

MIDTERM

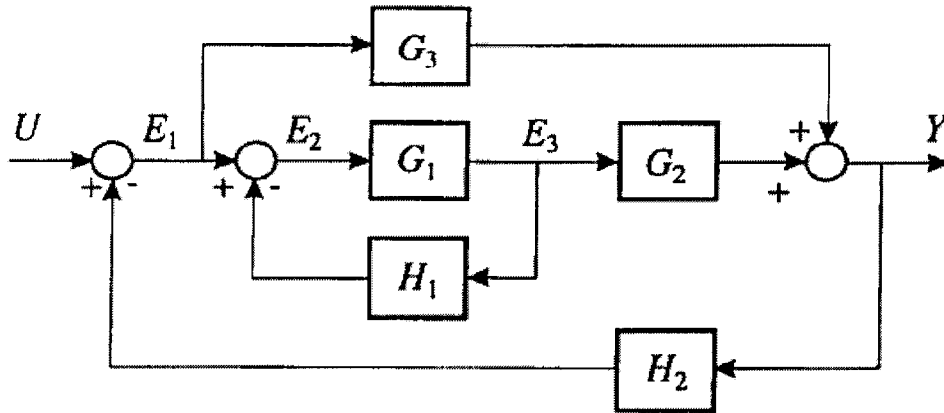
May 1, 2007

NAME: Solutions PID: _____

- One page (front and back) of your own handwritten notes.
- No graphing calculators
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 40 (to be normalized to account for 30% of your total grade in the class)
- Time: 1 hour 15 minutes.

Problem 1: Block Diagram Reduction (5 points)

Consider the following block diagram:



Using block diagram algebra, reduce the block diagram to find the transfer function $T = \frac{Y}{U}$. Simplify the transfer function as much as possible to receive full credit.

(a) Using block diagram algebra rules, this block diagram is simplified as shown in Figure 2.19. The required transfer function is given in Figure 2.19d.

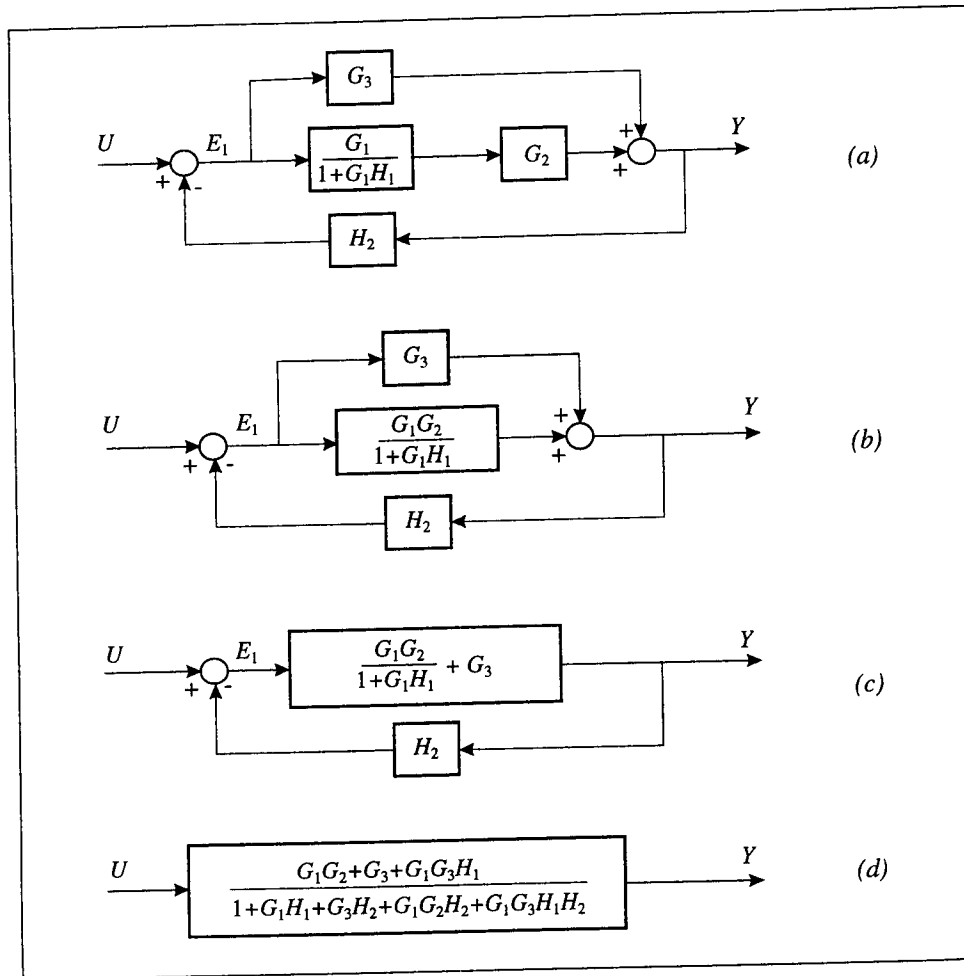
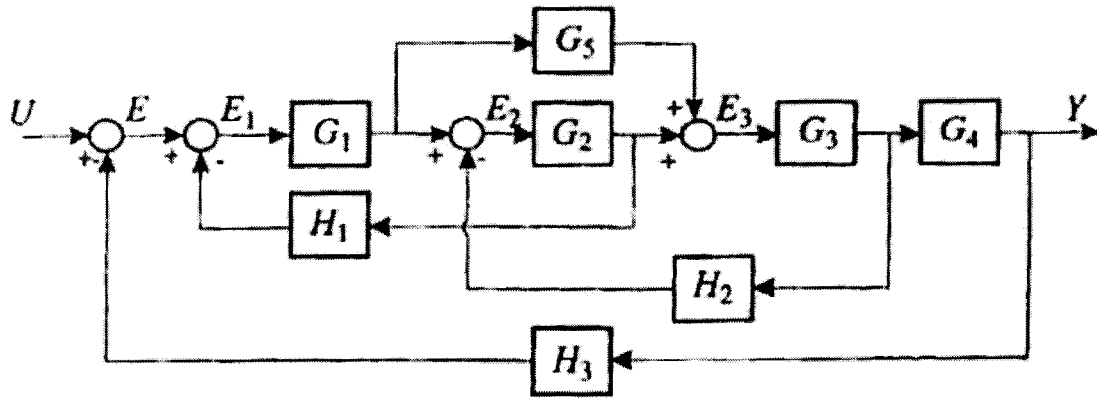


