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# SOLUTIONS TO SKILL-ASSESSMENT EXERCISES

## CHAPTER 2

### 2.1

The Laplace transform of  $t$  is  $\frac{1}{s^2}$  using Table 2.1, Item 3. Using Table 2.2, Item 4,  
$$F(s) = \frac{1}{(s+5)^2}.$$

### 2.2

Expanding  $F(s)$  by partial fractions yields:

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+3)^2} + \frac{D}{s+3}$$

where,

$$A = \frac{10}{(s+2)(s+3)^2} \Big|_{s \rightarrow 0} = \frac{5}{9} \quad B = \frac{10}{s(s+3)^2} \Big|_{s \rightarrow -2} = -5$$
$$C = \frac{10}{s(s+2)} \Big|_{s \rightarrow -3} = \frac{10}{3}, \text{ and } D = (s+3)^2 \frac{dF(s)}{ds} \Big|_{s \rightarrow -3} = \frac{40}{9}$$

Taking the inverse Laplace transform yields,

$$f(t) = \frac{5}{9} - 5e^{-2t} + \frac{10}{3}te^{-3t} + \frac{40}{9}e^{-3t}$$

### 2.3

Taking the Laplace transform of the differential equation assuming zero initial conditions yields:

$$s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$$

Collecting terms,

$$(s^3 + 3s^2 + 7s + 5)C(s) = (s^2 + 4s + 3)R(s)$$

Thus,

$$\frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

## 2.4

$$G(s) = \frac{C(s)}{R(s)} = \frac{2s + 1}{s^2 + 6s + 2}$$

Cross multiplying yields,

$$\frac{d^2c}{dt^2} + 6\frac{dc}{dt} + 2c = 2\frac{dr}{dt} + r$$

## 2.5

$$C(s) = R(s)G(s) = \frac{1}{s^2} * \frac{s}{(s+4)(s+8)} = \frac{1}{s(s+4)(s+8)} = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+8)}$$

where

$$A = \frac{1}{(s+4)(s+8)} \Big|_{s \rightarrow 0} = \frac{1}{32} \quad B = \frac{1}{s(s+8)} \Big|_{s \rightarrow -4} = -\frac{1}{16}, \text{ and}$$

$$C = \frac{1}{s(s+4)} \Big|_{s \rightarrow -8} = \frac{1}{32}$$

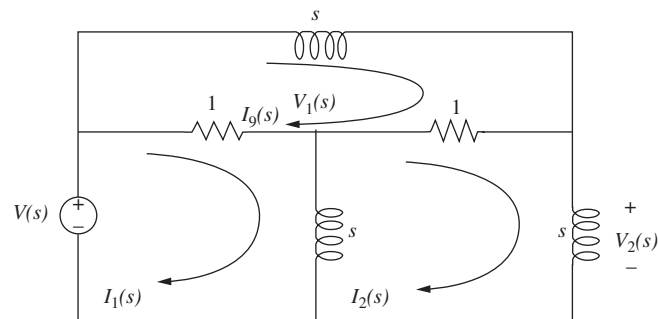
Thus,

$$c(t) = \frac{1}{32} - \frac{1}{16}e^{-4t} + \frac{1}{32}e^{-8t}$$

## 2.6

### Mesh Analysis

Transforming the network yields,



Now, writing the mesh equations,

$$\begin{aligned}(s+1)I_1(s) - sI_2(s) - I_3(s) &= V(s) \\ -sI_1(s) + (2s+1)I_2(s) - I_3(s) &= 0 \\ -I_1(s) - I_2(s) + (s+2)I_3(s) &= 0\end{aligned}$$

Solving the mesh equations for  $I_2(s)$ ,

$$I_2(s) = \frac{\begin{vmatrix} (s+1) & V(s) & -1 \\ -s & 0 & -1 \\ -1 & 0 & (s+2) \end{vmatrix}}{\begin{vmatrix} (s+1) & -s & -1 \\ -s & (2s+1) & -1 \\ -1 & -1 & (s+2) \end{vmatrix}} = \frac{(s^2 + 2s + 1)V(s)}{s(s^2 + 5s + 2)}$$

But,  $V_L(s) = sI_2(s)$

Hence,

$$V_L(s) = \frac{(s^2 + 2s + 1)V(s)}{(s^2 + 5s + 2)}$$

or

$$\frac{V_L(s)}{V(s)} = \frac{s^2 + 2s + 1}{s^2 + 5s + 2}$$

### Nodal Analysis

Writing the nodal equations,

$$\begin{aligned}\left(\frac{1}{s} + 2\right)V_1(s) - V_L(s) &= V(s) \\ -V_1(s) + \left(\frac{2}{s} + 1\right)V_L(s) &= \frac{1}{s}V(s)\end{aligned}$$

Solving for  $V_L(s)$ ,

$$V_L(s) = \frac{\begin{vmatrix} \left(\frac{1}{s} + 2\right) & V(s) \\ -1 & \frac{1}{s}V(s) \end{vmatrix}}{\begin{vmatrix} \left(\frac{1}{s} + 2\right) & -1 \\ -1 & \left(\frac{2}{s} + 1\right) \end{vmatrix}} = \frac{(s^2 + 2s + 1)V(s)}{(s^2 + 5s + 2)}$$

or

$$\frac{V_L(s)}{V(s)} = \frac{s^2 + 2s + 1}{s^2 + 5s + 2}$$

## 2.7

### Inverting

$$G(s) = -\frac{Z_2(s)}{Z_1(s)} = \frac{-100000}{(10^5/s)} = -s$$

**Noninverting**

$$G(s) = \frac{[Z_1(s) + Z_2(s)]}{Z_1(s)} = \frac{\left(\frac{10^5}{s} + 10^5\right)}{\left(\frac{10^5}{s}\right)} = s + 1$$

**2.8**

Writing the equations of motion,

$$(s^2 + 3s + 1)X_1(s) - (3s + 1)X_2(s) = F(s)$$

$$-(3s + 1)X_1(s) + (s^2 + 4s + 1)X_2(s) = 0$$

Solving for  $X_2(s)$ ,

$$X_2(s) = \frac{\begin{vmatrix} (s^2 + 3s + 1) & F(s) \\ -(3s + 1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2 + 3s + 1) & -(3s + 1) \\ -(3s + 1) & (s^2 + 4s + 1) \end{vmatrix}} = \frac{(3s + 1)F(s)}{s(s^3 + 7s^2 + 5s + 1)}$$

Hence,

$$\frac{X_2(s)}{F(s)} = \frac{(3s + 1)}{s(s^3 + 7s^2 + 5s + 1)}$$

**2.9**

Writing the equations of motion,

$$(s^2 + s + 1)\theta_1(s) - (s + 1)\theta_2(s) = T(s)$$

$$-(s + 1)\theta_1(s) + (2s + 2)\theta_2(s) = 0$$

where  $\theta_1(s)$  is the angular displacement of the inertia.

Solving for  $\theta_2(s)$ ,

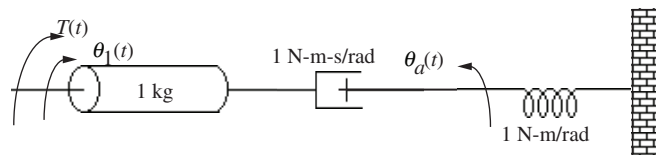
$$\theta_2(s) = \frac{\begin{vmatrix} (s^2 + s + 1) & T(s) \\ -(s + 1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2 + s + 1) & -(s + 1) \\ -(s + 1) & (2s + 2) \end{vmatrix}} = \frac{(s + 1)F(s)}{2s^3 + 3s^2 + 2s + 1}$$

From which, after simplification,

$$\theta_2(s) = \frac{1}{2s^2 + s + 1}$$

**2.10**

Transforming the network to one without gears by reflecting the 4 N-m/rad spring to the left and multiplying by  $(25/50)^2$ , we obtain,



Writing the equations of motion,

$$\begin{aligned}(s^2 + s)\theta_1(s) - s\theta_a(s) &= T(s) \\ -s\theta_1(s) + (s + 1)\theta_a(s) &= 0\end{aligned}$$

where  $\theta_1(s)$  is the angular displacement of the 1-kg inertia.

Solving for  $\theta_a(s)$ ,

$$\theta_a(s) = \frac{\begin{vmatrix} (s^2 + s) & T(s) \\ -s & 0 \end{vmatrix}}{\begin{vmatrix} (s^2 + s) & -s \\ -s & (s + 1) \end{vmatrix}} = \frac{sT(s)}{s^3 + s^2 + s}$$

From which,

$$\frac{\theta_a(s)}{T(s)} = \frac{1}{s^2 + s + 1}$$

But,  $\theta_2(s) = \frac{1}{2}\theta_a(s)$ .

Thus,

$$\frac{\theta_2(s)}{T(s)} = \frac{1/2}{s^2 + s + 1}$$

## 2.11

First find the mechanical constants.

$$J_m = J_a + J_L \left( \frac{1}{5} * \frac{1}{4} \right)^2 = 1 + 400 \left( \frac{1}{400} \right) = 2$$

$$D_m = D_a + D_L \left( \frac{1}{5} * \frac{1}{4} \right)^2 = 5 + 800 \left( \frac{1}{400} \right) = 7$$

Now find the electrical constants. From the torque-speed equation, set  $\omega_m = 0$  to find stall torque and set  $T_m = 0$  to find no-load speed. Hence,

$$\begin{aligned}T_{stall} &= 200 \\ \omega_{no-load} &= 25\end{aligned}$$

which,

$$\begin{aligned}\frac{K_t}{R_a} &= \frac{T_{stall}}{E_a} = \frac{200}{100} = 2 \\ K_b &= \frac{E_a}{\omega_{no-load}} = \frac{100}{25} = 4\end{aligned}$$

Substituting all values into the motor transfer function,

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_T}{R_a J_m}}{s \left( s + \frac{1}{J_m} \right) \left( D_m + \frac{K_T K_b}{R_a} \right)} = \frac{1}{s \left( s + \frac{15}{2} \right)}$$

where  $\theta_m(s)$  is the angular displacement of the armature.

Now  $\theta_L(s) = \frac{1}{20}\theta_m(s)$ . Thus,

$$\frac{\theta_L(s)}{E_a(s)} = \frac{1/20}{s \left( s + \frac{15}{2} \right)}$$

## 2.12

Letting

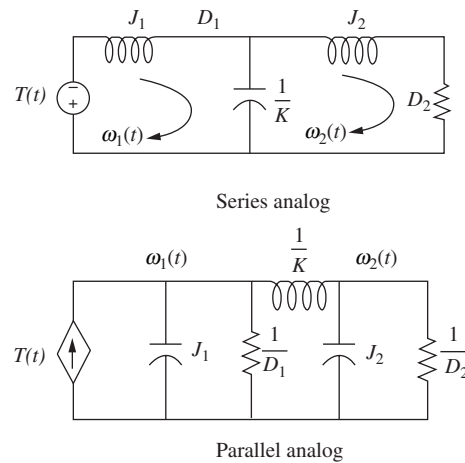
$$\theta_1(s) = \omega_1(s)/s$$

$$\theta_2(s) = \omega_2(s)/s$$

in Eqs. 2.127, we obtain

$$\begin{aligned} \left( J_1 s + D_1 + \frac{K}{s} \right) \omega_1(s) - \frac{K}{s} \omega_2(s) &= T(s) \\ -\frac{K}{s} \omega_1(s) + \left( J_2 s + D_2 + \frac{K}{s} \right) \omega_2(s) &= 0 \end{aligned}$$

From these equations we can draw both series and parallel analogs by considering these to be mesh or nodal equations, respectively.



## 2.13

Writing the nodal equation,

$$C \frac{dv}{dt} + i_r - 2 = i(t)$$

But,

$$\begin{aligned} C &= 1 \\ v &= v_o + \delta v \\ i_r &= e^{v_r} = e^v = e^{v_o + \delta v} \end{aligned}$$

Substituting these relationships into the differential equation,

$$\frac{d(v_o + \delta v)}{dt} + e^{v_o + \delta v} - 2 = i(t) \quad (1)$$

We now linearize  $e^v$ .

The general form is

$$f(v) - f(v_o) \approx \left. \frac{df}{dv} \right|_{v_o} \delta v$$

Substituting the function,  $f(v) = e^v$ , with  $v = v_o + \delta v$  yields,

$$e^{v_o+\delta v} - e^{v_o} \approx \left. \frac{de^v}{dv} \right|_{v_o} \delta v$$

Solving for  $e^{v_o+\delta v}$ ,

$$e^{v_o+\delta v} = e^{v_o} + \left. \frac{de^v}{dv} \right|_{v_o} \delta v = e^{v_o} + e^{v_o} \delta v$$

Substituting into Eq. (1)

$$\frac{d\delta v}{dt} + e^{v_o} + e^{v_o} \delta v - 2 = i(t) \quad (2)$$

Setting  $i(t) = 0$  and letting the circuit reach steady state, the capacitor acts like an open circuit. Thus,  $v_o = v_r$  with  $i_r = 2$ . But,  $i_r = e^{v_r}$  or  $v_r = \ln i_r$ .

Hence,  $v_o = \ln 2 = 0.693$ . Substituting this value of  $v_o$  into Eq. (2) yields

$$\frac{d\delta v}{dt} + 2\delta v = i(t)$$

Taking the Laplace transform,

$$(s + 2)\delta v(s) = I(s)$$

Solving for the transfer function, we obtain

$$\frac{\delta v(s)}{I(s)} = \frac{1}{s + 2}$$

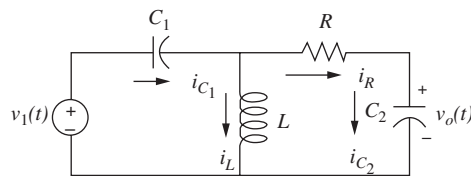
or

$$\frac{V(s)}{I(s)} = \frac{1}{s + 2} \text{ about equilibrium.}$$

## CHAPTER 3

### 3.1

Identifying appropriate variables on the circuit yields



Writing the derivative relations

$$C_1 \frac{dv_{C_1}}{dt} = i_{C_1}$$

$$L \frac{di_L}{dt} = v_L$$

$$C_2 \frac{dv_{C_2}}{dt} = i_{C_2}$$

(1)

Using Kirchhoff's current and voltage laws,

$$\begin{aligned}i_{C_1} &= i_L + i_R = i_L + \frac{1}{R}(v_L - v_{C_2}) \\v_L &= -v_{C_1} + v_i \\i_{C_2} &= i_R = \frac{1}{R}(v_L - v_{C_2})\end{aligned}$$

Substituting these relationships into Eqs. (1) and simplifying yields the state equations as

$$\begin{aligned}\frac{dv_{C_1}}{dt} &= -\frac{1}{RC_1}v_{C_1} + \frac{1}{C_1}i_L - \frac{1}{RC_1}v_{C_2} + \frac{1}{RC_1}v_i \\ \frac{di_L}{dt} &= -\frac{1}{L}v_{C_1} + \frac{1}{L}v_i \\ \frac{dv_{C_2}}{dt} &= -\frac{1}{RC_2}v_{C_1} - \frac{1}{RC_2}v_{C_2} + \frac{1}{RC_2}v_i\end{aligned}$$

where the output equation is

$$v_o = v_{C_2}$$

Putting the equations in vector-matrix form,

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{C_1} & -\frac{1}{RC_1} \\ -\frac{1}{L} & 0 & 0 \\ -\frac{1}{RC_2} & 0 & -\frac{1}{RC_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{L} \\ \frac{1}{RC_2} \end{bmatrix} v_i(t) \\ y &= [0 \quad 0 \quad 1] \mathbf{x}\end{aligned}$$

### 3.2

Writing the equations of motion

$$\begin{aligned}(s^2 + s + 1)X_1(s) - sX_2(s) &= F(s) \\ -sX_1(s) + (s^2 + s + 1)X_2(s) - X_3(s) &= 0 \\ -X_2(s) + (s^2 + s + 1)X_3(s) &= 0\end{aligned}$$

Taking the inverse Laplace transform and simplifying,

$$\begin{aligned}\ddot{x}_1 &= -\dot{x}_1 - x_1 + \dot{x}_2 + f \\ \ddot{x}_2 &= \dot{x}_1 - \dot{x}_2 - x_2 + x_3 \\ \ddot{x}_3 &= -\dot{x}_3 - x_3 + x_2\end{aligned}$$

Defining state variables,  $z_i$ ,

$$z_1 = x_1; z_2 = \dot{x}_1; z_3 = x_2; z_4 = \dot{x}_2; z_5 = x_3; z_6 = \dot{x}_3$$

Writing the state equations using the definition of the state variables and the inverse transform of the differential equation,

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \ddot{x}_1 = -\dot{x}_1 - x_1 + \dot{x}_2 + f = -z_2 - z_1 + z_4 + f \\ \dot{z}_3 &= \dot{x}_2 = z_4 \\ \dot{z}_4 &= \ddot{x}_2 = \dot{x}_1 - \dot{x}_2 - x_2 + x_3 = z_2 - z_4 - z_3 + z_5 \\ \dot{z}_5 &= \dot{x}_3 = z_6 \\ \dot{z}_6 &= \ddot{x}_3 = -\dot{x}_3 - x_3 + x_2 = -z_6 - z_5 + z_3\end{aligned}$$



The output is  $z_5$ . Hence,  $y = z_5$ . In vector-matrix form,

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(t); y = [0 \ 0 \ 0 \ 0 \ 1 \ 0] \mathbf{z}$$

### 3.3

First derive the state equations for the transfer function without zeros.

