \\ \text{poles}}} + \underbrace{\sum_{\ell=1}^{M} \frac{k_\ell}{s - p_\ell}}_{\substack{\text{forced} \\ \text{poles}}}$$

- where p_j (j=1, 2, . . . ,N) are the poles of T(s) and s = p_l (l=1, 2, . . . ,M) are the poles of X(s).

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DEFINITION OF A NETWORK FUNCTION

- The inverse transform of this expansion is:

$$y(t) = \underbrace{\sum_{j=1}^{N} k_j e^{p_j t}}_{\text{natural response}} + \underbrace{\sum_{\ell=1}^{M} k_\ell e^{p_\ell t}}_{\text{forced response}}$$

- The poles of T(s) lead to the natural response.
- In a **stable** circuit, the natural poles are all in the left half of the s plane, and all of the exponential terms in the natural response eventually decay to zero.
- The poles of X(s) lead to the forced response.
- In a **stable** circuit, those elements in the forced response that do not decay to zero are called the **steady-state response**.

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DEFINITION OF A NETWORK FUNCTION

- The complex frequencies in the **natural** response are determined by the circuit and do not depend on input.
- Conversely, the complex frequencies in the **forced** response are determined by the input and do not depend on the circuit.
- However, the **amplitude** of its part of the response depends on the residues in the partial-fraction expansion. These residues are influenced by all of the poles and zeros, whether forced or natural.
- Thus, the amplitudes of the forced and natural responses depend on an interaction between the poles and zeros of T(s) and X(s).

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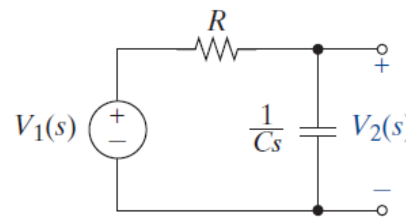
DEFINITION OF A NETWORK FUNCTION

Example 1:

A simple series RC circuit shown in is driven by a charging exponential source. If R=10 kΩ and C=0.01 μF, the network function is

$$T(s) = \frac{V_2(s)}{V_1(s)} = \frac{1/RC}{s + 1/RC} = \frac{10000}{s + 10000}$$



Find the zero-state response $v_2(t)$ when the input is $v_1(t)=10(1-e^{-5000t})u(t)$ V. Identify the natural and forced components of your answer.

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DEFINITION OF A NETWORK FUNCTION

- SOLUTION:
- The transform of the input signal is $$V_1(s) = \frac{10}{s} - \frac{10}{s+5000} = \frac{50000}{s(s+5000)}$$
- the transform of the response is $$V_2(s) = \frac{(50000)(10000)}{s(s+5000)(s+10000)}$$
- Expanding by partial fractions, $$V_2(s) = \underbrace{\frac{k_1}{s} + \frac{k_2}{s+5000}}_{\text{forced poles}} + \underbrace{\frac{k_3}{s+10000}}_{\text{natural pole}}$$
- ***The two forced poles came from the input charging exponential, while the natural pole came from the RC circuit via the network function.***

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DEFINITION OF A NETWORK FUNCTION

$$k_1 = \left. \frac{5 \times 10^8}{(s+5000)(s+10000)} \right|_{s=0} = 10$$

$$k_2 = \left. \frac{5 \times 10^8}{s(s+10000)} \right|_{s=-5000} = -20$$

$$k_3 = \left. \frac{5 \times 10^8}{s(s+5000)} \right|_{s=-1000} = 10$$

- Collectively the residues depend on all of the poles and zeros. The inverse transform yields the zero-state response as:

$$v_2(t) = \left(\underbrace{10e^{-10^4 t}}_{\text{natural response}} + \underbrace{10 - 20e^{-5\times 10^3 t}}_{\text{forced response}} \right) u(t)\ \text{V}$$

- The natural pole is s=-10,000 and is located in the left-half of the s plane causing the natural response to decay to zero.
- The forced poles are at **zero** and at s=-5000. The pole at s=-5000 will decay to zero leaving a steady-state response of 10 u(t).

