Solution of Assignment 2

Problem 1:

$$m_1\ddot{y}_1 + b_1(\dot{y}_1 - \dot{y}_2) + k_1 y_1 = u_1$$
$$m_2\ddot{y}_2 + b_1(\dot{y}_2 - \dot{y}_1) + k_2 y_2 = u_2$$

Problem 2:

The equations for the system are

$$m_1\ddot{x}_1 = -k_1 x_1 - b_1\dot{x}_1 - k_3(x_1 - x_2) + u$$
$$m_2\ddot{x}_2 = -k_2 x_2 - b_2\dot{x}_2 - k_3(x_2 - x_1)$$

Rewriting, we have

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1 x_1 + k_3 x_1 = k_3 x_2 + u$$
$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2 x_2 + k_3 x_2 = k_3 x_1$$

Assuming the zero initial condition and taking the Laplace transforms of these two equations, we obtsin

$$(m_1 s^2 + b_1 s + k_1 + k_3)X_1(s) = k_3 X_2(s) + U(s) \quad (1)$$

$$(m_2 s^2 + b_2 s + k_2 + k_3)X_2(s) = k_3 X_1(s) \quad (2)$$

By eliminating $X_2(s)$ from Equations (1) and (2), we get

$$(m_1 s^2 + b_1 s + k_1 + k_3)X_1(s) = \frac{k_3^2}{m_2 s^2 + b_2 s + k_2 + k_3} X_1(s) + U(s)$$

Hence

$$\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + b_2 s + k_2 + k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}$$

From Equation (2), we obtain

$$\frac{X_2(s)}{X_1(s)} = \frac{k_3}{m_2 s^2 + b_2 s + k_2 + k_3}$$

Hence

$$\frac{X_2(s)}{U(s)} = \frac{X_2(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)} = \frac{k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}$$

Problem 3:

The equations for the given circuit are as followw:

$$R_1 i_1 + L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = e_i$$

$$R_2 i_2 + \frac{1}{C}\int i_2\,dt + L\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) = 0$$

$$\frac{1}{C}\int i_2\,dt = e_o$$

Taking the Laplace transforms of these three equations, assuming zero initial conditions, gives

$$R_1 I_1(s) + L\left[sI_1(s) - sI_2(s)\right] = E_i(s) \tag{1}$$

$$R_2 I_2(s) + \frac{1}{Cs} I_2(s) + L\left[sI_2(s) - sI_1(s)\right] = 0 \tag{2}$$

$$\frac{1}{Cs} I_2(s) = E_o(s) \tag{3}$$

From Equation (2) we obtain

$$\left(R_2 + \frac{1}{Cs} + Ls\right) I_2(s) = Ls\, I_1(s)$$

or

$$I_2(s) = \frac{LCs^2}{LCs^2 + R_2Cs + 1} I_1(s) \tag{4}$$

Substituting Equation (4) into Equation (1), we get

$$\left(R_1 + Ls - Ls\,\frac{LCs^2}{LCs^2 + R_2Cs + 1}\right) I_1(s) = E_i(s)$$

or

$$\frac{LC(R_1 + R_2)s^2 + (R_1R_2C + L)s + R_1}{LCs^2 + R_2Cs + 1} I_1(s) = E_i(s) \tag{5}$$

From Equations (3) and (4), we have

$$\frac{Ls}{LCs^2 + R_2Cs + 1} I_1(s) = E_o(s) \tag{6}$$

From Equations (5) and (6), we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{Ls}{LC(R_1 + R_2)s^2 + (R_1R_2C + L)s + R_1}$$

Problem 4:

Since

$$R(s) = \frac{1}{s}$$

we have

$$Y(s) = \frac{4(s+50)}{s(s+20)(s+10)} .$$

The partial fraction expansion of $Y(s)$ is given by

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+20} + \frac{A_3}{s+10}$$

where

$$A_1 = 1 , \quad A_2 = 0.6 \text{ and } \quad A_3 = -1.6 .$$

Using the Laplace transform table, we find that

$$y(t) = 1 + 0.6e^{-20t} - 1.6e^{-10t} .$$

The final value is computed using the final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\left[\frac{4(s+50)}{s(s^2+30s+200)}\right] = 1 .$$

Problem 5:

$x_o = 1$

$y_o = 1 + 1.4(1)^3 = 2.4$

$$g(x) = y_o + \left.\frac{dy}{dx}\right|_{x_o}(x - x_o) = 2.4 + \left.(1.4 \times 3x^2)\right|_{x_o=1}(x-1)$$

$$\boxed{g(x) = 2.4 + 4.2(x-1)}$$

Problem 6:

$$R_1 i_1 + \frac{1}{C_1}\int i_1 dt + L_1 \frac{d(i_1 - i_2)}{dt} + R_2(i_1 - i_2) = v(t)$$

and

$$R_3 i_2 + \frac{1}{C_2}\int i_2 dt + R_2(i_2 - i_1) + L_1 \frac{d(i_2 - i_1)}{dt} = 0 \ .$$

