$V_1 = Z_{11} I_1 + Z_{12} I_2$ — ① $V_1 = V_g - I_1 Z_g$ — ③

$V_2 = Z_{21} I_1 + Z_{22} I_2$ — ② $V_2 = -Z_L I_2$ — ④

[1] $Z_{in} = \frac{V_1}{I_1}$

eq ② $V_2 = Z_{21} I_1 + Z_{22} I_2$

④ $-Z_L I_2 = Z_{21} I_1 + Z_{22} I_2$

$-(Z_L + Z_{22}) I_2 = Z_{21} I_1$

$$I_2 = \frac{-Z_{21}}{Z_L + Z_{22}} I_1 \quad \text{— (A)}$$

eq ① $V_1 = Z_{11} I_1 + Z_{12} I_2$ sub (A) in (1)

$$= Z_{11} I_1 - \frac{Z_{12} Z_{21}}{Z_L + Z_{22}} I_1$$

$$\therefore Z_{in} = \frac{V_1}{I_1} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_{22} + Z_L}$$

[2] $I_2 = ??$

from (A) $$I_1 = -\frac{Z_L + Z_{22}}{Z_{21}} I_2 \quad \text{— (B)}$$

eq ① $V_1 = Z_{11} I_1 + Z_{12} I_2$

eq ③ in ①

$$V_g - I_1 Z_g = Z_{11} I_1 + Z_{12} I_2$$

$$Z_{12} I_2 = V_g - (Z_g + Z_{11}) I_1 \quad \text{sub (B)}$$

$$\frac{Z_{21} Z_{12} I_2}{Z_{21}} = V_g + \frac{(Z_g + Z_{11})(Z_L + Z_{22})}{Z_{21}} I_2$$

$$\left(\frac{Z_{12} Z_{21} - (V_g + Z_{11})(Z_L + Z_{22})}{Z_{21}} \right) I_2 = V_g$$

$$\therefore I_2 = \frac{Z_{21} V_g}{Z_{12} Z_{21} - (V_g + Z_{11})(Z_L + Z_{22})}$$

3 $V_{Th} = ??$

$$V_{Th} = V_2 \Big|_{I_2 = 0}$$

$\therefore$ eq (1) $V_1 = Z_{11} I_1$ — (C)

eq (2) $V_2 \Big|_{I_2=0} = Z_{21} I_1$ — (D)

Sub (C) in (D)

$$V_{Th} = \frac{Z_{21}}{Z_{11}} V_1$$

from (3) $V_1 = V_g - I_1 Z_g$ sub in V_{Th}

$$V_{Th} = \frac{Z_{21}}{Z_{11}} (V_g - I_1 Z_g)$$

I_1 from D: $I_1 = \frac{V_{Th}}{Z_{21}}$

$$= \frac{Z_{21}}{Z_{11}} \left(V_g - \frac{V_{Th}}{Z_{21}} Z_g\right)$$

$$\therefore V_{Th}\left(1 + \frac{Z_{21}}{Z_{11}} \frac{Z_g}{Z_{21}}\right) = \frac{Z_{21}}{Z_{11}} V_g$$

$$V_{Th}\left(\frac{Z_{11} + Z_g}{Z_{11}}\right) = \frac{Z_{21}}{Z_{11}} V_g$$

$$\boxed{V_{Th} = \frac{Z_{21}}{Z_{11} + Z_g} V_g}$$

OR

$V_1 = Z_{11} I_1$

$V_g - I_1 Z_g = Z_{11} I_1$

$$\therefore \boxed{I_1 = \frac{V_g}{Z_{11} + Z_g}}$$

$$V_{Th} = \frac{Z_{21}}{Z_{11}} \left(V_g - \frac{Z_g V_g}{Z_{11} + Z_g}\right)$$

$$= V_g \left(\frac{Z_{21}}{Z_{11}} - \frac{Z_{21} Z_g}{Z_{11}(Z_{11} + Z_g)}\right)$$

$$= \frac{Z_{21}}{Z_{11}} V_g \left(1 - \frac{Z_g}{Z_{11} + Z_g}\right)$$

$$= \frac{Z_{21}}{Z_{11}} V_g \left(\frac{Z_{11}}{Z_{11} + Z_g}\right)$$

$$\boxed{V_{Th} = \frac{Z_{21}}{Z_{11} + Z_g} V_g}$$

[4] Thev. impedance = output impedance

$$Z_{Th} = \left.\frac{V_2}{I_2}\right|_{V_g=0}$$ V_g is S.C.