Figure 2.19: Block diagram simplification using block diagram algebra rules

Problem 2: Mason's Rule (8 points)

Use Mason's rule to find the transfer function $T = \frac{Y}{U}$, for the following system:



The corresponding signal flow graph is presented in Figure 2.22. In this example we have two paths and five loops. There are no nontouching loops.

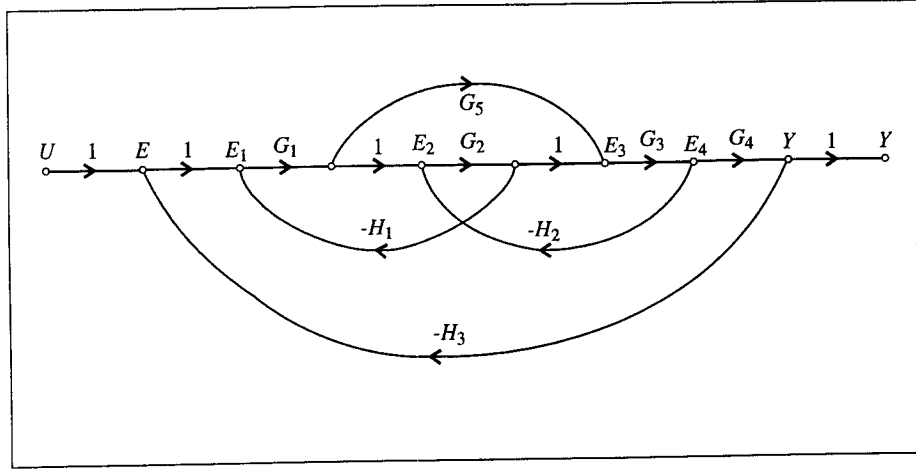


Figure 2.22: Signal flow graph for the system given in Figure 2.21

The corresponding path gains $P_i, i = 1, 2$, loop gains $L_j, j = 1, 2, \dots, 5$, signal flow graph determinant, and cofactors are given by