Taking the Laplace transform and using the fact that the initial voltage across C_2 is 10v yields

$$[R_1 + \frac{1}{C_1 s} + L_1 s + R_2]I_1(s) + [-R_2 - L_1 s]I_2(s) = 0$$

and

$$[-R_2 - L_1 s]I_1(s) + [L_1 s + R_3 + \frac{1}{C_2 s} + R_2]I_2(s) = -\frac{10}{s} \ .$$

Rewriting in matrix form we have

$$\begin{bmatrix} R_1 + \frac{1}{C_1 s} + L_1 s + R_2 & -R_2 - L_1 s \\ -R_2 - L_1 s & L_1 s + R_3 + \frac{1}{C_2 s} + R_2 \end{bmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} 0 \\ -10/s \end{pmatrix}$$

Solving for I_2 yields

$$\begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} L_1 s + R_3 + \frac{1}{C_2 s} + R_2 & R_2 + L_1 s \\ R_2 + L_1 s & R_1 + \frac{1}{C_1 s} + L_1 s + R_2 \end{bmatrix} \begin{pmatrix} 0 \\ -10/s \end{pmatrix}$$

or

$$I_2(s) = \frac{-10(R_1 + 1/C_1 s + L_1 s + R_2)}{s\Delta}$$

where

$$\Delta = (R_1 + \frac{1}{C_1 s} + L_1 s + R_2)(L_1 s + R_3 + \frac{1}{C_2 s} + R_2) - (R_2 + L_1 s)^2 \ .$$

(you can use direct substitution instead of the matrix form)

Problem 7:

a. Cross multiplying, $(s^2+5s+10)X(s) = 7F(s)$.

Taking the inverse Laplace transform, $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 10x = 7f$.

b. Cross multiplying after expanding the denominator, $(s^2+21s+110)X(s) = 15F(s)$.

Taking the inverse Laplace transform, $\frac{d^2x}{dt^2} + 21\frac{dx}{dt} + 110x = 15f$.

c. Cross multiplying, $(s^3+11s^2+12s+18)X(s) = (s+3)F(s)$.

Taking the inverse Laplace transform, $\frac{d^3x}{dt^3} + 11\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 18x = df/dt + 3f$.

Problem 8:

Let $X_1(s)$ be the displacement of the left member of the spring and $X_3(s)$ be the displacement of the mass.

Writing the equations of motion

$$2X_1(s) - 2X_2(s) = F(s)$$
$$-2X_1(s) + (5s+2)X_2(s) - 5sX_3(s) = 0$$
$$-5sX_2(s) + (10s^2 + 7s)X_3(s) = 0$$

Thus, $\frac{X_2(s)}{F(s)} = \frac{1}{10}\frac{(10s+7)}{s(5s+1)}$

Problem 9:

$$(s^2+6s+9)X_1(s) - (3s+5)X_2(s) = 0$$
$$-(3s+5)X_1(s) + (2s^2+5s+5)X_2(s) = F(s)$$

Thus $G(s) = X_1(s)/F(s) = \frac{(3s+5)}{2s^4+17s^3+44s^2+45s+20}$

Problem 10:

$$x_0 = 0$$

$$y_0 = e^0 = 1$$

$$g(x) = y_0 + \left.\frac{dy}{dx}\right|_{x=0} (x - x_0)$$

$$= 1 + \left.e^x\right|_{x=0} (x - 0)$$

$$\boxed{g(x) = 1 + x}$$

Problem 11:

$$m_1 \ddot{y}_1 + b_1(\dot{y}_1 - \dot{y}_2) + k_1 y_1 = u_1$$

$$m_2 \ddot{y}_2 + b_1(\dot{y}_2 - \dot{y}_1) + k_2 y_2 = u_2$$

Define

$$x_1 = y_1, \quad x_2 = \dot{y}_1, \quad x_3 = y_2, \quad x_4 = \dot{y}_2$$

Then

$$m_1 \dot{x}_2 + b_1(x_2 - x_4) + k_1 x_1 = u_1$$

$$m_2 \dot{x}_4 + b_1(x_4 - x_2) + k_2 x_3 = u_2$$

Hence

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{1}{m_1}\left[b_1(x_2 - x_4) + k_1 x_1\right] + \frac{1}{m_1} u_1$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{1}{m_2}\left[b_1(x_4 - x_2) + k_2 x_3\right] + \frac{1}{m_2} u_2$$

