Appendix E Bode Diagrams

As we have seen, the frequency response plot is a very important tool for analyzing a circuit's behavior. Up to this point, however, we have shown qualitative sketches of the frequency response without discussing how to create such diagrams. The most efficient method for generating and plotting the amplitude and phase data is to use a digital computer; we can rely on it to give us accurate numerical plots of $|H(j\omega)|$ and $\theta(j\omega)$ versus ω. However, in some situations, preliminary sketches using Bode diagrams can help ensure the intelligent use of the computer.

A Bode diagram, or plot, is a graphical technique that gives a feel for the frequency response of a circuit. These diagrams are named in recognition of the pioneering work done by H. W. Bode.[1] They are most useful for circuits in which the poles and zeros of $H(s)$ are reasonably well separated.

Like the qualitative frequency response plots seen thus far, a Bode diagram consists of two separate plots: One shows how the amplitude of $H(j\omega)$ varies with frequency, and the other shows how the phase angle of $H(j\omega)$ varies with frequency. In Bode diagrams, the plots are made on semilog graph paper for greater accuracy in representing the wide range of frequency values. In both the amplitude and phase plots, the frequency is plotted on the horizontal log scale, and the amplitude and phase angle are plotted on the linear vertical scale.

E.1 Real, First-Order Poles and Zeros

To simplify the development of Bode diagrams, we begin by considering only cases where all the poles and zeros of $H(s)$ are real and first order. Later we will present cases with complex and repeated poles and zeros. For our purposes, having a specific expression for $H(s)$ is helpful. Hence we base the discussion on

$$H(s) = \frac{K(s + z_1)}{s(s + p_1)}, \tag{E.1}$$

from which

$$H(j\omega) = \frac{K(j\omega + z_1)}{j\omega(j\omega + p_1)}. \tag{E.2}$$

The first step in making Bode diagrams is to put the expression for $H(j\omega)$ in a **standard form**, which we derive simply by dividing out the poles and zeros:

$$H(j\omega) = \frac{Kz_1(1 + j\omega/z_1)}{p_1(j\omega)(1 + j\omega/p_1)}. \tag{E.3}$$

[1] See H. W. Bode, *Network Analysis and Feedback Design* (New York: Van Nostrand, 1945).

Next we let K_o represent the constant quantity Kz_1/p_1, and at the same time we express $H(j\omega)$ in polar form:

$$H(j\omega) = \frac{K_o|1 + j\omega/z_1| \angle\psi_1}{|\omega| \angle 90^\circ |1 + j\omega/p_1| \angle\beta_1}$$

$$= \frac{K_o|1 + j\omega/z_1|}{|\omega||1 + j\omega/p_1|} \angle(\psi_1 - 90^\circ - \beta_1). \quad \text{(E.4)}$$

From Eq. E.4,

$$|H(j\omega)| = \frac{K_o|1 + j\omega/z_1|}{\omega|1 + j\omega/p_1|}, \quad \text{(E.5)}$$

$$\theta(\omega) = \psi_1 - 90^\circ - \beta_1. \quad \text{(E.6)}$$

By definition, the phase angles ψ_1 and β_1 are

$$\psi_1 = \tan^{-1}\omega/z_1; \quad \text{(E.7)}$$

$$\beta_1 = \tan^{-1}\omega/p_1. \quad \text{(E.8)}$$

The Bode diagrams consist of plotting Eq. E.5 (amplitude) and Eq. E.6 (phase) as functions of ω.

E.2 Straight-Line Amplitude Plots

The amplitude plot involves the multiplication and division of factors associated with the poles and zeros of $H(s)$. We reduce this multiplication and division to addition and subtraction by expressing the amplitude of $H(j\omega)$ in terms of a logarithmic value: the decibel (dB).[2] The amplitude of $H(j\omega)$ in decibels is

$$A_{\text{dB}} = 20 \log_{10}|H(j\omega)|. \quad \text{(E.9)}$$

To give you a feel for the unit of decibels, Table E.1 provides a translation between the actual value of several amplitudes and their values in decibels. Expressing Eq. E.5 in terms of decibels gives

$$A_{\text{dB}} = 20 \log_{10} \frac{K_o|1 + j\omega/z_1|}{\omega|1 + j\omega/p_1|}$$

$$= 20 \log_{10} K_o + 20 \log_{10}|1 + j\omega/z_1|$$

$$- 20 \log_{10} \omega - 20 \log_{10}|1 + j\omega/p_1|. \quad \text{(E.10)}$$

TABLE E.1 Actual Amplitudes and Their Decibel Values

A_{dB}	A	A_{dB}	A
0	1.00	30	31.62
3	1.41	40	100.00
6	2.00	60	10^3
10	3.16	80	10^4
15	5.62	100	10^5
20	10.00	120	10^6

[2] See Appendix D for more information regarding the decibel.

The key to plotting Eq. E.10 is to plot each term in the equation separately and then combine the separate plots graphically. The individual factors are easy to plot because they can be approximated in all cases by straight lines.

The plot of $20 \log_{10} K_o$ is a horizontal straight line because K_o is not a function of frequency. The value of this term is positive for $K_o > 1$, zero for $K_o = 1$, and negative for $K_o < 1$.

Two straight lines approximate the plot of $20 \log_{10}|1 + j\omega/z_1|$. For small values of ω, the magnitude $|1 + j\omega/z_1|$ is approximately 1, and therefore

$$20 \log_{10}|1 + j\omega/z_1| \rightarrow 0 \quad \text{as } \omega \rightarrow 0. \tag{E.11}$$

For large values of ω, the magnitude $|1 + j\omega/z_1|$ is approximately ω/z_1, and therefore

$$20 \log_{10}|1 + j\omega/z_1| \rightarrow 20 \log_{10}(\omega/z_1) \quad \text{as } \omega \rightarrow \infty. \tag{E.12}$$

On a log frequency scale, $20 \log_{10}(\omega/z_1)$ is a straight line with a slope of 20 dB/decade (a decade is a 10-to-1 change in frequency). This straight line intersects the 0 dB axis at $\omega = z_1$. This value of ω is called the **corner frequency**. Thus, on the basis of Eqs. E.11 and E.12, two straight lines can approximate the amplitude plot of a first-order zero, as shown in Fig. E.1.