$$\frac{X(s)}{R(s)} = \frac{1}{s^2 + 7s + 9}$$

Cross multiplying yields

$$(s^2 + 7s + 9)X(s) = R(s)$$

Taking the inverse Laplace transform assuming zero initial conditions, we get

$$\ddot{x} + 7\dot{x} + 9x = r$$

Defining the state variables as,

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Hence,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{x} = -7\dot{x} - 9x + r = -9x_1 - 7x_2 + r \end{aligned}$$

Using the zeros of the transfer function, we find the output equation to be,

$$c = 2\dot{x} + x = x_1 + 2x_2$$

Putting all equation in vector-matrix form yields,

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -9 & -7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \\ c &= [1 \ 2] \mathbf{x} \end{aligned}$$

### 3.4

The state equation is converted to a transfer function using

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \quad (1)$$

where

$$\mathbf{A} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{C} = [1.5 \ 0.625].$$

Evaluating  $(s\mathbf{I} - \mathbf{A})$  yields

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s + 4 & 1.5 \\ -4 & s \end{bmatrix}$$

Taking the inverse we obtain

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 4s + 6} \begin{bmatrix} s & -1.5 \\ 4 & s + 4 \end{bmatrix}$$

Substituting all expressions into Eq. (1) yields

$$G(s) = \frac{3s + 5}{s^2 + 4s + 6}$$

### 3.5

Writing the differential equation we obtain

$$\frac{d^2x}{dt^2} + 2x^2 = 10 + \delta f(t) \quad (1)$$

Letting  $x = x_o + \delta x$  and substituting into Eq. (1) yields

$$\frac{d^2(x_o + \delta x)}{dt^2} + 2(x_o + \delta x)^2 = 10 + \delta f(t) \quad (2)$$

Now, linearize  $x^2$ .

$$(x_o + \delta x)^2 - x_o^2 = \left. \frac{d(x^2)}{dx} \right|_{x_o} \delta x = 2x_o \delta x$$

from which

$$(x_o + \delta x)^2 = x_o^2 + 2x_o \delta x \quad (3)$$

Substituting Eq. (3) into Eq. (1) and performing the indicated differentiation gives us the linearized intermediate differential equation,

$$\frac{d^2 \delta x}{dt^2} + 4x_o \delta x = -2x_o^2 + 10 + \delta f(t) \quad (4)$$

The force of the spring at equilibrium is 10 N. Thus, since  $F = 2x^2$ ,  $10 = 2x_o^2$  from which

$$x_o = \sqrt{5}$$

Substituting this value of  $x_o$  into Eq. (4) gives us the final linearized differential equation.

$$\frac{d^2 \delta x}{dt^2} + 4\sqrt{5} \delta x = \delta f(t)$$

Selecting the state variables,

$$\begin{aligned} x_1 &= \delta x \\ x_2 &= \dot{\delta x} \end{aligned}$$

Writing the state and output equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{\delta x} = -4\sqrt{5}x_1 + \delta f(t) \\ y &= x_1 \end{aligned}$$

Converting to vector-matrix form yields the final result as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4\sqrt{5} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta f(t)$$

$$y = [1 \quad 0] \mathbf{x}$$

## CHAPTER 4

### 4.1

For a step input

$$C(s) = \frac{10(s+4)(s+6)}{s(s+1)(s+7)(s+8)(s+10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+7} + \frac{D}{s+8} + \frac{E}{s+10}$$

Taking the inverse Laplace transform,

$$c(t) = A + Be^{-t} + Ce^{-7t} + De^{-8t} + Ee^{-10t}$$

### 4.2

Since  $a = 50$ ,  $T_c = \frac{1}{a} = \frac{1}{50} = 0.02$  s;  $T_s = \frac{4}{a} = \frac{4}{50} = 0.08$  s; and  $T_r = \frac{2.2}{a} = \frac{2.2}{50} = 0.044$  s.

### 4.3

- Since poles are at  $-6 \pm j19.08$ ,  $c(t) = A + Be^{-6t} \cos(19.08t + \phi)$ .
- Since poles are at  $-78.54$  and  $-11.46$ ,  $c(t) = A + Be^{-78.54t} + Ce^{-11.4t}$ .
- Since poles are double on the real axis at  $-15$ ,  $c(t) = A + Be^{-15t} + Cte^{-15t}$ .
- Since poles are at  $\pm j25$ ,  $c(t) = A + B \cos(25t + \phi)$ .

### 4.4

- $\omega_n = \sqrt{400} = 20$  and  $2\zeta\omega_n = 12$ ;  $\therefore \zeta = 0.3$  and system is underdamped.
- $\omega_n = \sqrt{900} = 30$  and  $2\zeta\omega_n = 90$ ;  $\therefore \zeta = 1.5$  and system is overdamped.
- $\omega_n = \sqrt{225} = 15$  and  $2\zeta\omega_n = 30$ ;  $\therefore \zeta = 1$  and system is critically damped.
- $\omega_n = \sqrt{625} = 25$  and  $2\zeta\omega_n = 0$ ;  $\therefore \zeta = 0$  and system is undamped.

### 4.5

$$\omega_n = \sqrt{361} = 19 \text{ and } 2\zeta\omega_n = 16; \quad \therefore \zeta = 0.421.$$

$$\text{Now, } T_s = \frac{4}{\zeta\omega_n} = 0.5 \text{ s and } T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 0.182 \text{ s.}$$

From Figure 4.16,  $\omega_n T_r = 1.4998$ . Therefore,  $T_r = 0.079$  s.

$$\text{Finally, } \%os = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} * 100 = 23.3\%$$

## 4.6

- a. The second-order approximation is valid, since the dominant poles have a real part of  $-2$  and the higher-order pole is at  $-15$ , i.e. more than five-times further.
- b. The second-order approximation is not valid, since the dominant poles have a real part of  $-1$  and the higher-order pole is at  $-4$ , i.e. not more than five-times further.

## 4.7

- a. Expanding  $G(s)$  by partial fractions yields  $G(s) = \frac{1}{s} + \frac{0.8942}{s+20} - \frac{1.5918}{s+10} - \frac{0.3023}{s+6.5}$ .

But  $-0.3023$  is not an order of magnitude less than residues of second-order terms (term 2 and 3). Therefore, a second-order approximation is not valid.

- b. Expanding  $G(s)$  by partial fractions yields  $G(s) = \frac{1}{s} + \frac{0.9782}{s+20} - \frac{1.9078}{s+10} - \frac{0.0704}{s+6.5}$ .

But  $0.0704$  is an order of magnitude less than residues of second-order terms (term 2 and 3). Therefore, a second-order approximation is valid.

## 4.8

See Figure 4.31 in the textbook for the Simulink block diagram and the output responses.

## 4.9

- a. Since  $s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}$ ,  $(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}$ . Also,  $\mathbf{B}\mathbf{U}(s) = \begin{bmatrix} 0 \\ 1/(s+1) \end{bmatrix}$ .

The state vector is  $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)] = \frac{1}{(s+1)(s+2)(s+3)} \times \begin{bmatrix} 2(s^2 + 7s + 7) \\ s^2 - 4s - 6 \end{bmatrix}$ . The output is  $Y(s) = [1 \quad 3]\mathbf{X}(s) = \frac{5s^2 + 2s - 4}{(s+1)(s+2)(s+3)} = -\frac{0.5}{s+1} - \frac{12}{s+2} + \frac{17.5}{s+3}$ . Taking the inverse Laplace transform yields  $y(t) = -0.5e^{-t} - 12e^{-2t} + 17.5e^{-3t}$ .

- b. The eigenvalues are given by the roots of  $|s\mathbf{I} - \mathbf{A}| = s^2 + 5s + 6$ , or  $-2$  and  $-3$ .

## 4.10

- a. Since  $(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -2 \\ 2 & s+5 \end{bmatrix}$ ,  $(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 5s + 4} \begin{bmatrix} s+5 & 2 \\ -2 & s \end{bmatrix}$ . Taking the

Laplace transform of each term, the state transition matrix is given by

$$\Phi(t) = \begin{bmatrix} \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{-4t} & -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{bmatrix}$$

- b. Since  $\Phi(t - \tau) = \begin{bmatrix} \frac{4}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)} & \frac{2}{3}e^{-(t-\tau)} - \frac{2}{3}e^{-4(t-\tau)} \\ -\frac{2}{3}e^{-(t-\tau)} + \frac{2}{3}e^{-4(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)} + \frac{4}{3}e^{-4(t-\tau)} \end{bmatrix}$  and

$$\mathbf{B}\mathbf{u}(\tau) = \begin{bmatrix} 0 \\ e^{-2\tau} \end{bmatrix}, \quad \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau) = \begin{bmatrix} \frac{2}{3}e^{-\tau}e^{-t} - \frac{2}{3}e^{2\tau}e^{-4t} \\ -\frac{1}{3}e^{-\tau}e^{-t} + \frac{4}{3}e^{2\tau}e^{-4t} \end{bmatrix}.$$

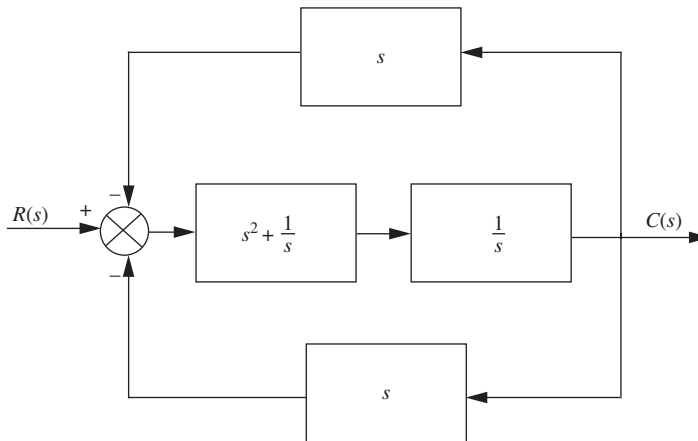
$$\text{Thus, } \mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau) \mathbf{B}\mathbf{u}(\tau) d\tau = \begin{bmatrix} \frac{10}{3}e^{-t} - e^{-2t} - \frac{4}{3}e^{-4t} \\ -\frac{5}{3}e^{-t} + e^{-2t} + \frac{8}{3}e^{-4t} \end{bmatrix}.$$

$$\text{c. } y(t) = [2 \quad 1]\mathbf{x} = 5e^{-t} - e^{-2t}$$

## CHAPTER 5

### 5.1

Combine the parallel blocks in the forward path. Then, push  $\frac{1}{s}$  to the left past the pickoff point.



Combine the parallel feedback paths and get  $2s$ . Then, apply the feedback formula, simplify, and get,  $T(s) = \frac{s^3 + 1}{2s^4 + s^2 + 2s}$ .

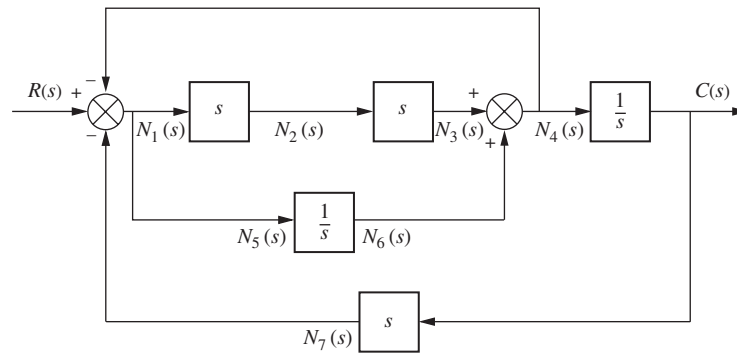
### 5.2

Find the closed-loop transfer function,  $T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{16}{s^2 + as + 16}$ , where and  $G(s) = \frac{16}{s(s+a)}$  and  $H(s) = 1$ . Thus,  $\omega_n = 4$  and  $2\zeta\omega_n = a$ , from which  $\zeta = \frac{a}{8}$ .

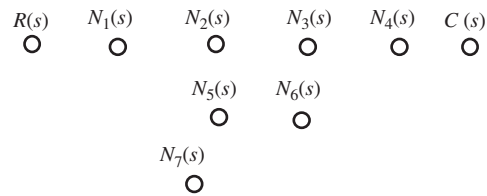
But, for 5% overshoot,  $\zeta = \frac{-\ln\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%}{100}\right)}} = 0.69$ . Since,  $\zeta = \frac{a}{8}$ ,  $a = 5.52$ .

5.3

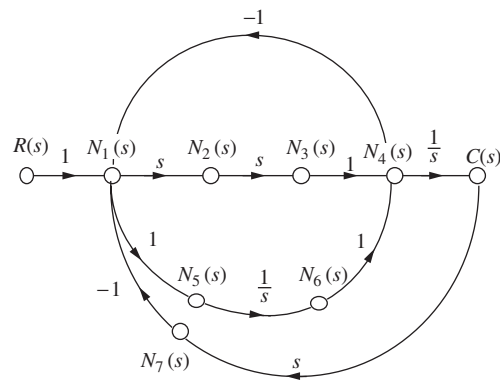
Label nodes.



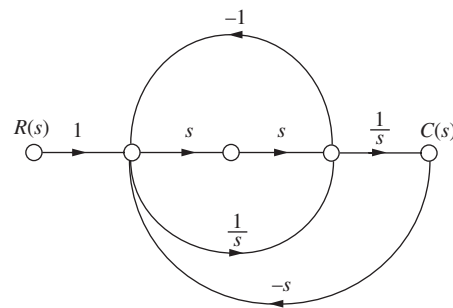
Draw nodes.



Connect nodes and label subsystems.



Eliminate unnecessary nodes.



## 5.4

Forward-path gains are  $G_1G_2G_3$  and  $G_1G_3$ .

Loop gains are  $-G_1G_2H_1$ ,  $-G_2H_2$ , and  $-G_3H_3$ .

Nontouching loops are  $[-G_1G_2H_1][-G_3H_3] = G_1G_2G_3H_1H_3$  and

$[-G_2H_2][-G_3H_3] = G_2G_3H_2H_3$ .

Also,  $\Delta = 1 + G_1G_2H_1 + G_2H_2 + G_3H_3 + G_1G_2G_3H_1H_3 + G_2G_3H_2H_3$ .

Finally,  $\Delta_1 = 1$  and  $\Delta_2 = 1$ .

Substituting these values into  $T(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$  yields

$$T(s) = \frac{G_1(s)G_3(s)[1 + G_2(s)]}{[1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)][1 + G_3(s)H_3(s)]}$$

## 5.5

The state equations are,

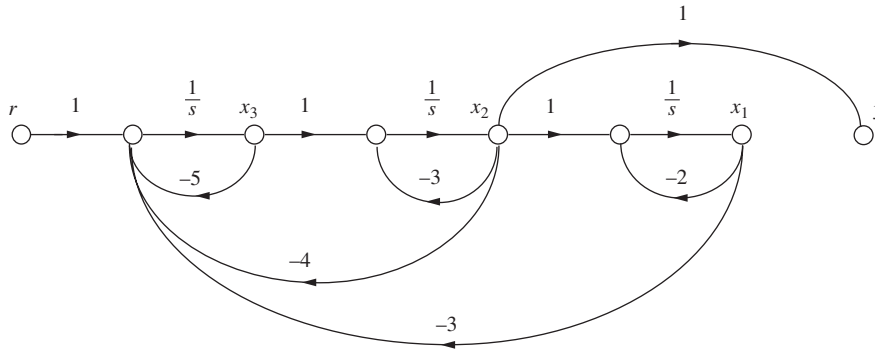
$$\dot{x}_1 = -2x_1 + x_2$$

$$\dot{x}_2 = -3x_2 + x_3$$

$$\dot{x}_3 = -3x_1 - 4x_2 - 5x_3 + r$$

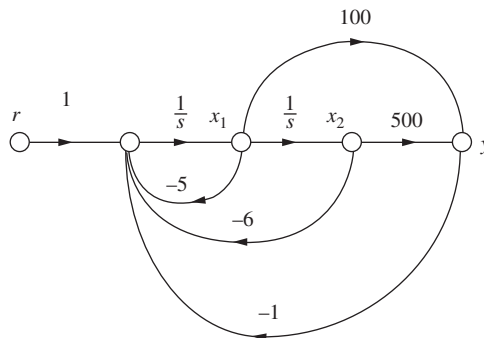
$$y = x_2$$

Drawing the signal-flow diagram from the state equations yields



## 5.6

From  $G(s) = \frac{100(s+5)}{s^2+5s+6}$  we draw the signal-flow graph in controller canonical form and add the feedback.



Writing the state equations from the signal-flow diagram, we obtain

$$\mathbf{x} = \begin{bmatrix} -105 & -506 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$$y = [100 \quad 500] \mathbf{x}$$

### 5.7

From the transformation equations,

$$\mathbf{P}^{-1} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}$$

Taking the inverse,

$$\mathbf{P} = \begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix}$$

Now,

$$\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}$$

$$\mathbf{P}^{-1} \mathbf{B} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \end{bmatrix}$$

$$\mathbf{C} \mathbf{P} = [1 \quad 4] \begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix} = [0.8 \quad -1.4]$$

Therefore,

$$\dot{\mathbf{z}} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix} \mathbf{z} + \begin{bmatrix} -3 \\ -11 \end{bmatrix} u$$

$$y = [0.8 \quad -1.4] \mathbf{z}$$

### 5.8

First find the eigenvalues.

$$|\lambda \mathbf{I} - \mathbf{A}| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda - 1 & -3 \\ 4 & \lambda + 6 \end{bmatrix} \right| = \lambda^2 + 5\lambda + 6$$

From which the eigenvalues are  $-2$  and  $-3$ .

