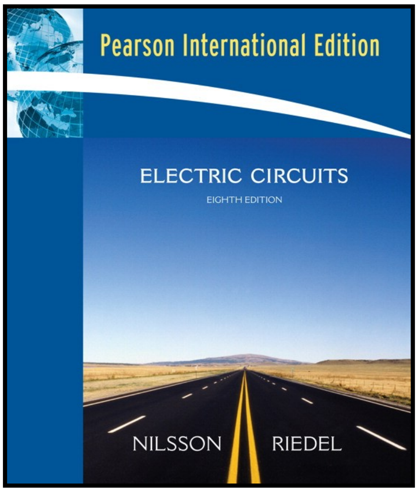


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# ELECTRIC CIRCUITS

EIGHTH EDITION

PEARSON Education Asia

## CHAPTER 18

# Two-Port Circuits
# Part 1

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CONTENTS

- 18.1 **The Terminal Equations**
- 18.2 **The Two-Port Parameters**
- 18.3 **Analysis of the Terminated Two-Port Circuit**
- 18.4 **Interconnected Two-Port Circuits**

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## CHAPTER OBJECTIVES

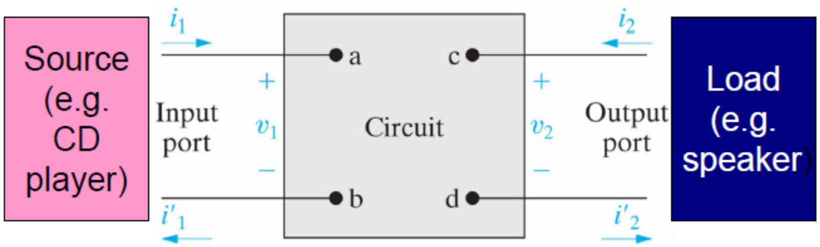
1. Be able to calculate any set of two-port parameters with any of the following methods:
   1. Circuit analysis;
   2. Measurements made on a circuit;
   3. Converting from another set of two-port parameters using Table 18.1.
2. Be able to analyze a terminated two-port circuit to find currents, voltages, impedances, and ratios of interest using Table 18.2.
3. Know how to analyze a cascade interconnection of two-port circuits.



## Motivation

- Thévenin and Norton equivalent circuits are used in representing the contribution of a circuit to one specific pair of terminals.
- Usually, a signal is fed into one pair of terminals (input port), processed by the system, then extracted at a second pair of terminals (output port). It would be convenient to relate the ***v/i*** at one port to the ***v/i*** at the other port without knowing the element values and how they are connected inside the "**black box**".

# How to model the "black box"?




- We will see that a two-port circuit can be modeled by a **2*2 matrix** to relate the *v/i* variables, where the four matrix elements can be obtained by performing 2 experiments.

# Restrictions of the model

- No energy stored within the circuit.
- No independent source.
- Each port is not a current source or sink, i.e.

$i_1 = i'_1, i_2 = i'_2.$

- No inter-port connection, i.e. between ac, ad, bc, bd.

## Key points

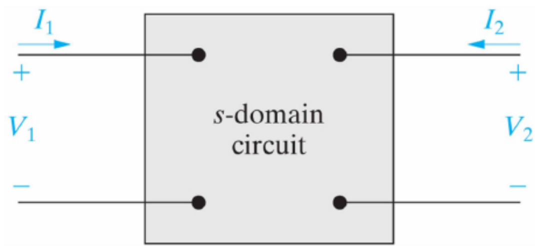
- How to calculate the **6** possible 2*2 matrices of a two-port circuit?
- How to find the **4** simultaneous equations in solving a terminated two-port circuit?
- How to find the total 2*2 matrix of a circuit consisting of **interconnected** two-port circuits?

# Section 18.1

# The Terminal Equations

# s-domain model

- The most general description of a two-port circuit is carried out in the s-domain.




- Any 2 out of the 4 variables $\{V_1, I_1, V_2, I_2\}$ can be determined by the other 2 variables and 2 simultaneous equations.