The plot of $-20 \log_{10} \omega$ is a straight line having a slope of -20 dB/decade that intersects the 0 dB axis at $\omega = 1$. Two straight lines approximate the plot of $-20 \log_{10}|1 + j\omega/p_1|$. Here the two straight lines

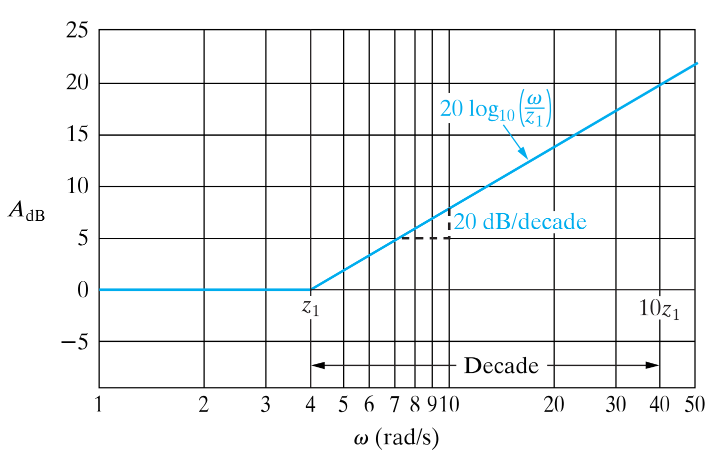

Figure E.1 ▲ A straight-line approximation of the amplitude plot of a first-order zero.

intersect on the 0 dB axis at $\omega = p_1$. For large values of ω, the straight line $20 \log_{10}(\omega/p_1)$ has a slope of -20 dB/decade. Figure E.2 shows the straight-line approximation of the amplitude plot of a first-order pole.

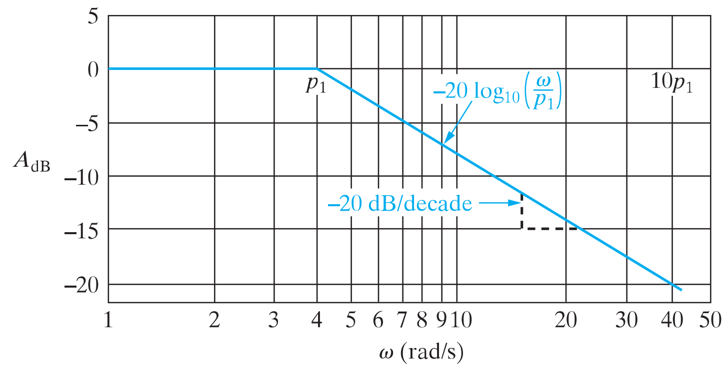

Figure E.2 ▲ A straight-line approximation of the amplitude plot of a first-order pole.

Figure E.3 shows a plot of Eq. E.10 for $K_o = \sqrt{10}$, $z_1 = 0.1$ rad/s, and $p_1 = 5$ rad/s. Each term in Eq. E.10 is labeled on Fig. E.3, so you can verify that the individual terms sum to create the resultant plot, labeled $20 \log_{10}|H(j\omega)|$.

Example E.1 illustrates the construction of a straight-line amplitude plot for a transfer function characterized by first-order poles and zeros.

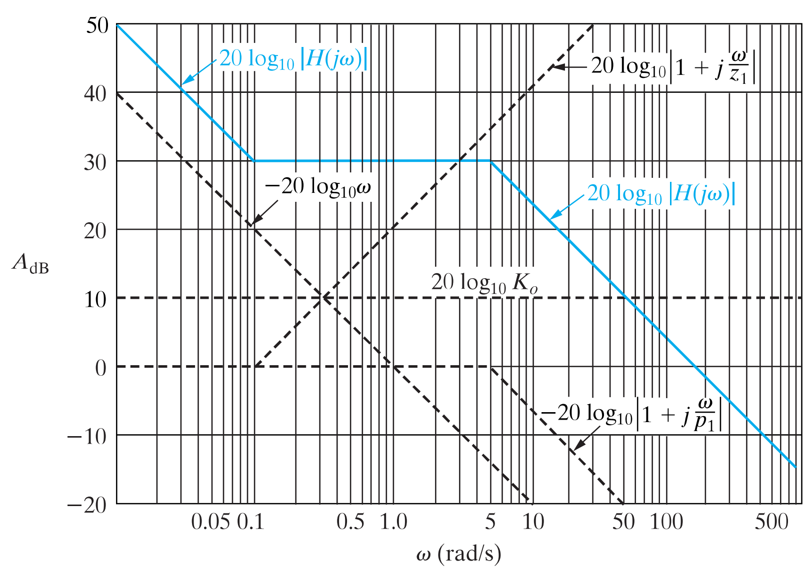

Figure E.3 ▲ A straight-line approximation of the amplitude plot for Eq. E.10.

Example E.1

For the circuit in Fig. E.4:

a) Compute the transfer function, $H(s)$.

b) Construct a straight-line approximation of the Bode amplitude plot.

c) Calculate $20\log_{10}|H(j\omega)|$ at $\omega = 50$ rad/s and $\omega = 1000$ rad/s.

d) Plot the values computed in (c) on the straight-line graph; and

e) Suppose that $v_i(t) = 5\cos(500t + 15^\circ)$ V, and then use the Bode plot you constructed to predict the amplitude of $v_o(t)$ in the steady state.

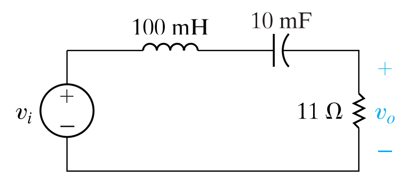

Figure E.4 ▲ The circuit for Example E.1.

Solution

a) Transforming the circuit in Fig. E.4 into the s-domain and then using s-domain voltage division gives

$$H(s) = \frac{(R/L)s}{s^2 + (R/L)s + \frac{1}{LC}}.$$

Substituting the numerical values from the circuit, we get

$$H(s) = \frac{110s}{s^2 + 110s + 1000} = \frac{110s}{(s + 10)(s + 100)}.$$

b) We begin by writing $H(j\omega)$ in standard form:

$$H(j\omega) = \frac{0.11\, j\omega}{[1 + j(\omega/10)][1 + j(\omega/100)]}.$$

The expression for the amplitude of $H(j\omega)$ in decibels is

$$A_{\mathrm{dB}} = 20\log_{10}|H(j\omega)|$$

$$= 20\log_{10} 0.11 + 20\log_{10}|j\omega|$$

$$- 20\log_{10}\left|1 + j\frac{\omega}{10}\right| - 20\log_{10}\left|1 + j\frac{\omega}{100}\right|.$$

