Solution of Homework 3

Problem 1:

a. $C(s) = \frac{5}{s(s+5)} = \frac{1}{s} - \frac{1}{s+5}$. Therefore, $c(t) = 1 - e^{-5t}$.

Also, $T = \frac{1}{5}$, $T_r = \frac{2.2}{a} = \frac{2.2}{5} = 0.44$, $T_s = \frac{4}{a} = \frac{4}{5} = 0.8$.

b. $C(s) = \frac{20}{s(s+20)} = \frac{1}{s} - \frac{1}{s+20}$. Therefore, $c(t) = 1 - e^{-20t}$. Also, $T = \frac{1}{20}$,

$T_r = \frac{2.2}{a} = \frac{2.2}{20} = 0.11$, $T_s = \frac{4}{a} = \frac{4}{20} = 0.2$.

Program:

```
'(a)'
num=5;
den=[1 5];
Ga=tf(num,den)
subplot(1,2,1)
step(Ga)
title('(a)')
'(b)'
num=20;
den=[1 20];
Gb=tf(num,den)
subplot(1,2,2)
step(Gb)
title('(b)')
```

Problem 2:

Writing the equation of motion,

$$(Ms^2 + 6s)X(s) = F(s)$$

Thus, the transfer function is,

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + 6s}$$

Differentiating to yield the transfer function in terms of velocity,

$$\frac{sX(s)}{F(s)} = \frac{1}{Ms + 6} = \frac{1/M}{s + \frac{6}{M}}$$

Thus, the settling time, T_s, and the rise time, T_r, are given by

$$T_s = \frac{4}{6/M} = \frac{2}{3}M = 0.667M; \quad T_r = \frac{2.2}{6/M} = \frac{1.1}{3}M = 0.367M$$

Program:

```
Clf
M=1
num=1/M;
den=[1 6/M];
G=tf(num,den)
step(G)
pause
M=2
num=1/M;
den=[1 6/M];
G=tf(num,den)
step(G)
```

Problem 3:

a. Pole: -2; $c(t) = A + Be^{-2t}$; first-order response.

b. Poles: -3, -6; $c(t) = A + Be^{-3t} + Ce^{-6t}$; overdamped response.

c. Poles: -10, -20; Zero: -7; $c(t) = A + Be^{-10t} + Ce^{-20t}$; overdamped response.

d. Poles: $(-3+j3\sqrt{15})$, $(-3-j3\sqrt{15})$; $c(t) = A + Be^{-3t}\cos(3\sqrt{15}\ t + \phi)$; underdamped.

e. Poles: j3, -j3; Zero: -2; $c(t) = A + B\cos(3t + \phi)$; undamped.

f. Poles: -10, -10; Zero: -5; $c(t) = A + Be^{-10t} + Cte^{-10t}$; critically damped.

Problem 4:

The equation of motion is: $(Ms^2+f_vs+K_s)X(s) = F(s)$. Hence, $\frac{X(s)}{F(s)} = \frac{1}{Ms^2+f_vs+K_s} = \frac{1}{s^2+s+5}$.

The step response is now evaluated: $X(s) = \frac{1}{s(s^2+s+5)} = \frac{1/5}{s} - \frac{\frac{1}{5}s + \frac{1}{5}}{(s+\frac{1}{2})^2+\frac{19}{4}} =$

$$\frac{\frac{1}{5}(s+\frac{1}{2}) + \frac{1}{5\sqrt{19}}\frac{\sqrt{19}}{2}}{(s+\frac{1}{2})^2 + \frac{19}{4}} .$$

Taking the inverse Laplace transform, $x(t) = \frac{1}{5} - \frac{1}{5}e^{-0.5t}\left(\cos\frac{\sqrt{19}}{2}t + \frac{1}{\sqrt{19}}\sin\frac{\sqrt{19}}{2}t\right)$

$$= \frac{1}{5}\left[1 - 2\sqrt{\frac{5}{19}}\, e^{-0.5t}\cos\left(\frac{\sqrt{19}}{2}t - 12.92^\circ\right)\right].$$