DEFINITION OF A NETWORK FUNCTION

TEST SIGNALS

Impulse input ➡ impulse response

Step input ➡ step response

sinusoidal input ➡ frequency response

The impulse response is of great importance because it contains all of the information needed to calculate the response due to any other input.
The step response is important because it describes how a circuit response transitions from one state to another.

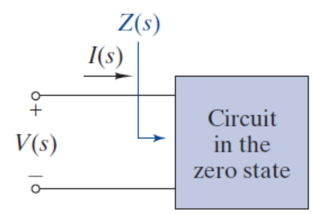
NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

- The two major types of network functions are **driving-point impedance** and **transfer functions**.
- A driving-point impedance relates the voltage and current at a pair of terminals called a port.
- The driving-point impedance Z(s) of the one-port circuit is defined as

driving-point impedance
input impedance
equivalent impedance

$$Z(s) = \frac{V(s)}{I(s)}$$

The term **driving point** means that the circuit is driven at one port and the response is observed at the same port.



A one-port circuit.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

$$Z(s) = \frac{V(s)}{I(s)}$$

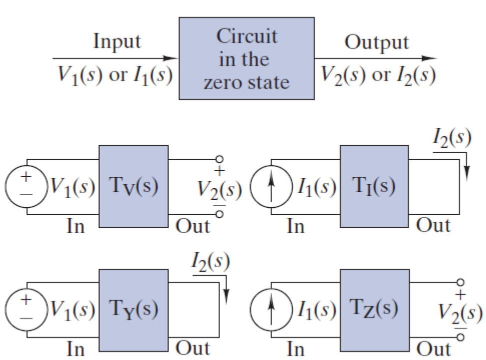
- When the one port is driven by a **current source**, the response is **V(s)=Z(s)I(s)** and the natural frequencies in the response are the poles of impedance Z(s).
- When the one port is driven by a **voltage source**, the response is **$I(s)=[Z(s)]^{-1}V(s)$** and the natural frequencies in the response are the poles of 1/Z(s); that is, the zeros of Z(s).
- In other words, *the driving-point impedance is a network function whether upside down or right side up*.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

- The driving-point impedance seen at a **pair** of terminals *determines the loading effects* that result when those terminals are connected to another circuit.
- When two circuits are connected together, these loading effects can alter the responses observed when the same two circuits operated in isolation.
- In an **analysis** situation it is important to be able to predict the response changes that occur when one circuit loads another.
- In **design** situations it is important to know when the circuits can be designed separately and then interconnected without encountering loading effects that alter their designed performance.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

- A **transfer function** (sometimes called **forward transfer function**) relates an input and response (or output) at different ports in the circuit.



$$T_V(s) = \text{Voltage Transfer Function} = \frac{V_2(s)}{V_1(s)}$$ dimensionless

$$T_I(s) = \text{Current Transfer Function} = \frac{I_2(s)}{I_1(s)}$$ dimensionless

$$T_Y(s) = \text{Transfer Admittance} = \frac{I_2(s)}{V_1(s)}$$ siemens

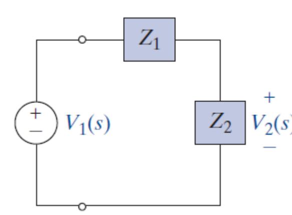
$$T_Z(s) = \text{Transfer Impedance} = \frac{V_2(s)}{I_1(s)}$$ ohms

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

- There are, of course, **reverse transfer functions** that relate inputs at port 2 to outputs at port 1.
- It is important to realize that a transfer function is valid only for a specified input port and output port. For example, **the voltage transfer function $T_V(s)=V_2(s)/V_1(s)$** relates the input voltage applied at port 1 to the voltage response observed at the output port. The **reverse voltage transfer function** for signal transmission from output to input **is not $1/T_V(s)$**.
- Unlike driving-point impedance, transfer functions are **not** network functions when they are turned upside down.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