# Six possible sets of terminal equations (1)

$$\begin{cases} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}; \ [Z] \text{ is the impedance matrix;} \\ \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}; \ [Y] = [Z]^{-1} \text{ is the admittance matrix;} \end{cases}$$

$$\begin{cases} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}; [A] \text{ is a transmission matrix;} \\ \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} b_{11} & -b_{12} \\ b_{21} & -b_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}; \ [B] = [A]^{-1} \text{ is a transmission matrix;} \end{cases}$$

## Six possible sets of terminal equations (2)

$$\begin{cases} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}; [H] \text{ is a hybrid matrix;} \\ \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}; \ [G] = [H]^{1} \text{ is a hybrid matrix;} \end{cases}$$

- Which set is chosen depends on which variables are given. E.g. If the source voltage and current $\{V_1, I_1\}$ are given, choosing transmission matrix [*B*] in the analysis.

# Section 18.2

# The Two-Port Parameters

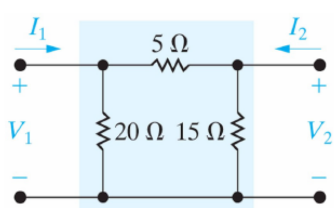
1. Calculation of matrix [*Z*]
2. Relations among 6 matrixes

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \Omega, \qquad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \Omega,$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \Omega, \qquad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} \Omega.$$

# Example 18.1: Finding [*Z*]

Find the impedance matrix [*Z*] (Z parameters) for a given resistive circuit (not a “black box”):




$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

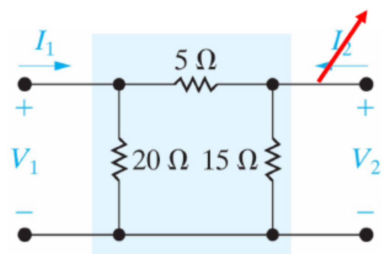
By definition, $z_{11}=(V_1/I_1)$ when $I_2=0$, i.e. the input impedance when port 2 is open. $\Rightarrow z_{11}=$ $(20\ \Omega)//(20\ \Omega)=10\ \Omega$.

# Example 18.1: Finding [Z]

- By definition, $z_{21}=(V_2/I_1)$ when $I_2=0$, i.e. the transfer impedance when port 2 is open.
- When port 2 is open:

$$\begin{cases} V_2 = \dfrac{15\,\Omega}{5\,\Omega+15\,\Omega}V_1 = 0.75V_1, \\ \dfrac{V_1}{I_1} = z_{11} = 10\,\Omega, \Rightarrow I_1 = \dfrac{V_1}{10\,\Omega}, \end{cases}$$

$$\Rightarrow z_{21} = \frac{V_2}{I_1} = \frac{0.75V_1}{V_1/(10\,\Omega)} = 7.5\,\Omega.$$

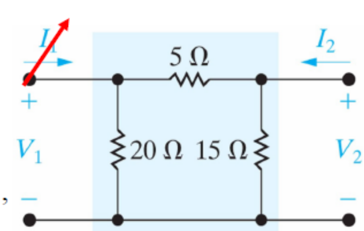



# Example 18.1: Finding [Z]

- By definition, $z_{22}=(V_2/I_2)$ when $I_1=0$, i.e. the output impedance when port 1 is open. $\Rightarrow z_{22}=$ $(15\ \Omega)//(25\ \Omega)=9.375\ \Omega$.
- $z_{12}=(V_1/I_2)$ when $I_1=0$, $\Rightarrow$

$$\begin{cases} V_1 = \dfrac{20\,\Omega}{20\,\Omega+5\,\Omega}V_2 = 0.8V_2, \\ \dfrac{V_2}{I_2} = z_{22} = 9.375\,\Omega, \Rightarrow I_2 = \dfrac{V_2}{9.375\,\Omega}, \end{cases}$$

$$\Rightarrow z_{12} = \frac{V_1}{I_2} = \frac{0.8V_2}{V_2/(9.375\,\Omega)} = 7.5\,\Omega.$$



# Comments

- When the circuit is well known, calculation of [$Z$] by circuit analysis methods shows the physical meaning of each matrix element.
- When the circuit is a "black box", we can perform 2 test experiments to get [$Z$]:

(1) Open port 2, apply a current $I_1$ to port 1, measure the input voltage $V_1$ and output voltage $V_2$ .

(2) Open port 1, apply a current $I_2$ to port 2, measure the terminal voltages $V_1$ and $V_2$ .