Problem 5:

$$X(s) = \frac{100^2}{s(s^2+100s+100^2)} = \frac{1}{s} - \frac{s+100}{(s+50)^2+7500} = \frac{1}{s} - \frac{(s+50)+50}{(s+50)^2+7500} = \frac{1}{s} - \frac{(s+50)+\frac{50}{\sqrt{7500}}\sqrt{7500}}{(s+50)^2+7500}$$

Therefore, $x(t) = 1 - e^{-50t}\left(\cos\sqrt{7500}\ t + \frac{50}{\sqrt{7500}}\ \sin\sqrt{7500}\ t\right)$

$= 1 - \frac{2}{\sqrt{3}}\ e^{-50t}\cos\left(50\sqrt{3}\ t - \tan^{-1}\frac{1}{\sqrt{3}}\right)$

Problem 6:

a. $\omega_n^2 = 16$ r/s, $2\zeta\omega_n = 3$. Therefore $\zeta = 0.375$, $\omega_n = 4$. $T_s = \frac{4}{\zeta\omega_n} = 2.667$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$ 0.8472 s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 28.06$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1) = 1.4238$; therefore, $T_r = 0.356$ s.

b. $\omega_n^2 = 0.04$ r/s, $2\zeta\omega_n = 0.02$. Therefore $\zeta = 0.05$, $\omega_n = 0.2$. $T_s = \frac{4}{\zeta\omega_n} = 400$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$ 15.73 s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 85.45$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 5.26$ s.

c. $\omega_n^2 = 1.05 \times 10^7$ r/s, $2\zeta\omega_n = 1.6 \times 10^3$. Therefore $\zeta = 0.247$, $\omega_n = 3240$. $T_s = \frac{4}{\zeta\omega_n} = 0.005$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.001$ s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 44.92$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 3.88\text{x}10^{-4}$ s.

Problem7 :

$$\textbf{a. } \zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.56,\ \omega_n = \frac{4}{\zeta T_s} = 11.92. \text{ Therefore, poles} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$= -6.67 \pm j9.88$.

$$\textbf{b. } \zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.591,\ \omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = 0.779.$$

Therefore, poles $= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -0.4605 \pm j0.6283$.

c. $\zeta\omega_n = \frac{4}{T_s} = 0.571,\ \omega_n\sqrt{1-\zeta^2} = \frac{\pi}{T_p} = 1.047$. Therefore, poles $= -0.571 \pm j1.047$.

Problem 8:

$$\text{Re} = \frac{4}{T_s} = 4; \quad \zeta = \frac{-\ln(12.3/100)}{\sqrt{\pi^2 + \ln^2(12.3/100)}} = 0.5549$$

$$\text{Re} = \zeta\omega_n = 0.5549\omega_n = 4; \quad \therefore \omega_n = 7.21$$

$$\text{Im} = \omega_n\sqrt{1-\zeta^2} = 6$$

$$\therefore G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{51.96}{s^2 + 8s + 51.96}$$

Problem 9:

a. Writing the equation of motion yields, $(5s^2 + 5s + 28)X(s) = F(s)$

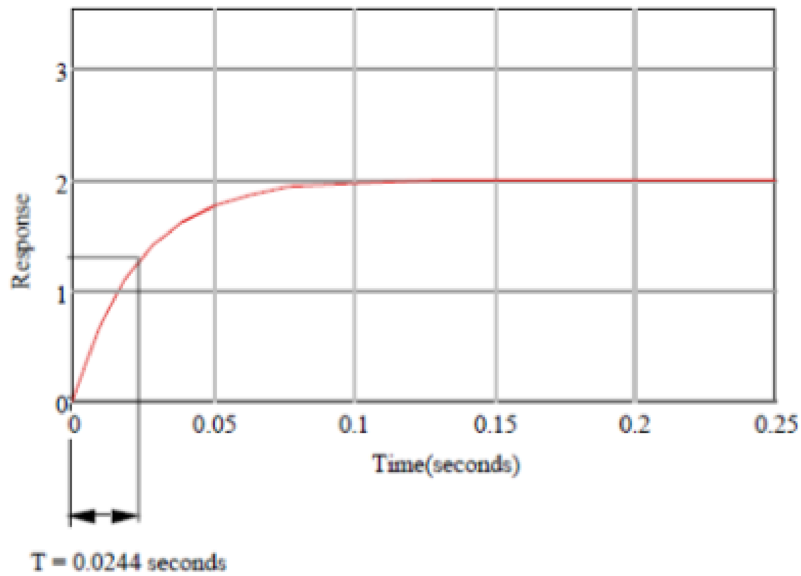
Solving for the transfer function,

$$\frac{X(s)}{F(s)} = \frac{1/5}{s^2 + s + \frac{28}{5}}$$

b. $\omega_n^2 = 28/5$ r/s, $2\zeta\omega_n = 1$. Therefore $\zeta = 0.211$, $\omega_n = 2.37$. $T_s = \frac{4}{\zeta\omega_n} = 8.01$ s; $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$ 1.36 s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 50.7\ \%$; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 0.514$ s.

Problem 10:

a. Measuring the time constant from the graph, $T = 0.0244$ seconds.

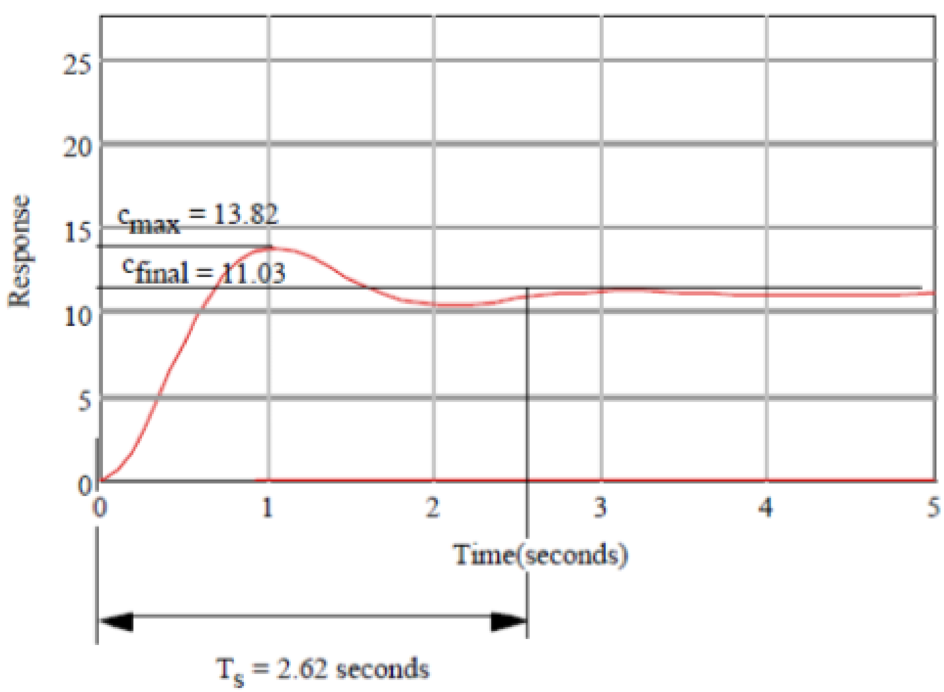


Estimating a first-order system, $G(s) = \frac{K}{s+a}$. But, $a = 1/T = 40.984$, and $\frac{K}{a} = 2$. Hence, $K = 81.967$.

Thus,

$$G(s) = \frac{81.967}{s+40.984}$$

b. Measuring the percent overshoot and settling time from the graph: $\%OS = (13.82-11.03)/11.03 = 25.3\%$,



and $T_s = 2.62$ seconds. Estimating a second-order system, we use Eq. (4.39) to find $\zeta = 0.4$, and Eq. (4.42) to find $\omega_n = 3.82$. Thus, $G(s) = \frac{K}{s^2+2\zeta\omega_n s+\omega_n^2}$. Since $C_{final} = 11.03$, $\frac{K}{\omega_n^2} = 11.03$. Hence,

K = 160.95. Substituting all values,

$$G(s) = \frac{160.95}{s^2+3.056s+14.59}$$

c. From the graph, %OS = 40%. Using Eq. (4.39), $\zeta = 0.28$. Also from the graph, $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 4$. Substituting $\zeta = 0.28$, we find $\omega_n = 0.818$.

Thus,

$$G(s) = \frac{K}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{0.669}{s^2+0.458s+0.669}.$$