Figure E.5 shows the straight-line plot. Each term contributing to the overall amplitude is identified.

c) We have

$$H(j50) = \frac{0.11(j50)}{(1 + j5)(1 + j0.5)}$$

$$= 0.9648\angle{-15.25^\circ},$$

$$20\log_{10}|H(j50)| = 20\log_{10} 0.9648$$

$$= -0.311 \text{ dB};$$

$$H(j1000) = \frac{0.11(j1000)}{(1 + j100)(1 + j10)}$$

$$= 0.1094\angle{-83.72^\circ};$$

$$20\log_{10} 0.1094 = -19.22 \text{ dB}.$$

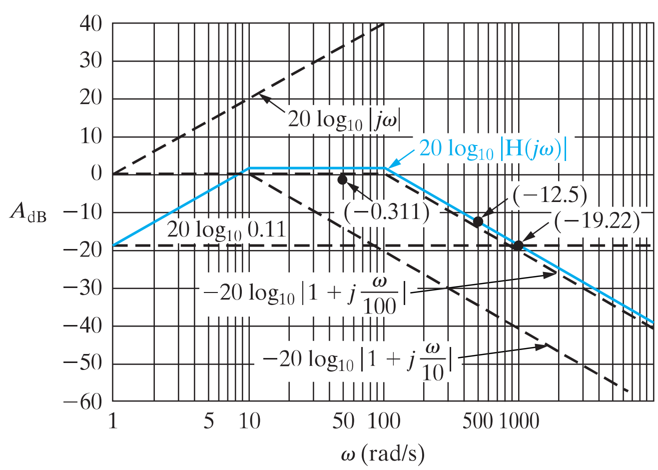

Figure E.5 ▲ The straight-line amplitude plot for the transfer function of the circuit in Fig. E.4.

d) See Fig. E.5.

e) As we can see from the Bode plot in Fig. E.5, the value of A_{dB} at $\omega = 500$ rad/s is approximately -12.5 dB. Therefore,

$$|A| = 10^{(-12.5/20)} = 0.24$$

and

$$V_{mo} = |A|V_{mi} = (0.24)(5) = 1.19 \text{ V}.$$

We can compute the actual value of $|H(j\omega)|$ by substituting $\omega = 500$ into the equation for $|H(j\omega)|$:

$$H(j500) = \frac{0.11(j500)}{(1 + j50)(1 + j5)} = 0.22\angle{-77.54^\circ}.$$

Thus, the actual output voltage magnitude for the specified signal source at a frequency of 500 rad/s is

$$V_{mo} = |A|V_{mi} = (0.22)(5) = 1.1 \text{ V}.$$

E.3 More Accurate Amplitude Plots

We can make the straight-line plots for first-order poles and zeros more accurate by correcting the amplitude values at the corner frequency, one half the corner frequency, and twice the corner frequency. At the corner frequency, the actual value in decibels is

$$\begin{aligned} A_{dB_c} &= \pm 20 \log_{10}|1 + j1| \\ &= \pm 20 \log_{10}\sqrt{2} \\ &\approx \pm 3 \text{ dB}. \end{aligned} \tag{E.13}$$

The actual value at one half the corner frequency is

$$\begin{aligned} A_{dB_{c/2}} &= \pm 20 \log_{10}\left|1 + j\frac{1}{2}\right| \\ &= \pm 20 \log_{10}\sqrt{5/4} \\ &\approx \pm 1 \text{ dB}. \end{aligned} \tag{E.14}$$

At twice the corner frequency, the actual value in decibels is

$$\begin{aligned} A_{dB_{2c}} &= \pm 20 \log_{10}|1 + j2| \\ &= \pm 20 \log_{10}\sqrt{5} \\ &\approx \pm 7 \text{ dB}. \end{aligned} \tag{E.15}$$

In Eqs. E.13–E.15, the plus sign applies to a first-order zero, and the minus sign applies to a first-order pole. The straight-line approximation of the amplitude plot gives 0 dB at the corner and one half the corner frequencies, and ± 6 dB at twice the corner frequency. Hence the corrections are ± 3 dB at the corner frequency and ± 1 dB at both one half the corner frequency and twice the corner frequency. Figure E.6 summarizes these corrections.

A 2-to-1 change in frequency is called an **octave**. A slope of 20 dB/decade is equivalent to 6.02 dB/octave, which for graphical purposes is equivalent to 6 dB/octave. Thus the corrections enumerated correspond to one octave below and one octave above the corner frequency.

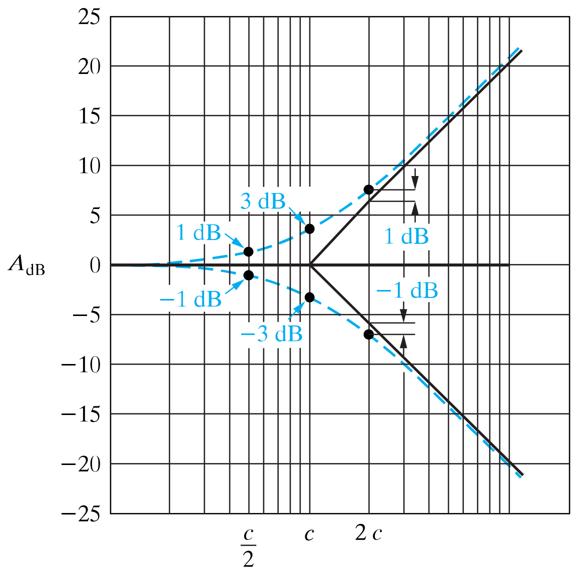

Figure E.6 ▲ Corrected amplitude plots for a first-order zero and pole.

If the poles and zeros of $H(s)$ are well separated, inserting these corrections into the overall amplitude plot and achieving a reasonably accurate curve is relatively easy. However, if the poles and zeros are close together, the overlapping corrections are difficult to evaluate, and you're better off using the straight-line plot as a first estimate of the amplitude characteristic. Then use a computer to refine the calculations in the frequency range of interest.

E.4 Straight-Line Phase Angle Plots

We can also make phase angle plots by using straight-line approximations. The phase angle associated with the constant K_o is zero, and the phase angle associated with a first-order zero or pole at the origin is a constant $\pm 90^\circ$. For a first-order zero or pole not at the origin, the straight-line approximations are as follows:

- For frequencies less than one tenth the corner frequency, the phase angle is assumed to be zero.
- For frequencies greater than 10 times the corner frequency, the phase angle is assumed to be $\pm 90^\circ$.
- Between one tenth the corner frequency and 10 times the corner frequency, the phase angle plot is a straight line that goes through 0° at one-tenth the corner frequency, $\pm 45^\circ$ at the corner frequency, and $\pm 90^\circ$ at 10 times the corner frequency.