The admittance parameters

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \text{S}, \qquad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \text{S},$$

$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \text{S}, \qquad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} \text{S}.$$

**transmission** parameters

$$b_{11} = \left.\frac{V_2}{V_1}\right|_{I_1=0}, \qquad b_{12} = -\left.\frac{V_2}{I_1}\right|_{V_1=0} \Omega,$$

$$b_{21} = \left.\frac{I_2}{V_1}\right|_{I_1=0} \text{S}, \qquad b_{22} = -\left.\frac{I_2}{I_1}\right|_{V_1=0}.$$

**transmission** parameters

$$a_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0}, \qquad a_{12} = -\left.\frac{V_1}{I_2}\right|_{V_2=0} \Omega,$$

$$a_{21} = \left.\frac{I_1}{V_2}\right|_{I_2=0} \text{S}, \qquad a_{22} = -\left.\frac{I_1}{I_2}\right|_{V_2=0}.$$

**hybrid** parameters

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \Omega, \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0},$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}, \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} \text{S}.$$

**hybrid** parameters

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} \text{S}, \qquad g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0},$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}, \qquad g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} \Omega.$$

**Example 18.2** **Finding the *a* Parameters from Measurements**

The following measurements pertain to a two-port circuit operating in the sinusoidal steady state. With port 2 open, a voltage equal to $150\cos 4000t$ V is applied to port 1. The current into port 1 is $25\cos(4000t - 45^\circ)$ A, and the port 2 voltage is $100\cos(4000t + 15^\circ)$ V. With port 2 short-circuited, a voltage equal to $30\cos 4000t$ V is applied to port 1. The current into port 1 is $1.5\cos(4000t + 30^\circ)$ A, and the current into port 2 is $0.25\cos(4000t + 150^\circ)$ A. Find the *a* parameters that can describe the sinusoidal steady-state behavior of the circuit.

**Solution**

The first set of measurements gives

$$\mathbf{V}_1 = 150\angle 0^\circ \text{ V}, \qquad \mathbf{I}_1 = 25\angle -45^\circ \text{ A},$$

$$\mathbf{V}_2 = 100\angle 15^\circ \text{ V}, \qquad \mathbf{I}_2 = 0 \text{ A}.$$

From Eqs. 18.12,

$$a_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{I_2=0} = \frac{150\angle 0^\circ}{100\angle 15^\circ} = 1.5\angle -15^\circ,$$

$$a_{21} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{I_2=0} = \frac{25\angle -45^\circ}{100\angle 15^\circ} = 0.25\angle -60^\circ \text{S}.$$

The second set of measurements gives

$$\mathbf{V}_1 = 30\angle 0^\circ \text{ V}, \qquad \mathbf{I}_1 = 1.5\angle 30^\circ \text{A},$$

$$\mathbf{V}_2 = 0 \text{ V}, \qquad \mathbf{I}_2 = 0.25\angle 150^\circ \text{ A}.$$

Therefore

$$a_{12} = -\left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{V_2=0} = \frac{-30\angle 0^\circ}{0.25\angle 150^\circ} = 120\angle 30^\circ \ \Omega,$$

$$a_{21} = -\left.\frac{\mathbf{I}_1}{\mathbf{I}_2}\right|_{V_2=0} = \frac{-1.5\angle 30^\circ}{0.25\angle 150^\circ} = 6\angle 60^\circ.$$

# Relations among the 6 matrixes

- If we know one matrix, we can derive all the others analytically (Table 18.1).
- $[Y]=[Z]^{-1}$, $[B]=[A]^{-1}$, $[G]=[H]^{-1}$, elements between mutually inverse matrixes can be easily related.
- E.g.

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta y}\begin{bmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{bmatrix},$$

where $\Delta y \equiv \det[Y] = y_{11}y_{22} - y_{12}y_{21}$.

# Represent [*Z*] by elements of [*A*] (1)

- $[Z]$ and $[A]$ are not mutually inverse, relation between their elements are less explicit.
- By definitions of $[Z]$ and $[A]$,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix},$$

  the independent variables of $[Z]$ and $[A]$ are $\{I_1, I_2\}$ and $\{V_2, I_2\}$, respectively.
- Key of matrix transformation: Representing the distinct independent variable $V_2$ by $\{I_1, I_2\}$.