Now use  $\mathbf{A} \mathbf{x}_i = \lambda \mathbf{x}_i$  for each eigenvalue,  $\lambda$ .

Thus,

$$\begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For  $\lambda = -2$ ,

$$3x_1 + 3x_2 = 0$$

$$-4x_1 - 4x_2 = 0$$

Thus  $x_1 = -x_2$

For  $\lambda = -3$

$$4x_1 + 3x_2 = 0$$

$$-4x_1 - 3x_2 = 0$$



Thus  $x_1 = -x_2$  and  $x_1 = -0.75x_2$ ; from which we let

$$\mathbf{P} = \begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix}$$

Taking the inverse yields

$$\mathbf{P}^{-1} = \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix}$$

Hence,

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 18.39 \\ 20 \end{bmatrix}$$

$$\mathbf{C}\mathbf{P} = [1 \quad 4] \begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix} = [-2.121 \quad 2.6]$$

Finally,

$$\begin{aligned} \dot{\mathbf{z}} &= \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 18.39 \\ 20 \end{bmatrix} u \\ \mathbf{y} &= [-2.121 \quad 2.6] \mathbf{z} \end{aligned}$$

## CHAPTER 6

### 6.1

Make a Routh table.

$s^7$	3	6	7	2
$s^6$	9	4	8	6
$s^5$	4.666666667	4.333333333	0	0
$s^4$	-4.35714286	8	6	0
$s^3$	12.90163934	6.426229508	0	0
$s^2$	10.17026684	6	0	0
$s^1$	-1.18515742	0	0	0
$s^0$	6	0	0	0

Since there are four sign changes and no complete row of zeros, there are four right half-plane poles and three left half-plane poles.

### 6.2

Make a Routh table. We encounter a row of zeros on the  $s^3$  row. The even polynomial is contained in the previous row as  $-6s^4 + 0s^2 + 6$ . Taking the derivative yields

$-24s^3 + 0s$ . Replacing the row of zeros with the coefficients of the derivative yields the  $s^3$  row. We also encounter a zero in the first column at the  $s^2$  row. We replace the zero with  $\varepsilon$  and continue the table. The final result is shown now as

$s^6$	1	-6	-1	6	
$s^5$	1	0	-1	0	
$s^4$	6	0	6	0	
$s^3$	-24	0	0	0	ROZ
$s^2$	$\varepsilon$	6	0	0	
$s^1$	$144/\varepsilon$	0	0	0	
$s^0$	6	0	0	0	

There is one sign change below the even polynomial. Thus the even polynomial (4<sup>th</sup> order) has one right half-plane pole, one left half-plane pole, and 2 imaginary axis poles. From the top of the table down to the even polynomial yields one sign change. Thus, the rest of the polynomial has one right half-plane root, and one left half-plane root. The total for the system is two right half-plane poles, two left half-plane poles, and 2 imaginary poles.

### 6.3

$$\text{Since } G(s) = \frac{K(s+20)}{s(s+2)(s+3)}, T(s) = \frac{G(s)}{1+G(s)} = \frac{K(s+20)}{s^3 + 5s^2 + (6+K)s + 20K}$$

Form the Routh table.

$s^3$	1	$(6+K)$
$s^2$	5	$20K$
$s^1$	$\frac{30-15K}{5}$	
$s^0$	$\frac{5}{20K}$	

From the  $s^1$  row,  $K < 2$ . From the  $s^0$  row,  $K > 0$ . Thus, for stability,  $0 < K < 2$ .

### 6.4

First find

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}| &= \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ 1 & 7 & 1 \\ -3 & 4 & -5 \end{bmatrix} \right| = \left| \begin{array}{ccc} (s-2) & -1 & -1 \\ -1 & (s-7) & -1 \\ 3 & -4 & (s+5) \end{array} \right| \\ &= s^3 - 4s^2 - 33s + 51 \end{aligned}$$

Now form the Routh table.

$s^3$	1	-33
$s^2$	-4	51
$s^1$	-20.25	
$s^0$	51	

There are two sign changes. Thus, there are two rhp poles and one lhp pole.

## CHAPTER 7

## 7.1

a. First check stability.

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{10s^2 + 500s + 6000}{s^3 + 70s^2 + 1375s + 6000} = \frac{10(s + 30)(s + 20)}{(s + 26.03)(s + 37.89)(s + 6.085)}$$

Poles are in the lhp. Therefore, the system is stable. Stability also could be checked via Routh-Hurwitz using the denominator of  $T(s)$ . Thus,

$$\begin{aligned} 15u(t) : e_{step}(\infty) &= \frac{15}{1 + \lim_{s \rightarrow 0} G(s)} = \frac{15}{1 + \infty} = 0 \\ 15tu(t) : e_{ramp}(\infty) &= \frac{15}{\lim_{s \rightarrow 0} sG(s)} = \frac{15}{\frac{10 \cdot 20 \cdot 30}{25 \cdot 35}} = 2.1875 \\ 15t^2u(t) : e_{parabola}(\infty) &= \frac{15}{\lim_{s \rightarrow 0} s^2 G(s)} = \frac{30}{0} = \infty, \text{ since } L[15t^2] = \frac{30}{s^3} \end{aligned}$$

b. First check stability.

$$\begin{aligned} T(s) &= \frac{G(s)}{1 + G(s)} = \frac{10s^2 + 500s + 6000}{s^5 + 110s^4 + 3875s^3 + 4.37e04s^2 + 500s + 6000} \\ &= \frac{10(s + 30)(s + 20)}{(s + 50.01)(s + 35)(s + 25)(s^2 - 7.189e - 04s + 0.1372)} \end{aligned}$$

From the second-order term in the denominator, we see that the system is unstable. Instability could also be determined using the Routh-Hurwitz criteria on the denominator of  $T(s)$ . Since the system is unstable, calculations about steady-state error cannot be made.

## 7.2

a. The system is stable, since

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{1000(s + 8)}{(s + 9)(s + 7) + 1000(s + 8)} = \frac{1000(s + 8)}{s^2 + 1016s + 8063}$$

and is of Type 0. Therefore,

$$K_p = \lim_{s \rightarrow 0} G(s) = \frac{1000 \cdot 8}{7 \cdot 9} = 127; \quad K_v = \lim_{s \rightarrow 0} sG(s) = 0;$$

$$\text{and } K_a = \lim_{s \rightarrow 0} s^2 G(s) = 0$$

b.

$$e_{step}(\infty) = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)} = \frac{1}{1 + 127} = 7.8e - 03$$

$$e_{ramp}(\infty) = \frac{1}{\lim_{s \rightarrow 0} sG(s)} = \frac{1}{0} = \infty$$

$$e_{parabola}(\infty) = \frac{1}{\lim_{s \rightarrow 0} s^2 G(s)} = \frac{1}{0} = \infty$$

**7.3**

System is stable for positive  $K$ . System is Type 0. Therefore, for a step input  $e_{step}(\infty) = \frac{1}{1 + K_p} = 0.1$ . Solving for  $K_p$  yields  $K_p = 9 = \lim_{s \rightarrow 0} G(s) = \frac{12K}{14 \cdot 18}$ ; from which we obtain  $K = 189$ .

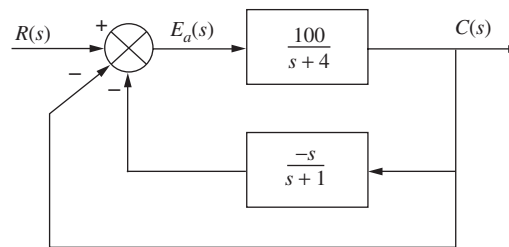
**7.4**

System is stable. Since  $G_1(s) = 1000$ , and  $G_2(s) = \frac{(s+2)}{(s+4)}$ ,

$$e_D(\infty) = -\frac{1}{\lim_{s \rightarrow 0} \frac{1}{G_2(s)} + \lim_{s \rightarrow 0} G_1(s)} = \frac{1}{2 + 1000} = -9.98e - 04$$

**7.5**

System is stable. Create a unity-feedback system, where  $H_e(s) = \frac{1}{s+1} - 1 = \frac{-s}{s+1}$ . The system is as follows:



Thus,

$$G_e(s) = \frac{G(s)}{1 + G(s)H_e(s)} = \frac{\frac{100}{(s+4)}}{1 - \frac{100s}{(s+1)(s+4)}} = \frac{100(s+1)}{s^2 - 95s + 4}$$

Hence, the system is Type 0. Evaluating  $K_p$  yields

$$K_p = \frac{100}{4} = 25$$

The steady-state error is given by

$$e_{step}(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 + 25} = 3.846e - 02$$

## 7.6

$$\text{Since } G(s) = \frac{K(s+7)}{s^2+2s+10}, e(\infty) = \frac{1}{1+K_p} = \frac{1}{1+\frac{7K}{10}} = \frac{10}{10+7K}.$$

Calculating the sensitivity, we get

$$S_{e:K} = \frac{K}{e} \frac{\partial e}{\partial K} = \frac{K}{\left(\frac{10}{10+7K}\right)} \frac{(-10)7}{(10+7K)^2} = -\frac{7K}{10+7K}$$

## 7.7

Given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3 & -6 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \mathbf{C} = [1 \quad 1]; R(s) = \frac{1}{s}.$$

Using the final value theorem,

$$\begin{aligned} e_{step}(\infty) &= \lim_{s \rightarrow 0} sR(s)[1 - \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}] = \lim_{s \rightarrow 0} \left[ 1 - [1 \quad 1] \begin{bmatrix} s & -1 \\ 3 & s+6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] \\ &= \lim_{s \rightarrow 0} \left[ 1 - [1 \quad 1] \frac{\begin{bmatrix} s+6 & 1 \\ -3s & s \end{bmatrix}}{s^2+6s+3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \lim_{s \rightarrow 0} \frac{s^2+5s+2}{s^2+6s+3} = \frac{2}{3} \end{aligned}$$

Using input substitution,

$$\begin{aligned} e_{step}(\infty) &= 1 + \mathbf{C}\mathbf{A}^{-1}\mathbf{B} = 1 - [1 \quad 1] \begin{bmatrix} 0 & 1 \\ -3 & -6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 1 + [1 \quad 1] \frac{\begin{bmatrix} -6 & -1 \\ 3 & 0 \end{bmatrix}}{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 + [1 \quad 1] \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix} = \frac{2}{3} \end{aligned}$$

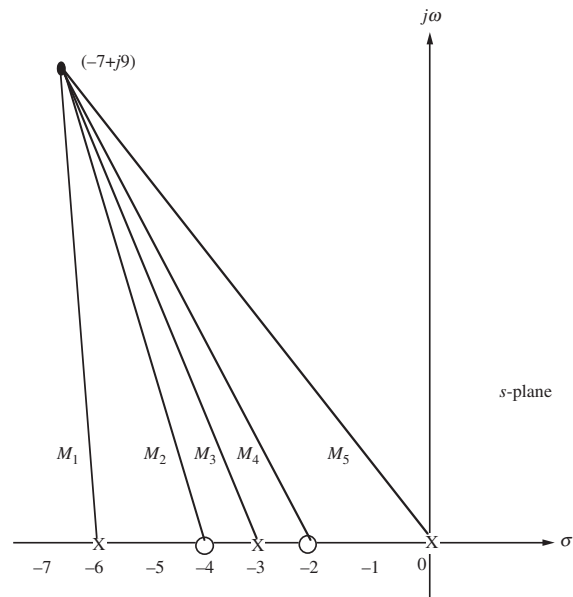
## CHAPTER 8

## 8.1

a.

$$\begin{aligned} F(-7+j9) &= \frac{(-7+j9+2)(-7+j9+4)0.0339}{(-7+j9)(-7+j9+3)(-7+j9+6)} = \frac{(-5+j9)(-3+j9)}{(-7+j9)(-4+j9)(-1+j9)} \\ &= \frac{(-66-j72)}{(944-j378)} = -0.0339 - j0.0899 = 0.096 \angle -110.7^\circ \end{aligned}$$

b. The arrangement of vectors is shown as follows:

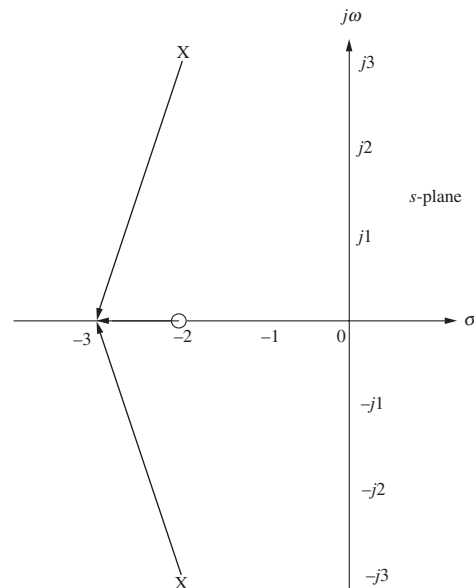


From the diagram,

$$\begin{aligned}
 F(-7 + j9) &= \frac{M_2 M_4}{M_1 M_3 M_5} = \frac{(-3 + j9)(-5 + j9)}{(-1 + j9)(-4 + j9)(-7 + j9)} = \frac{(-66 - j72)}{(944 - j378)} \\
 &= -0.0339 - j0.0899 = 0.096 \angle -110.7^\circ
 \end{aligned}$$

## 8.2

a. First draw the vectors.



From the diagram,

$$\Sigma \text{ angles} = 180^\circ - \tan^{-1}\left(\frac{-3}{-1}\right) - \tan^{-1}\left(\frac{-3}{1}\right) = 180^\circ - 108.43^\circ + 108.43^\circ = 180^\circ.$$

b. Since the angle is  $180^\circ$ , the point is on the root locus.

$$\text{c. } K = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}} = \frac{(\sqrt{1^2 + 3^2})(\sqrt{1^2 + 3^2})}{1} = 10$$

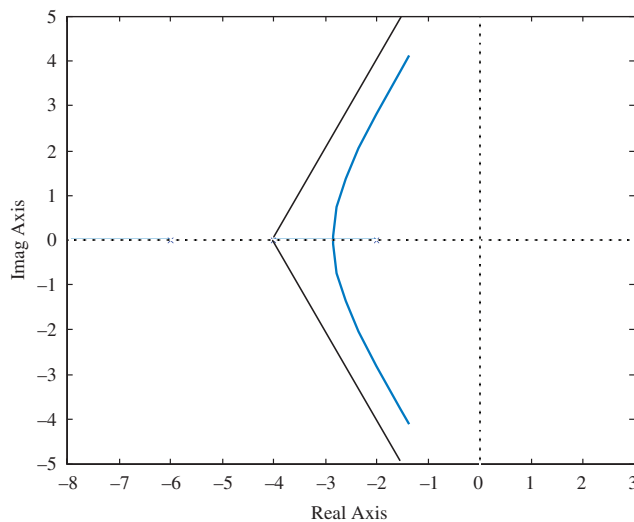
### 8.3

First, find the asymptotes.

$$\sigma_a = \frac{\Sigma \text{poles} - \Sigma \text{zeros}}{\# \text{poles} - \# \text{zeros}} = \frac{(-2 - 4 - 6) - (0)}{3 - 0} = -4$$

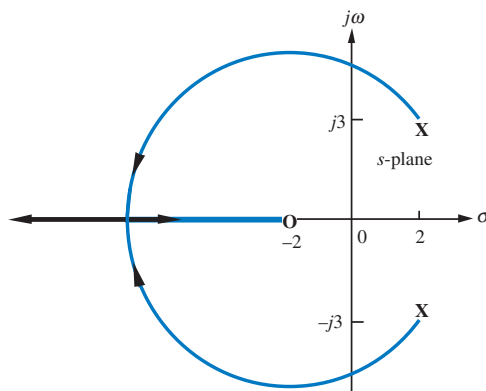
$$\theta_a = \frac{(2k + 1)\pi}{3} = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

Next draw root locus following the rules for sketching.



### 8.4

a.



- b. Using the Routh-Hurwitz criteria, we first find the closed-loop transfer function.

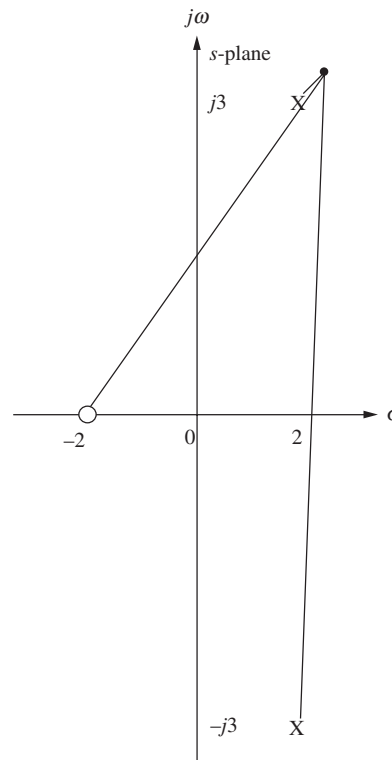
$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{K(s + 2)}{s^2 + (K - 4)s + (2K + 13)}$$

Using the denominator of  $T(s)$ , make a Routh table.

$s^2$	1	$2K + 13$
$s^1$	$K - 40$	0
$s^0$	$2K + 13$	0

We get a row of zeros for  $K = 4$ . From the  $s^2$  row with  $K = 4$ ,  $s^2 + 21 = 0$ . From which we evaluate the imaginary axis crossing at  $\sqrt{21}$ .