$$P_1 = G_1G_2G_3G_4, \quad P_2 = G_1G_5G_3G_4$$

$$L_1 = -G_1G_2H_1, \quad L_2 = -G_2G_3H_2, \quad L_3 = G_1G_5G_3H_2G_2H_1$$

$$L_4 = -G_1G_5G_3G_4H_3, \quad L_5 = -G_1G_2G_3G_4H_3$$

$$\Delta_1 = 1, \quad \Delta_2 = 1$$

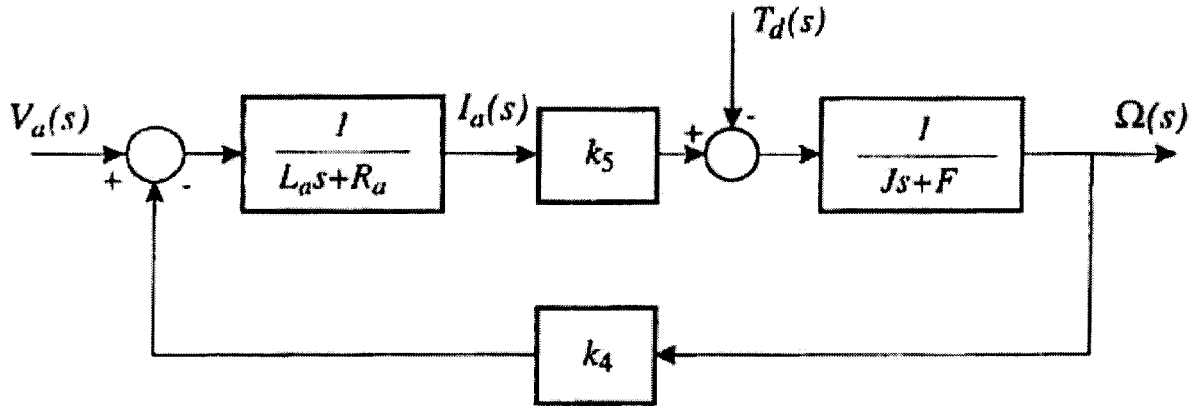
$$\Delta = 1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3G_4H_3 + G_1G_5G_4H_3 - G_1G_5G_3H_2G_2H_1$$

so that the closed-loop transfer function for this system is obtained as

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3G_4 + G_1G_3G_4G_5}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3G_4H_3 + G_1G_3G_4G_5H_3 - G_1G_2G_3G_5H_1H_2}$$

Problem 3: Time Domain Specifications and Sensitivity (12 points)

The following block diagram models an armature-controlled DC motor.



- (a) (4 points) Find the transfer function $G(s)$ from $T_d(s)$ to $\Omega(s)$, and the transfer function $H(s)$ from $V_a(s)$ to $\Omega(s)$. Simplify the transfer functions as much as possible to receive full credit.
- (b) (3 points) Given $L_a = 4, R_a = 2, J = \frac{1}{4}, F = \frac{1}{2}, k_4 = 7$, calculate the sensitivity of the closed-loop transfer function with respect to changes in k_5 .
- (c) (2 points) Determine k_5 such that the overshoot is $M_p = e^{-\frac{\pi}{\sqrt{3}}}$.
- (d) (3 points) What are the settling time, rise time, and peak time for $H(s)$ determined in (c)?

$$\text{(a)} \quad G(s) = \frac{1}{Js + F} \cdot \frac{1}{1 + \frac{k_4 k_5}{(Js + F)(L_a s + R_a)}} = \frac{L_a s + R_a}{(Js + F)(L_a s + R_a) + k_4 k_5}$$

$$G(s) = \frac{L_a s + R_a}{L_a J s^2 + (R_a J + L_a F) s + R_a F + k_4 k_5}$$

$$H(s) = \frac{k_s}{(L_a s + R_a)(J s + F)} = \frac{k_s}{(L_a s + R_a)(J s + F) + k_y k_s}$$

$$H(s) = \frac{k_s}{L_a J s^2 + (R_a J + L_a F) s + R_a F + k_y k_s}$$

$$H(s) = \frac{k_s}{(4)(\frac{1}{4}) s^2 + ((2)(\frac{1}{4}) + (4)(\frac{1}{2})) s + (2)(\frac{1}{2}) + 7k_s} = \frac{k_s}{s^2 + \frac{5}{2} s + 1 + 7k_s} = \frac{A(s)}{B(s)}$$

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$$\frac{dH}{dk_s} = \frac{B(s) - 7k_s}{[B(s)]^2}$$

$$S_{k_s}^H = \frac{k_s B(s)}{A(s)} \left(\frac{B(s) - 7k_s}{[B(s)]^2} \right) = \frac{B(s) - 7k_s}{B(s)}$$

$$S_{k_s}^H = \frac{s^2 + \frac{5}{2} s + 1}{s^2 + \frac{5}{2} s + 7k_s + 1}$$

$$c) M_p = e^{-\pi/\sqrt{3}} = e^{-\pi \cdot 3/\sqrt{1-3^2}}$$

$$\frac{1}{\sqrt{3}} = \frac{3}{\sqrt{1-3^2}}$$

$$[\sqrt{1-3^2} = \sqrt{3} \cdot 3]^2$$

$$1-3^2 = 3 \cdot 3^2$$

$$\frac{1}{4} = 3^2 \Rightarrow 3 = \frac{1}{2}$$

$$2 \cdot 3 \omega_n = \frac{5}{2} \Rightarrow \omega_n = \frac{5}{2}$$

$$\omega_n^2 = 7k_s + 1 \Rightarrow k_s = \frac{\omega_n^2 - 1}{7} = \frac{\frac{25}{4} - 1}{7} = \frac{21/4}{7} = \frac{3}{4}$$