# Represent [*Z*] by elements of [*A*] (2)

- By definitions of $[A]$ and $[Z]$,

$$\begin{cases} V_1 = a_{11}V_2 - a_{12}I_2 \cdots (1) \\ I_1 = a_{21}V_2 - a_{22}I_2 \cdots (2) \end{cases}$$

$$(2) \Rightarrow V_2 = \frac{1}{a_{21}} I_1 + \frac{a_{22}}{a_{21}} I_2 = z_{21}I_1 + z_{22}I_2 \cdots (3),$$

$$(1),(3) \Rightarrow V_1 = a_{11}\left(\frac{1}{a_{21}} I_1 + \frac{a_{22}}{a_{21}} I_2\right) - a_{12}I_2$$

$$= \frac{a_{11}}{a_{21}} I_1 + \left(\frac{a_{11}a_{22}}{a_{21}} - a_{12}\right) I_2 = z_{11}I_1 + z_{12}I_2 \cdots (4)$$

$$\Rightarrow \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \frac{1}{a_{21}} \begin{bmatrix} a_{11} & \Delta a \\ 1 & a_{22} \end{bmatrix}, \text{ where } \Delta a \equiv \det[A]$$

**Example 18.3** **Finding *h* Parameters from Measurements and Table 18.1**

Two sets of measurements are made on a two-port resistive circuit. The first set is made with port 2 open, and the second set is made with port 2 short-circuited. The results are as follows:

| Port 2 Open | Port 2 Short-Circuited |
|---|---|
| $V_1 = 10$ mV | $V_1 = 24$ mV |
| $I_1 = 10\ \mu$A | $I_1 = 20\ \mu$A |
| $V_2 = -40$ V | $I_2 = 1$ mA |

Find the *h* parameters of the circuit.

**Solution**

We can find $h_{11}$ and $h_{21}$ directly from the short-circuit test:

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

$$= \frac{24 \times 10^{-3}}{20 \times 10^{-6}} = 1.2\ \text{k}\Omega,$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

$$= \frac{10^{-3}}{20 \times 10^{-6}} = 50.$$

The parameters $h_{12}$ and $h_{22}$ cannot be obtained directly from the open-circuit test. However, a check of Eqs. 18.7–18.15 indicates that the four $a$ parameters can be derived from the test data. Therefore, $h_{12}$ and $h_{22}$ can be obtained through the conversion table. Specifically,

$$h_{12} = \frac{\Delta a}{a_{22}}$$

$$h_{22} = \frac{a_{21}}{a_{22}}.$$

The $a$ parameters are

$$a_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{10 \times 10^{-3}}{-40} = -0.25 \times 10^{-3},$$

$$a_{21} = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{10 \times 10^{-6}}{-40} = -0.25 \times 10^{-6}\ \text{S},$$

$$a_{12} = -\left.\frac{V_1}{I_2}\right|_{V_2=0} = -\frac{24 \times 10^{-3}}{10^{-3}} = -24\ \Omega,$$

$$a_{22} = -\left.\frac{I_1}{I_2}\right|_{V_2=0} = -\frac{20 \times 10^{-6}}{10^{-3}} = -20 \times 10^{-3}.$$

The numerical value of $\Delta a$ is

$$\Delta a = a_{11}a_{22} - a_{12}a_{21}$$

$$= 5 \times 10^{-6} - 6 \times 10^{-6} = -10^{-6}.$$

Thus

$$h_{12} = \frac{\Delta a}{a_{22}}$$

$$= \frac{-10^{-6}}{-20 \times 10^{-3}} = 5 \times 10^{-5},$$

$$h_{22} = \frac{a_{21}}{a_{22}}$$

$$= \frac{-0.25 \times 10^{-6}}{-20 \times 10^{-3}} = 12.5\ \mu\text{S}.$$

# Reciprocal Two-Port Circuits

- If a two-port circuit is **reciprocal**, the following relationships exist among the port parameters:

$$z_{12} = z_{21}, \quad (18.28)$$

$$y_{12} = y_{21}, \quad (18.29)$$

$$a_{11}a_{22} - a_{12}a_{21} = \Delta a = 1, \quad (18.30)$$

$$b_{11}b_{22} - b_{12}b_{21} = \Delta b = 1, \quad (18.31)$$

$$h_{12} = -h_{21}, \quad (18.32)$$

$$g_{12} = -g_{21}. \quad (18.33)$$

# Reciprocal Two-Port Circuits

- A two-port circuit is reciprocal if the interchange of an ideal voltage source at one port with an ideal ammeter at the other port produces the same ammeter reading.