DETERMINING NETWORK FUNCTIONS



$$V_2(s) = \left[\frac{Z_2(s)}{Z_1(s)+Z_2(s)}\right] V_1(s)$$

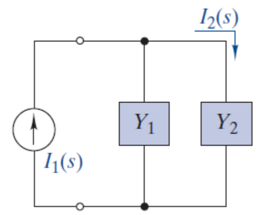
- Therefore, the voltage transfer function of a voltage divider circuit is

$$T_V(s) = \frac{V_2(s)}{V_1(s)} = \frac{Z_2(s)}{Z_1(s)+Z_2(s)}$$

- By **series equivalence**, the driving-point impedance at the input of the voltage divider is $Z_{EQ}(s)=Z_1(s)+Z_2(s)$.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

DETERMINING NETWORK FUNCTIONS



$$T_I(s) = \frac{I_2(s)}{I_1(s)} = \frac{Y_2(s)}{Y_1(s) + Y_2(s)} = \frac{Z_1(s)}{Z_1(s) + Z_2(s)}$$

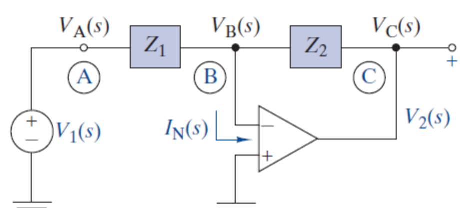
- By **parallel equivalence** the driving-point impedance at the input of the current divider is $Z_{EQ}(s)=1/(Y_1(s)+Y_2(s))$.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

DETERMINING NETWORK FUNCTIONS

- The voltage transfer function

Inverting amplifier



$$\frac{V_B(s) - V_A(s)}{Z_1(s)} + \frac{V_B(s) - V_C(s)}{Z_2(s)} = 0$$

$$T_V(s) = \frac{V_2(s)}{V_1(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

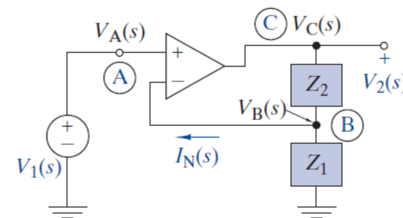
- The driving-point impedance at the input to the inverting circuit is

$$Z_{IN}(s) = \frac{V_1(s)}{[V_A(s) - V_B(s)]/Z_1(s)}$$

- But $V_A(s)=V_1(s)$ and $V_B(s)=0$; hence the input impedance is $\mathbf{Z_{IN}(s)=Z_1(s)}$, and should be a loading consideration.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

Noninverting amplifier



DETERMINING NETWORK FUNCTIONS

- The voltage transfer function

$$T_V(s) = \frac{V_2(s)}{V_1(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$

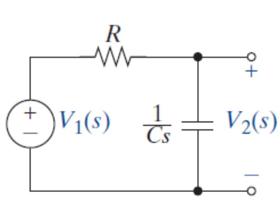
- The ideal OPAMP draws no current at its input terminals, so theoretically **the input impedance** of the noninverting circuit is **infinite** making it useful to solve loading issues.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

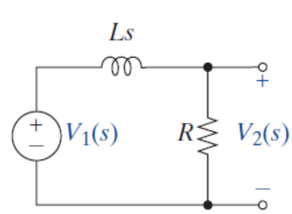
Example 2:

a) Find the transfer functions of the following circuits.

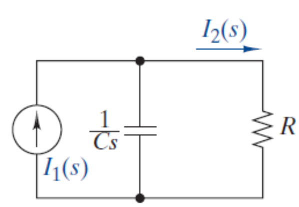
b) Find the driving-point impedances seen by the input sources in these circuits.