In all these cases, the plus sign applies to the first-order zero and the minus sign to the first-order pole. Figure E.7 depicts the straight-line approximation for a first-order zero and pole. The dashed curves show the exact variation of the phase angle as the frequency varies. Note how closely the

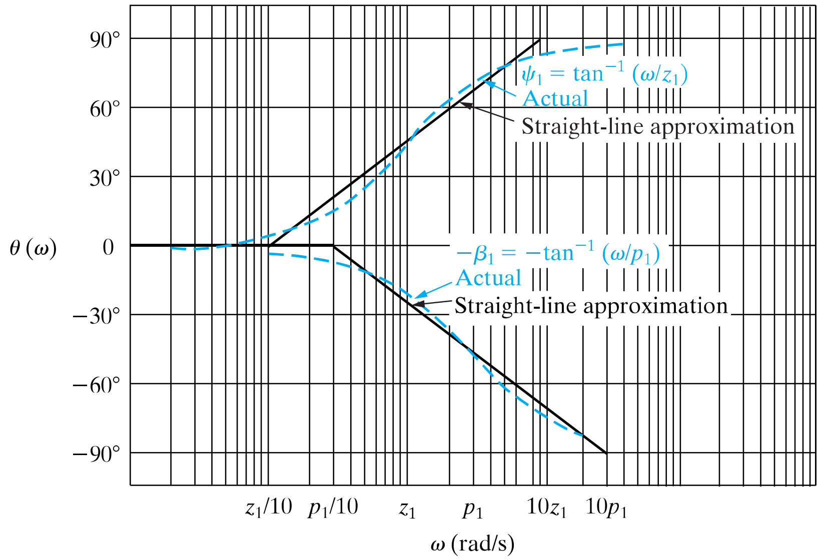

Figure E.7 ▲ Phase angle plots for a first-order zero and pole.

straight-line plot approximates the actual variation in phase angle. The maximum deviation between the straight-line plot and the actual plot is approximately 6°.

Figure E.8 depicts the straight-line approximation of the phase angle of the transfer function given by Eq. B.1. Equation B.6 gives the equation for the phase angle; the plot corresponds to $z_1 = 0.1$ rad/s, and $p_1 = 5$ rad/s.

An illustration of a phase angle plot using a straight-line approximation is given in Example E.2.

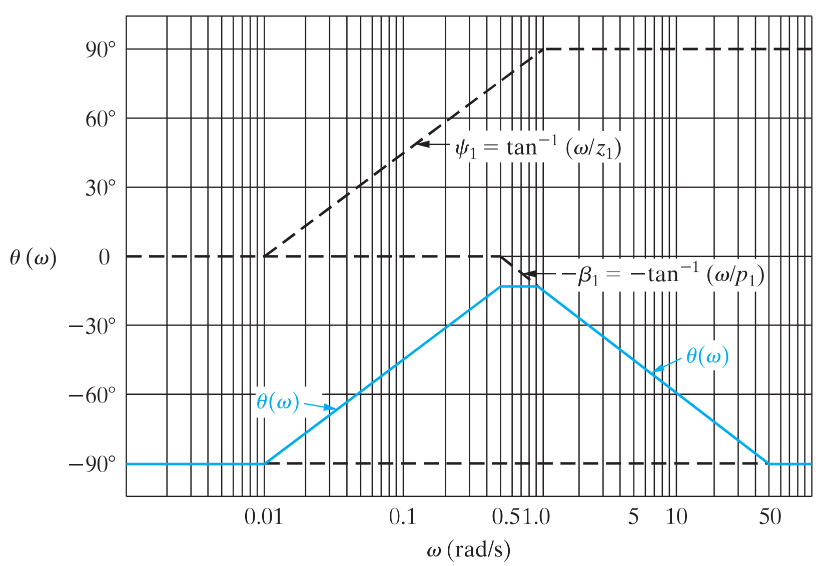

Figure E.8 ▲ A straight-line approximation of the phase angle plot for Eq. B.1.

Example E.2

a) Make a straight-line phase angle plot for the transfer function in Example E.1.

b) Compute the phase angle $\theta(\omega)$ at $\omega = 50, 500,$ and 1000 rad/s.

c) Plot the values of (b) on the diagram of (a).

d) Using the results from Example E.1(e) and (b) of this example, compute the steady-state output voltage if the source voltage is given by $v_i(t) = 10\cos(500t - 25^\circ)$ V.

Solution

a) From Example E.1,

$$H(j\omega) = \frac{0.11(j\omega)}{[1 + j(\omega/10)][1 + j(\omega/100)]}$$

$$= \frac{0.11|j\omega|}{|1 + j(\omega/10)||1 + j(\omega/100)|} \underline{/(\psi_1 - \beta_1 - \beta_2)}.$$

Therefore,

$$\theta(\omega) = \psi_1 - \beta_1 - \beta_2,$$

where $\psi_1 = 90^\circ$, $\beta_1 = \tan^{-1}(\omega/10)$, and $\beta_2 = \tan^{-1}(\omega/100)$. Figure E.9 depicts the straight-line approximation of $\theta(\omega)$.

b) We have

$$H(j50) = 0.96\underline{/-15.25^\circ},$$

$$H(j500) = 0.22\underline{/-77.54^\circ},$$

$$H(j1000) = 0.11\underline{/-83.72^\circ}.$$

Thus,

$$\theta(j50) = -15.25^\circ,$$

$$\theta(j500) = -77.54^\circ,$$

and

$$\theta(j1000) = -83.72^\circ.$$

c) See Fig. E.9.

d) We have

$$V_{mo} = |H(j500)|V_{mi}$$
$$= (0.22)(10)$$
$$= 2.2 \text{ V},$$

and

$$\theta_o = \theta(\omega) + \theta_i$$
$$= -77.54^\circ - 25^\circ$$
$$= -102.54^\circ.$$

Thus,

$$v_o(t) = 2.2\cos(500t - 102.54^\circ) \text{ V}.$$

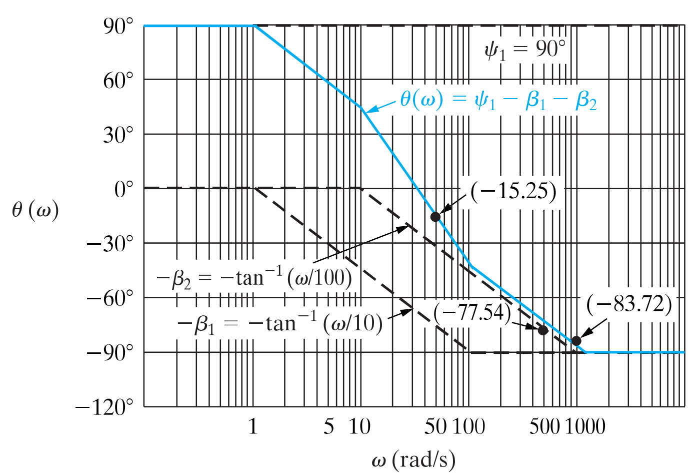

Figure E.9 ▲ A straight-line approximation of $\theta(\omega)$ for Example E.2.