- c. From part (b),  $K = 4$ .
- d. Searching for the minimum gain to the left of  $-2$  on the real axis yields  $-7$  at a gain of 18. Thus the break-in point is at  $-7$ .
- e. First, draw vectors to a point  $\varepsilon$  close to the complex pole.

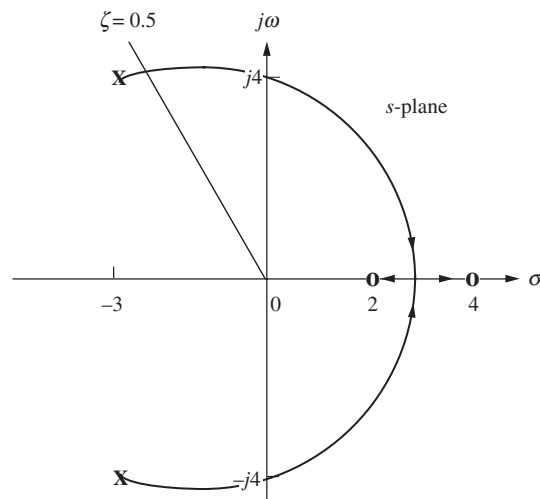


At the point  $\varepsilon$  close to the complex pole, the angles must add up to zero. Hence, angle from zero – angle from pole in 4<sup>th</sup> quadrant – angle from pole in 1<sup>st</sup> quadrant =  $180^\circ$ , or  $\tan^{-1}\left(\frac{3}{4}\right) - 90^\circ - \theta = 180^\circ$ . Solving for the angle of departure,  $\theta = -233.1$ .



## 8.5

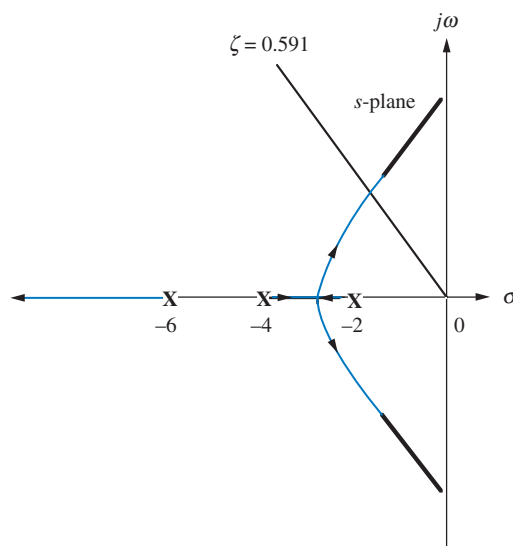
a.



- b. Search along the imaginary axis and find the  $180^\circ$  point at  $s = \pm j4.06$ .
- c. For the result in part (b),  $K = 1$ .
- d. Searching between 2 and 4 on the real axis for the minimum gain yields the break-in at  $s = 2.89$ .
- e. Searching along  $\zeta = 0.5$  for the  $180^\circ$  point we find  $s = -2.42 + j4.18$ .
- f. For the result in part (e),  $K = 0.108$ .
- g. Using the result from part (c) and the root locus,  $K < 1$ .

## 8.6

a.



b. Searching along the  $\zeta = 0.591$  (10% overshoot) line for the  $180^\circ$  point yields  $-2.028 + j2.768$  with  $K = 45.55$ .

c.  $T_s = \frac{4}{|\text{Re}|} = \frac{4}{2.028} = 1.97 \text{ s}$ ;  $T_p = \frac{\pi}{|\text{Im}|} = \frac{\pi}{2.768} = 1.13 \text{ s}$ ;  $\omega_n T_r = 1.8346$  from the rise-time chart and graph in Chapter 4. Since  $\omega_n$  is the radial distance to the pole,  $\omega_n = \sqrt{2.028^2 + 2.768^2} = 3.431$ . Thus,  $T_r = 0.53 \text{ s}$ ; since the system is Type 0,  $K_p = \frac{K}{2 \cdot 4 \cdot 6} = \frac{45.55}{48} = 0.949$ . Thus,

$$e_{step}(\infty) = \frac{1}{1 + K_p} = 0.51.$$

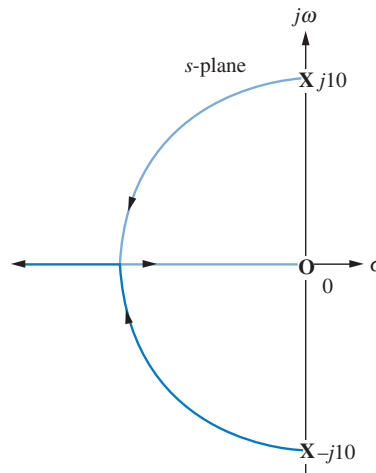
d. Searching the real axis to the left of  $-6$  for the point whose gain is 45.55, we find  $-7.94$ . Comparing this value to the real part of the dominant pole,  $-2.028$ , we find that it is not five times further. The second-order approximation is not valid.

### 8.7

Find the closed-loop transfer function and put it the form that yields  $p_i$  as the root locus variable. Thus,

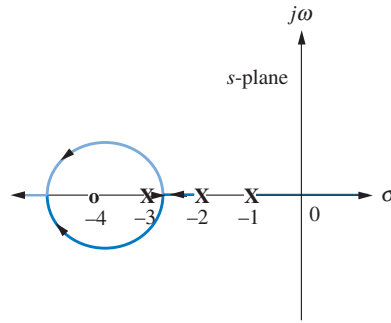
$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{100}{s^2 + p_i s + 100} = \frac{100}{(s^2 + 100) + p_i s} = \frac{100}{s^2 + 100 + \frac{p_i s}{s^2 + 100}}$$

Hence,  $KG(s)H(s) = \frac{p_i s}{s^2 + 100}$ . The following shows the root locus.



### 8.8

Following the rules for plotting the root locus of positive-feedback systems, we obtain the following root locus:



## 8.9

The closed-loop transfer function is  $T(s) = \frac{K(s+1)}{s^2 + (K+2)s + K}$ . Differentiating the denominator with respect to  $K$  yields

$$2s \frac{\partial s}{\partial K} + (K+2) \frac{\partial s}{\partial K} + (s+1) = (2s+K+2) \frac{\partial s}{\partial K} + (s+1) = 0$$

Solving for  $\frac{\partial s}{\partial K}$ , we get  $\frac{\partial s}{\partial K} = \frac{-(s+1)}{(2s+K+2)}$ . Thus,  $S_{s:K} = \frac{K}{s} \frac{\partial s}{\partial K} = \frac{-K(s+1)}{s(2s+K+2)}$ .

Substituting  $K = 20$  yields  $S_{s:K} = \frac{-10(s+1)}{s(s+11)}$ .

Now find the closed-loop poles when  $K = 20$ . From the denominator of  $T(s)$ ,  $s_{1,2} = -21.05, -0.95$ , when  $K = 20$ .

For the pole at  $-21.05$ ,

$$\Delta s = s(S_{s:K}) \frac{\Delta K}{K} = -21.05 \left( \frac{-10(-21.05+1)}{-21.05(-21.05+11)} \right) 0.05 = -0.9975.$$

For the pole at  $-0.95$ ,

$$\Delta s = s(S_{s:K}) \frac{\Delta K}{K} = 0.95 \left( \frac{-10(-0.95+1)}{-0.95(-0.95+11)} \right) 0.05 = -0.0025.$$

## CHAPTER 9

## 9.1

a. Searching along the 15% overshoot line, we find the point on the root locus at  $-3.5 + j5.8$  at a gain of  $K = 45.84$ . Thus, for the uncompensated system,  $K_v = \lim_{s \rightarrow 0} sG(s) = K/7 = 45.84/7 = 6.55$ .

Hence,  $e_{ramp\_uncompensated}(\infty) = 1/K_v = 0.1527$ .

b. Compensator zero should be 20x further to the left than the compensator pole.

Arbitrarily select  $G_c(s) = \frac{(s+0.2)}{(s+0.01)}$ .

c. Insert compensator and search along the 15% overshoot line and find the root locus at  $-3.4 + j5.63$  with a gain,  $K = 44.64$ . Thus, for the compensated system,

$$K_v = \frac{44.64(0.2)}{(7)(0.01)} = 127.5 \text{ and } e_{ramp\_compensated}(\infty) = \frac{1}{K_v} = 0.0078.$$

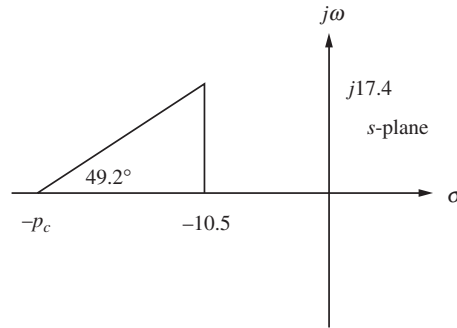
$$\text{d. } \frac{e_{ramp\_uncompensated}}{e_{ramp\_compensated}} = \frac{0.1527}{0.0078} = 19.58$$

## 9.2

- a. Searching along the 15% overshoot line, we find the point on the root locus at  $-3.5 + j5.8$  at a gain of  $K = 45.84$ . Thus, for the uncompensated system,

$$T_s = \frac{4}{|Re|} = \frac{4}{3.5} = 1.143 \text{ s.}$$

- b. The real part of the design point must be three times larger than the uncompensated pole's real part. Thus the design point is  $3(-3.5) + j3(5.8) = -10.5 + j17.4$ . The angular contribution of the plant's poles and compensator zero at the design point is  $130.8^\circ$ . Thus, the compensator pole must contribute  $180^\circ - 130.8^\circ = 49.2^\circ$ . Using the following diagram,



we find  $\frac{17.4}{p_c - 10.5} = \tan 49.2^\circ$ , from which,  $p_c = 25.52$ . Adding this pole, we find the gain at the design point to be  $K = 476.3$ . A higher-order closed-loop pole is found to be at  $-11.54$ . This pole may not be close enough to the closed-loop zero at  $-10$ . Thus, we should simulate the system to be sure the design requirements have been met.

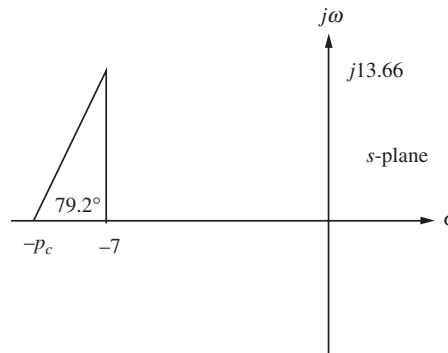
## 9.3

- a. Searching along the 20% overshoot line, we find the point on the root locus at  $-3.5 + j6.83$  at a gain of  $K = 58.9$ . Thus, for the uncompensated system,

$$T_s = \frac{4}{|Re|} = \frac{4}{3.5} = 1.143 \text{ s.}$$

- b. For the uncompensated system,  $K_v = \lim_{s \rightarrow 0} sG(s) = K/7 = 58.9/7 = 8.41$ . Hence,  $e_{ramp\_uncompensated}(\infty) = 1/K_v = 0.1189$ .
- c. In order to decrease the settling time by a factor of 2, the design point is twice the uncompensated value, or  $-7 + j13.66$ . Adding the angles from the plant's poles and the compensator's zero at  $-3$  to the design point, we obtain  $-100.8^\circ$ . Thus, the compensator pole must contribute  $180^\circ - 100.8^\circ = 79.2^\circ$ . Using the following

diagram,



we find  $\frac{13.66}{p_c - 7} = \tan 79.2^\circ$ , from which,  $p_c = 9.61$ . Adding this pole, we find the gain at the design point to be  $K = 204.9$ .

Evaluating  $K_v$  for the lead-compensated system:

$$K_v = \lim_{s \rightarrow 0} sG(s)G_{lead} = K(3)/[(7)(9.61)] = (204.9)(3)/[(7)(9.61)] = 9.138.$$

$K_v$  for the uncompensated system was 8.41. For a 10x improvement in steady-state error,  $K_v$  must be  $(8.41)(10) = 84.1$ . Since lead compensation gave us  $K_v = 9.138$ , we need an improvement of  $84.1/9.138 = 9.2$ . Thus, the lag compensator zero should be 9.2x further to the left than the compensator pole. Arbitrarily select  $G_c(s) = \frac{(s + 0.092)}{(s + 0.01)}$ .

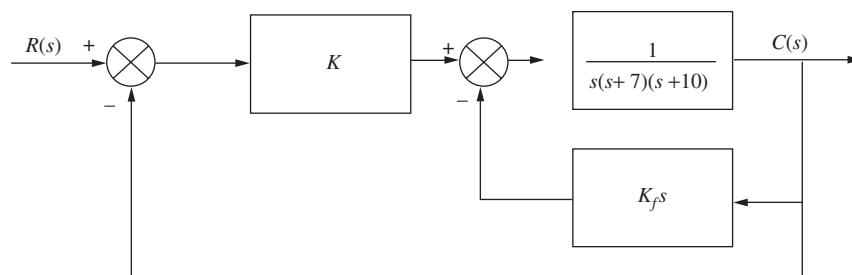
Using all plant and compensator poles, we find the gain at the design point to be  $K = 205.4$ . Summarizing the forward path with plant, compensator, and gain yields

$$G_e(s) = \frac{205.4(s + 3)(s + 0.092)}{s(s + 7)(9.61)(s + 0.01)}.$$

Higher-order poles are found at  $-0.928$  and  $-2.6$ . It would be advisable to simulate the system to see if there is indeed pole-zero cancellation.

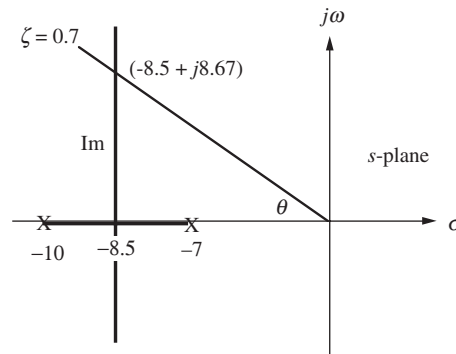
## 9.4

The configuration for the system is shown in the figure below.



Minor-Loop Design:

For the minor loop,  $G(s)H(s) = \frac{K_f}{(s+7)(s+10)}$ . Using the following diagram, we find that the minor-loop root locus intersects the 0.7 damping ratio line at  $-8.5 + j8.67$ . The imaginary part was found as follows:  $\theta = \cos^{-1} \zeta = 45.57^\circ$ . Hence,  $\frac{\text{Im}}{8.5} = \tan 45.57^\circ$ , from which  $\text{Im} = 8.67$ .



The gain,  $K_f$ , is found from the vector lengths as

$$K_f = \sqrt{1.5^2 + 8.67^2} \sqrt{1.5^2 + 8.67^2} = 77.42$$

Major-Loop Design:

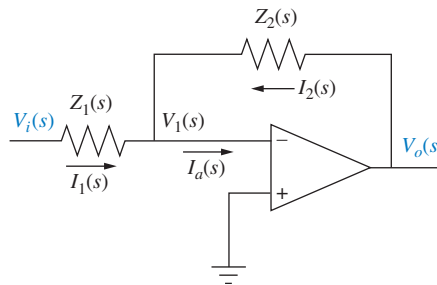
Using the closed-loop poles of the minor loop, we have an equivalent forward-path transfer function of

$$G_e(s) = \frac{K}{s(s+8.5+j8.67)(s+8.5-j8.67)} = \frac{K}{s(s^2+17s+147.4)}$$

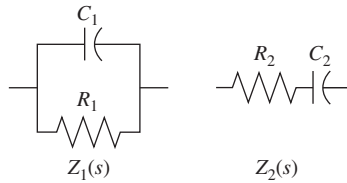
Using the three poles of  $G_e(s)$  as open-loop poles to plot a root locus, we search along  $\zeta = 0.5$  and find that the root locus intersects this damping ratio line at  $-4.34 + j7.51$  at a gain,  $K = 626.3$ .

## 9.5

a. An active PID controller must be used. We use the circuit shown in the following figure:



where the impedances are shown below as follows:



Matching the given transfer function with the transfer function of the PID controller yields

$$G_c(s) = \frac{(s + 0.1)(s + 5)}{s} = \frac{s^2 + 5.1s + 0.5}{s} = s + 5.1 + \frac{0.5}{s}$$

$$= - \left[ \left( \frac{R_2}{R_1} + \frac{C_1}{C_2} \right) + R_2 C_1 s + \frac{1}{R_1 C_2} \right]$$

Equating coefficients

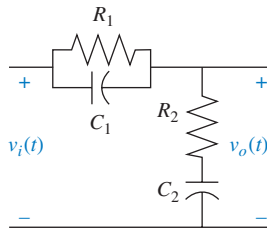
$$\frac{1}{R_1 C_2} = 0.5 \quad (1)$$

$$R_2 C_1 = 1 \quad (2)$$

$$\left( \frac{R_2}{R_1} + \frac{C_1}{C_2} \right) = 5.1 \quad (3)$$

In Eq. (2) we arbitrarily let  $C_1 = 10^{-5}$ . Thus,  $R_2 = 10^5$ . Using these values along with Eqs. (1) and (3) we find  $C_2 = 100 \mu F$  and  $R_1 = 20 \text{ k}\Omega$ .