When $V_g = 0$, eq (3) becomes

$$V_1 = -I_1 Z_g$$

sub in (1)

$$-I_1 Z_g = Z_{11} I_1 + Z_{12} I_2$$

$$I_1 (Z_{11} + Z_g) = -Z_{12} I_2$$

$$\boxed{I_1 = -\frac{Z_{12}}{Z_{11} + Z_g} I_2} \quad (E)$$

sub (E) in (2)

$$\left.V_2\right|_{V_g=0} = Z_{21} I_1 + Z_{22} I_2$$

$$= \left(\frac{-Z_{21} Z_{12}}{Z_{11} + Z_g} + Z_{22}\right) I_2$$

$$\therefore \boxed{\left.\frac{V_2}{I_2}\right|_{V_g=0} = Z_{Th} = Z_{22} - \frac{Z_{12} Z_{21}}{Z_{11} + Z_g}}$$

[5] current gain $\frac{I_2}{I_1}$

Directly from (A) OR (B)

$$\boxed{\frac{I_2}{I_1} = \frac{-Z_{21}}{Z_L + Z_{22}}}$$

6 Voltage gain ($\frac{V_2}{V_1}$)

from (2) $V_2 = Z_{21} I_1 + Z_{22} I_2$

from (4)

$$I_2 = \frac{-V_2}{Z_L} \quad \text{— (X)}$$

Sub X in (2)

$$V_2 = Z_{21} I_1 - \frac{Z_{22}}{Z_L} V_2 \quad \text{— (F)}$$

From (1)

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

Sub X in (1)

$$V_1 = Z_{11} I_1 - \frac{Z_{12}}{Z_L} V_2$$

$$I_1 = \frac{V_1}{Z_{11}} + \frac{Z_{12}}{Z_{11} Z_L} V_2 \quad \text{— (G)}$$

Sub (G) in (F)

$$V_2 = Z_{21}\left(\frac{V_1}{Z_{11}} + \frac{Z_{12}}{Z_{11} Z_L} V_2\right) - \frac{Z_{22}}{Z_L} V_2$$

$$= \frac{Z_{21}}{Z_{11}} V_1 + \frac{Z_{12} Z_{21}}{Z_{11} Z_L} V_2 - \frac{Z_{22}}{Z_L} V_2$$

$$V_2\left(1 - \frac{Z_{12} Z_{21}}{Z_{11} Z_L} + \frac{Z_{22}}{Z_L}\right) = \frac{Z_{21}}{Z_{11}} V_1$$

$$V_2\left(\frac{Z_{11} Z_L}{Z_{11} Z_L} - \frac{Z_{12} Z_{21}}{Z_{11} Z_L} + \frac{Z_{11} Z_{22}}{Z_{11} Z_L}\right) = \frac{Z_{21}}{Z_{11}} V_1$$

$$V_2\left(\frac{Z_{11} Z_L + Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{11} Z_L}\right) = \frac{Z_{21}}{Z_{11}} V_1$$

$$\frac{V_2}{V_1} = \frac{Z_{21} Z_L}{Z_{11} Z_L + \underbrace{Z_{11} Z_{22} - Z_{12} Z_{21}}_{\Delta Z}}$$

$$= \frac{Z_{21} Z_L}{Z_{11} Z_L + \Delta Z}$$

7 The Voltage ratio $\frac{V_2}{V_g}$

Combine (1), (3), (4) to find I_1

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_g - I_1 Z_g = Z_{11} I_1 + Z_{12}\left(\frac{-V_2}{Z_L}\right)$$

$$I_1 (Z_g + Z_{11}) = V_g + \frac{Z_{12}}{Z_L} V_2$$

$$\boxed{I_1 = \frac{V_g}{Z_g + Z_{11}} + \frac{Z_{12}}{Z_L (Z_g + Z_{11})} V_2} \quad \text{(H)}$$

Combine (4) & (H) with (2)

eq(2) $\quad V_2 = Z_{21} I_1 + Z_{22} I_2$

$$V_2 = \frac{Z_{21}}{Z_g + Z_{11}} V_g + \frac{Z_{12} Z_{21}}{Z_L (Z_g + Z_{11})} V_2 - \frac{Z_{22}}{Z_L} V_2$$

$$V_2\left(1 + \frac{Z_{22}}{Z_L}\frac{(Z_g + Z_{11})}{(Z_g + Z_{11})} - \frac{Z_{12} Z_{21}}{Z_L (Z_g + Z_{11})}\right) = \frac{Z_{21}}{Z_g + Z_{11}} V_g$$

$$V_2\left(\frac{Z_L (Z_g + Z_{11}) + Z_{22}(Z_g + Z_{11}) - Z_{12} Z_{21}}{Z_L (Z_g + Z_{11})}\right) = \frac{Z_{21}}{(Z_g + Z_{11})} V_g$$

$$\boxed{\frac{V_2}{V_g} = \frac{Z_{21} Z_L}{Z_L (Z_g + Z_{11}) + Z_{22}(Z_g + Z_{11}) - Z_{12} Z_{21}}}$$

OR

$$\boxed{\frac{V_2}{V_g} = \frac{Z_{21} Z_L}{(Z_{11} + Z_g)(Z_{22} + Z_L) - Z_{12} Z_{21}}}$$