E.5 Bode Diagrams: Complex Poles and Zeros

Complex poles and zeros in the expression for $H(s)$ require special attention when you make amplitude and phase angle plots. Let's focus on the contribution that a pair of complex poles makes to the amplitude and phase angle plots. Once you understand the rules for handling complex poles, their application to a pair of complex zeros becomes apparent.

The complex poles and zeros of $H(s)$ always appear in conjugate pairs. The first step in making either an amplitude or a phase angle plot of a transfer function that contains complex poles is to combine the conjugate pair into a single quadratic term. Thus, for

$$H(s) = \frac{K}{(s + \alpha - j\beta)(s + \alpha + j\beta)}, \tag{E.16}$$

we first rewrite the product $(s + \alpha - j\beta)(s + \alpha + j\beta)$ as

$$(s + \alpha)^2 + \beta^2 = s^2 + 2\alpha s + \alpha^2 + \beta^2. \tag{E.17}$$

When making Bode diagrams, we write the quadratic term in a more convenient form:

$$s^2 + 2\alpha s + \alpha^2 + \beta^2 = s^2 + 2\zeta\omega_n s + \omega_n^2. \tag{E.18}$$

A direct comparison of the two forms shows that

$$\omega_n^2 = \alpha^2 + \beta^2 \tag{E.19}$$

and

$$\zeta\omega_n = \alpha. \tag{E.20}$$

The term ω_n is the corner frequency of the quadratic factor, and ζ is the damping coefficient of the quadratic term. The critical value of ζ is 1. If $\zeta < 1$, the roots of the quadratic factor are complex, and we use Eq. E.18 to represent the complex poles. If $\zeta \geq 1$, we factor the quadratic factor into $(s + p_1)(s + p_2)$ and then plot amplitude and phase in accordance with the discussion previously. Assuming that $\zeta < 1$, we rewrite Eq. E.16 as

$$H(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{E.21}$$

We then write Eq. E.21 in standard form by dividing through by the poles and zeros. For the quadratic term, we divide through by ω_n, so

$$H(s) = \frac{K}{\omega_n^2}\frac{1}{1 + (s/\omega_n)^2 + 2\zeta(s/\omega_n)}, \tag{E.22}$$

from which

$$H(j\omega) = \frac{K_o}{1 - (\omega^2/\omega_n^2) + j(2\zeta\omega/\omega_n)}, \tag{E.23}$$

where

$$K_o = \frac{K}{\omega_n^2}.$$

Before discussing the amplitude and phase angle diagrams associated with Eq. E.23, for convenience we replace the ratio ω/ω_n by a new variable, u. Then

$$H(j\omega) = \frac{K_o}{1 - u^2 + j2\zeta u}. \tag{E.24}$$

Now we write $H(j\omega)$ in polar form:

$$H(j\omega) = \frac{K_o}{|(1 - u^2) + j2\zeta u|\underline{/\beta_1}}, \tag{E.25}$$

from which

$$A_{\text{dB}} = 20\log_{10}|H(j\omega)|$$
$$= 20\log_{10}K_o - 20\log_{10}|(1-u^2) + j2\zeta u|, \quad \text{(E.26)}$$

and

$$\theta(\omega) = -\beta_1 = -\tan^{-1}\frac{2\zeta u}{1-u^2}. \quad \text{(E.27)}$$

E.6 Amplitude Plots

The quadratic factor contributes to the amplitude of $H(j\omega)$ by means of the term $-20\log_{10}|1-u^2+j2\zeta u|$. Because $u=\omega/\omega_n$, $u \to 0$ as $\omega \to 0$, and $u \to \infty$ as $\omega \to \infty$. To see how the term behaves as ω ranges from 0 to ∞, we note that

$$-20\log_{10}|(1-u^2)+j2\zeta u| = -20\log_{10}\sqrt{(1-u^2)^2+4\zeta^2u^2}$$
$$= -10\log_{10}[u^4+2u^2(2\zeta^2-1)+1], \quad \text{(E.28)}$$

as $u \to 0$,

$$-10\log_{10}[u^4+2u^2(2\zeta^2-1)+1] \to 0, \quad \text{(E.29)}$$

and as $u \to \infty$,

$$-10\log_{10}[u^4+2u^2(2\zeta^2-1)+1] \to -40\log_{10}u. \quad \text{(E.30)}$$

From Eqs. E.29 and E.30, we conclude that the approximate amplitude plot consists of two straight lines. For $\omega < \omega_n$, the straight line lies along the 0 dB axis, and for $\omega > \omega_n$, the straight line has a slope of -40 dB/decade. These two straight lines join on the 0 dB axis at $u=1$ or $\omega=\omega_n$. Figure E.10 shows the straight-line approximation for a quadratic factor with $\zeta < 1$.

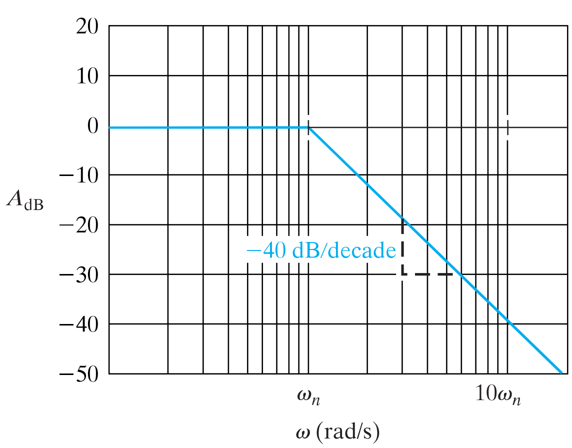

Figure E.10 ▲ The amplitude plot for a pair of complex poles.