- b.** The lag-lead compensator can be implemented with the following passive network, since the ratio of the lead pole-to-zero is the inverse of the ratio of the lag pole-to-zero:



Matching the given transfer function with the transfer function of the passive lag-lead compensator yields

$$G_c(s) = \frac{(s + 0.1)(s + 2)}{(s + 0.01)(s + 20)} = \frac{(s + 0.1)(s + 2)}{s^2 + 20.01s + 0.2}$$

$$= \frac{\left( s + \frac{1}{R_1 C_1} \right) \left( s + \frac{1}{R_2 C_2} \right)}{s^2 + \left( \frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} + \frac{1}{R_2 C_1} \right) s + \frac{1}{R_1 R_2 C_1 C_2}}$$

Equating coefficients

$$\frac{1}{R_1 C_1} = 0.1 \quad (1)$$

$$\frac{1}{R_2 C_2} = 0.1 \quad (2)$$

$$\left( \frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} + \frac{1}{R_2 C_1} \right) = 20.01 \quad (3)$$

Substituting Eqs. (1) and (2) in Eq. (3) yields

$$\frac{1}{R_2 C_1} = 17.91 \quad (4)$$

Arbitrarily letting  $C_1 = 100 \mu F$  in Eq. (1) yields  $R_1 = 100 k\Omega$ .

Substituting  $C_1 = 100 \mu F$  into Eq. (4) yields  $R_2 = 558 k\Omega$ .

Substituting  $R_2 = 558 k\Omega$  into Eq. (2) yields  $C_2 = 900 \mu F$ .

## CHAPTER 10

### 10.1

a.

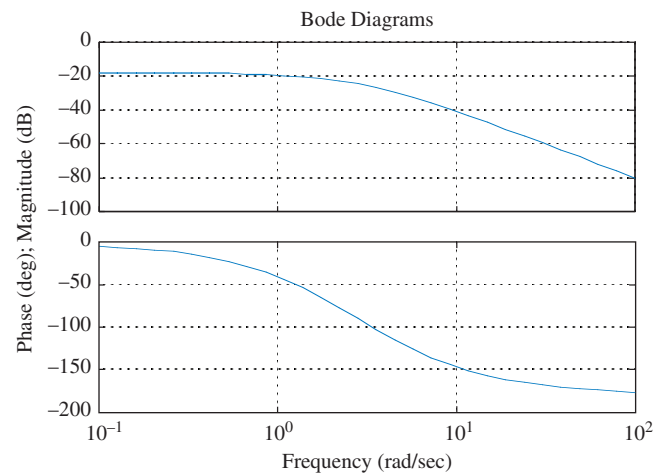
$$G(s) = \frac{1}{(s+2)(s+4)}; \quad G(j\omega) = \frac{1}{(8+\omega^2) + j6\omega}$$

$$M(\omega) = \sqrt{(8-\omega^2)^2 + (6\omega)^2}$$

$$\text{For } \omega < \sqrt{8}, \phi(\omega) = -\tan^{-1}\left(\frac{6\omega}{8-\omega^2}\right).$$

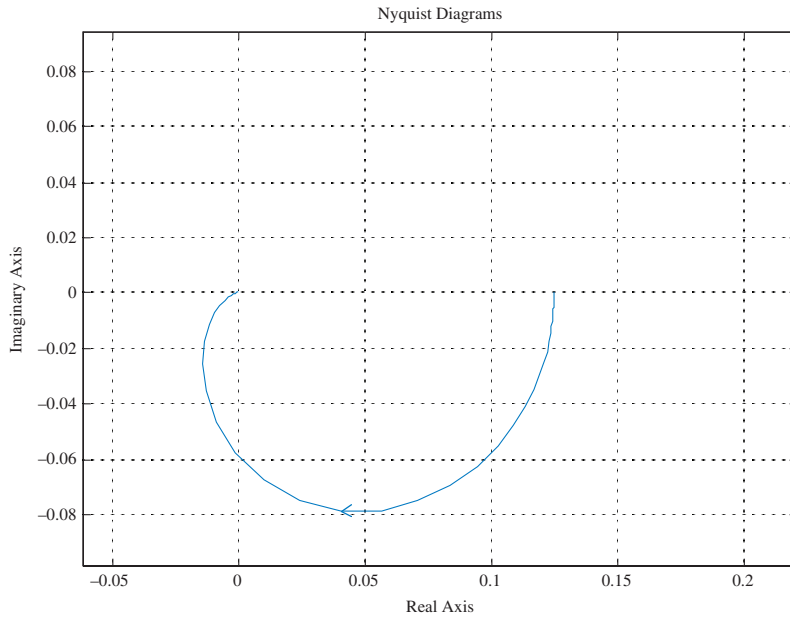
$$\text{For } \omega > \sqrt{8}, \phi(\omega) = -\left(\pi + \tan^{-1}\left[\frac{6\omega}{8-\omega^2}\right]\right).$$

b.

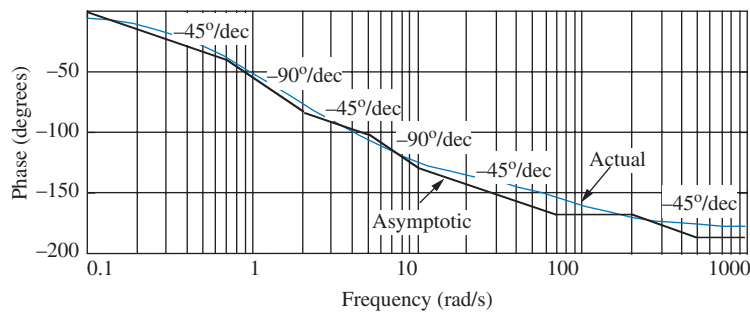
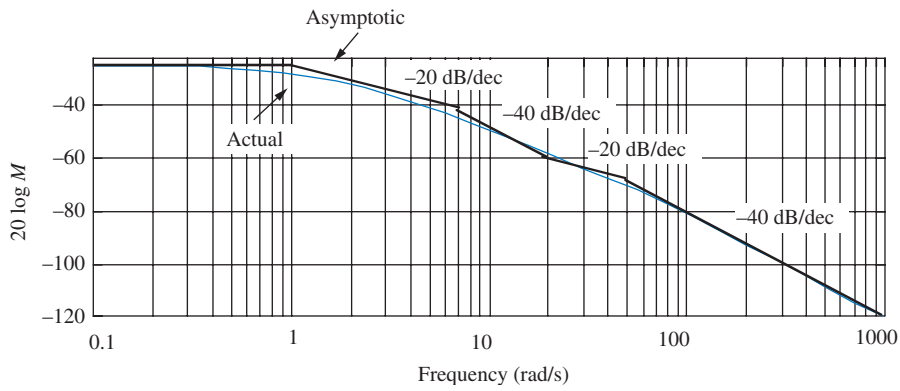




c.

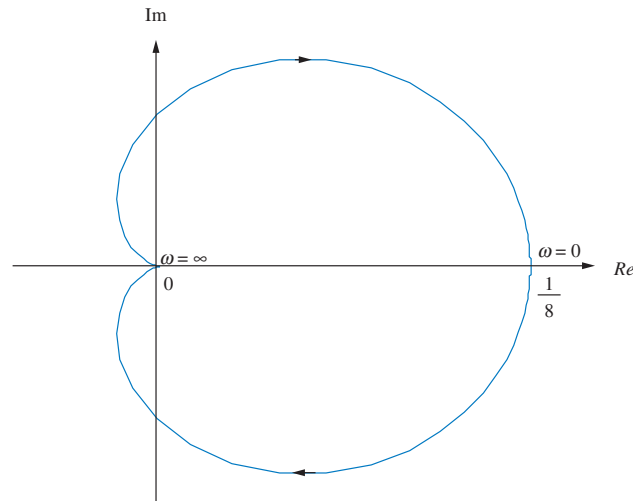


10.2



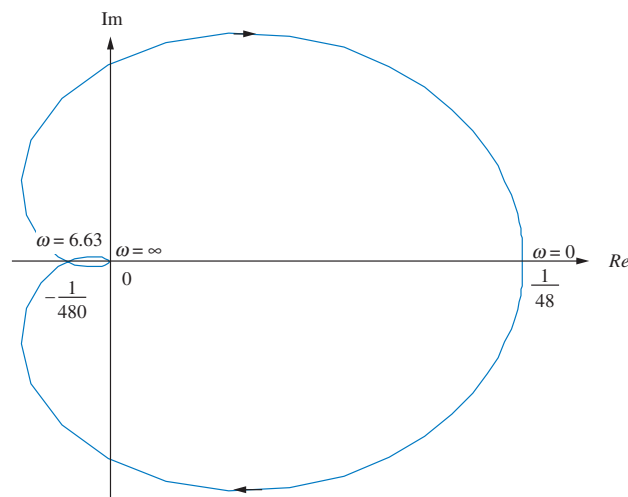
## 10.3

The frequency response is  $1/8$  at an angle of zero degrees at  $\omega = 0$ . Each pole rotates  $90^\circ$  in going from  $\omega = 0$  to  $\omega = \infty$ . Thus, the resultant rotates  $-180^\circ$  while its magnitude goes to zero. The result is shown below.



## 10.4

a. The frequency response is  $1/48$  at an angle of zero degrees at  $\omega = 0$ . Each pole rotates  $90^\circ$  in going from  $\omega = 0$  to  $\omega = \infty$ . Thus, the resultant rotates  $-270^\circ$  while its magnitude goes to zero. The result is shown below.



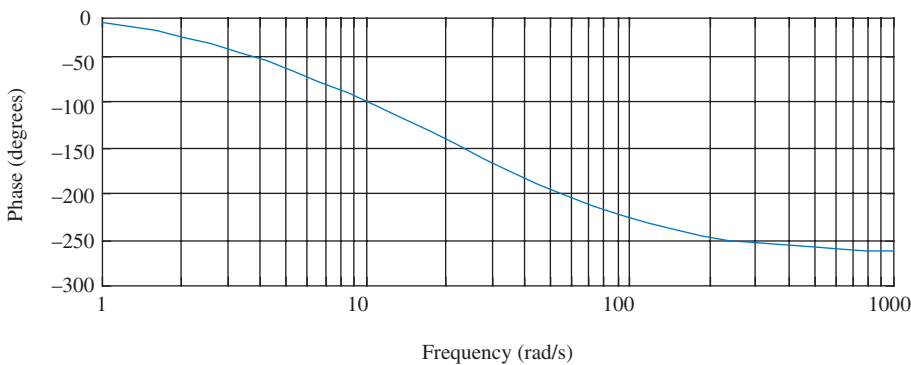
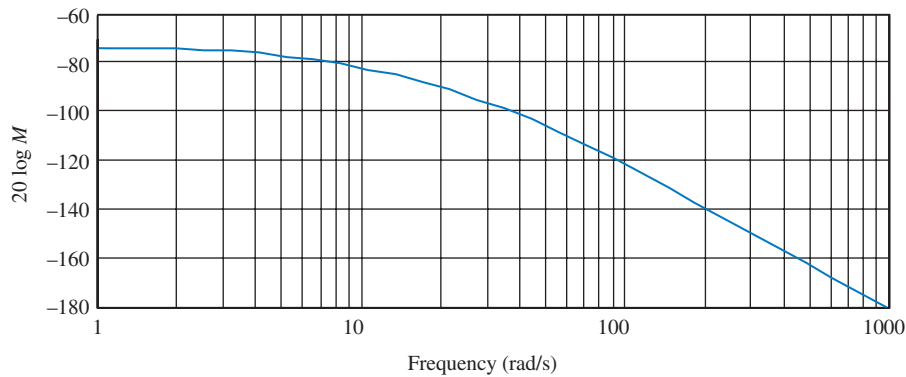
b. Substituting  $j\omega$  into  $G(s) = \frac{1}{(s+2)(s+4)(s+6)} = \frac{1}{s^3 + 12s^2 + 44s + 48}$  and simplifying, we obtain  $G(j\omega) = \frac{(48 - 12\omega^2) - j(44\omega - \omega^3)}{\omega^6 + 56\omega^4 + 784\omega^2 + 2304}$ . The Nyquist diagram crosses the real axis when the imaginary part of  $G(j\omega)$  is zero. Thus, the Nyquist diagram crosses the real axis at  $\omega^2 = 44$ , or  $\omega = \sqrt{44} = 6.63$  rad/s. At this frequency  $G(j\omega) = -\frac{1}{480}$ . Thus, the system is stable for  $K < 480$ .

## 10.5

If  $K = 100$ , the Nyquist diagram will intersect the real axis at  $-100/480$ . Thus,  $G_M = 20 \log \frac{480}{100} = 13.62$  dB. From Skill-Assessment Exercise Solution 10.4, the  $180^\circ$  frequency is  $6.63$  rad/s.

## 10.6

a.



- b. The phase angle is  $180^\circ$  at a frequency of  $36.74$  rad/s. At this frequency the gain is  $-99.67$  dB. Therefore,  $20 \log K = 99.67$ , or  $K = 96,270$ . We conclude that the system is stable for  $K < 96,270$ .
- c. For  $K = 10,000$ , the magnitude plot is moved up  $20 \log 10,000 = 80$  dB. Therefore, the gain margin is  $99.67 - 80 = 19.67$  dB. The  $180^\circ$  frequency is  $36.7$  rad/s. The gain curve crosses  $0$  dB at  $\omega = 7.74$  rad/s, where the phase is  $87.1^\circ$ . We calculate the phase margin to be  $180^\circ - 87.1^\circ = 92.9^\circ$ .

## 10.7

Using  $\zeta = \frac{-\ln(\%/100)}{\sqrt{\pi^2 + \ln^2(\%/100)}}$ , we find  $\zeta = 0.456$ , which corresponds to 20% over-

shoot. Using  $T_s = 2$ ,  $\omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 5.79$  rad/s.

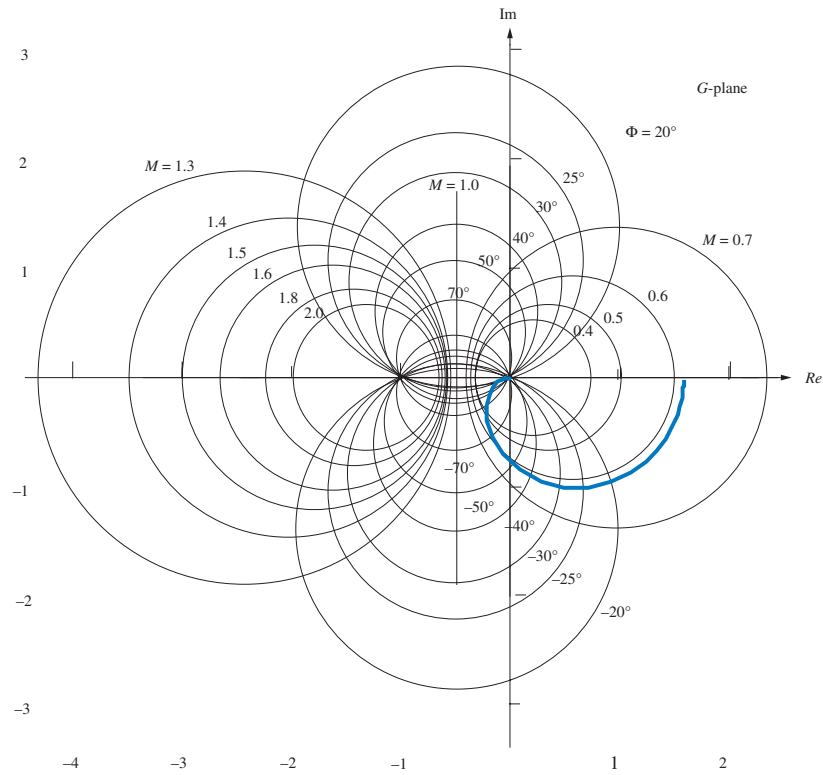
10.8

For both parts find that  $G(j\omega) = \frac{160}{27} * \frac{(6750000 - 101250\omega^2) + j1350(\omega^2 - 1350)\omega}{\omega^6 + 2925\omega^4 + 1072500\omega^2 + 25000000}$ .

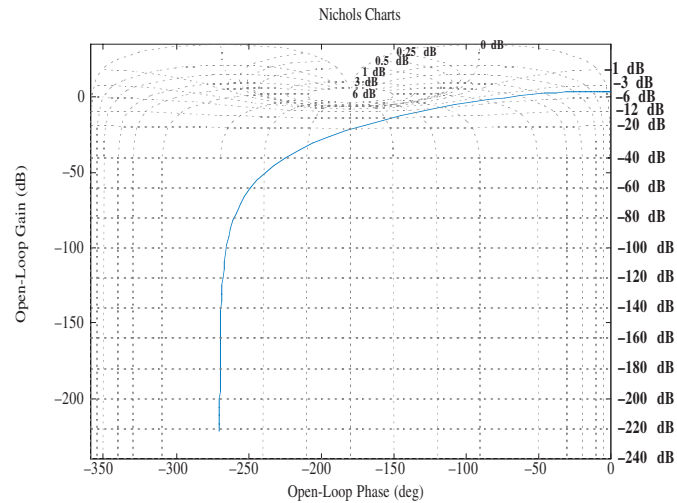
For a range of values for  $\omega$ , superimpose  $G(j\omega)$  on the a. M and N circles, and on the

b. Nichols chart.