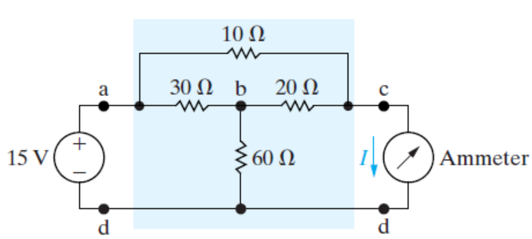



Figure 18.4 ▲ A reciprocal two-port circuit.

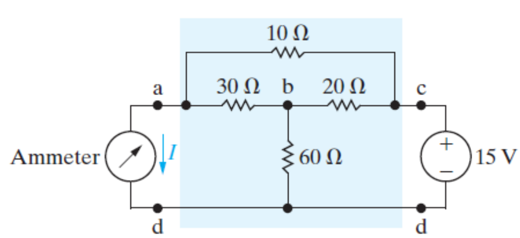



Figure 18.5 ▲ The circuit shown in Fig. 18.4, with the voltage source and ammeter interchanged.

# Reciprocal Two-Port Circuits

- Consider, for example, the resistive circuit shown in Fig. 18.4.When a voltage source of 15 V is applied to port ad, it produces a current of 1.75 A in the ammeter at port cd. The ammeter current is easily determined once we know the voltage $V_{bd}$. Thus:

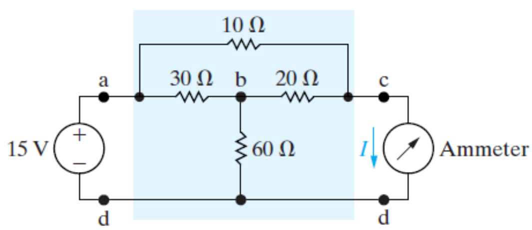



Figure 18.4 ▲ A reciprocal two-port circuit.

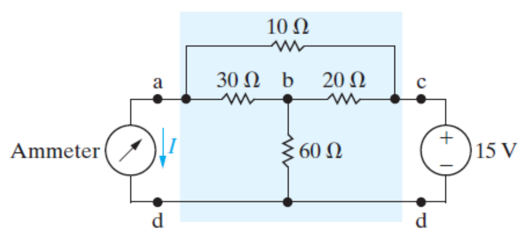



Figure 18.5 ▲ The circuit shown in Fig. 18.4, with the voltage source and ammeter interchanged.

# Reciprocal Two-Port Circuits

$$\frac{V_{bd}}{60} + \frac{V_{bd} - 15}{30} + \frac{V_{bd}}{20} = 0, \quad (18.34)$$

and $V_{bd} = 5$ V. Therefore

$$I = \frac{5}{20} + \frac{15}{10} = 1.75 \text{ A}. \quad (18.35)$$

**If the voltage source and ammeter are interchanged, the ammeter will still read 1.75 A. We verify this by solving the circuit shown in Fig. 18.5:**

$$\frac{V_{bd}}{60} + \frac{V_{bd}}{30} + \frac{V_{bd} - 15}{20} = 0. \quad (18.36)$$

From Eq. 18.36, $V_{bd} = 7.5$ V. The current $I_{ad}$ equals

$$I_{ad} = \frac{7.5}{30} + \frac{15}{10} = 1.75 \text{ A}. \quad (18.37)$$

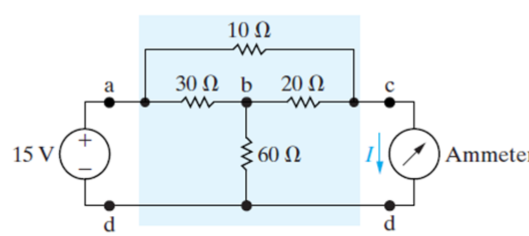



Figure 18.4 ▲ A reciprocal two-port circuit.