E.7 Correcting Straight-Line Amplitude Plots

Correcting the straight-line amplitude plot for a pair of complex poles is not as easy as correcting a first-order real pole, because the corrections depend on the damping coefficient ζ. Figure E.11 shows the effect of ζ on the amplitude plot. Note that as ζ becomes very small, a large peak in the amplitude occurs in the neighborhood of the corner frequency $\omega_n(u = 1)$. When $\zeta \geq 1/\sqrt{2}$, the corrected amplitude plot lies entirely below the straight-line approximation. For sketching purposes, the straight-line amplitude plot can be corrected by locating four points on the actual curve. These four points correspond to (1) one half the corner frequency, (2) the frequency at which the amplitude reaches its peak value, (3) the corner frequency, and (4) the frequency at which the amplitude is zero. Figure E.12 shows these four points.

At one half the corner frequency (point 1), the actual amplitude is

$$A_{\mathrm{dB}}(\omega_n/2) = -10\log_{10}(\zeta^2 + 0.5625). \quad \text{(E.31)}$$

The amplitude peaks (point 2) at a frequency of

$$\omega_p = \omega_n\sqrt{1 - 2\zeta^2}, \quad \text{(E.32)}$$

and it has a peak amplitude of

$$A_{\mathrm{dB}}(\omega_p) = -10\log_{10}[4\zeta^2(1 - \zeta^2)]. \quad \text{(E.33)}$$

At the corner frequency (point 3), the actual amplitude is

$$A_{\mathrm{dB}}(\omega_n) = -20\log_{10}2\zeta. \quad \text{(E.34)}$$

The corrected amplitude plot crosses the 0 dB axis (point 4) at

$$\omega_o = \omega_n\sqrt{2(1 - 2\zeta^2)} = \sqrt{2}\omega_p. \quad \text{(E.35)}$$

The derivations of Eqs. E.31, E.34, and E.35 follow from Eq. E.28. Evaluating Eq. E.28 at $u = 0.5$ and $u = 1.0$, respectively, yields Eqs. E.31 and E.34 Equation E.35 corresponds to finding the value of u that makes $u^4 + 2u^2(2\zeta^2 - 1) + 1 = 1$. The derivation of Eq. E.32 requires differentiating Eq. E.28 with respect to u and then finding the value of u where the derivative is zero. Equation E.33 is the evaluation of Eq. E.28 at the value of u found in Eq. E.32.

Example E.3 illustrates the amplitude plot for a transfer function with a pair of complex poles.

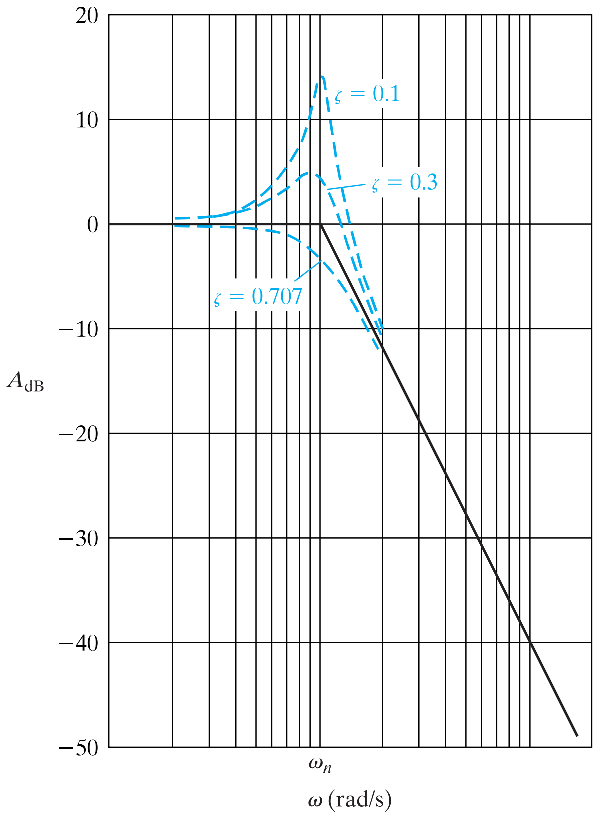

Figure E.11 ▲ The effect of ζ on the amplitude plot.

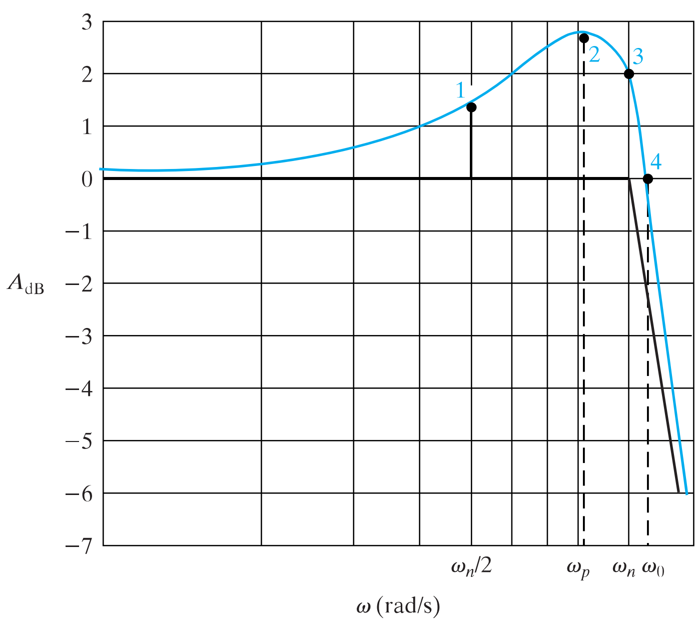

Figure E.12 ▲ Four points on the corrected amplitude plot for a pair of complex poles.

Example E.3

Compute the transfer function for the circuit shown in Fig. E.13.

a) What is the value of the corner frequency in radians per second?

b) What is the value of K_o?

c) What is the value of the damping coefficient?

d) Make a straight-line amplitude plot ranging from 10 to 500 rad/s.

e) Calculate and sketch the actual amplitude in decibels at $\omega_n/2, \omega_p, \omega_n$, and ω_o.

f) From the straight-line amplitude plot, describe the type of filter represented by the circuit in Fig. E.13 and estimate its cutoff frequency, ω_c.

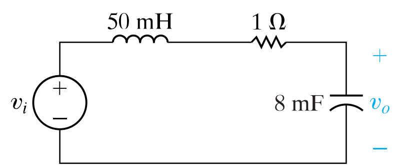

Figure E.13 ▲ The circuit for Example E.3.