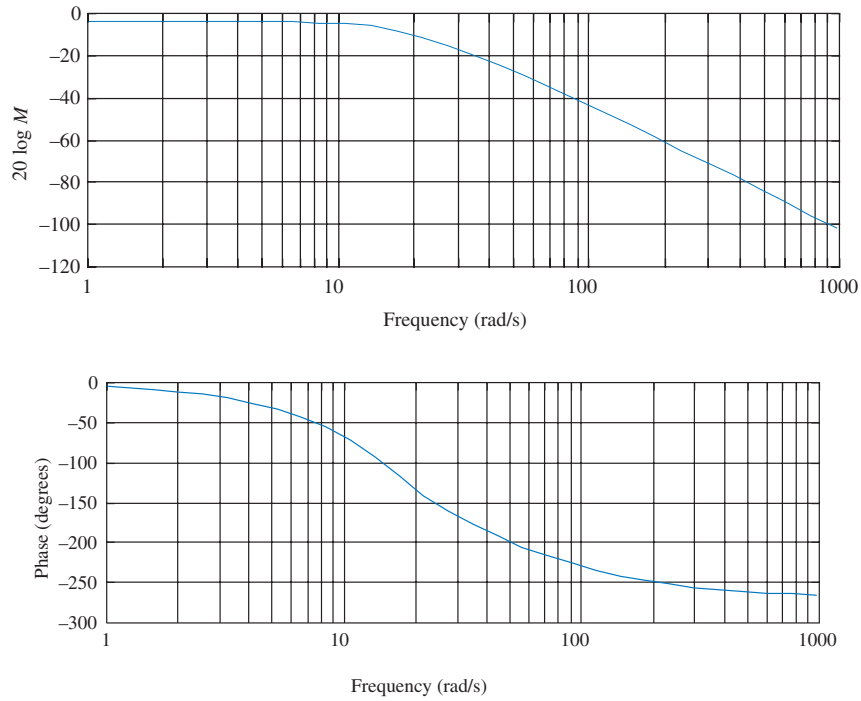
a.



b.

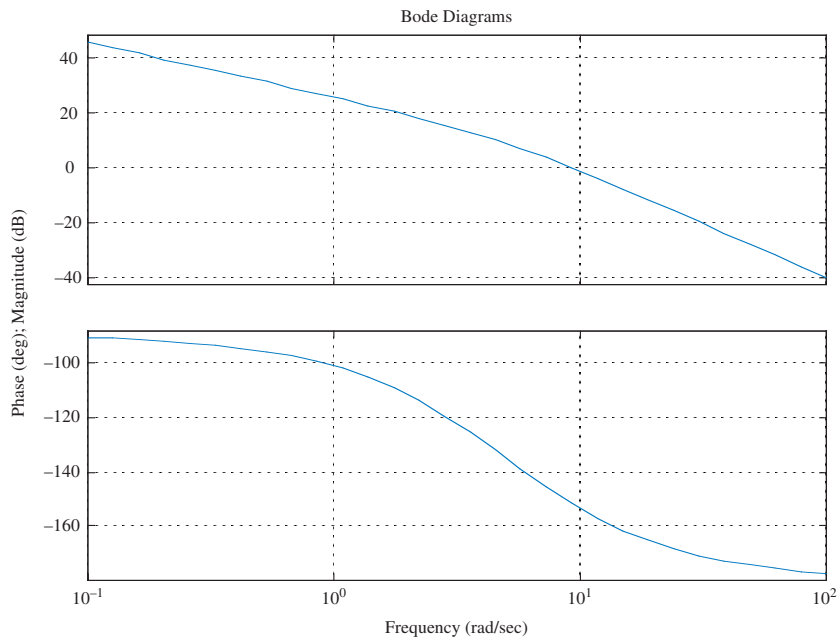


Plotting the closed-loop frequency response from **a.** or **b.** yields the following plot:



### 10.9

The open-loop frequency response is shown in the following figure:



The open-loop frequency response is  $-7$  at  $\omega = 14.5$  rad/s. Thus, the estimated bandwidth is  $\omega_{WB} = 14.5$  rad/s. The open-loop frequency response plot goes through zero dB at a frequency of 9.4 rad/s, where the phase is  $151.98^\circ$ . Hence, the phase margin is  $180^\circ - 151.98^\circ = 28.02^\circ$ . This phase margin corresponds to

$$\zeta = 0.25. \text{ Therefore, } \%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 44.4\%,$$

$$T_s = \frac{4}{\omega_{BW}\zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 1.64 \text{ s and}$$

$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 0.33 \text{ s}$$

### 10.10

The initial slope is 40 dB/dec. Therefore, the system is Type 2. The initial slope intersects 0 dB at  $\omega = 9.5$  rad/s. Thus,  $K_a = 9.5^2 = 90.25$  and  $K_p = K_v = \infty$ .

### 10.11

a. Without delay,  $G(j\omega) = \frac{10}{j\omega(j\omega + 1)} = \frac{10}{\omega(-\omega + j)}$ , from which the zero dB frequency is found as follows:  $M = \frac{10}{\omega\sqrt{\omega^2 + 1}} = 1$ . Solving for  $\omega$ ,  $\omega\sqrt{\omega^2 + 1} = 10$ , or after squaring both sides and rearranging,  $\omega^4 + \omega^2 - 100 = 0$ . Solving for the roots,  $\omega^2 = -10.51, 9.51$ . Taking the square root of the positive root, we find the 0 dB frequency to be 3.08 rad/s. At this frequency, the phase angle,  $\phi = -\angle(-\omega + j) = -\angle(-3.08 + j) = -162^\circ$ . Therefore the phase margin is  $180^\circ - 162^\circ = 18^\circ$ .

b. With a delay of 0.1 s,

$$\begin{aligned} \phi &= -\angle(-\omega + j) - \omega T = -\angle(-3.08 + j) - (3.08)(0.1)(180/\pi) = -162 - 17.65 \\ &= -179.65^\circ. \end{aligned}$$

Therefore the phase margin is  $180^\circ - 179.65^\circ = 0.35^\circ$ . Thus, the system is stable.

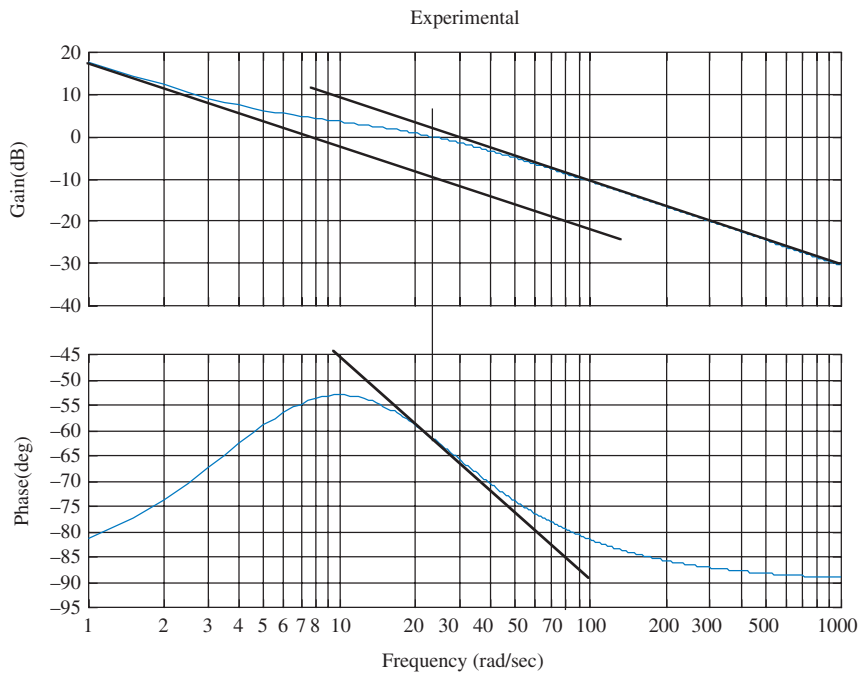
c. With a delay of 3 s,

$$\begin{aligned} \phi &= -\angle(-\omega + j) - \omega T = -\angle(-3.08 + j) - (3.08)(3)(180/\pi) = -162^\circ - 529.41^\circ \\ &= -691.41^\circ = 28.59 \text{ deg.} \end{aligned}$$

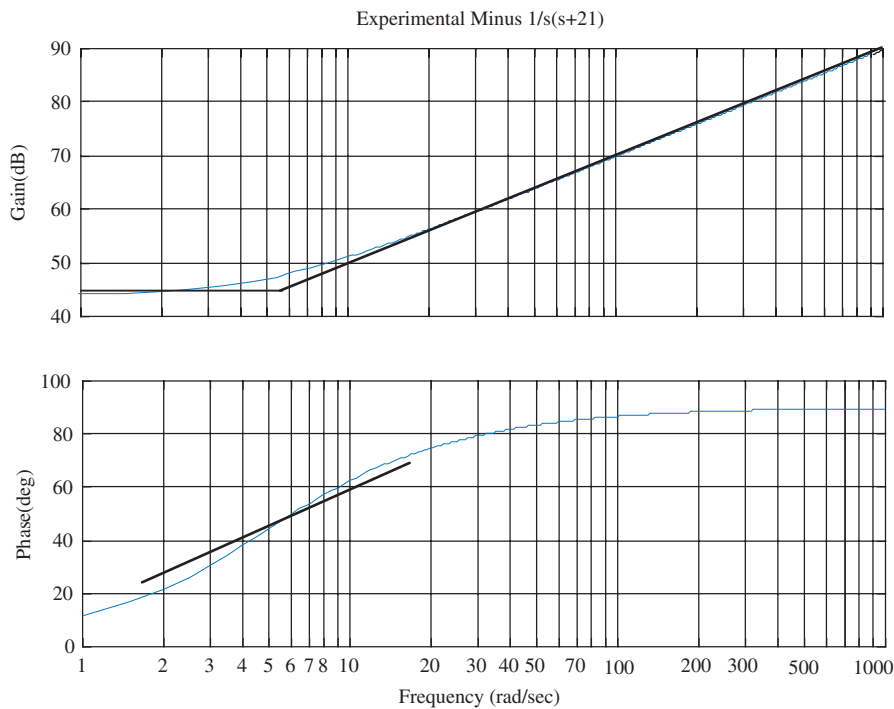
Therefore the phase margin is  $28.59 - 180 = -151.41$  deg. Thus, the system is unstable.

### 10.12

Drawing judiciously selected slopes on the magnitude and phase plot as shown below yields a first estimate.



We see an initial slope on the magnitude plot of  $-20$  dB/dec. We also see a final  $-20$  dB/dec slope with a break frequency around  $21$  rad/s. Thus, an initial estimate is  $G_1(s) = \frac{1}{s(s+21)}$ . Subtracting  $G_1(s)$  from the original frequency response yields the frequency response shown below.



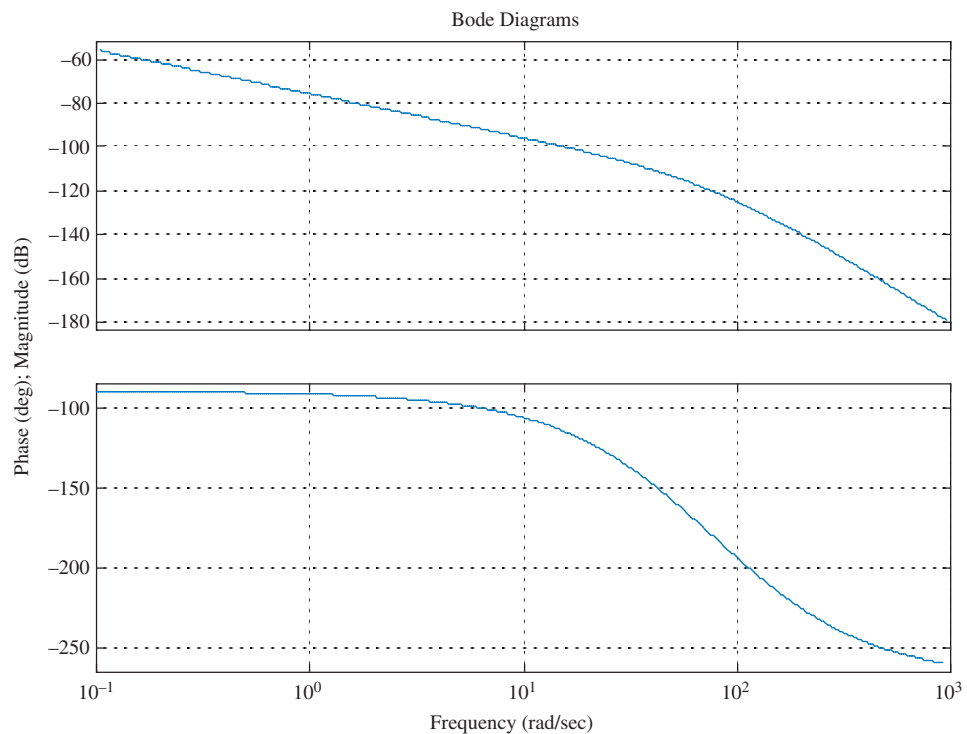
Drawing judiciously selected slopes on the magnitude and phase plot as shown yields a final estimate. We see first-order zero behavior on the magnitude and phase plots with a break frequency of about 5.7 rad/s and a dc gain of about 44 dB =  $20 \log(5.7K)$ , or  $K = 27.8$ . Thus, we estimate  $G_2(s) = 27.8(s + 7)$ . Thus,  $G(s) = G_1(s)G_2(s) = \frac{27.8(s + 5.7)}{s(s + 21)}$ . It is interesting to note that the original problem was developed from

$$G(s) = \frac{30(s + 5)}{s(s + 20)}$$

## CHAPTER 11

### 11.1

The Bode plot for  $K = 1$  is shown below.

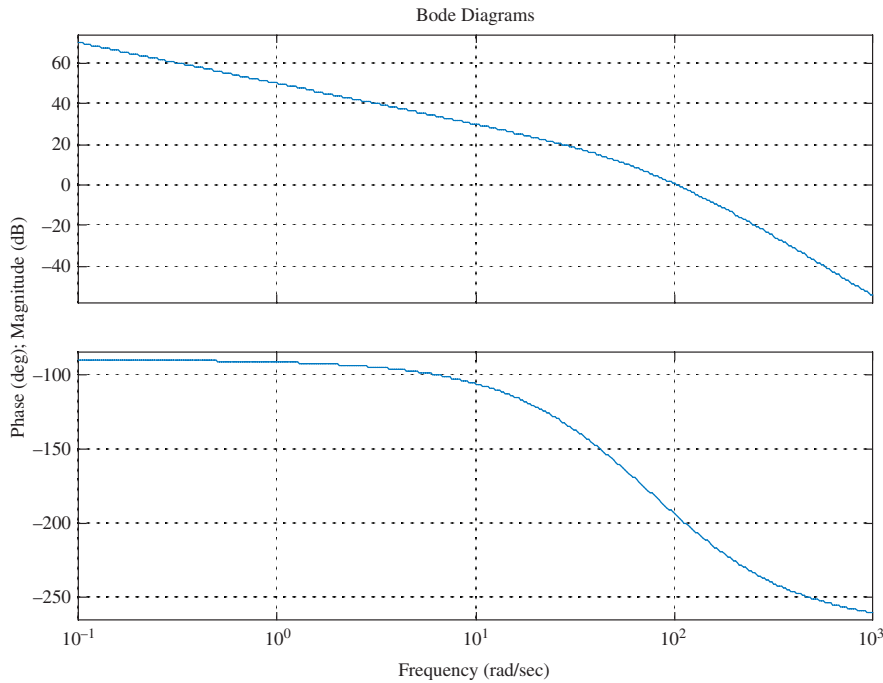


A 20% overshoot requires  $\zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.456$ . This damping ratio implies a phase margin of 48.10, which is obtained when the phase angle =  $-180 + 48.10 = 131.9^\circ$ . This phase angle occurs at  $\omega = 27.6$  rad/s. The magnitude at this frequency is  $5.15 \times 10^{-6}$ . Since the magnitude must be unity  $K = \frac{1}{5.15 \times 10^{-6}} = 194,200$ .

### 11.2

To meet the steady-state error requirement,  $K = 1,942,000$ . The Bode plot for this gain is shown below.