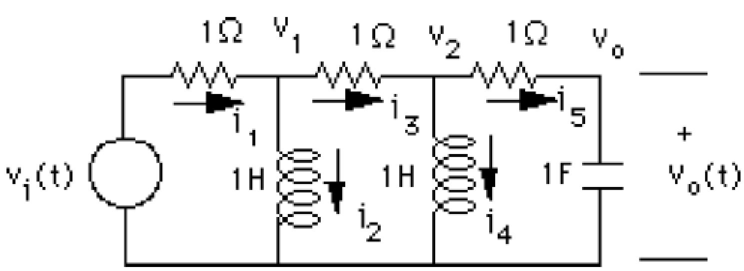
or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{b_1}{m_1} & 0 & \frac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b_1}{m_2} & -\frac{k_2}{m_2} & -\frac{b_1}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Problem 12:

Add the branch currents and node voltages to the network.



Write the differential equation for each energy storage element.

$$\frac{di_2}{dt} = v_1$$
$$\frac{di_4}{dt} = v_2$$
$$\frac{dv_o}{dt} = i_5$$

Therefore, the state vector is $\mathbf{x} = \begin{bmatrix} i_2 \\ i_4 \\ v_o \end{bmatrix}$

Now obtain v_1, v_2, and i_5 in terms of the state variables. First find i_1 in terms of the state variables.

$-v_i + i_1 + i_3 + i_5 + v_o = 0$

But $i_3 = i_1 - i_2$ and $i_5 = i_3 - i_4$. Thus,

$-v_i + i_1 + (i_1 - i_2) + (i_3 - i_4) + v_o = 0$

Making the substitution for i_3 yields

$$-v_i + i_1 + (i_1 - i_2) + ((i_1 - i_2) - i_4) + v_o = 0$$

Solving for i_1

$$i_1 = \frac{2}{3}i_2 + \frac{1}{3}i_4 - \frac{1}{3}v_o + \frac{1}{3}v_i$$

Thus,

$$v_1 = v_i - i_1 = -\frac{2}{3}i_2 - \frac{1}{3}i_4 + \frac{1}{3}v_o + \frac{2}{3}v_i$$

Also,

$$i_3 = i_1 - i_2 = -\frac{1}{3}i_2 + \frac{1}{3}i_4 - \frac{1}{3}v_o + \frac{1}{3}v_i$$

and

$$i_5 = i_3 - i_4 = -\frac{1}{3}i_2 - \frac{2}{3}i_4 - \frac{1}{3}v_o + \frac{1}{3}v_i$$

Finally,

$$v_2 = i_5 + v_o = -\frac{1}{3}i_2 - \frac{2}{3}i_4 + \frac{2}{3}v_o + \frac{1}{3}v_i$$

Using v_1, v_2, and i_5, the state equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

Problem 13:

$$\dddot{y} + 4\ddot{y} + 6\dot{y} + 8y = 20\,u(t)$$

3 states :

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \end{aligned}\right\} \quad \begin{aligned} \dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} = x_3 \\ \dot{x}_3 &= \dddot{y} = 20u(t) - 4\ddot{y} - 6\dot{y} - 8y \\ &= 20u(t) - 4x_3 - 6x_2 - 8x_1 \end{aligned}$$

$$y = x_1$$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} u(t)$$

$$[y] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\, u(t)$$

Problem 14:

a. Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -13 & -5 & -1 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 10 & 8 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

b. Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -8 & -13 & -9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 6 & 7 & 12 & 2 & 1 \end{bmatrix} \mathbf{x}$$

Problem 15:

a. $$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 10 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

$$\begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1} = \frac{1}{s(s+4)+3} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix} = \frac{1}{s^2+4s+3} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+4s+3} \begin{bmatrix} 10 & 0 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+4s+3} \begin{bmatrix} 10(s+4) & 10 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{10}{s^2+4s+3}}$$

b.

$G(s)=\mathbf{C}(s\mathbf{I}\text{-}\mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 3 & -5 & 2 \\ 1 & -8 & 7 \\ -3 & -6 & 2 \end{bmatrix}; \; \mathbf{B} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}; \; \mathbf{C} = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s^3+3s^2+19s-133} \begin{bmatrix} (s^2+6s+26) & -(5s+2) & (2s-19) \\ (s-23) & (s^2-5s+12) & (7s-19) \\ -(3s+30) & -(6s-33) & (s^2+5s-19) \end{bmatrix}$$

Therefore, $G(s) = \dfrac{23s^2-48s-7}{s^3+3s^2+19s-133}$.