Solution

Transform the circuit in Fig. E.13 to the s-domain and then use s-domain voltage division to get

$$H(s) = \frac{\frac{1}{LC}}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}}.$$

Substituting the component values,

$$H(s) = \frac{2500}{s^2 + 20s + 2500}.$$

a) From the expression for $H(s)$, $\omega_n^2 = 2500$; therefore, $\omega_n = 50$ rad/s.

b) By definition, K_o is $2500/\omega_n^2$, or 1.

c) The coefficient of s equals $2\zeta\omega_n$; therefore

$$\zeta = \frac{20}{2\omega_n} = 0.20.$$

d) See Fig. E.14.

e) The actual amplitudes are

$$A_{\text{dB}}(\omega_n/2) = -10\log_{10}(0.6025) = 2.2 \text{ dB},$$

$$\omega_p = 50\sqrt{0.92} = 47.96 \text{ rad/s},$$

$$A_{\text{dB}}(\omega_p) = -10\log_{10}(0.16)(0.96) = 8.14 \text{ dB},$$

$$A_{\text{dB}}(\omega_n) = -20\log_{10}(0.4) = 7.96 \text{ dB},$$

$$\omega_o = \sqrt{2}\omega_p = 67.82 \text{ rad/s},$$

$$A_{\text{dB}}(\omega_o) = 0 \text{ dB}.$$

Figure E.14 shows the corrected plot.

f) It is clear from the amplitude plot in Fig. E.14 that this circuit acts as a low-pass filter. At the cutoff frequency, the magnitude of the transfer function, $|H(j\omega_c)|$, is 3 dB less than the maximum magnitude. From the corrected plot, the cutoff frequency appears to be about 55 rad/s, almost the same as that predicted by the straight-line Bode diagram.

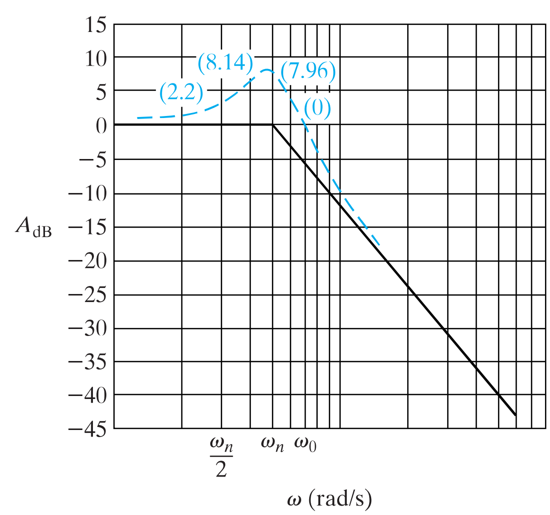

Figure E.14 ▲ The amplitude plot for Example E.3.

E.8 Phase Angle Plots

The phase angle plot for a pair of complex poles is a plot of Eq. E.27. The phase angle is zero at zero frequency and is -90° at the corner frequency. It approaches -180° as $\omega(u)$ becomes large. As in the case of the amplitude plot, ζ is important in determining the exact shape of the phase angle plot. For small values of ζ, the phase angle changes rapidly in the vicinity of the corner frequency. Figure E.15 shows the effect of ζ on the phase angle plot.

We can also make a straight-line approximation of the phase angle plot for a pair of complex poles. We do so by drawing a line tangent to the phase angle curve at the corner frequency and extending this line until it intersects with the 0° and -180° lines. The line tangent to the phase angle curve at -90° has a slope of $-2.3/\zeta$ rad/decade ($-132/\zeta$ degrees/decade), and it intersects the 0° and -180° lines at $u_1 = 4.81^{-\zeta}$ and $u_2 = 4.81^{\zeta}$, respectively. Figure E.16 depicts the straight-line approximation for $\zeta = 0.3$ and shows the actual phase angle plot. Comparing the straight-line approximation to the actual curve indicates that the approximation is reasonable in the vicinity of the corner frequency. However, in the neighborhood of u_1 and u_2, the error is quite large. In Example E.4, we summarize our discussion of Bode diagrams.

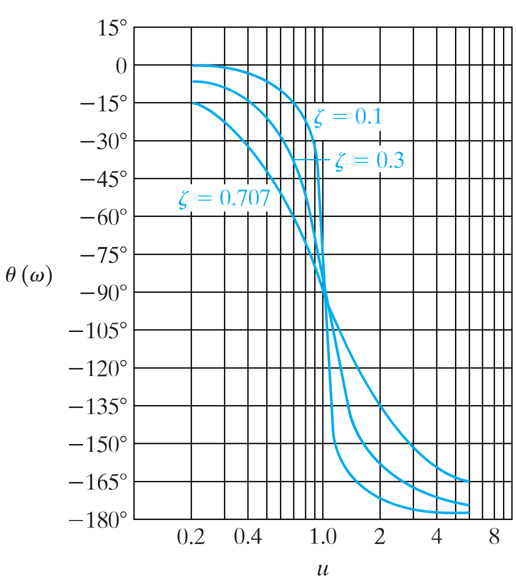

Figure E.15 ▲ The effect of ζ on the phase angle plot.

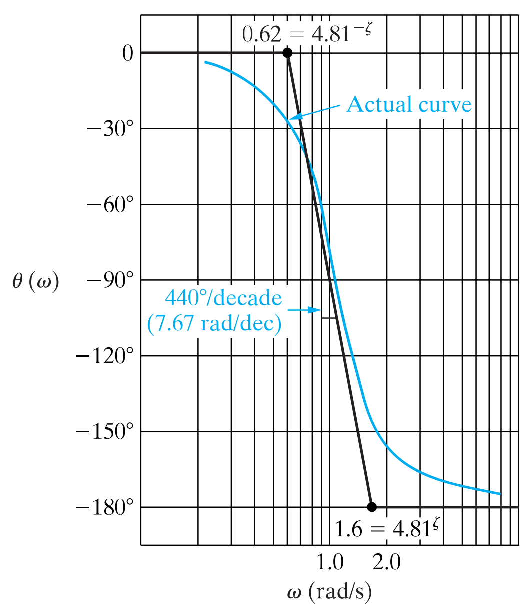

Figure E.16 ▲ A straight-line approximation of the phase angle for a pair of complex poles.