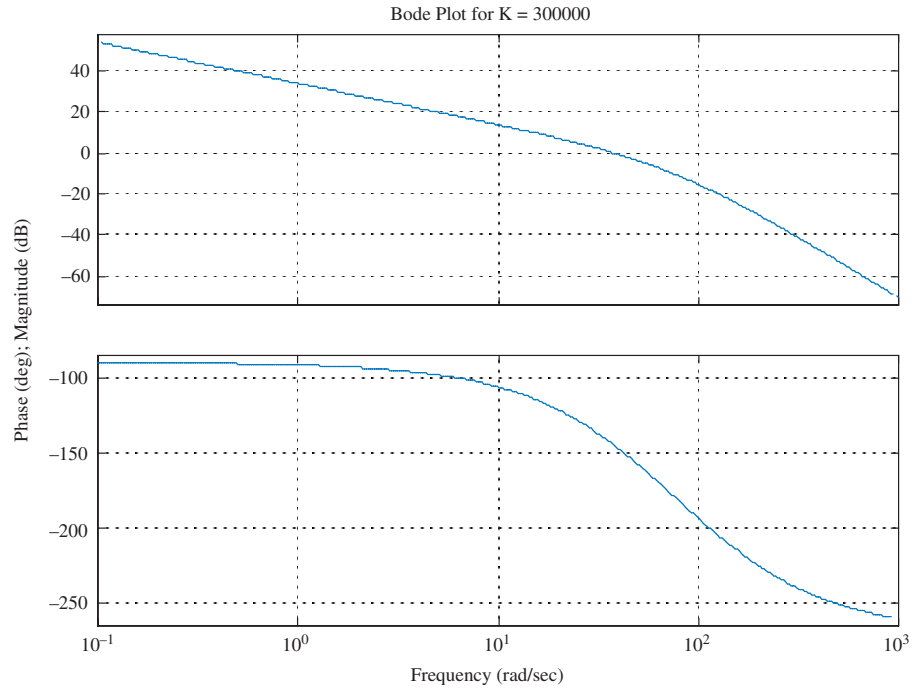
A 20% overshoot requires  $\zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.456$ . This damping ratio

implies a phase margin of  $48.1^\circ$ . Adding  $10^\circ$  to compensate for the phase angle contribution of the lag, we use  $58.1^\circ$ . Thus, we look for a phase angle of  $-180^\circ + 58.1^\circ = -121.9^\circ$ . The frequency at which this phase occurs is 20.4 rad/s. At this frequency the magnitude plot must go through zero dB. Presently, the magnitude plot is 23.2 dB. Therefore draw the high frequency asymptote of the lag compensator at  $-23.2$  dB/dec slope. Insert a break at  $0.1(20.4) = 2.04$  rad/s. At this frequency, draw  $-23.2$  dB/dec slope until it intersects 0 dB. The frequency of intersection will be the low frequency break or 0.141 rad/s. Hence the compensator is  $G_c(s) = K_c \frac{(s + 2.04)}{(s + 0.141)}$ , where the gain is chosen to yield 0 dB at low frequencies, or  $K_c = 0.141/2.04 = 0.0691$ . In summary,

$$G_c(s) = 0.0691 \frac{(s + 2.04)}{(s + 0.141)} \text{ and } G(s) = \frac{1,942,000}{s(s + 50)(s + 120)}$$

### 11.3

A 20% overshoot requires  $\zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.456$ . The required bandwidth is then calculated as  $\omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 57.9$  rad/s. In order to meet the steady-state error requirement of  $K_v = 50 = \frac{K}{(50)(120)}$ , we calculate  $K = 300,000$ . The uncompensated Bode plot for this gain is shown below.



The uncompensated system's phase margin measurement is taken where the magnitude plot crosses 0 dB. We find that when the magnitude plot crosses 0 dB, the phase angle is  $-144.8^\circ$ . Therefore, the uncompensated system's phase margin is  $-180^\circ + 144.8^\circ = 35.2^\circ$ . The required phase margin based on the required damping ratio is

$$\Phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}} = 48.1^\circ. \text{ Adding a } 10^\circ \text{ correction factor, the}$$

required phase margin is  $58.1^\circ$ . Hence, the compensator must contribute  $\phi_{\max} = 58.1^\circ - 35.2^\circ = 22.9^\circ$ . Using  $\phi_{\max} = \sin^{-1} \frac{1 - \beta}{1 + \beta}$ ,  $\beta = \frac{1 - \sin \phi_{\max}}{1 + \sin \phi_{\max}} = 0.44$ .

The compensator's peak magnitude is calculated as  $M_{\max} = \frac{1}{\sqrt{\beta}} = 1.51$ . Now find the

frequency at which the uncompensated system has a magnitude  $1/M_{\max}$ , or  $-3.58$  dB. From the Bode plot, this magnitude occurs at  $\omega_{\max} = 50$  rad/s. The compensator's zero

is at  $z_c = \frac{1}{T}$ .  $\omega_{\max} = \frac{1}{T\sqrt{\beta}}$ . Therefore,  $z_c = 33.2$ .

The compensator's pole is at  $p_c = \frac{1}{\beta T} = \frac{z_c}{\beta} = 75.4$ . The compensator gain is chosen to yield unity gain at dc.

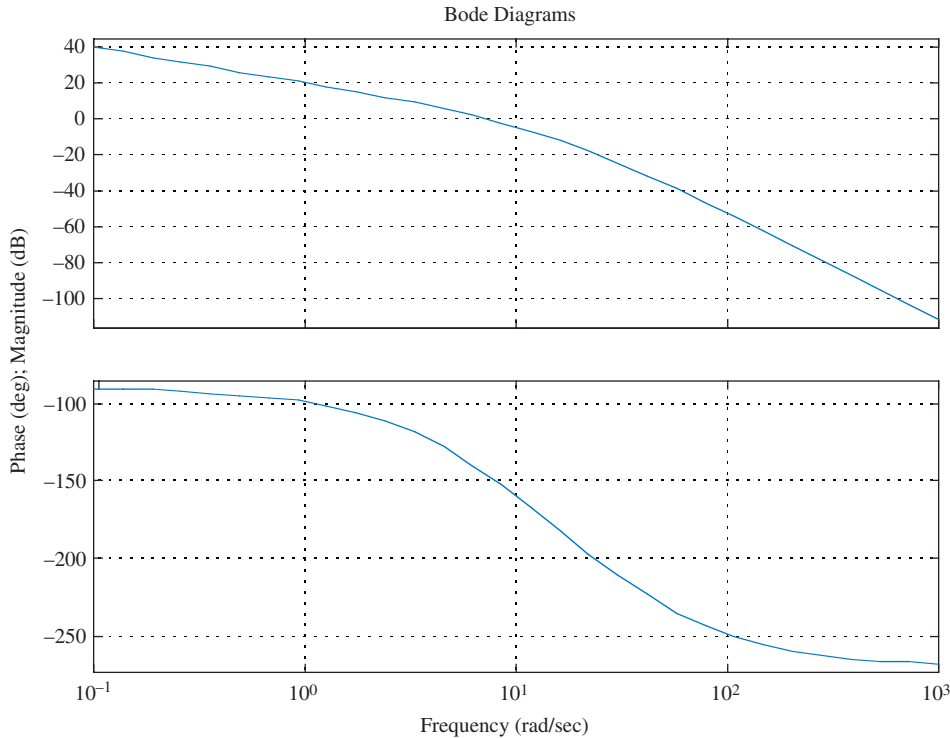
Hence,  $K_c = 75.4/33.2 = 2.27$ . Summarizing,  $G_c(s) = 2.27 \frac{(s + 33.2)}{(s + 75.4)}$ , and  $G(s) = \frac{300,000}{s(s + 50)(s + 120)}$ .

#### 11.4

A 10% overshoot requires  $\zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.591$ . The required bandwidth is

then calculated as  $\omega_{BW} = \frac{\pi}{T_p \sqrt{1 - \zeta^2}} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 7.53$  rad/s.

In order to meet the steady-state error requirement of  $K_v = 10 = \frac{K}{(8)(30)}$ , we calculate  $K = 2400$ . The uncompensated Bode plot for this gain is shown below.



Let us select a new phase-margin frequency at  $0.8\omega_{BW} = 6.02$  rad/s. The required phase margin based on the required damping ratio is  $\Phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}$  is  $58.6^\circ$ . Adding a  $5^\circ$  correction factor, the required phase margin is  $63.6^\circ$ . At 6.02 rad/s, the new phase-margin frequency, the phase angle is— which represents a phase margin of  $180^\circ - 138.3^\circ = 41.7^\circ$ . Thus, the lead compensator must contribute  $\phi_{\max} = 63.6^\circ - 41.7^\circ = 21.9^\circ$ .

Using  $\phi_{\max} = \sin^{-1} \frac{1 - \beta}{1 + \beta}$ ,  $\beta = \frac{1 - \sin\phi_{\max}}{1 + \sin\phi_{\max}} = 0.456$ .

We now design the lag compensator by first choosing its higher break frequency one decade below the new phase-margin frequency, that is,  $z_{lag} = 0.602$  rad/s. The lag compensator's pole is  $p_{lag} = \beta z_{lag} = 0.275$ . Finally, the lag compensator's gain is  $K_{lag} = \beta = 0.456$ .

Now we design the lead compensator. The lead zero is the product of the new phase margin frequency and  $\sqrt{\beta}$ , or  $z_{lead} = 0.8\omega_{BW}\sqrt{\beta} = 4.07$ . Also,  $p_{lead} = \frac{z_{lead}}{\beta} = 8.93$ . Finally,  $K_{lead} = \frac{1}{\beta} = 2.19$ . Summarizing,

$$G_{lag}(s) = 0.456 \frac{(s + 0.602)}{(s + 0.275)}; \quad G_{lead}(s) = 2.19 \frac{(s + 4.07)}{(s + 8.93)}; \quad \text{and } K = 2400.$$

## CHAPTER 12

## 12.1

We first find the desired characteristic equation. A 5% overshoot requires

$$\zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.69. \text{ Also, } \omega_n = \frac{\pi}{T_p \sqrt{1 - \zeta^2}} = 14.47 \text{ rad/s. Thus, the}$$

characteristic equation is  $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 19.97s + 209.4$ . Adding a pole at  $-10$  to cancel the zero at  $-10$  yields the desired characteristic equation,  $(s^2 + 19.97s + 209.4)(s + 10) = s^3 + 29.97s^2 + 409.1s + 2094$ . The compensated system matrix in phase-variable form is

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_1) & -(36 + k_2) & -(15 + k_3) \end{bmatrix}. \text{ The characteristic equation for this}$$

system is  $|s\mathbf{I} - (\mathbf{A} - \mathbf{BK})| = s^3 + (15 + k_3)s^2 + (36 + k_2)s + (k_1)$ . Equating coefficients of this equation with the coefficients of the desired characteristic equation yields the gains as

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3] = [2094 \quad 373.1 \quad 14.97].$$

## 12.2

The controllability matrix is  $\mathbf{C}_M = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & -9 \\ 1 & -1 & 16 \end{bmatrix}$ . Since

$|\mathbf{C}_M| = 80$ ,  $\mathbf{C}_M$  is full rank, that is, rank 3. We conclude that the system is controllable.

## 12.3

First check controllability. The controllability matrix is  $\mathbf{C}_{Mz} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] =$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -17 \\ 1 & -9 & 81 \end{bmatrix}. \text{ Since } |\mathbf{C}_{Mz}| = -1, \mathbf{C}_{Mz} \text{ is full rank, that is, rank 3. We conclude that}$$

the system is controllable. We now find the desired characteristic equation. A 20%

$$\text{overshoot requires } \zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.456. \text{ Also, } \omega_n = \frac{4}{\zeta T_s} = 4.386 \text{ rad/s.}$$

Thus, the characteristic equation is  $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 4s + 19.24$ . Adding a pole at  $-6$  to cancel the zero at  $-6$  yields the resulting desired characteristic equation,

$$(s^2 + 4s + 19.24)(s + 6) = s^3 + 10s^2 + 43.24s + 115.45.$$

Since  $G(s) = \frac{(s + 6)}{(s + 7)(s + 8)(s + 9)} = \frac{s + 6}{s^3 + 24s^2 + 191s + 504}$ , we can write the

phase-variable representation as  $\mathbf{A}_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -504 & -191 & -24 \end{bmatrix}; \mathbf{B}_p = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \mathbf{C}_p =$

$\begin{bmatrix} 6 & 1 & 0 \end{bmatrix}$ . The compensated system matrix in phase-variable form is  $\mathbf{A}_p - \mathbf{B}_p \mathbf{K}_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(504 + k_1) & -(191 + k_2) & -(24 + k_3) \end{bmatrix}$ . The characteristic equation for this system is  $|s\mathbf{I} - (\mathbf{A}_p - \mathbf{B}_p \mathbf{K}_p)| = s^3 + (24 + k_3)s^2 + (191 + k_2)s + (504 + k_1)$ . Equating coefficients of this equation with the coefficients of the desired characteristic equation yields the gains as  $\mathbf{K}_p = [k_1 \ k_2 \ k_3] = [-388.55 \ -147.76 \ -14]$ . We now develop the transformation matrix to transform back to the  $z$ -system.

$$\mathbf{C}_{Mz} = \begin{bmatrix} \mathbf{B}_z & \mathbf{A}_z \mathbf{B}_z & \mathbf{A}_z^2 \mathbf{B}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -17 \\ 1 & -9 & 81 \end{bmatrix} \text{ and}$$

$$\mathbf{C}_{Mp} = \begin{bmatrix} \mathbf{B}_p & \mathbf{A}_p \mathbf{B}_p & \mathbf{A}_p^2 \mathbf{B}_p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -24 \\ 1 & -24 & 385 \end{bmatrix}.$$

Therefore,

$$\mathbf{P} = \mathbf{C}_{Mz} \mathbf{C}_{Mp}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -17 \\ 1 & -9 & 81 \end{bmatrix} \begin{bmatrix} 191 & 24 & 1 \\ 24 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 56 & 15 & 1 \end{bmatrix}$$

$$\text{Hence, } \mathbf{K}_z = \mathbf{K}_p \mathbf{P}^{-1} = [-388.55 \ -147.76 \ -14] \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 49 & -15 & 1 \end{bmatrix}$$

$$= [-40.23 \ 62.24 \ -14].$$

#### 12.4

For the given system  $\mathbf{e}_x = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}_x = \begin{bmatrix} -(24 + l_1) & 1 & 0 \\ -(191 + l_2) & 0 & 1 \\ -(504 + l_3) & 0 & 0 \end{bmatrix} \mathbf{e}_x$ . The characteristic polynomial is given by  $|[s\mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C})]| = s^3 + (24 + l_1)s^2 + (191 + l_2)s + (504 + l_3)$ . Now we find the desired characteristic equation. The dominant poles from Skill-Assessment Exercise 12.3 come from  $(s^2 + 4s + 19.24)$ . Factoring yields  $(-2 + j3.9)$  and  $(-2 - j3.9)$ . Increasing these poles by a factor of 10 and adding a third pole 10 times the real part of the dominant second-order poles yields the desired characteristic polynomial,  $(s + 20 + j39)(s + 20 - j39)(s + 200) = s^3 + 240s^2 + 9921s + 384200$ . Equating coefficients of the desired characteristic equation to the system's characteristic

equation yields  $\mathbf{L} = \begin{bmatrix} 216 \\ 9730 \\ 383696 \end{bmatrix}$ .

#### 12.5

The observability matrix is  $\mathbf{O}_M = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ -64 & -80 & -78 \\ 674 & 848 & 814 \end{bmatrix}$ , where

$$\mathbf{A}^2 = \begin{bmatrix} 25 & 28 & 32 \\ -7 & -4 & -11 \\ 77 & 95 & 94 \end{bmatrix}.$$

The matrix is of full rank, that is, rank 3, since

$|\mathbf{O}_M| = -1576$ . Therefore the system is observable.

## 12.6

The system is represented in cascade form by the following state and output equations:

$$\dot{\mathbf{z}} = \begin{bmatrix} -7 & 1 & 0 \\ 0 & -8 & 1 \\ 0 & 0 & -9 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0] \mathbf{z}$$

The observability matrix is  $\mathbf{O}_{\mathbf{Mz}} = \begin{bmatrix} \mathbf{C}_z \\ \mathbf{C}_z \mathbf{A}_z \\ \mathbf{C}_z \mathbf{A}_z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 49 & -15 & 1 \end{bmatrix}$ , where  $\mathbf{A}_z^2 = \begin{bmatrix} 49 & -15 & 1 \\ 0 & 64 & -17 \\ 0 & 0 & 81 \end{bmatrix}$ . Since  $G(s) = \frac{1}{(s+7)(s+8)(s+9)} = \frac{1}{s^3 + 24s^2 + 191s + 504}$ ,

we can write the observable canonical form as

$$\dot{\mathbf{x}} = \begin{bmatrix} -24 & 1 & 0 \\ -191 & 0 & 1 \\ -504 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0] \mathbf{x}$$

The observability matrix for this form is  $\mathbf{O}_{\mathbf{Mx}} = \begin{bmatrix} \mathbf{C}_x \\ \mathbf{C}_x \mathbf{A}_x \\ \mathbf{C}_x \mathbf{A}_x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -24 & 1 & 0 \\ 385 & -24 & 1 \end{bmatrix}$ ,

where  $\mathbf{A}_x^2 = \begin{bmatrix} 385 & -24 & 1 \\ 4080 & -191 & 0 \\ 12096 & 504 & 0 \end{bmatrix}$ .