$$k_s = \frac{3}{4}$$

$$d) t_r = \frac{1.8}{\omega_n} = \frac{1.8}{(5/2)} = 0.72$$

$$t_r = 0.72$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{(5/2) \sqrt{1-\frac{1}{2}^2}} = 1.45$$

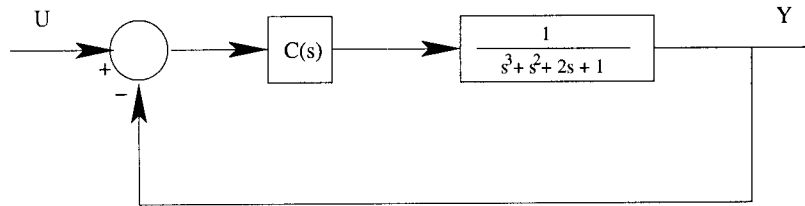
$$t_p = 1.45$$

$$3 \omega_n t_s = 4.6 \Rightarrow t_s = \frac{4.6}{(\frac{1}{2})(\frac{5}{2})} = 3.68$$

$$t_s = 3.68$$

Problem 4: PID Control (10 points)

Consider the following system:



- (a) (3 points) Using Routh's criterion, find the gain K_u that would make the feedback system marginally stable.
- (b) (1 point) Using Ziegler-Nichols tuning, find the gain k_p needed for plain P (proportional) control.
- (c) (3 points) The ultimate period corresponding to K_u in (a) for this plant can be shown to be $P_u = \sqrt{2}\pi$. What should then be the PI and PID control gains?
- (d) (3 points) Show that indeed $P_u = \sqrt{2}\pi$. (Note that this part is the trickiest part of the whole exam.)

(a)
$$\frac{Y}{U} = \frac{K}{s^3 + s^2 + 2s + 1 + K}$$

s^3	1	2
s^2	1	$1+K$
s^1	$1-K$	
s^0	$1+K$	

$\leftarrow K < 1 \Rightarrow K_u = 1$

(b) $K_p = \frac{1}{2} K_u = 0.5 \Rightarrow K_p = 0.5$

(c) $P_u = \sqrt{2} \pi$

PI: $K_p = 0.45 K_u = 0.45(1)$
 $T_I = \frac{P_u}{1.2} = \frac{\sqrt{2}\pi}{1.2} = 3.70$

$K_p = 0.45$
 $T_I = 3.70$

~~①~~ PID: $k_p = 0.6 k_u = 0.6(1) \Rightarrow k_p = 0.6$
 $T_I = \frac{1}{2} P_u = \frac{\sqrt{2}\pi}{2} \Rightarrow T_I = 2.22$
 $T_D = \frac{1}{8} P_u = \frac{\sqrt{2}\pi}{8} \Rightarrow T_D = 0.56$

① $T(s) = \frac{k_u}{s^3 + s^2 + 2s + 1 + k_u}$
 $= \frac{k_u}{(s + \sigma)(s^2 + \omega_n^2)}$
 $= \frac{k_u}{s^3 + \sigma s^2 + \omega_n^2 s + \sigma \omega_n^2}$

$$\sigma = 1$$

$$\omega_n^2 = 2$$

$$\sigma \omega_n^2 = 1 + k_u = 2$$

$$\sigma = 1$$

$$\omega_n = \sqrt{2}$$

$$P_u = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$$

Problem 5: Stability (5 points)

Are the following polynomials stable? If not, how many eigenvalues in the right-half plane do they have?

(a) (2 point)

$$p_a(s) = s^3 + 2s^2 + 2s + 1$$

(b) (3 points)

$$p_b(s) = s^6 + s^5 + 3s^4 + 2s^3 + s^2 + 2s + 1$$

(a)

s^3	1	2	
s^2	2	1	
s^1	3/2	0	
s^0	1		

\Rightarrow stable

(b)

s^6	1	3	1	1	
s^5	1	2	2		
s^4	1	-1	1		
s^3	3	1			
s^2	-4/3	1			
s^1	13/4				
s^0	1				

\Rightarrow Unstable
2 in RHP