C1



C2

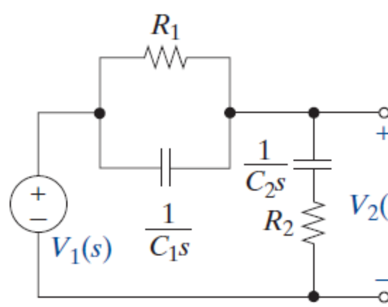


C3

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

Example 3:

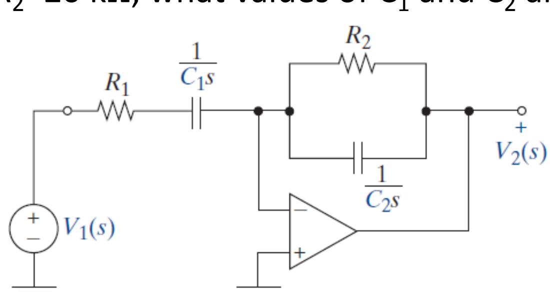
a) Find the input impedance seen by the voltage source.
b) Find the voltage transfer function $T_V(s)=V_2(s)/V_1(s)$ of the circuit.
c) Locate the poles and zeros of $T_V(s)$ for R_1=10 kΩ, R_2=20 kΩ, C_1=0.1 µF, and C_2=0.05 µF.



NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

Example 4: (Design Example)

Find the driving-point impedance seen by the voltage source in the following circuit. Find the voltage transfer function $T_V(s)=V_2(s)/V_1(s)$ of the circuit. The poles of $T_V(s)$ are located at p_1=-1000 rad/s and p_2=-5000 rad/s. If $R_1=R_2$=20 kΩ, what values of C_1 and C_2 are required?

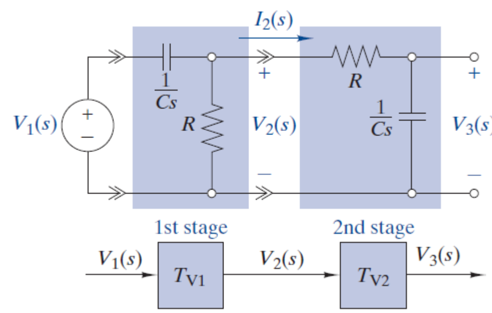


NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

$$T_V(s) = T_{V1}(s)T_{V2}(s) \cdots T_{Vk}(s)$$

- where $T_{V1}, T_{V2}, \ldots, T_{Vk}$ are the voltage transfer functions of the individual stages when operated separately.
- It is important to understand when the chain rule applies since it greatly simplifies the analysis and design of cascade circuits.



NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

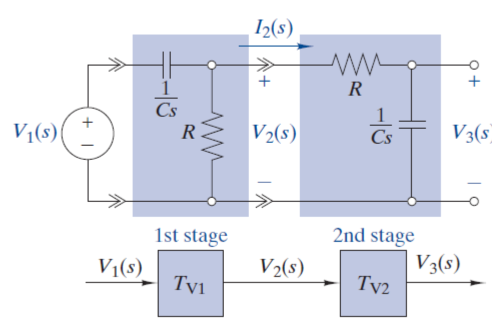
THE CASCADE CONNECTION AND THE CHAIN RULE

$$T_{V1}(s) = \frac{R}{R + 1/Cs} = \frac{RCs}{RCs + 1}$$

$$T_{V2}(s) = \frac{1/Cs}{R + 1/Cs} = \frac{1}{RCs + 1}$$

$$T_V(s) = \frac{V_3(s)}{V_1(s)} = \left(\frac{V_2(s)}{V_1(s)}\right)\left(\frac{V_3(s)}{V_2(s)}\right) = (T_{V1}(s))\,(T_{V2}(s))$$

$$= \underbrace{\left(\frac{RCs}{RCs+1}\right)}_{\substack{\text{first}\\\text{stage}}} \underbrace{\left(\frac{1}{RCs+1}\right)}_{\substack{\text{second}\\\text{stage}}} = \underbrace{\frac{RCs}{(RCs)^2 + 2RCs + 1}}_{\text{overall}}$$



NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

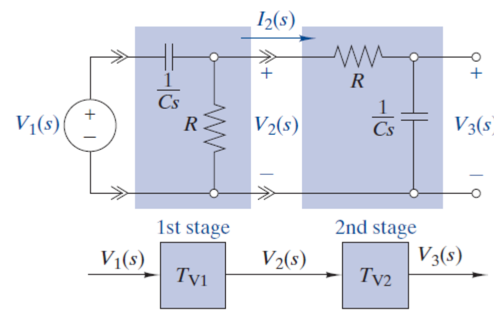
THE CASCADE CONNECTION AND THE CHAIN RULE

- However, in Example 11–5, the overall transfer function of this circuit was found to be

$$T_V(s) = \frac{RCs}{(RCs)^2 + 3RCs + 1}$$

- which disagrees with the chain rule result in

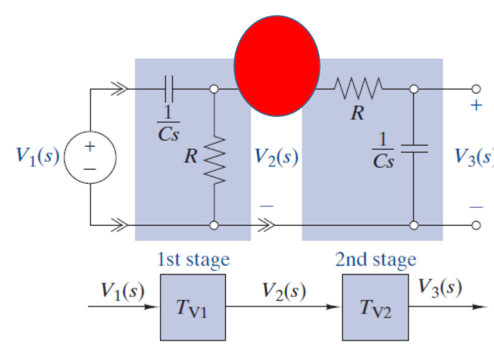
$$T_V(s) = \frac{RCs}{\underbrace{(RCs)^2 + 2RCs + 1}_{\text{overall}}}$$



NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

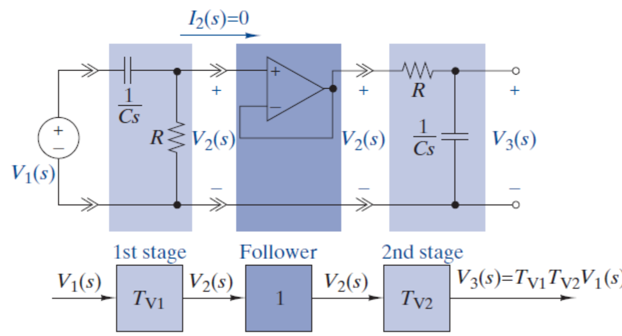
- The reason for the discrepancy is that when they are connected in cascade, the second circuit "**loads**" the first circuit.
- That is, the voltage-divider rule requires that the interface current $I_2(s)$ be zero.
- The no-load condition $I_2(s) = 0$ applies when the stages operate separately, but when connected in cascade, the interface current is not zero.
- The chain rule does not apply here because **loading** caused by the second stage changes the transfer function of the first stage.



NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

- the loading problem goes away when an OP AMP **voltage follower** is inserted between the RC circuit stages.
- With this modification the chain rule applies because the voltage follower isolates the two circuits, thereby solving the loading problem.



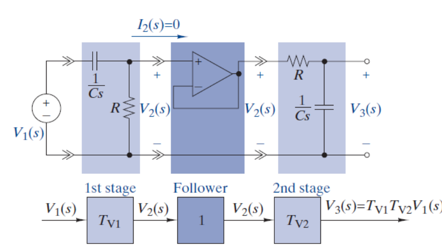
NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

- In the s domain, loading causes the transfer function of a circuit to change when it drives the input of another circuit.
- **In a cascade connection, loading does not occur at an interface if:**

1. The output (Thevenin) impedance of the **driving** stage is zero, **or**
2. The input impedance of the **driven** stage is infinite.

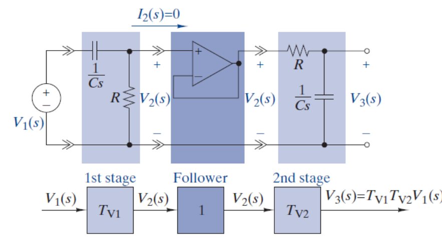
- The **voltage follower** is an example of a stage that meets both criteria (1) and (2).



NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

- In general, an **inverting OP AMP stage** meets criterion (1) but not criterion (2), while a **voltage-divider stage** meets neither criteria.