We next find the desired characteristic equation. A 10% overshoot requires

$$\zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.591. \text{ Also, } \omega_n = \frac{4}{\zeta T_s} = 67.66 \text{ rad/s. Thus, the character-}$$

istic equation is  $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 80s + 4578.42$ . Adding a pole at  $-400$ , or 10 times the real part of the dominant second-order poles, yields the resulting desired characteristic equation,  $(s^2 + 80s + 4578.42)(s + 400) = s^3 + 480s^2 + 36580s + 1.831 \times 10^6$ . For the system represented in observable canonical form  $\mathbf{e}_x =$

$$(\mathbf{A}_x - \mathbf{L}_x \mathbf{C}_x) \mathbf{e}_x = \begin{bmatrix} -(24 + l_1) & 1 & 0 \\ -(191 + l_2) & 0 & 1 \\ -(504 + l_3) & 0 & 0 \end{bmatrix} \mathbf{e}_x. \text{ The characteristic polynomial is given}$$

by  $||[s\mathbf{I} - (\mathbf{A}_x - \mathbf{L}_x \mathbf{C}_x)]|| = s^3 + (24 + l_1)s^2 + (191 + l_2)s + (504 + l_3)$ . Equating coefficients of the desired characteristic equation to the system's characteristic equation

$$\text{yields } \mathbf{L}_x = \begin{bmatrix} 456 \\ 36,389 \\ 1,830,496 \end{bmatrix}.$$

Now, develop the transformation matrix between the observer canonical and cascade forms.

$$\begin{aligned} \mathbf{P} = \mathbf{O}_{Mz}^{-1} \mathbf{O}_{Mx} &= \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 49 & -15 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -24 & 1 & 0 \\ 385 & -24 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 56 & 15 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -24 & 1 & 0 \\ 385 & -24 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -17 & 1 & 0 \\ 81 & -9 & 1 \end{bmatrix}. \end{aligned}$$

Finally,

$$\mathbf{L}_z = \mathbf{P} \mathbf{L}_x = \begin{bmatrix} 1 & 0 & 0 \\ -17 & 1 & 0 \\ 81 & -9 & 1 \end{bmatrix} \begin{bmatrix} 456 \\ 36,389 \\ 1,830,496 \end{bmatrix} = \begin{bmatrix} 456 \\ 28,637 \\ 1,539,931 \end{bmatrix} \approx \begin{bmatrix} 456 \\ 28,640 \\ 1,540,000 \end{bmatrix}.$$

## 12.7

We first find the desired characteristic equation. A 10% overshoot requires

$$\zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.591$$

Also,  $\omega_n = \frac{\pi}{T_p \sqrt{1 - \zeta^2}} = 1.948$  rad/s. Thus, the characteristic equation is  $s^2 +$

$2\zeta\omega_n s + \omega_n^2 = s^2 + 2.3s + 3.79$ . Adding a pole at  $-4$ , which corresponds to the original system's zero location, yields the resulting desired characteristic equation,  $(s^2 + 2.3s + 3.79)(s + 4) = s^3 + 6.3s^2 + 13s + 15.16$ .

$$\text{Now, } \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{K}) & \mathbf{B}\mathbf{K}_e \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r; \text{ and } y = [\mathbf{C} \ 0] \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{A} - \mathbf{B}\mathbf{K} &= \begin{bmatrix} 0 & 1 \\ -7 & -9 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \ k_2] = \begin{bmatrix} 0 & 1 \\ -7 & -9 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -(7+k_1) & -(9+k_2) \end{bmatrix} \\ \mathbf{C} &= [4 \ 1] \\ \mathbf{B}\mathbf{K}_e &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} k_e = \begin{bmatrix} 0 \\ k_e \end{bmatrix} \end{aligned}$$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -(7+k_1) & -(9+k_2) & k_e \\ -4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r; \text{ and } y = [4 \ 1 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix}.$$

Finding the characteristic equation of this system yields

$$\begin{aligned} \left| s\mathbf{I} - \begin{bmatrix} (\mathbf{A} - \mathbf{BK}) & \mathbf{BK}_e \\ -\mathbf{C} & 0 \end{bmatrix} \right| &= \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -(7+k_1) & -(9+k_2) & k_e \\ -4 & -1 & 0 \end{bmatrix} \right| \\ &= \left| \begin{bmatrix} s & -1 & 0 \\ (7+k_1) & s+(9+k_2) & -k_e \\ 4 & 1 & s \end{bmatrix} \right| = s^3 + (9+k_2)s^2 + (7+k_1+k_e)s + 4k_e \end{aligned}$$

Equating this polynomial to the desired characteristic equation,

$$s^3 + 6.3s^2 + 13s + 15.16 = s^3 + (9+k_2)s^2 + (7+k_1+k_e)s + 4k_e$$

Solving for the  $k$ 's,

$$\mathbf{K} = [2.21 \quad -2.7] \text{ and } k_e = 3.79.$$

## CHAPTER 13

### 13.1

$$f(t) = \sin(\omega kT); f^*(t) = \sum_{k=0}^{\infty} \sin(\omega kT) \delta(t - kT);$$

$$\begin{aligned} F^*(s) &= \sum_{k=0}^{\infty} \sin(\omega kT) e^{-kTs} = \sum_{k=0}^{\infty} \frac{(e^{j\omega kT} - e^{-j\omega kT}) e^{-kTs}}{2j} \\ &= \frac{1}{2j} \sum_{k=0}^{\infty} (e^{T(s-j\omega)})^{-k} - (e^{T(s+j\omega)})^{-k} \\ \text{But, } \sum_{k=0}^{\infty} x^{-k} &= \frac{1}{1-x^{-1}} \end{aligned}$$

Thus,

$$\begin{aligned} F^*(s) &= \frac{1}{2j} \left[ \frac{1}{1 - e^{-T(s-j\omega)}} - \frac{1}{1 - e^{-T(s+j\omega)}} \right] = \frac{1}{2j} \left[ \frac{e^{-Ts} e^{j\omega T} - e^{-Ts} e^{-j\omega T}}{1 - (e^{-Ts} e^{j\omega T} - e^{-Ts} e^{-j\omega T}) + e^{-2Ts}} \right] \\ &= e^{-Ts} \left[ \frac{\sin(\omega T)}{1 - e^{-Ts} 2\cos(\omega T) + e^{-2Ts}} \right] = \frac{z^{-1} \sin(\omega T)}{1 - 2z^{-1} \cos(\omega T) + z^{-2}} \end{aligned}$$

### 13.2

$$F(z) = \frac{z(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)}$$

$$\frac{F(z)}{z} = \frac{(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)}$$

$$= 46.875 \frac{z}{z-0.5} - 114.75 \frac{z}{z-0.7} + 68.875 \frac{z}{z-0.9}$$

$$F(z) = 46.875 \frac{z}{z-0.5} - 114.75 \frac{z}{z-0.7} + 68.875 \frac{z}{z-0.9},$$

$$f(kT) = 46.875(0.5)^k - 114.75(0.7)^k + 68.875(0.9)^k$$



**13.3**

Since  $G(s) = (1 - e^{-Ts}) \frac{8}{s(s+4)}$ ,

$$G(z) = (1 - z^{-1})z \left\{ \frac{8}{s(s+4)} \right\} = \frac{z-1}{z} z \left\{ \frac{A}{s} + \frac{B}{s+4} \right\} = \frac{z-1}{z} z \left\{ \frac{2}{s} + \frac{2}{s+4} \right\}.$$

Let  $G_2(s) = \frac{2}{s} + \frac{2}{s+4}$ . Therefore,  $g_2(t) = 2 - 2e^{-4t}$ , or  $g_2(kT) = 2 - 2e^{-4kT}$ .

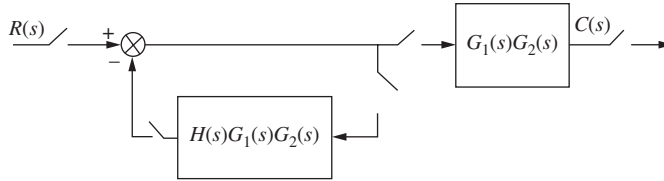
$$\text{Hence, } G_2(z) = \frac{2z}{z-1} - \frac{2z}{z-e^{-4T}} = \frac{2z(1-e^{-4T})}{(z-1)(z-e^{-4T})}.$$

$$\text{Therefore, } G(z) = \frac{z-1}{z} G_2(z) = \frac{2(1-e^{-4T})}{(z-e^{-4T})}.$$

$$\text{For } T = \frac{1}{4}s, G(z) = \frac{1.264}{z-0.3679}.$$

**13.4**

Add phantom samplers to the input, feedback after  $H(s)$ , and to the output. Push  $G_1(s)G_2(s)$ , along with its input sampler, to the right past the pickoff point and obtain the block diagram shown below.



$$\text{Hence, } T(z) = \frac{G_1 G_2(z)}{1 + H G_1 G_2(z)}.$$

**13.5**

Let  $G(s) = \frac{20}{s+5}$ . Let  $G_2(s) = \frac{G(s)}{s} = \frac{20}{s(s+5)} = \frac{4}{s} - \frac{4}{s+5}$ . Taking the inverse

Laplace transform and letting  $t = kT$ ,  $g_2(kT) = 4 - 4e^{-5kT}$ . Taking the z-transform

$$\text{yields } G_2(z) = \frac{4z}{z-1} - \frac{4z}{z-e^{-5T}} = \frac{4z(1-e^{-5T})}{(z-1)(z-e^{-5T})}.$$

$$\text{Now, } G(z) = \frac{z-1}{z} G_2(z) = \frac{4(1-e^{-5T})}{(z-e^{-5T})}. \text{ Finally, } T(z) = \frac{G(z)}{1+G(z)} = \frac{4(1-e^{-5T})}{z-5e^{-5T}+4}.$$

The pole of the closed-loop system is at  $5e^{-5T} - 4$ . Substituting values of  $T$ , we find that the pole is greater than 1 if  $T > 0.1022$  s. Hence, the system is stable for  $0 < T < 0.1022$  s.

**13.6**

Substituting  $z = \frac{s+1}{s-1}$  into  $D(z) = z^3 - z^2 - 0.5z + 0.3$ , we obtain  $D(s) = s^3 - 8s^2 - 27s - 6$ . The Routh table for this polynomial is shown below.

$s^3$	1	-27
$s^2$	-8	-6
$s^1$	-27.75	0
$s^0$	-6	0

Since there is one sign change, we conclude that the system has one pole outside the unit circle and two poles inside the unit circle. The table did not produce a row of zeros and thus, there are no  $j\omega$  poles. The system is unstable because of the pole outside the unit circle.

### 13.7

Defining  $G(s)$  as  $G_1(s)$  in cascade with a zero-order-hold,

$$G(s) = 20(1 - e^{-Ts}) \left[ \frac{(s+3)}{s(s+4)(s+5)} \right] = 20(1 - e^{-Ts}) \left[ \frac{3/20}{s} + \frac{1/4}{(s+4)} - \frac{2/5}{(s+5)} \right].$$

Taking the  $z$ -transform yields

$$G(z) = 20(1 - z^{-1}) \left[ \frac{(3/20)z}{z-1} + \frac{(1/4)z}{z - e^{-4T}} - \frac{(2/5)z}{z - e^{-5T}} \right] = 3 + \frac{5(z-1)}{z - e^{-4T}} - \frac{8(z-1)}{z - e^{-5T}}.$$

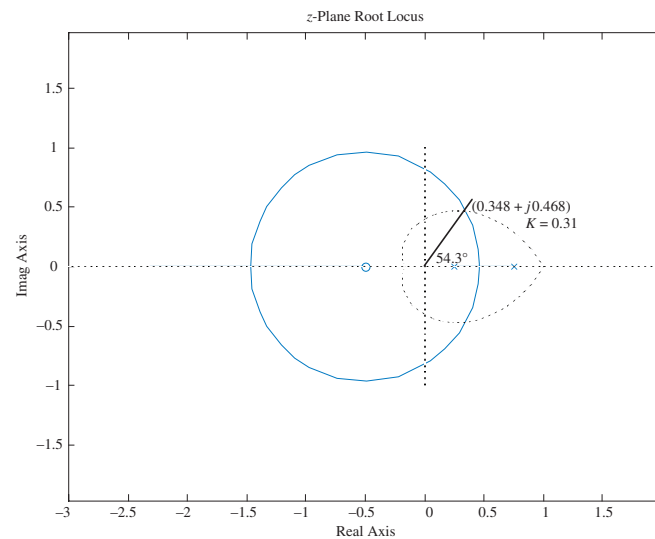
Hence for  $T = 0.1$  second,  $K_p = \lim_{z \rightarrow 1} G(z) = 3$ , and  $K_v = \frac{1}{T} \lim_{z \rightarrow 1} (z-1)G(z) = 0$ , and  $K_a = \frac{1}{T^2} \lim_{z \rightarrow 1} (z-1)^2 G(z) = 0$ . Checking for stability, we find that the system is stable

for  $T = 0.1$  second, since  $T(z) = \frac{G(z)}{1 + G(z)} = \frac{1.5z - 1.109}{z^2 + 0.222z - 0.703}$  has poles inside the

unit circle at  $-0.957$  and  $+0.735$ . Again, checking for stability, we find that the system is unstable for  $T = 0.5$  second, since  $T(z) = \frac{G(z)}{1 + G(z)} = \frac{3.02z - 0.6383}{z^2 + 2.802z - 0.6272}$  has poles inside and outside the unit circle at  $+0.208$  and  $-3.01$ , respectively.

### 13.8

Draw the root locus superimposed over the  $\zeta = 0.5$  curve shown below. Searching along a  $54.3^\circ$  line, which intersects the root locus and the  $\zeta = 0.5$  curve, we find the point  $0.587 \angle 54.3^\circ = (0.348 + j0.468)$  and  $K = 0.31$ .

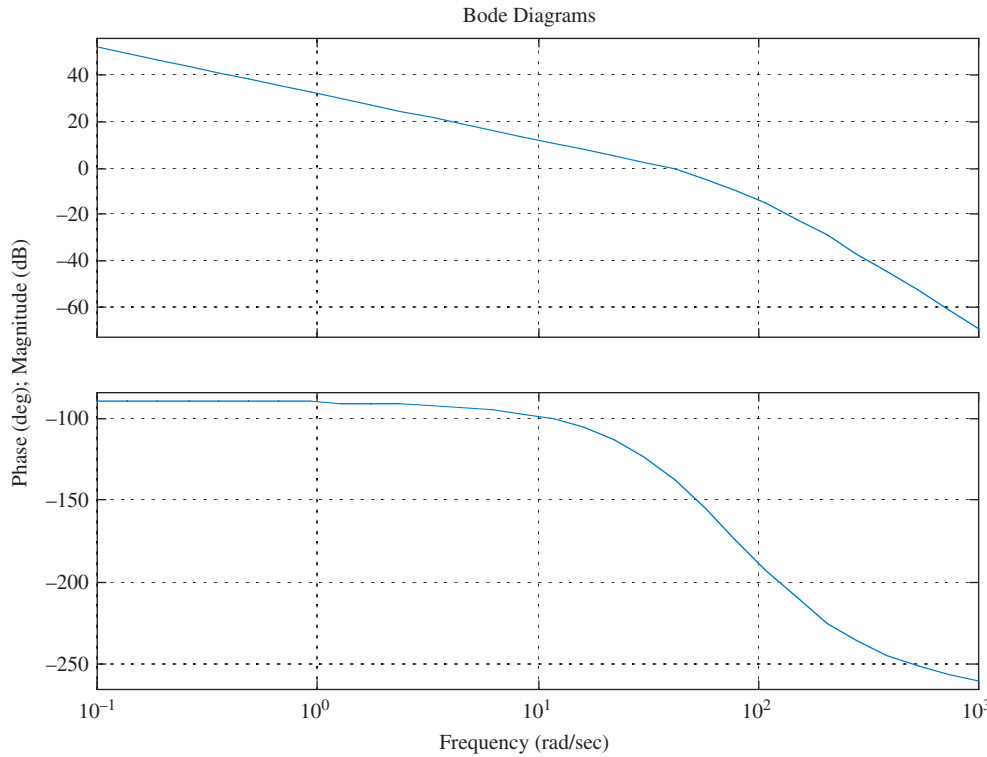


## 13.9

Let

$$G_e(s) = G(s)G_c(s) = \frac{100K}{s(s+36)(s+100)} \frac{2.38(s+25.3)}{(s+60.2)}$$

$$= \frac{342720(s+25.3)}{s(s+36)(s+100)(s+60.2)}.$$

The following shows the frequency response of  $G_e(j\omega)$ .

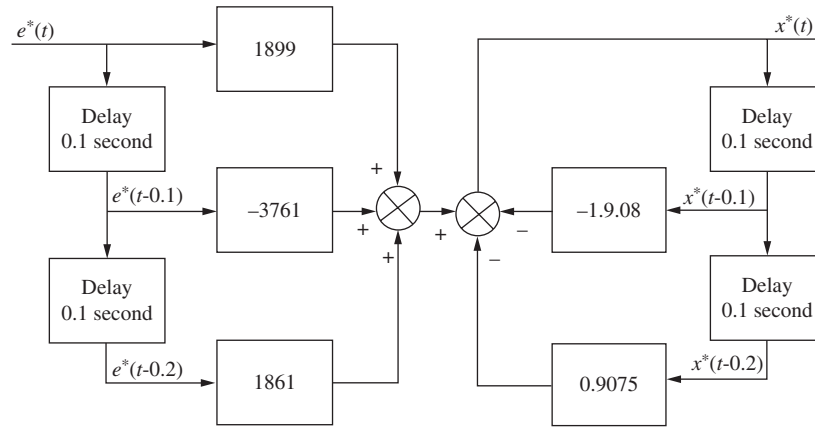
We find that the zero dB frequency,  $\omega_{\Phi_M}$ , for  $G_e(j\omega)$  is 39 rad/s. Using Astrom's guideline the value of  $T$  should be in the range,  $0.15/\omega_{\Phi_M} = 0.0038$  second to  $0.5/\omega_{\Phi_M} = 0.0128$  second. Let us use  $T = 0.001$  second. Now find the Tustin transformation for the compensator. Substituting  $s = \frac{2(z-1)}{T(z+1)}$  into  $G_c(s) = \frac{2.38(s+25.3)}{(s+60.2)}$  with  $T = 0.001$  second yields

$$G_c(z) = 2.34 \frac{(z - 0.975)}{(z - 0.9416)}.$$

## 13.10

$G_c(z) = \frac{X(z)}{E(z)} = \frac{1899z^2 - 3761z + 1861}{z^2 - 1.908z + 0.9075}$ . Cross-multiply and obtain  $(z^2 - 1.908z + 0.9075)X(z) = (1899z^2 - 3761z + 1861)E(z)$ . Solve for the highest power of  $z$  operating on the output,  $X(z)$ , and obtain  $z^2X(z) = (1899z^2 - 3761z + 1861)$

$E(z) - (-1.908z + 0.9075)X(z)$ . Solving for  $X(z)$  on the left-hand side yields  $X(z) = (1899 - 3761z^{-1} + 1861z^{-2})E(z) - (-1.908z^{-1} + 0.9075z^{-2})X(z)$ . Finally, we implement this last equation with the following flow chart:



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