Example E.4

a) Compute the transfer function for the circuit shown in Fig. E.17.

b) Make a straight-line amplitude plot of $20\log_{10}|H(j\omega)|$.

c) Use the straight-line amplitude plot to determine the type of filter represented by this circuit and then estimate its cutoff frequency.

d) What is the actual cutoff frequency?

e) Make a straight-line phase angle plot of $H(j\omega)$.

f) What is the value of $\theta(\omega)$ at the cutoff frequency from (c)?

g) What is the actual value of $\theta(\omega)$ at the cutoff frequency?

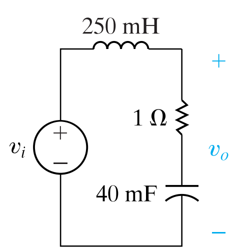

Figure E.17 ▲ The circuit for Example E.4.

Solution

a) Transform the circuit in Fig. E.17 to the s-domain and then perform s-domain voltage division to get

$$H(s) = \frac{\frac{R}{L}s + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.$$

Substituting the component values from the circuit gives

$$H(s) = \frac{4(s + 25)}{s^2 + 4s + 100}.$$

b) The first step in making Bode diagrams is to put $H(j\omega)$ in standard form. Because $H(s)$ contains a quadratic factor, we first check the value of ζ. We find that $\zeta = 0.2$ and $\omega_n = 10$, so

$$H(s) = \frac{s/25 + 1}{1 + (s/10)^2 + 0.4(s/10)},$$

from which

$$H(j\omega) = \frac{|1 + j\omega/25|\underline{/\psi_1}}{|1 - (\omega/10)^2 + j0.4(\omega/10)|\underline{/\beta_1}}.$$

Note that for the quadratic factor, $u = \omega/10$. The amplitude of $H(j\omega)$ in decibels is

$$A_{\text{dB}} = 20\log_{10}|1 + j\omega/25| - 20\log_{10}\left[\left|1 - \left(\frac{\omega}{10}\right)^2 + j0.4\left(\frac{\omega}{10}\right)\right|\right],$$

and the phase angle is

$$\theta(\omega) = \psi_1 - \beta_1,$$

where

$$\psi_1 = \tan^{-1}(\omega/25),$$

$$\beta_1 = \tan^{-1}\frac{0.4(\omega/10)}{1 - (\omega/10)^2}.$$

Figure E.18 shows the amplitude plot.

c) From the straight-line amplitude plot in Fig. E.18, this circuit acts as a low-pass filter. At the cutoff frequency, the amplitude of $H(j\omega)$ is 3 dB less than the amplitude in the passband. From the plot, we predict that the cutoff frequency is approximately 13 rad/s.

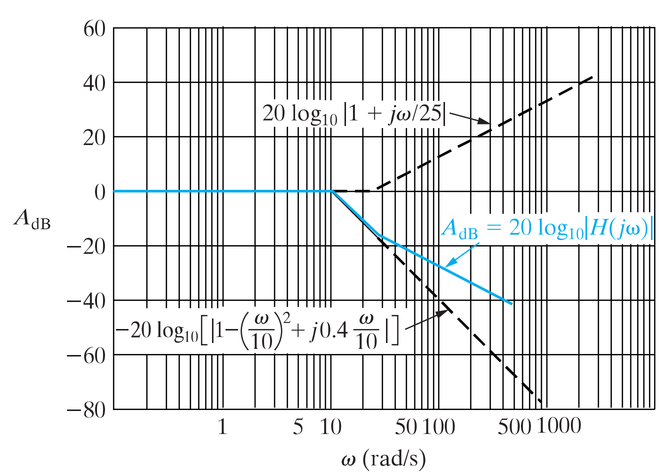

Figure E.18 ▲ The amplitude plot for Example E.4.

d) To solve for the actual cutoff frequency, replace s with $j\omega$ in $H(s)$, compute the expression for $|H(j\omega)|$, set $|H(j\omega_c)| = (1/\sqrt{2})\ H_{max} = 1/\sqrt{2}$, and solve for ω_c. First,

$$H(j\omega) = \frac{4(j\omega) + 100}{(j\omega)^2 + 4(j\omega) + 100}.$$

Then,

$$|H(j\omega_c)| = \frac{\sqrt{(4\omega_c)^2 + 100^2}}{\sqrt{(100 - \omega_c^2)^2 + (4\omega_c)^2}} = \frac{1}{\sqrt{2}}.$$

Solving for ω_c gives us

$$\omega_c = 16 \text{ rad/s}.$$

e) Figure E.19 shows the phase angle plot. Note that the straight-line segment of $\theta(\omega)$ between 1.0 and 2.5 rad/s does not have the same slope as the segment between 2.5 and 100 rad/s.

f) From the phase angle plot in Fig. E.19, we estimate the phase angle at the cutoff frequency of 16 rad/s to be -65°.

g) We can compute the exact phase angle at the cutoff frequency by substituting $s = j16$ into the transfer function $H(s)$:

$$H(j16) = \frac{4(j16 + 25)}{(j16)^2 + 4(j16) + 100}.$$

Computing the phase angle, we see

$$\theta(\omega_c) = \theta(j16) = -125.0^\circ.$$

Note the large error in the predicted angle. In general, straight-line phase angle plots do not give satisfactory results in the frequency band where the phase angle is changing. The straight-line phase angle plot is useful only in predicting the general behavior of the phase angle, not in estimating actual phase angle values at particular frequencies.

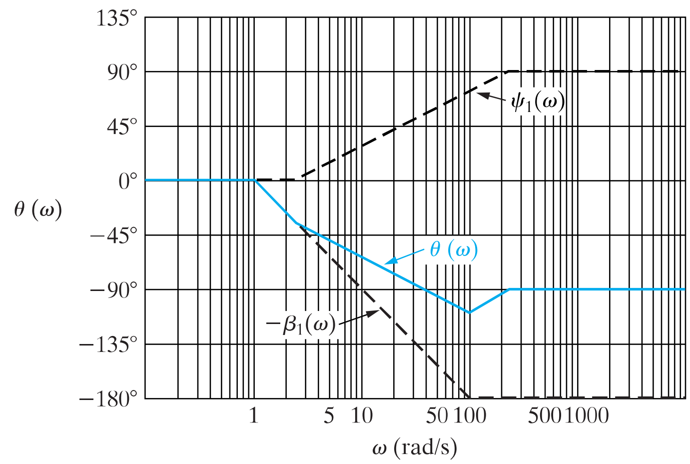

Figure E.19 ▲ The phase angle plot for Example E.4.