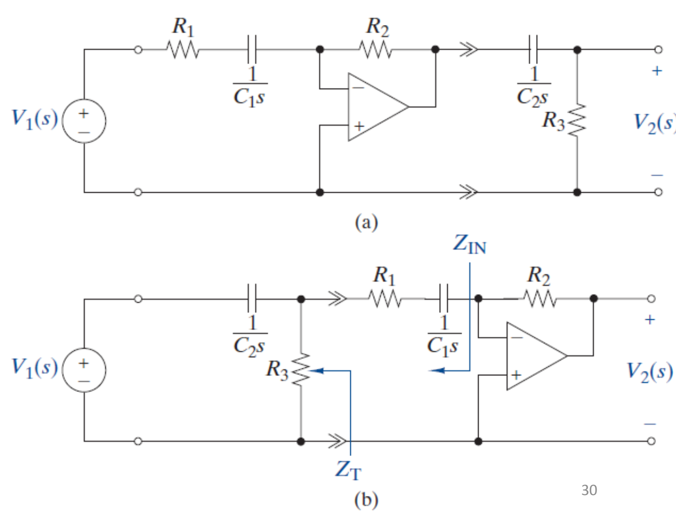
- When analyzing or designing a cascade connection, it is important to recognize situations in which the chain rule applies.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

EVALUATION EXAMPLE 7:

The two cascade connections involving the same two stages but with their positions reversed. Do either of these connections involve loading? If not, use the chain rule to find the overall transfer function.



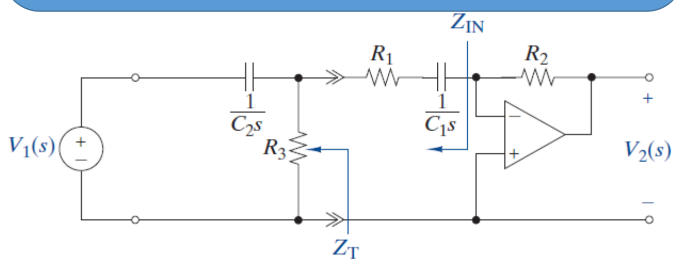
NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

SOLUTION:

Both circuits involve a cascade connection of a voltage-divider stage and an inverting amplifier stage.

The version in Figure 11–17(a) **does not involve loading** because the output impedance of the first stage is zero. Hence, connecting the second-stage voltage divider does not load the first stage and the chain rule applies.



(b)

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

The transfer function of the inverting amplifier stage is

$$T_{V1}(s) = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2}{R_1 + 1/C_1 s} = -\frac{R_2 C_1 s}{R_1 C_1 s + 1}$$

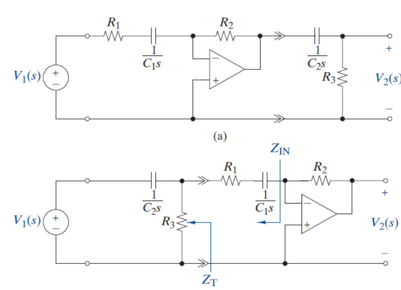
The second stage is a voltage divider whose transfer function is

$$T_{V2}(s) = \frac{Z_2(s)}{Z_2(s) + Z_1(s)} = \frac{R_3}{R_3 + 1/C_2 s} = \frac{R_3 C_2 s}{R_3 C_2 s + 1}$$

and the chain rule yields the overall transfer function as

$$T_V(s) = T_{V1}(s) \times T_{V2}(s) = \frac{-R_2 C_1 s}{R_1 C_1 s + 1} \times \frac{R_3 C_2 s}{R_3 C_2 s + 1}$$

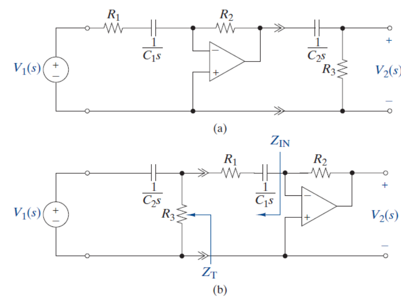
$$= \frac{-R_2 C_1 R_3 C_2 s^2}{R_1 C_1 R_3 C_2 s^2 + (R_1 C_1 + R_3 C_2) s + 1}$$



NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

The cascade connection in Figure 11–17(b) interchanges the positions of the two stages. **The loading occurs in this case** because the first stage is a voltage divider with a nonzero output (Thevenin) impedance of