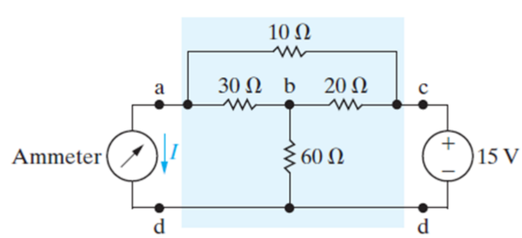



Figure 18.5 ▲ The circuit shown in Fig. 18.4, with the voltage source and ammeter interchanged.

# Reciprocal Two-Port Circuits

- A two-port circuit is also reciprocal if the interchange of an ideal current source at one port with an ideal voltmeter at the other port produces the same voltmeter reading. For a reciprocal two-port circuit, only three calculations or measurements are needed to determine a set of parameters.
- A reciprocal two-port circuit is **symmetric** if its ports can be interchanged without disturbing the values of the terminal currents and voltages.

# Reciprocal Two-Port Circuits

- Figure 18.6 shows four examples of symmetric two-port circuits.
- In such circuits, the following additional relationships exist among the port parameters:

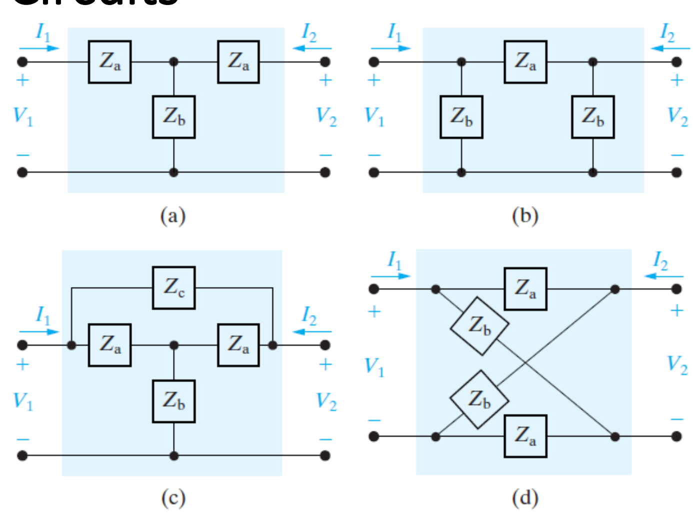



**Figure 18.6** ▲ Four examples of symmetric two-port circuits. (a) A symmetric tee. (b) A symmetric pi. (c) A symmetric bridged tee. (d) A symmetric lattice.

# Reciprocal Two-Port Circuits

$$z_{11} = z_{22}, \quad (18.38)$$

$$y_{11} = y_{22}, \quad (18.39)$$

$$a_{11} = a_{22}, \quad (18.40)$$

$$b_{11} = b_{22}, \quad (18.41)$$

$$h_{11}h_{22} - h_{12}h_{21} = \Delta h = 1, \quad (18.42)$$

$$g_{11}g_{22} - g_{12}g_{21} = \Delta g = 1. \quad (18.43)$$

**For a symmetric reciprocal network, only two calculations or measurements are necessary to determine all the two-port parameters.**

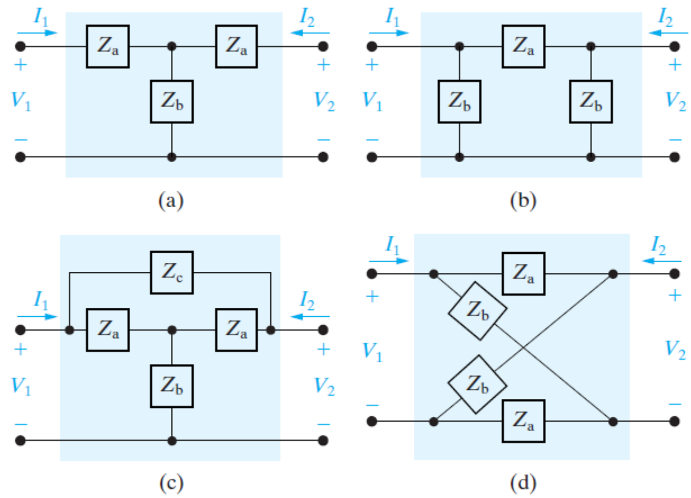



Figure 18.6 ▲ Four examples of symmetric two-port circuits. (a) A symmetric tee. (b) A symmetric pi. (c) A symmetric bridged tee. (d) A symmetric lattice.