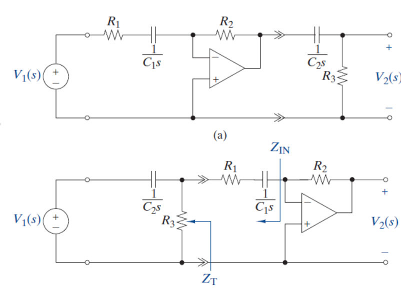


$$Z_T = \frac{1}{\dfrac{1}{R_3} + C_2 s} = \frac{R_3}{R_3 C_2 s + 1}$$

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

and the inverting amplifier in the second stage has a finite input impedance of



$$Z_{IN} = Z_T + \left(R_1 + \frac{1}{C_1 s}\right) = \frac{R_3}{R_3 C_2 s + 1} + \frac{R_1 C_1 s + 1}{C_1 s} = \frac{C_1 R_3 s + (R_1 C_1 s + 1)(R_3 C_2 s + 1)}{C_1 s (R_3 C_2 s + 1)}$$

$$T_V(s) = \frac{V_2(s)}{V_T(s)} \times \frac{V_T(s)}{V_1(s)} = -\frac{Z_2}{Z_{IN}} \times \frac{R_3 C_2 s}{R_3 C_2 s + 1} \text{ since } V_1(s) = V_T(s)\left(\frac{R_3 C_2 s + 1}{R_3 C_2 s}\right)$$

This results in a voltage transfer function of

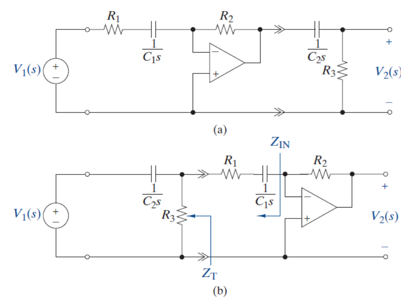
$$T_V(s) = \frac{-R_2 C_1 R_3 C_2 s^2}{R_1 C_1 R_3 C_2 s^2 + (R_1 C_1 + R_3 C_2 + R_3 C_1)s + 1}$$

which is not equal to $T_{V1}(s)$ X $T_{V2}(s)$. The chain rule does not apply to this connection since the second stage loads the first stage.

NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

- if R1=R2=R3=10 kΩ, C1=10 μF and C2=1 μF. the transfer function for the circuit in Figure 11–17(a) is
$$T_{V1} = -\frac{s^2}{(s+10)(s+100)}$$
- For the circuit in Figure 11–17(b), the transfer function is
$$T_{V2} = -\frac{s^2}{(s+4.88)(s+205)}$$
- The locations of the poles are quite different for the two circuits. Avoiding unintentional loading is a sign of experienced circuit designers.

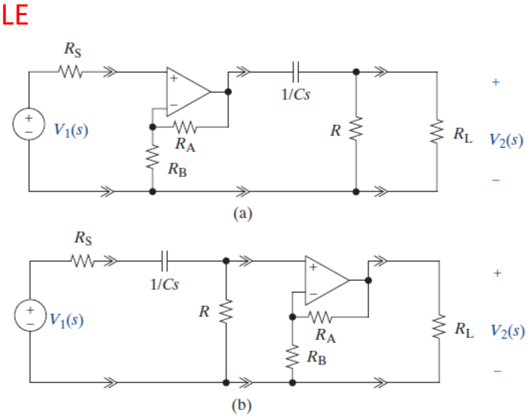


NETWORK FUNCTIONS OF ONE- AND TWO-PORT CIRCUITS

THE CASCADE CONNECTION AND THE CHAIN RULE

Evaluation Exercise 11–8

Figure 11–18 shows two cascade connections involving the same two stages but with their positions reversed. Does either of these connections involve loading? Find their voltage transfer functions and, if loading is present, determine the condition necessary to minimize the effect.



- ✓ 11–3 NETWORK FUNCTIONS AND IMPULSE RESPONSE
- ✓ 11–4 NETWORK FUNCTIONS AND STEP RESPONSE
- ✓ 11–5 NETWORK FUNCTIONS AND SINUSOIDAL STEADY-STATE RESPONSE
- ✓ 11–6 IMPULSE RESPONSE AND CONVOLUTION